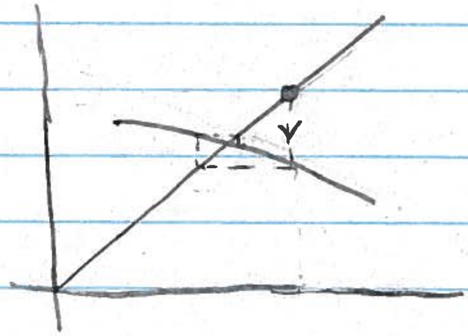
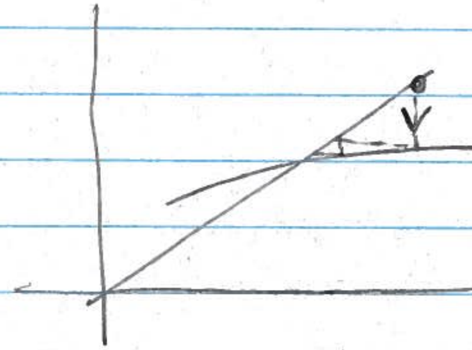
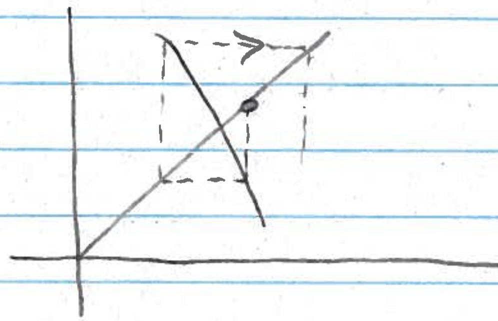
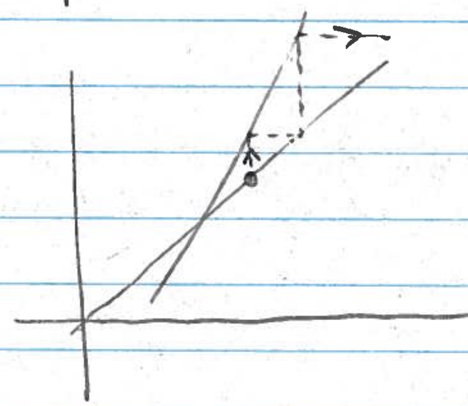
Lec 5 **Newton Iteration**

Recall: fixed pt. iteration

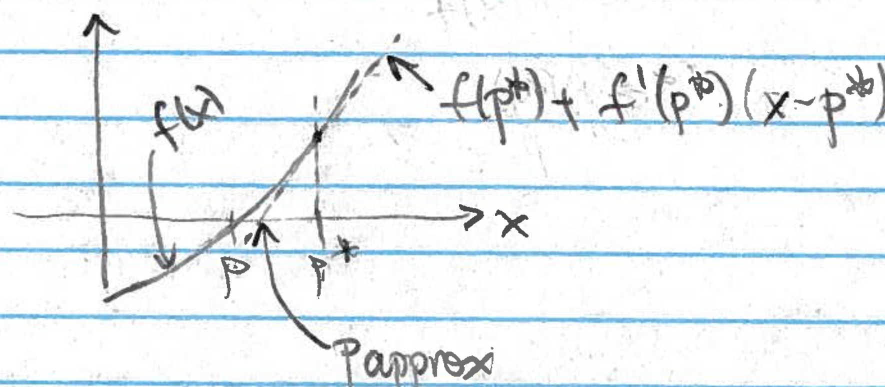
stable
$|\text{slope}| < 1$

unstable
$|\text{slope}| > 1$

Motivation for Newton:



$$f(p) = f(p^*) + f'(p^*)(p - p^*)$$

$$\uparrow$$

$$0$$

$$+ \frac{1}{2} f''(\xi(p^*)) (p - p^*)^2$$

$\hookrightarrow$ neglect if $p^* \simeq p$

ie. $0 \simeq f(p^*) + f'(p^*)(p - p^*)$

or $0 = f(p^*) + f'(p^*)(P_{approx} - p^*)$

Solve for P_{approx}:

$$f'(p^*) P_{approx} = f'(p^*) p^* - f(p^*)$$

$$\Rightarrow \quad P_{approx} = p^* - \frac{f(p^*)}{f'(p^*)}$$

This relation inspires the Newton Method, which is:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

Proposition:

Given: $g(p)=p$

$|g'(p)| = P < 1.$

W.T.S.: (1) $|g'(x)| \leq K < 1$ if $x \in [p-\delta, p+\delta]$

(2) g maps $[p-\delta, p+\delta]$ onto itself for some p.

for some δ.

Pf

(1) $|g'(x)|$ is cts $\Rightarrow$

Given $\varepsilon > 0$ $\exists\, \delta$ s.t.

$$\big|\, |g'(x)| - P \,\big| < \varepsilon \qquad (*)$$

when

$|x - p| \leq \delta.$

Choose $\varepsilon = (1-P)/2$

$x \in [p-\delta, p+\delta]$

Then

$(*) \Rightarrow \quad P - \frac{1-P}{2} \leq |g'(x)| \leq P + \frac{1-P}{2}$

$\Rightarrow \quad |g'(x)| \leq \frac{1+P}{2} < 1 \quad (**)$

(2) $\exists\ \xi(p)$ s.t.

$$g(x) = \underbrace{g(p)}_{p} + g'(\xi(p))(x-p)$$

$$\Rightarrow \quad |g(x)-p| = |g'(\xi(p))|\,|x-p|$$

$$< |x-p| \quad \text{by } (**)$$

$$< \delta \quad \text{since } |x-p| < \delta$$

ie. $g(x) \in [p-\delta, p+\delta]$.

Apply proposition to Newton:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$\Rightarrow \quad g(p) = p - \frac{f(p)}{f'(p)}$$

ie. $g(p) = p$ (a)

Also:

$$g' = 1 - \frac{f'}{f'} + \frac{ff''}{(f')^2}$$

$$= \frac{ff''}{(f')^2}$$

$$\Rightarrow \quad g'(p) = 0 \quad (\text{since } f(p)=0)$$

in particular: $|g'(p)| < 1$. (b)

$$(a),(b) \xrightarrow[\text{prop}^{\underline{n}}]{} \left\{ \begin{array}{l} |g'| < 1 \\ g: [a,b] \to [a,b] \end{array} \right\} \xrightarrow[\text{Lec 4}]{\text{Thm 2}} \left\{ \begin{array}{l} \text{Newton} \\ \text{converges} \\ \text{to zero} \\ \text{of } f(x) \\ \text{in some} \\ \text{interval} \end{array} \right.$$

Moreover: rate of convergence is (by Thm 2, Lec 4):

$$g'(x) \propto f(x) \to 0 \text{ as } x \to p.$$

ie. speed of convergence increases!