Lec 8

Revisit motivating example

Gaussian Elimination

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

"zero" the elements below diagonal in 1st column using elementary row operations:

$$\begin{array}{c} \\ R_2 - \frac{1}{1}R_1 \\ \\ \end{array} \begin{array}{ccc|c} \overset{\text{PIVOT}}{\boxed{1}} & -1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0. \end{array}$$

zero elements below diagonal in 2nd column:

$$\begin{array}{c} \\ \\ R_3 - \frac{1}{-2}R_2 \end{array} \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & \overset{\text{PIVOT}}{\boxed{-2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

Backward Substitution:

System of equations is now in triangular form and be solved by backward substitution:

R_3: $0 \cdot c = 0$ (nothing to learn here)

R_2: $-2b + 0 \cdot c = 0 \Rightarrow b = 0$

R_1: $1 \cdot a - 1 \cdot (0) + 1 \cdot c = 0$

$\Rightarrow a = -c$

... as before.

<u>Example</u> (zero pivot elements)

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -8 \\ 2 & -2 & 3 & -20 \\ 1 & 1 & 1 & -2 \end{array}\right]$$

pivot for column 2 = 0!

$$\begin{array}{l} \\ R_2 - \frac{2}{1}R_1 \\ R_3 - \frac{1}{1}R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 8 \\ 0 & \boxed{0} & -1 & -4 \\ 0 & 2 & -1 & 6 \end{array}\right]$$

switch/exchange

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 8 \\ 0 & 2 & -1 & 6 \\ 0 & 0 & -1 & -4 \end{array}\right]$$

General procedure for Gaussian Eliminat$^{\underline{n}}$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ & & \vdots & & b_2 \\ & & & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} & b_j \\ & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

Provided $a_{11} \neq 0$, do:

$$R_j \leftarrow R_j - \frac{a_{j1}}{a_{11}} R_1$$

yielding a <u>new</u> augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & & & & \\ | & a_{j2} & \cdots & a_{jn} & b_j \\ 0 & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

Note: entries in rows 2–n are expected to have changed.

Repeat, ie. for $i = 2, \ldots, n$:

if $a_{ii} == 0$ $\quad R_i \leftrightarrow R_j$ where j is first index after i s.t. $a_{ji} \neq 0$.

$$R_j \leftarrow R_j - \frac{a_{ji}}{a_{ii}} R_i \qquad j = i+1, \dots, n.$$

Note 1: a_{ji}/a_{ii} is called a "multiplier".

Note 2: OPTIONAL: multipliers can be stored in the lower part of the upper-triangular matrix that results from Gaussian (Forward) elimination, in case more than one system $Ax=b$ needs to be solved w/ same matrix A.

General procedure for Backsubstitution:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$+ a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$$
$$a_{n,n}x_n = b_n$$

$$a_{nn}x_n = b_n$$

$$a_{n-1,n-1}x_{n-1} = b_{n-1} - a_{n-1,n}x_n$$

$$\vdots$$

$$a_{11}x_1 = b_1 - a_{12}x_2 - \cdots - a_{1n}x_n$$

ie.

$$x_i = \frac{1}{a_{ii}}\left[b_i - \sum_{j=i+1}^{n} a_{ij}\, x_j\right]$$

for $i = n, n-1, \ldots, 2, 1$.

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cf. Algorithm 6.1 in the TEXT.