

# **Week 11: Elliptic equations, non linearities and implicit methods**

**Solving an elliptic equation via relaxation, dealing with non linearities and use of implicit integration methods in time**

**Dr K Clough, Topics in Scientific computing, Autumn term 2023**

# Announcements

1. Coursework is out! I will say a few words now, but I will provide much less help than in the last example. I STRONGLY advise you to finish the labs and review the walkthrough videos as these will help a lot.
2. Please fill in the Module Experience Survey - I will give you some time at the end of the lecture.

# Plan for today

1. Classification of PDEs - revision and the higher dimensional case
2. Non linearities - revision of the ODE case and extension to PDEs
3. Implicit integration methods - revision of the ODE case and extension to PDEs
4. Solving elliptic equations as a boundary value problem via relaxation, the non linear Poisson equation as an example

# Classification of second order PDEs

Consider the most general second order PDE for 1 dependent variable with 2 independent variables:

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F = 0$$

The equation is classified based on the discriminant  $\Delta = B^2 - 4AC$ :

$\Delta < 0$  Elliptic

$\Delta = 0$  Parabolic

$\Delta > 0$  Hyperbolic

# The three kinds - elliptic, parabolic, hyperbolic

$$1. \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + S$$

$$2. \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$3. \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$$

Which is which?

# The three kinds - elliptic, parabolic, hyperbolic

1. 
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + S$$

*Parabolic - decaying and spreading out*

2. 
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

*Hyperbolic - wave-y*

3. 
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$$

*Elliptic - smooth deformation by source  $f$*

# What about higher dimensional systems?

$$\frac{\partial^2 u}{\partial t^2} - 2v \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} - 4 \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} = f(u, v)$$

$$\frac{\partial^2 v}{\partial t^2} - v \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + 3 \frac{\partial^2 u}{\partial y^2} + 6 \frac{\partial^2 v}{\partial y^2} = g(u, v)$$

Now have a coupled system of PDEs for 2 dependent variables (u,v), and 3 independent variables (t,x,y)...

# Formulation as a matrix first order system

First consider the case with only 2 independent variables, but several dependent ones. The goal is to get the equations into a first order form so that we have something that looks like:

$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \end{bmatrix} = \overbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}^M \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} E \\ F \end{bmatrix}$$



# Formulation as a matrix first order system

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$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \end{bmatrix} = \overbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}^M \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} E \\ F \end{bmatrix}$$

We will then be able to classify the PDE and check its well posedness by considering properties of the matrix  $M$ , specifically:

*If all eigenvalues of the characteristic matrix  $M$  are purely real, the system of PDEs is hyperbolic, whereas if some are imaginary then it is elliptic.*

# Formulation as a matrix first order system

The goal is to get the equations into a first order form so that we have something that looks like:

$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathbf{M} \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} E \\ F \end{bmatrix}$$

If all eigenvalues of the characteristic matrix  $M$  are purely real, the system of PDEs is hyperbolic.

- If a system of PDEs is hyperbolic, and the principal matrix does not have a complete set of eigenvectors, the system is called ***weakly hyperbolic***. Such systems are ***not well posed***.
- If a system of PDEs is hyperbolic, and the principal matrix has a complete set of eigenvectors, the system is called ***strongly hyperbolic***. Such systems are ***well posed***.
- If the principal matrix of a hyperbolic system of PDEs has all of its eigenvalues distinct, then the system is called ***strictly hyperbolic*** (*includes strongly hyperbolic systems*).
- If the principal matrix of the system of PDEs is hermitian, i.e.,  $M = (M^T)^*$ , then the system is called ***symmetric hyperbolic*** (*includes strictly hyperbolic systems*).

# The wave equation as an example

Consider trying to make the wave equation purely first order:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

We define:

$$v = \frac{\partial u}{\partial t} \quad \text{and} \quad \psi = \frac{\partial u}{\partial x}$$

What system of equations do we get?

# The wave equation as an example

Consider trying to make the wave equation purely first order:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

We define:

$$v = \frac{\partial u}{\partial t} \quad \text{and} \quad \psi = \frac{\partial u}{\partial x}$$

Then we obtain the following system:

$$\frac{\partial v}{\partial t} = c^2 \frac{\partial \psi}{\partial x}, \quad \frac{\partial \psi}{\partial t} = \frac{\partial^2 u}{\partial t \partial x} = \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial t} = v$$

How to write  
this in matrix  
form?

# The wave equation as an example

Then we obtain the following system:

$$\frac{\partial v}{\partial t} = c^2 \frac{\partial \psi}{\partial x}, \quad \frac{\partial \psi}{\partial t} = \frac{\partial^2 u}{\partial t \partial x} = \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial t} = v$$

Which we can write as:

$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c^2 \\ 0 & 1 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \\ \psi \end{bmatrix} + \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}$$

# The wave equation as an example

$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \\ \psi \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c^2 \\ 0 & 1 & 0 \end{bmatrix}}^M \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \\ \psi \end{bmatrix} + \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}$$

M is the characteristic matrix.

The eigenvalues are the solutions of  $\det(M - \lambda I) = 0$ , ie:

$$\lambda(\lambda^2 - c^2) = 0$$

What do we conclude?

# The wave equation as an example

$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \\ \psi \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c^2 \\ 0 & 1 & 0 \end{bmatrix}}^M \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \\ \psi \end{bmatrix} + \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}$$

M is the characteristic matrix.

The eigenvalues are the solutions of  $\det(M - \lambda I) = 0$

$\lambda = 0, \pm |c|$  so the equation is hyperbolic  
(also strictly hyperbolic, and therefore well posed)

# Note on higher numbers of independent variables

Now consider the case with  $d$  independent variables. The goal is to get the equations into a first order form so that we have something that looks like:

$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \end{bmatrix} = \sum_{i=1}^{i=d} \begin{bmatrix} A_i \end{bmatrix} \frac{\partial}{\partial x^i} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} E \\ F \end{bmatrix}$$

We now construct the characteristic matrix  $M$  as follows:

$$M = \sum_{i=1}^{i=d} A_i k^i \quad \text{where } k^i \text{ is a unit vector in some norm } |k^i| = 1$$

The classification then follows as in the previous  $d=1$  example.



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1. ~~Classification of PDEs – revision and the higher dimensional case~~
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4. Solving elliptic equations as a boundary value problem via relaxation, the non linear Poisson equation as an example

# Ordinary differential equations

What terms make this ODEs non linear?

$$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 + \sin(y) + y - 1 = \sin(t)$$

# Ordinary differential equations

Coefficient of the derivative is another derivative of  $y$ , not a function of  $t$ , so non linear

$$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 + \sin(y) + y - 1 = \sin(t)$$

This term is ok -  $t$  is the independent variable!

Sinusoid can be expressed as a power series in  $y$ , that is

$$\sin(y) = y - \frac{y^3}{3!} \dots \text{ so is non linear}$$

# Partial differential equations

For the general PDE:

$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \end{bmatrix} = \sum_{i=1}^{i=d} \begin{bmatrix} A_i \end{bmatrix} \frac{\partial}{\partial x^i} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} E \\ F \end{bmatrix}$$

The equation is:

- **Linear** if the components of  $A_i$  and E and F are just numbers or functions of the independent variables
- **Quasi linear** if the components of  $A_i$  and E and F include  $\vec{u}$
- Truly **non linear** if the components of  $A_i$  and E and F include derivatives of  $\vec{u}$

# The three kinds - linear, quasilinear, non linear

$$1. \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial T}{\partial y} \frac{\partial^2 T}{\partial x^2} + T \frac{\partial^2 T}{\partial y^2} + \sin(t)$$

$$2. \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + T \frac{\partial^2 T}{\partial y^2} + \sin(T) + t$$

Which is which?

$$3. \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = t^3 + t$$

# The three kinds - linear, quasilinear, non linear

1.  $\frac{\partial T}{\partial t} = \alpha \frac{\partial T}{\partial y} \frac{\partial^2 T}{\partial x^2} + T \frac{\partial^2 T}{\partial y^2} + \sin(t)$  *Non linear*

2.  $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + T \frac{\partial^2 T}{\partial y^2} + \sin(T) + t$  *Quasi linear*

3.  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = t^3 + t$  *Linear*

# Recall: Well posed problems

- Theorems in mathematics guarantee the (local) well-posedness of **linear and quasi-linear\*** **strongly hyperbolic\*** and **parabolic** PDEs.
- Elliptic PDEs do not admit a well-posed IVP. This does not (necessarily) mean they cannot be solved, just that another method may be required.



\*Now you know what these terms mean

# How to implement non linear equations?

In the tutorial you will solve the quasi-linear parabolic equation:

$$\frac{\partial T}{\partial t} = -\tau \left[ \frac{\partial^2 T}{\partial x^2} - e^{-\alpha T^2} \right]$$

Numerically this is simple! We just add the non linear part into time derivative that is used in `solve_ivp()` or any other method to integrate.



# Plan for today

1. ~~Classification of PDEs – revision and the higher dimensional case~~
2. ~~Non-linearities – revision of the ODE case and extension to PDEs~~
3. Implicit integration methods - revision of the ODE case and extension to PDEs (including non linear ones!)
4. Solving elliptic equations as a boundary value problem via relaxation, the non linear Poisson equation as an example

# Explicit versus implicit methods - ODEs

An explicit method is one where the variable we want at the next step  $y_{k+1}$  can be written explicitly in terms of quantities we know at the current step  $y_k, t_k$ , e.g.

$$y_{k+1} = y_k + h f(y_k, t_k) \quad \text{“forward Euler - explicit”}$$

Implicit methods will instead result in equations where we cannot easily isolate and solve for the quantity we want, e.g.

$$y_{k+1} = y_k + h f(y_{k+1}, t_{k+1}) \quad \text{“backward Euler - implicit”}$$

# How to implement implicit integration?

In the tutorial you will solve a quasi-linear parabolic equation:

$$\frac{\partial T}{\partial t} = -\tau \left[ \frac{\partial^2 T}{\partial x^2} - e^{-\alpha T^2} \right]$$

Since this is non linear, we cannot easily solve for the unknown function at the next timestep.

# Iterate and hope for the best...

## ***Algorithm:***

Initially use the last value as a first try:

$$T_{k+1}^{guess(0)} = T_k + h \left. \frac{\partial T}{\partial t} \right|_k$$

Now use this new guess as the value and repeat:

$$T_{k+1}^{guess(i)} = T_k + h \left. \frac{\partial T}{\partial t} \right|_{k+1}^{guess(i-1)}$$

I stop when:

$$| T_{k+1}^{guess(i)} - T_{k+1}^{guess(i-1)} | < \epsilon$$

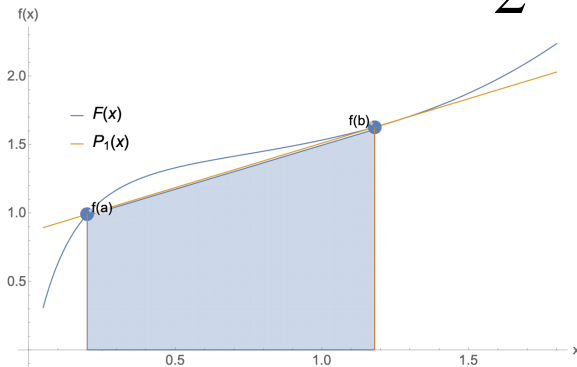
# Implicit methods

In the tutorial you will use the simplest one to integrate a parabolic equation:

$$y_{k+1} = y_k + h f(y_{k+1}, t_{k+1}) \quad \text{“implicit (backward) Euler method”}$$

Another one is the trapezoidal integration method we learned, applied to the derivatives:

$$y_{k+1} = y_k + \frac{h}{2} \left( f(y_{k+1}, t_{k+1}) + f(y_k, t_k) \right) \quad \text{“trapezoidal rule”}$$



$$\text{where } f(y, t) = \frac{\partial y}{\partial t}$$

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# How do we solve elliptic equations?

In the tutorial we want to solve the quasi-linear elliptic equation:

$$\frac{\partial^2 T}{\partial x^2} = e^{-\alpha T^2} \quad \text{subject to the boundary conditions}$$
$$T(x = 0) = 0 \text{ and } T(x = L) = 0$$

This is a boundary value problem, since we are told the values at the start and end of the interval of interest.

How can we solve this?

# How do we solve elliptic equations?

What if instead we solve:

$$\frac{\partial T}{\partial t} = \tau \left[ \frac{\partial^2 T}{\partial x^2} - e^{-\alpha T^2} \right] \text{ subject to the boundary conditions}$$

$T(x = 0) = 0 \text{ and } T(x = L) = 0 ?$

How do we know what to use as  
an initial condition?

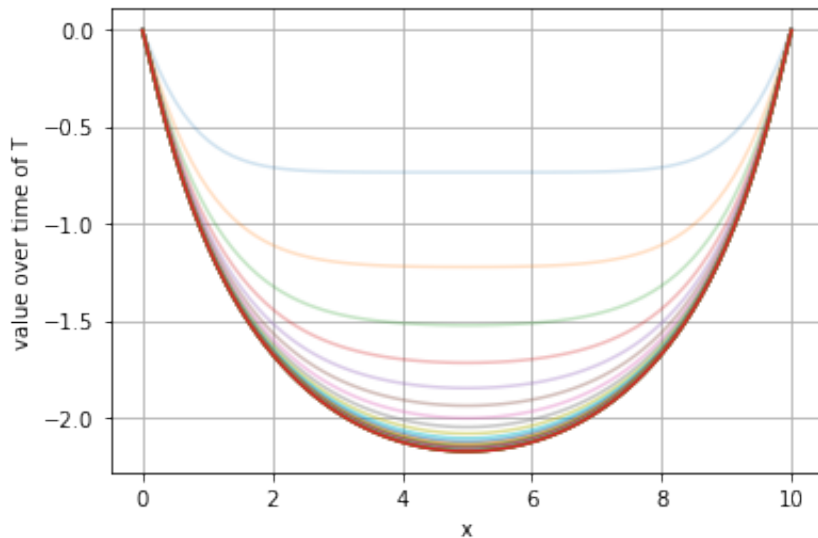


# How do we solve elliptic equations?

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The solution “relaxes” into the solution ,

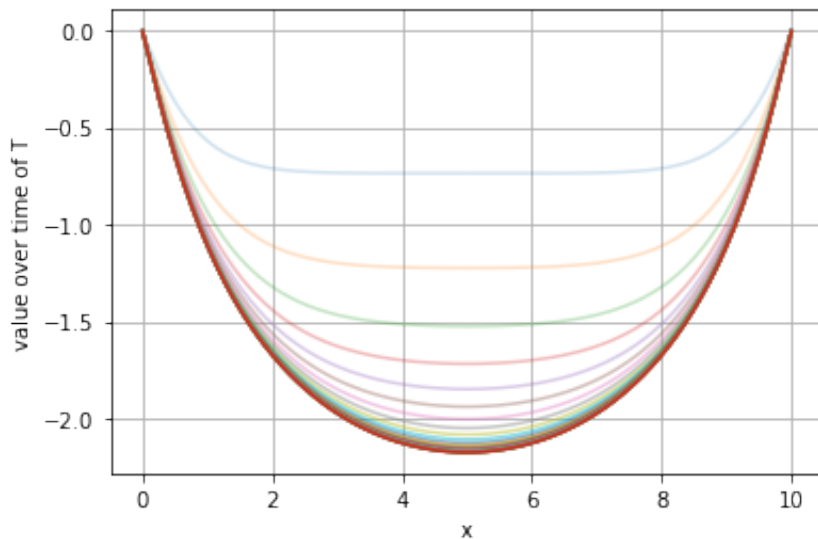
When  $\frac{\partial T}{\partial t} = 0$ , the original equation is solved

# How do we solve elliptic equations?

If instead we solve:

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The solution “relaxes” into the solution,

When  $\frac{\partial T}{\partial t} = 0$ , the original equation is solved

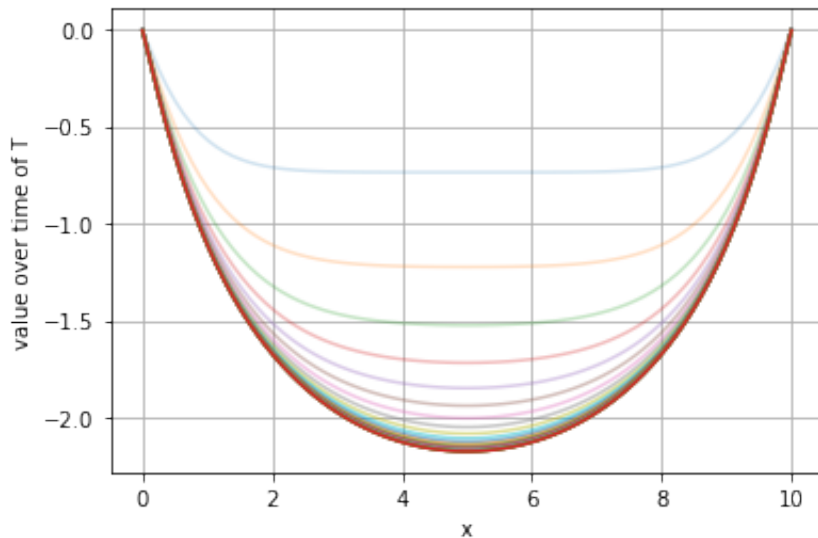
What is the role of  $\tau$ ?

# How do we solve elliptic equations?

If instead we solve:

$$\frac{\partial T}{\partial t} = \tau \left[ \frac{\partial^2 T}{\partial x^2} - e^{-\alpha T^2} \right] \text{ subject to the boundary conditions}$$

$$T(x = 0) = 0 \text{ and } T(x = L) = 0$$



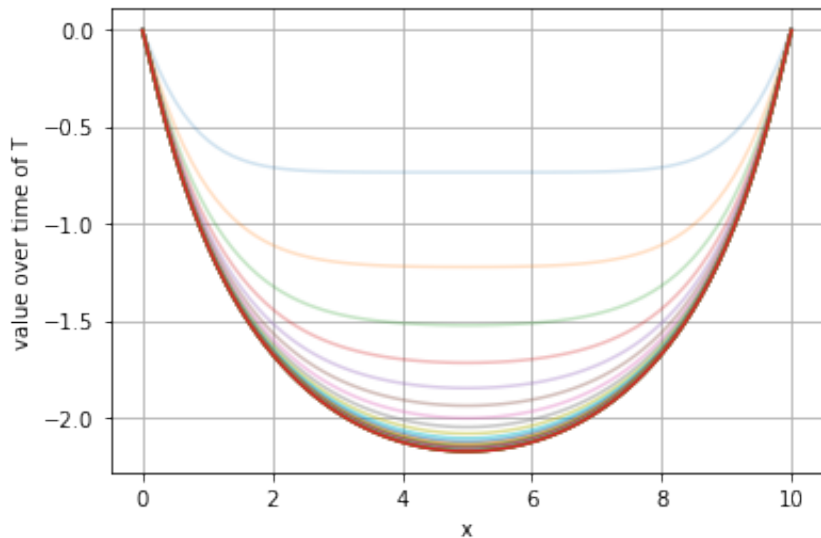
The parameter  $\tau$  is an (unphysical) time parameter that controls the speed at which the solution is approached. However, by the same stability arguments as the CFL condition, it cannot be made arbitrarily large.

# How do we solve elliptic equations?

If instead we solve:

$$\frac{\partial T}{\partial t} = \tau \left[ \frac{\partial^2 T}{\partial x^2} - e^{-\alpha T^2} \right] \text{ subject to the boundary conditions}$$

$$T(x = 0) = 0 \text{ and } T(x = L) = 0$$



This will work provided that the initial guess is “sufficiently close” to the actual solution.

This method is highly inefficient, but can be made more efficient using multigrid methods.

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***You now know everything you need to complete the coursework, well done, and good luck!***

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