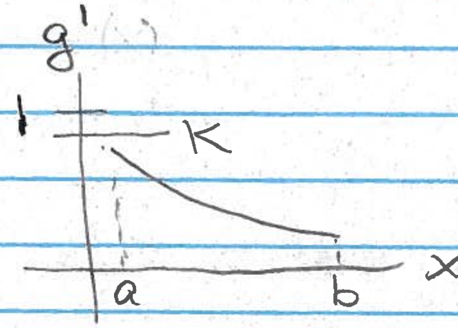
Lec 6 Convergence -1-

Recall : Bisection :

$$p_n = p + O(\tfrac{1}{2^n})$$

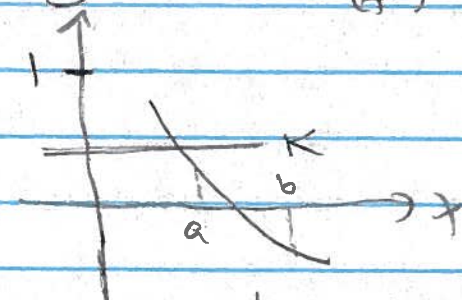
Fixed-point :



$$p_n = p + O(K^n)$$

Newton :

$$g' = ff''/(f')^2$$



$$p_n = p + O(K^n)$$

make K smaller by making [a,b] smaller.

We may say that Newton is a fixed-point scheme that converges faster than an arbitrary fixed-point scheme.

In this lecture, we will develop a different spin on this fundamental fact.

Consider:

$$p_n \to p.$$

$$\frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lambda$$

Case 1: $\alpha = 1 \quad \lambda < 1$

$$|p_{n+1}-p| < |p_n - p|.$$

This says that p_{n+1} is closer to p than p_n, which is good.

Case 2: $\alpha = 2 \quad \lambda < 1$

$$|p_{n+1}-p| < |p_n-p|^2 \ll |p_n-p|$$

if n so large that $|p_n-p| \ll 1$.

So: $|p_{n+1}-p| \ll |p_n-p|$

which is even better than Case 1!

<u>Thm 2</u> (special fixed-pt scheme)

Let $p = (a+b)/2$ be a fixed pt of $g \in C^2[a,b]$.

Let $g'(p) = 0$ and $|g'(x)| \leq K < 1$ in $[a,b]$

Then, for $p_0 \in [a,b]$, $p_n = g(p_{n-1})$ <u>converges</u>(1) @ least <u>quadratically</u>(2) to p with asymptotic rate constant

$$\lambda = \frac{1}{2}|g''(p)|$$

Pf

(1) W.T.S. $p_n \to p$. Need to show: $g: [a,b] \to [a,b]$

Let $x \in [a,b]$.

M.V.T $\Rightarrow \exists$ $x < \xi < p$ s.t.

$$g(x) = g(p) + g'(\xi)(x-p)$$
$$= \underbrace{p + g'(\xi)(x-p)}_{RHS}$$

Now:

$$-|z| \leq z \leq |z|$$

is always true. In particular,

$$-|g'(\xi)||x-p| \leq g'(\xi)(x-p) \leq |g'(\xi)||x-p|$$

Add $p = \frac{a+b}{2}$ to get

$$\frac{a+b}{2} - |g'(\xi)||x-p| \leq g(x) \leq \frac{a+b}{2} + |g'(\xi)||x-p|$$

Now $|x-p| \leq \frac{b-a}{2}$ (and equivalently $-\frac{b-a}{2} \leq -|x-p|$) so that

$$\frac{a+b}{2} - \frac{b-a}{2} \leq g(x) \leq \frac{a+b}{2} + \frac{b-a}{2}$$

or $\quad a \leq g(x) \leq b$

or $\quad x \in [a,b] \Rightarrow g(x) \in [a,b]$.

Thus $p_n \to p$.

(2) Taylor $\Rightarrow$

$$g(x) = g(p) + g'(p)(x-p) + \tfrac{1}{2} g''(\xi)(x-p)^2$$

or $$g(x) = p + \tfrac{1}{2} g''(\xi)(x-p)^2 \qquad (*)$$

where $\quad \xi \in [x,p]$ or $[p,x]$.

Choose $x = p_n$ $(\xi = \xi_n)$ and rewrite $(*)$

$$\frac{|p_{n+1}-p|}{|p_n-p|^2} = \frac{1}{2}|g''(\xi_n)|$$

Then, because $\xi_n \to p$, we have

$$\frac{|p_{n+1}-p|}{|p_n-p|^2} \to \frac{1}{2}|g''(p)|.$$