Lecture 4 Fixed-point iteration

Ex Fixed pts of $g(x) = 1+\sqrt{x}$, $x \geq 0$

Sol^n

$$g(x) = x$$

$$\Rightarrow \quad 1+\sqrt{x} = x$$

Put $y = \sqrt{x}$, then we have

$$1+y = y^2$$

$$\Rightarrow \quad y^2 - y - 1 = 0$$

$$\Rightarrow \quad y = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

But $y = \sqrt{x} \Rightarrow y > 0 \Rightarrow$

$$y = \frac{1+\sqrt{5}}{2}$$

$$\Rightarrow \quad x = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \text{fixed pt. of } g(x).$$

Theorem 1

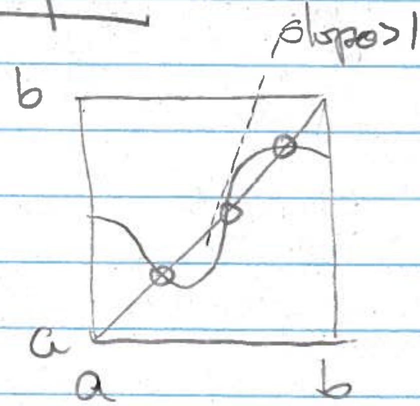
1. If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one fixed pt.

2. If, in addition, $g'(x)$ exists on (a,b) and there is a $0<K<1$ with

$$|g'(x)| \leq K \qquad \forall x \in (a,b)$$

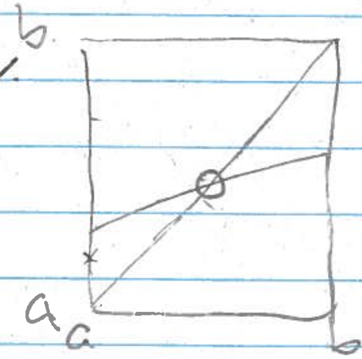
then there is exactly one fixed pt in $[a,b]$.

Examples

1/ 

3 fixed pts

Note: |slope| is <u>not</u> everywhere less than 1.

2/ 

unique fixed pt

Note: |slope| is everywhere < 1.

Pf of Thm 1

1. If $g(a)=a$ or $g(b)=b$, then g has a fixed pt at it's endpoints.

If $g(a)\neq a$ & $g(b)\neq b$, since $g(x)\in[a,b]$, then

$$g(a)>a\ ,\quad g(b)<b$$

$$\Rightarrow\quad g(a)-a>0\ ,\quad g(b)-b<0$$

$\Rightarrow$ $h(x)=g(x)-x$ satisfies $h(a)h(b)<0$.

$\Rightarrow$ $\exists\ p$ s.t. $h(p)=0$ by I.V.T.

(Since 0 lies between $h(a)$ and $h(b)$)

$\Rightarrow$ $\exists\ p$ st. $g(p)=p$.

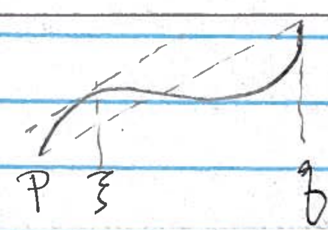
2. We are given that $|g'(x)|\leq K<1$.

Suppose that fixed pt. not unique

$\Rightarrow$ $\exists\ p\neq q$ st. $g(p)=p$, $g(q)=q$

Then by M.V.T $\exists\ \xi$ st

$$\frac{g(p)-g(q)}{p-q}=g'(\xi)$$



$\Rightarrow$ ie $|p-q| = |g(p)-g(q)|$

ie $\qquad = |g'(\xi)|\,|p-q|$

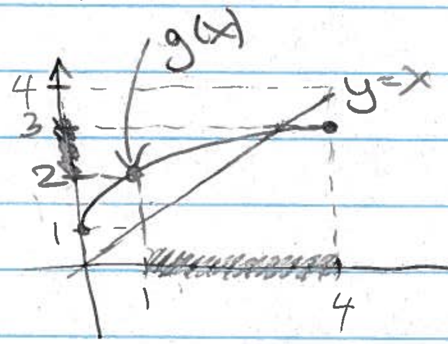
$\qquad < |p-q|$

which is a contradiction.

Ex Uniqueness of fixed pt. of $g(x) = 1+\sqrt{x}$ on $[1,4]$.

<u>Sol^n</u>

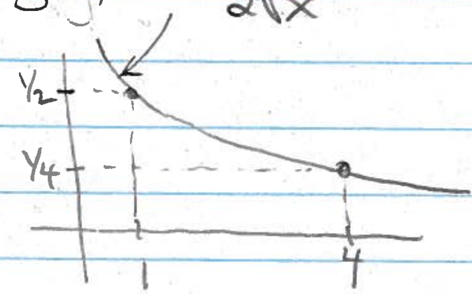
(1)



$\Rightarrow g(x) \in [1,4]$ when $x \in [1,4]$

(2) $g'(x) = \dfrac{1}{2\sqrt{x}}$

$\Rightarrow$



$\Rightarrow |g'(x)| < 1$

Theorem 2

Let $g \in C[a,b]$, $g(x) \in [a,b]$, g' exists, and $\exists$ $0 < K < 1$ st

$$|g'(x)| \leq K$$

for $x \in (a,b)$. Then, for any $p_0 \in [a,b]$ the sequence

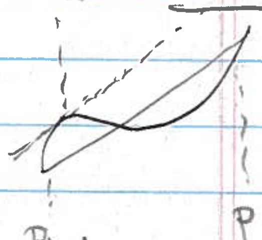
$$p_n = g(p_{n-1}), \quad n \geq 1$$

converges to p, the (unique) fixed pt. of g in $[a,b]$.

Pf:

$$|p_n - p| = |g(p_{n-1}) - g(p)|$$

ξ_n MVT!



$$\Rightarrow \frac{g(p_{n-1}) - g(p)}{p_{n-1} - p} = g'(\xi_n)$$

$$= |g'(\xi_n)|\,|p_{n-1} - p|, \quad \xi_n \in [p_{n-1}, p] \text{ or } [p, p_{n-1}]$$

$$\leq K\,|p_{n-1} - p|.$$

$$\leq K \cdot K\,|p_{n-2} - p|$$

$$\vdots$$

$$\leq K^n\,|p_0 - p|.$$

$$\leq K^n\,(b-a)$$

Since $K<1$, we get:

$$\lim_{n\to\infty} |p_n - p| = 0.$$

or $\quad p_n = p + O(K^n) \quad , K<1$

Ex Find the zeros of $f(x) = x^4 + 2x^2 - x - 3$
Use fixed pt iteration.

Sol^n:

Method 1 $\quad f(x) = 0$

$$\Rightarrow x^4 + 2x^2 - x - 3 = 0.$$

$$\Rightarrow x^4 + 2x^2 - 3 = x.$$

$$\Rightarrow g_1(x) = x$$

where $g_1(x) = x^4 + 2x^2 - 3.$

Method 2: $\quad f(x) = 0$

$$\Rightarrow x^4 + 2x^2 - x - 3 = 0.$$

$$\Rightarrow x^4 = 3 + x - 2x^2$$

$$\Rightarrow x = (3 + x - 2x^2)^{1/4} = g_2(x)$$

Derivatives

Meth 1

$g_1' = 4x^3 + 4x$

$\Rightarrow g_1'(1) = 8$

$\Rightarrow |g_1'(p)| \simeq 8 > 1.$

Meth 2

$g_2' = \frac{1}{4}(3 + x - 2x^2)^{-3/4}(1 - 4x)$

$g_2'(1) = \frac{1}{4}(2)^{-3/4}(-3)$

$\simeq -0.446$

$\Rightarrow |g_2'(p)| < 1.$