Lec 10

When Gaussian Elim^n fails:

RANK DEFICIENCY

Consider the systems:

$$x_1 + x_2 + x_3 = 4 \qquad\qquad x_1 + x_2 + x_3 = 4$$
$$2x_1 + 2x_2 + x_3 = 6 \qquad\qquad 2x_1 + 2x_2 + x_3 = 4$$
$$x_1 + x_2 + 2x_3 = 6, \qquad\qquad x_1 + x_2 + 2x_3 = 6$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 2 & 6 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - \frac{2}{1}R_1 \\ R_3 - \frac{1}{1}R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{l} \\ R_2 - \frac{2}{1}R_1 \\ R_3 - \frac{1}{1}R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

- $x_3 = 2$
- x_2 = undetermined
- $x_1 + x_2 + x_3 = 4$

  $x_1 = 2 - x_2$

$\Rightarrow$ infinitely many sol^{n}s.

Contradict:
- $x_3 = 2$
- $x_3 = 4$, x_2 unknown
- $x_1 + x_2 + x_3 = 4$

$\Rightarrow$ no sol^n.

Either way, there is no unique sol^n.

This is a symptom of the fact that the A in $Ax = b$ is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

which is "rank-deficient".

Recall that the rank of a matrix is the dimension of the vector space spanned by its row vectors. This space is also spanned by row vectors resulting from elementary row operations.

i.e. the "row space" of A is spanned by the vectors:

$$[1\ 1\ 1],\ [0,0,-1],\ [0,0,1].$$

Since the last two vectors are multiples of each other, the dim. of the row space is 2. Since $2<3$, A is "rank-deficient" (instead of "full rank", which would occur if $\dim(\text{row}(A))=3$).

Another result in linear algebra is that (square) rank-deficient matrices are singular (non-invertible). This explains the two examples above: if A were invertible, then

$$A^{-1}\begin{bmatrix}4\\6\\6\end{bmatrix} \quad \text{and} \quad A^{-1}\begin{bmatrix}4\\4\\6\end{bmatrix}$$

would be the unique sol^{n}s.

Summary When pivot AND ALL ELEMENTS BELOW IT are zero, this is a strong indication that no unique sol^{n} exists. We therefore chose to stop Gaussian elimination when this occurs.

Similar situation occurs if $a_{nn} = 0$ after Gaussian Eliminat$^{\underline{n}}$:

$$\begin{matrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{matrix}$$

$$\Rightarrow \quad \begin{aligned} 0x_3 &= b_3 \\ -x_2 - x_3 &= b_2 \\ x_1 + x_2 + x_3 &= b_1 \end{aligned}$$

<u>case 1</u>: $b_3 = 0$

$\Rightarrow x_3 = \alpha$ (anything)

$x_2 = -b_2 - \alpha$

$x_1 = b_1 - (-b_2 - \alpha) - \alpha$

$= b_1 + b_2$

<u>case 2</u> $b_3 \neq 0$

$\Rightarrow$ no sol$^{\underline{n}}$

Either way, there is no unique sol$^{\underline{n}}$; again because A is rank-deficient.

SMALL PIVOT ELEMENTS

The algo we have described can still run into trouble, even if the pivot is not zero, but merely small.

For example, consider the system:

$$\varepsilon x_1 + x_2 = 1$$
$$x_1 + x_2 = 2 \qquad \varepsilon \text{ small}, \ \varepsilon \neq 0$$

<u>algo</u>:

$$\left[\begin{array}{cc|c} \varepsilon & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]$$

$$\begin{array}{r} \\ R_2 - \frac{1}{\varepsilon}R_1 \end{array} \left[\begin{array}{cc|c} \varepsilon & 1 & 1 \\ 0 & 1-\frac{1}{\varepsilon} & 2-\frac{1}{\varepsilon} \end{array}\right]$$

$$x_2 = \frac{2-\frac{1}{\varepsilon}}{1-\frac{1}{\varepsilon}} \quad (*) \longrightarrow$$

<u>analytics</u>

$$x_2 \simeq 1 \text{ if } \varepsilon \ll 1$$

$$\varepsilon x_1 + x_2 = 1$$

$$x_1 = \frac{1-x_2}{\varepsilon}. \quad (**) \longrightarrow \quad x_1 = \frac{1-\left[\frac{2-1/\varepsilon}{1-1/\varepsilon}\right]}{\varepsilon}$$

$$= \frac{1}{\varepsilon} \cdot \frac{1-1/\varepsilon-2+1/\varepsilon}{1-1/\varepsilon}$$

$$= \frac{-1}{\varepsilon-1} \sim 1 \ (\varepsilon \ll 1)$$

But how would the computer evaluate x_1, x_2 when $\varepsilon \ll 1$?

Suppose 3-digit precision and $\varepsilon = 10^{-4}$.

Let's compute x_1, x_2 according to $(*), (**)$:

First, we need to compute

$$\frac{1}{\varepsilon} = 10^4 \overset{\text{mach}}{=} 0.100 \times 10^5$$
$$2 \overset{\text{mach.}}{=} 0.200 \times 10^1$$

$$
\begin{aligned}
2 - \frac{1}{\varepsilon} &= (0.00002 - 0.10000) \times 10^5 \\
&= -0.09998 \times 10^5 \\
&\overset{\text{mach.}}{=} -0.100 \times 10^5 \\
&= -\frac{1}{\varepsilon}
\end{aligned}
$$

Similarly

$$1 - \frac{1}{\varepsilon} \overset{\text{mach.}}{=} -\frac{1}{\varepsilon}$$

$(*) \Rightarrow$ $x_2 \overset{\text{mach}}{=} 1$, which is a good approx

However,

$$1 - x_2 \overset{\text{mach}}{=} 1 - 1 = 0$$

$$\varepsilon \overset{\text{mach}}{=} 10^{-4} \neq 0.$$

(**) $\Rightarrow x_1 \overset{\text{mach}}{=} 0$ BAD! (correct $x_1 \simeq 1$).

$\downarrow$

100% relative error