Demystifying Structural Equation Modeling

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Contents

Introduction

Welcome to our book on structural equation modeling!

[THIS BOOK IS A WORK IN PROGRESS. FEEL FREE TO PERUSE WHATEVER CONTENT YOU FIND HERE, BUT THE FINAL VERSION WILL NOT BE READY UNTIL SOMETIME IN 2023.]

If you want, you can also download this book as a PDF or EPUB file. Be aware that the print versions are missing some of the richer formatting of the online version.

Some history

In 2016, Jonathan and Sean embarked upon a bold experiment, asking the question, "Is is possible to teach structural equation modeling (SEM) to undergraduates with little statistical background?" To make things even more exciting, we attempted to do so in a special topics course lasting only one month during our May Term at Westminster College (Salt Lake City, UT).

In such an endeavor, we had to temper our expectations, of course. The goal was not to produce competent practitioners who would subsequently go on to do serious research using SEM techniques. We were quite happy that, at the end of May, we had undergraduates who were able to put together a simple final project that required them to find some data, posit a model, fit the model in R, interpret the output, and check a few model fit statistics. Some exposure to the topic and some appreciation of its power were satisfying enough. In fact, we think we got a little more out of it than that: we are reasonably confident that most of our students had developed—at that point right after taking the course—the ability to read a research article with an SEM model and have at least some idea what the article was talking about. We called it a win!

We repeated the experiment with some modifications to our materials and pedagogy in 2018. By that point, it was clear that finding textbooks and articles to assign to students was challenging. There are some great books out there,

but they are mostly aimed at graduate students. Even the ones labeled "introductory" were often far from that for the typical undergraduate with limited statistical training.

We decided that we could write our own textbook that would fill this hole in the literature. The book that follows is the fruit of our efforts.

Sean was granted sabbatical in Spring 2020 and proposed to use that time to start writing the book in preparation for running the May Term course again in May, 2020. And, well, we all know how that went…

Once the pandemic subsided enough for us to offer the course in person again, we attempted it again in May, 2022. [TO BE CONTINUED]

Our philosophy

As we mentioned before, our motivation for writing the book was driven by the difficulty we had finding readings for the students. Perhaps that begs the question, should one even try teaching a topic as "difficult" or "advanced" as structural equation modeling to the audience we had? To be sure, the traditional approaches already on the market seem to assume a lot more background than we had at our disposal. And the books that claim to assume less background…well, sometimes they require more than they let on.[1]

The prerequisite for our class at Westminster College is an intro stats class that covers pretty standard material for such a course: hypothesis testing and confidence intervals for one and two proportions, one and two means (and paired means), ANOVA, chi-squared, and simple linear regression. In some technical sense, very little of that material is truly required to understand our book. Having said that, though, some prior exposure to statistical ideas is helpful for motivating a rationale for building the kinds of models we teach in our course.

We also benefit in that our intro course introduces students to R. For those lacking R background, the first five chapters here [FIX THIS LINK ONCE THE DATA 220 BOOK IS ALSO FULLY ONLINE] should suffice as a basic introduction to R and R Markdown, graphing with `ggplot`, and some basic `tidyverse` stuff about tibbles and data manipulation. We try hard to give lots of fully worked-out code examples in this book that students should be able to copy, paste, and modify slightly to meet their own modeling needs. But know that we make no attempt to be language agnostic here; R is the one and only tool we use.

To respect our students, we made some very deliberate choices about the way our book would be structured.

[1] "Oh, you don't know anything about matrix algebra or maximum likelihood fitting algorithms? No problem. Go read three or four pages in an appendix and then you'll be 'prepared' to read this book."

- *Make the book free and open source.*

Students have enough trouble in their lives without being subjected to the extortionate practices of most textbook publishers. Not only is this book freely available online, it's also published under a permissive open source license (the MIT license) that allows folks to "use, copy, modify, merge, publish, distribute, sublicense, and/or sell" their own versions of the book as desired. Furthermore, any derivative of the book must also abide by the same open standards. So our book is both *libre* and *gratis* (or, in more common parlance, "free as in speech" and "free as in beer").

- *Start from scratch.*

Explain everything from the beginning in terms that are as simple as possible. Some of the first few chapters may look like review for students. Even if it is, of course, that review gives students confidence to tackle upcoming new material. But you might be surprised at some of the novel ways we explain seemingly familiar concepts. All the exposition has an eye toward direct application in later chapters, so what might seem a little idiosyncratic at first is motivated by a desire to smooth the pathways into later concepts.

- *Incorporate active learning into everything.*

The chapters are structured to work as templates for classroom experiences. They intersperse conceptual explanation with activities designed to reinforce those concepts and lead students to important conclusions. These learning activities will appear framed in blue boxes [CHANGE THIS IF WE ESTABLISH A CUSTOM CALLOUT] like this:

Hey, kids! Stop and do this activity here!

- *Do the math and do it well.*

One common thread we see in a lot of SEM books is a tendency to sweep most of the math under the rug. The intention comes from a good place; mathematics can appear intimidating and, therefore, may seem to serve as a deterrent to learning. To be sure, there are some complex mathematical ideas in SEM that are inaccessible to our audience. At the same time—and, in fairness, this may be due to Sean's bias as a mathematician—we truly believe that the mathematics, carefully explained and continually reinforced, can illuminate student understanding. The more mathy sections may need additional instructor support for students without a strong math background. But all it takes is some relatively straightforward algebra to nail down some concepts that most books ignore. A good example of this is investing time in the rules for manipulating variances

and covariances. This allows students to calculate the "model-implied matrix" that is only cryptically referenced in most textbooks. However, we do skip the math sometimes. For example, a lot of the math behind model fit indices is left unexplained. At the very least, we hope to be transparent about our choices to include or exclude certain mathematical details.

- *Use "nice" data.*

Finding data is hard, so we rely a lot on data sets that other textbooks and R package authors make available (with due attribution, of course). To keep things simple for this course, we work almost exclusively with cross-sectional, numerical (quantitative) data. [MODIFY THIS IF WE END UP WORKING WITH BINARY CATEGORICAL EXOGENOUS VARIABLES (CODED 0/1) AT SOME POINT.]

- *Be careful about diagrams.*

Learning about complex models induces a sizable cognitive load. Shortcuts in diagrams tend to confuse students. For example, if error terms are truly latent variables, they should be drawn as circles and not hidden, even if an advanced practitioner "knows" they're there. Variances and covariances among exogenous variables should always appear as well. We take the time to build up a consistent pictographic representation of every part of a model. (Each chapter is introduced with an archetypal diagram that illustrates that chapter's content.) Then we stick to that representation throughout the book.

- *Be careful about notation.*

While it may be the industry standard, LISREL notation is needlessly complex for undergraduate students. We take a consistent and simple approach to notation that represents all variables using UPPERCASE names and all parameter values using lowercase names. Abstract variables tend to be called something like X when exogenous and Y when endogenous. Real-world variables have contextually meaningful names. For those interested in reading the research literature, we have included an appendix describing LISREL notation.

Course structure

We use this book to teach a 2-credit-hour course. (Even though it's a special topics course in our May Term, the number of contact hours for students is equivalent to a semester-long, 2-credit-hour course.)

[ADD INFO HERE AS WE DECIDE HOW MUCH IS REASONABLE TO COVER. IF WE WANT THE BOOK TO BE USABLE IN A 4-CREDIT-HOUR COURSE, WHAT ADDITIONAL MATERIAL SHOULD WE CONSIDER INCLUDING?]

Onward and upward

We hope you enjoy our textbook. Please send us your feedback!

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–Sean Raleigh (sraleigh@westminstercollege.edu)

Chapter 1

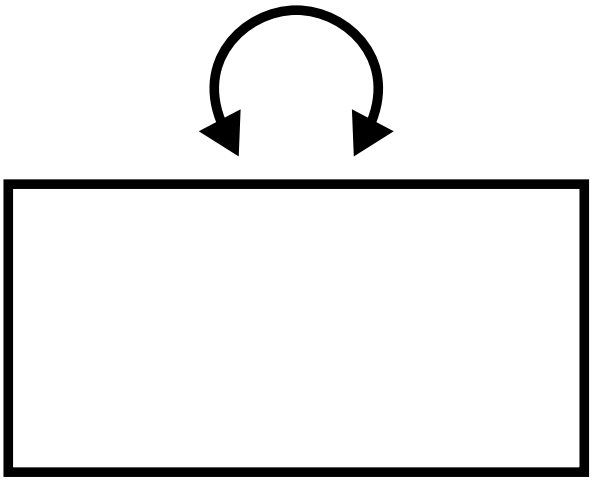
Variables and measurement

1.1 First section

[SOMEWHERE NEED TO MENTION "CONSTANT" VARIABLES, OR VARIABLES THAT TAKE ONLY ONE VALUE.]

Chapter 2

Variance



2.1 A quick refresher on the mean

Most of us were taught how to calculate the mean of a variable way back in elementary school: add up all the numbers and divide by the size of the group of numbers. In a statistics context, we often use a "bar" to indicate the mean of a variable; in other words, if a variable is called X, the mean is denoted $\overline{X}$. Remembering that we always use n to represent the sample size, the formula is

$$\overline{X} = \frac{\sum X}{n}$$

(In case you forgot, the Greek letter Sigma Σ stands for "sum" and means "add up all values of the thing that follows".)

Here is a small data set we'll use throughout this chapter as a simple example we can work "by hand":

3, 4, 5, 6, 6, 7, 8, 9

Calculate the mean of this set of eight numbers.

2.2 Calculating variance

Variance is a quantity meant to capture information about how spread out data is.

Let's build it up step by step.

The first thing to note about spread is that we don't care how large or small the numbers are in any absolute sense. We only care how large or small they are *relative to each other*.

Look at the numbers from the earlier exercise:

3, 4, 5, 6, 6, 7, 8, 9

What if we had the following numbers instead?

1003, 1004, 1005, 1006, 1006, 1007, 1008, 1009

Explain why any reasonable measure of "spread" should be the same for both groups of numbers.

One way to measure how large or small a number is relative to the whole set is to measure the distance of each number to the mean.

Recall that the mean of the following numbers is 6:

3, 4, 5, 6, 6, 7, 8, 9

Create a new list of eight numbers that measures the distance between each of the above numbers and the mean. In other words, subtract 6 from each of the above numbers.

Some of the numbers in your new list should be negative, some should be zero, and some should be positive. Why does that make sense? In other words, what does it mean when a number is negative, zero, or positive?

If the original set of numbers is called X, then what you've just calculated is a new list $\left(X - \overline{X}\right)$. Let's start organizing this into a table:

X	$\left(X - \overline{X}\right)$
3	-3
4	-2
5	-1

X	$(X - \overline{X})$
6	0
6	0
7	1
8	2
9	3

The numbers in the second columns are "deviations" from the mean.

One way you might measure "spread" is to look at the average deviation. After all, if the deviations represent the distances to the mean, a set with large spread will have large deviations and a set with small spread will have small deviations.

Go ahead and take the average (mean) of the numbers in the second column above.

Uh, oh! You should have calculated zero. Explain why you will always get zero, no matter what set of numbers you start with.

The idea of the "average deviation" seems like it should work, but it clearly doesn't. How do we fix the idea?

Hopefully, you identified that having negative deviations was a problem because they canceled out the positive deviations. But if all the deviations were positive, that wouldn't be an issue any more.

There are two ways of making numbers positive:

- Taking absolute values

We could just take the absolute value and make all the values positive. There are some statistical procures that do just that,[1] but we're going to take a slightly different approach…

- Squaring

If we square each value, they all become positive.

Taking the absolute value is conceptually easier, but there are some historical and mathematical reasons why squaring is a little better.[2]

Square each of the numbers from the second column of the table above. This will calculate a new list $\left(X - \overline{X}\right)^2$

[1] This leads to the "mean absolute deviation" or MAD.

[2] If you know calculus, you might think why the square function is much better behaved than the absolute value function.

Putting the new numbers into our previous table:

X	$(X - \overline{X})$	$(X - \overline{X})^2$
3	-3	9
4	-2	4
5	-1	1
6	0	0
6	0	0
7	1	1
8	2	4
9	3	9

Now take the average (mean) of the numbers in the third column above.

The number you got (should be 3.5) is *almost* what we call the variance. There's only one more annoying wrinkle.

When you took the mean of the last column of numbers, you added them all up and divided by 8 since there are 8 numbers in the list. But for some fairly technical mathematical reasons, we actually don't want to divide by 8. Instead, we divide by one less than that number; in other words, we divide by 7.[3]

Re-do the math above, but divide by 7 instead of dividing by 8.

The number you found is the *variance*, written as $Var(X)$. The full formula is

$$Var(X) = \frac{\sum (X - \overline{X})^2}{n - 1}$$

As a one-liner, the formula may look a little intimidating, but when you break it down step by step as we did above, it's not so bad.

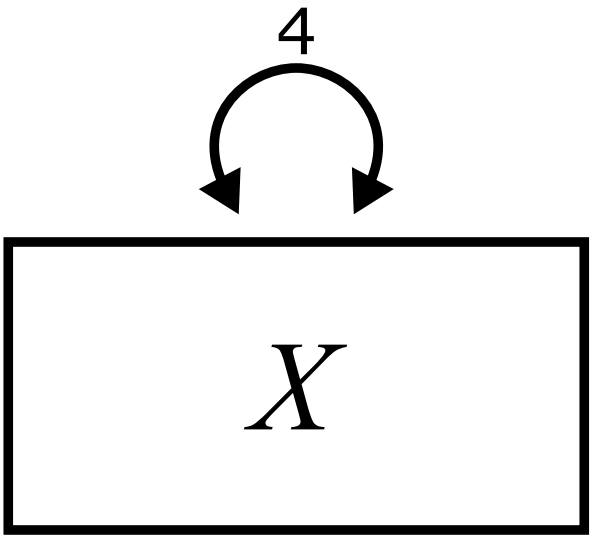
Here is the full calculation in the table:

X	$(X - \overline{X})$	$(X - \overline{X})^2$
3	-3	9
4	-2	4
5	-1	1
6	0	0
6	0	0
7	1	1
8	2	4
9	3	9

[3]For more information on that, search the internet for "sample variance unbiased"

X	$(X - \overline{X})$	$(X - \overline{X})^2$
		Sum: 28
		Variance:
		$28/7 = \boxed{4}$

In our diagrams, the variance of a variable is indicated by a curved, double-headed arrow, labeled with the value of the variance, like this:



Using the tabular approach, calculate the variance of the following set of numbers:

4, 3, 7, 2, 9, 4, 6

Consider the following two sets of numbers:

A) 1, 2, 5, 8, 9

B) 1, 4, 5, 6, 9

Without doing any calculations, which of the sets has the larger variance?

Once you've decided, then calculate the variance for both sets and check your answer.

2.3 Calculating variance in R

Once we've done it by hand a few times to make sure we understand how the formula works, from here on out we can let R do the work for us:

```
X1 <- c(3, 4, 5, 6, 6, 7, 8, 9)
var(X1)
```

```
## [1] 4
```

```
X2 <- c(4, 3, 7, 2, 9, 4, 6)
var(X2)
```

```
## [1] 6
```

```
X3 <- c(1, 2, 5, 8, 9)
var(X3)
```

```
## [1] 12.5
```

```
X4 <- c(1, 4, 5, 6, 9)
var(X4)
```

```
## [1] 8.5
```

This is also easier for real-world data that is not highly engineered to produce whole numbers:

```
PlantGrowth$weight
```

```
##  [1] 4.17 5.58 5.18 6.11 4.50 4.61 5.17 4.53 5.33 5.14 4.81 4.17 4.41 3.59 5.87
## [16] 3.83 6.03 4.89 4.32 4.69 6.31 5.12 5.54 5.50 5.37 5.29 4.92 6.15 5.80 5.26
```

```
var(PlantGrowth$weight)
```

```
## [1] 0.49167
```

2.4 Variance rules

In this course, we will need to be able to calculate the variance of various combinations of variables. For example, if X_1 and X_2 are two variables, we can create a new variable $X_1 + X_2$ by adding up the values of the two variables. What is the variance of $X_1 + X_2$?

But before we answer that, let's establish the first rule.

- **Rule 1**

Suppose that C is a "constant" variable, meaning that it always has the same value (rather than being a variable that could contain lots of different numbers). Then,

$$Var(C) = 0$$

Why is **Rule 1** true? You can either reason through this conceptually, based on how you understand what variance is supposed to measure, or you can do a sample calculation. (Make a table starting with a column that contains many copies of only a single number and work through the calculation.)

Now, back to the example at the beginning of the section of finding the variance of $X_1 + X_2$.

- **Rule 2**

If X_1 and X_2 are independent, then

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2)$$

We're not going to get into a formal definition of *independence* here. For now, it suffices to think of the intuitive definition you may already have in your head of what it means for two things to be independent. The idea is that, to be independent, X_1 and X_2 should have nothing to do with each other. Knowing a value of one variable should not give you any information about values of the other. In the next chapter, we'll say more about this rule.

It's important to note that **Rule 2** is an abstract mathematical rule that holds *in theory*. When we have actual data, however, we know that statistics won't always match their theoretical values. For example, even if a true population mean is 42, samples drawn from that population will have sample means that are *close* to 42, but likely not exactly 42.[4]

[4]The exact distribution of sample means around a true population value is something you probably learned about in an intro stats course. Sample means follow a Student t distribution.

Let's test this out. Below, we will define two new variables using random numbers.

A quick note about random numbers first: when we ask a computer to give us random numbers, it's not going to give us *truly* random numbers. The algorithms are designed to give us numbers that have all the correct statistical properties of random numbers without actually being random. These are called *pseudo-random* numbers. We can use this fact to our benefit. The `set.seed` command below tells the computer to start generating numbers in a very specific way. Anyone else using R (the same *version* of R) who gives their machine the same "seed" will generate the same list of numbers. This makes our work "reproducible": you will be able to reproduce the results here in this book on your own machine.

The variable `X5` below is normally distributed with mean 1 and standard deviation 2. (If you don't remember standard deviation from intro stats, we talk about it in the next section.) The next variable `X6` is normally distributed with mean 4 and standard deviation 3. These are independent because the definition of `X5` does not depend on any way on the definition of `X6` and vice versa.

The sample sizes (2000) are large enough that we should get pretty close to the theoretically correct results here.

```
set.seed(10101)
X5 <- rnorm(2000, mean = 1, sd = 2)
X6 <- rnorm(2000, mean = 4, sd = 3)
```

```
head(X5)
```

```
## [1] -0.7535339 -0.4927789  3.7518296  1.4751639  1.2172549  3.4054426
```

```
head(X6)
```

```
## [1] 2.297279 4.856377 6.661822 1.309892 2.270882 3.827944
```

Use R to calculate the variances of `X5` and `X6` separately. Then use R to add the two numbers you just obtained (the sum of the two variances). Finally, use R to calculate the variance of the sum of the two variables.

Here's an example to help think about this intuitively.

Suppose someone comes along and offers to give you a random amount of money, some number between \$0 and \$100.[5] If the variance is a measure of spread, then it stands to reason that variance reflects something about how uncertain you

[5] To be more concrete, the values are uniformly distributed, meaning that any number between 0 and 100 is equally likely.

are about how much money you will have after this transaction. On average, you expect about \$50, but you know that the actual amount of money you will receive can vary greatly.

Okay, now a second person comes along and offers you the same deal, a random dollar gift between \$0 and \$100.[6] At the end of both transactions, how much money will you have? On average, maybe about \$100, but what about your uncertainty? Because the total amount is the result of two random gifts, you are even less sure how close to \$100 you might be. The range of possible values is now \$0 to \$200.[7] Your uncertainty is greater overall.

Of course, all this explains is why the variance of the sum of two variables is larger than the variance of either variable individually. The fact that the variance of the sum of two independent variables is *exactly* the sum of the variances has to be shown mathematically. But hopefully, the intuition is clear.

The next rule is a consequence of the first two rules, so we will not give it a special number

$$Var\left(X+C\right)=Var\left(X\right)$$

Can you apply **Rule 2** followed by **Rule 1** to see mathematically why $Var\left(X+C\right)=Var\left(X\right)$?

This assumes that a constant is independent of any other variable? Intuitively speaking, why is this true?

What is the intuition behind the statement $Var\left(X+C\right)=Var\left(X\right)$? In other words, can you explain the rule to someone in terms of what it means about shifting the values of a data set up or down by a constant amount?

Rule 3 is similar to **Rule 2**, but it's quite counter-intuitive:

- **Rule 3**

If X_1 and X_2 are independent, then

$$Var\left(X_1-X_2\right)=Var\left(X_1\right)+Var\left(X_2\right)$$

It is very common for students to think that a minus sign on the left would translate into a minus sign on the right.[8]

[6] Apparently you live in a town with very generous strangers.

[7] To be clear, though, the probabilities are no longer uniform between 0 and 200. To get near 0, you would have to be unlucky twice, and to get near 200 you would have to get lucky twice. But there are lots of possible outcomes that would result in you having around 100.

[8] This results from many years of developing a Pavlovian response to anything that looks like the distributive law from algebra.

What gives?

Let's return to our example of strangers giving you money.[9] The first person still offers you a random amount between \$0 and \$100. But, now, the second person is a robber, and forces you to give them a random dollar value between \$0 and \$100 (of their choosing, of course). How much money do you expect to have after these two events? On average, \$0. (The first person gives you, on average, \$50, and the second person takes away, on average, \$50.) But how certain are you about that amount?

Imagine a world in which the wrong rule prevailed. What if $Var(X_1 - X_2)$ were truly the difference of the two variances. But $Var(X_1)$ and $Var(X_2)$ are the same in this scenario. (Although one person is giving money and one is taking, our uncertainty about the dollar amount is the same in both cases.) And this implies

$$Var(X_1) - Var(X_2) = 0$$

Can this be true? Zero variance means "no spread" which means exact certainty of the value. (Remember **Rule 1**?) Are you 100% confident that you will end both transactions with exactly \$0? No way!

In fact, the amount of money you end up with ranges from -\$100 up to \$100. This is a larger range than in either transaction individually. Our uncertainty has grown because there are two random processes in play, just like in the scenario with two beneficent strangers. In fact, the width of the range of possibilities is the same in both scenarios: \$0 to \$200 and -\$100 to \$100 both span a range of \$200.

The next rule, unfortunately, does not have a great intuitive explanation. It will make a little more sense in the next chapter, and we'll revisit it then.

- **Rule 4**

If a is any number,

$$Var(aX) = a^2 Var(X)$$

If you go back to the table, imagine multiplying every number in the first column by a. Every number in the second column will still have a factor of a. But when you square those values, every number in the third column will have a factor of a^2. That's the gist of the rule anyway. But, again, there's not much intuition about why that makes sense.

We can, at least, check empirically that the rule works.

We'll use X_5 as we defined it above, a normally distributed variable with mean 1 and standard deviation 2. The variance of the data is about 4:

[9] Actually, that sounds a little creepy when put like that.

```
var(X5)
```

```
## [1] 4.15763
```

Let's use $a = 3$.

In R, calculate $Var\left(3X_5\right)$. (Don't forget that in R, you can't just type `3 X5`. You have to explicitly include the multiplication sign: `3 * X5`.)

Now try calculating $3Var\left(X_5\right)$. You'll see that you don't get the right answer.

But now try $9Var\left(X_5\right)$. That should work.

And that's all the variance rules we'll need!

2.5 Standard deviation

The variance is nice because it obeys all the above rules. The one big downside is that it's not very interpretable.

For example, think of the scenario with people giving/taking money. In that case, the values were measures in units of dollars.

If X is measured in dollars, what are the units of measurement of $\overline{X}$? That seems sensible, right?

What are the units of $\left(X - \overline{X}\right)$? Still sensible, right? (It's not a problem that some of these values will be positive and other negative. Negative dollars still make sense. Just think about your student loans.)

Okay, now here's where things get weird. What are the units of $\left(X - \overline{X}\right)^2$? This no longer makes sense.

Variance is *nearly* the average of a bunch of squared deviations, so for a variable measured in dollars, the units of variance would be "squared dollars", whatever that is.

Variances are not really interpretable directly. How do we make them more interpretable? Well, if variance has "squared" units, we can take the square root to get back to the natural units we started with.

And this is called the standard deviation, $SD(X)$.

$$SD(X) = \sqrt{\frac{\sum \left(X - \overline{X}\right)^2}{n-1}}$$

Or, said more simply,

$$SD(X) = \sqrt{Var(X)}$$

Equivalently,

$$Var(X) = SD(X)^2$$

Often, if more concise notation is required, we write s_X for $SD(X)$.

$$s_X = \sqrt{Var(X)}$$
$$Var(X) = s_X^2$$

Due to its interpretability, an intro stats class will focus far more on the standard deviation than on the variance. The downside is that the mathematical rules aren't so nice for standard deviations. For example, what is

$$SD\left(X_1 + X_2\right)?$$

You can work through the definition to see that

$$SD\left(X_1 + X_2\right) = \sqrt{SD\left(X_1\right)^2 + SD\left(X_2\right)^2}$$

But, eww, that's gross.

The constant multiple rule works out nice, though.

For any number a, what is $SD(aX)$? Finish the following calculation until you can simplify it and get back something involving just $SD(X)$:

$$SD(aX) = \sqrt{Var(aX)} \tag{2.1}$$
$$= \quad ??? \tag{2.2}$$

Be careful! What happens if a is a negative number? Standard deviations (like variances) should always be non-negative.

A convenient way to express the fact that the coefficient will always come out positive is the following:

$$SD(aX) = |a|\, SD(X)$$

For SEM, we will focus almost exclusively on variance and switch to standard deviation for only two reasons:

1. We need to communicate something about spread in meaningful units.

2. We need to standardize variables. (See Section 2.7 below.)

Although the standard deviance is just the square root of the variance, it is worth knowing the R command to calculate it. It's just `sd`. For example:

```
sd(PlantGrowth$weight)
```

```
## [1] 0.7011918
```

You can see below that `sd` did the right thing:

```
sqrt(var(PlantGrowth$weight))
```

```
## [1] 0.7011918
```

2.6 Mean centering data

Many of the statistical techniques taught in an intro stats course focus on learning about the means of variables. Structural equation modeling is a little different in that it is more focused on the explaining the variability of data—how changes in one or more variables predict changes in other variables.[10]

A habit we'll start forming now is to mean center all our variables. We do this by subtracting the mean of a variable from all its values.

Let's use X_6 as we defined it before, a normally distributed variable with mean 4 and standard deviation 3. How do we interpret the values of $X_6 - \overline{X_6}$? (Remember, this is just the second column in our variance tables earlier.)

If we shift all the X_6 values to the left by $\overline{X_6}$ units, what is the mean of the new list of numbers?

Let's verify this in R. We'll use the "suffix" `mc` to indicate a mean-centered variable.

```
X6_mc <- X6 - mean(X6)
mean(X6_mc)
```

```
## [1] 2.851573e-16
```

[10]There are tools in SEM for working with means as well. WILL WE COVER THIS IN A FUTURE CHAPTER?

Why does this answer not exactly agree with the "theoretical" answer you came up with in a few lines above? (If you don't already know, the `e-16` in the expression above is scientific notation and means "times 10^{-16}. That's a really small number!)

Take a guess about the variance of `X6_mc`. Verify your guess in R.

So the good news is that **mean centering preserves the variance**. While the mean will be shifted to be 0, the variance does not change, so any statistical model we build that analyzes the variance will not be affecting by mean-centering.

2.7 Standardizing data

After we've mean centered the data, we can go one step further and divide by the standard deviation. This results in something often called a *z-score*. The process of converting variables from their original units to z-scores is called *standardizing* the data.

$$Z = \frac{(X - \overline{X})}{SD(X)}$$

What happens if you try to standardize a variable that is constant? (Hint: think about the denominator of the fraction defining the z-score.)

Why is it useful to standardize variables? One reason is that it remove the units of measurement to facilitate comparisons between variables. Suppose X represents height in inches. The numerator $(X - \overline{X})$ has units of inches. The standard deviation $SD(X)$ also has units of inches. So when you divide, the units go away and the z-score is left without units, sometimes called a "dimensionless quantity".

Suppose a female in the United States is 6 feet tall (72 inches). Suppose a female in China is 5'8 tall (68 inches). In absolute terms, the American woman is taller than the Chinese woman. But what if we're interested in knowing which woman is taller *relative* to their respective population?

The mean height for an American woman is 65" with a standard deviation of 3.5" The mean height for a Chinese woman is 62" with a standard deviation of 2.5". (These numbers aren't perfectly correct, but they're probably close-ish.)

Calculate the z-scores for both these women.

Which woman is taller relative to their population?

Although z-scores don't technically have units, we can think of them as measuring how many standard deviations a value lies above or below the mean.

What is the z-score for a value that equals the mean?

What is the meaning of a negative z-score?

The z-score for the American woman was 2. This means that her height measures two standard deviations above the mean.

For real-world data, we will use technology to do this. Here are some temperature measurements from New York in 1974. (These are daily highs across a six-month period.)

```
airquality$Temp
```

```
##   [1] 67 72 74 62 56 66 65 59 61 69 74 69 66 68 58 64 66 57 68 62 59 73 61 61 57
##  [26] 58 57 67 81 79 76 78 74 67 84 85 79 82 87 90 87 93 92 82 80 79 77 72 65 73
##  [51] 76 77 76 76 76 75 78 73 80 77 83 84 85 81 84 83 83 88 92 92 89 82 73 81 91
##  [76] 80 81 82 84 87 85 74 81 82 86 85 82 86 88 86 83 81 81 81 82 86 85 87 89 90
## [101] 90 92 86 86 82 80 79 77 79 76 78 78 77 72 75 79 81 86 88 97 94 96 94 91 92
## [126] 93 93 87 84 80 78 75 73 81 76 77 71 71 78 67 76 68 82 64 71 81 69 63 70 77
## [151] 75 76 68
```

We calculate the mean and standard deviation:

```
mean(airquality$Temp)
```

```
## [1] 77.88235
```

```
sd(airquality$Temp)
```

```
## [1] 9.46527
```

This is an average high of about 78 degrees Fahrenheit with a standard deviation of about 9.5 degrees Fahrenheit.

If we just subtract the mean, we get mean-centered data.

```
airquality$Temp - mean(airquality$Temp)
```

```
##   [1] -10.8823529  -5.8823529  -3.8823529 -15.8823529 -21.8823529 -11.8823529
##   [7] -12.8823529 -18.8823529 -16.8823529  -8.8823529  -3.8823529  -8.8823529
##  [13] -11.8823529  -9.8823529 -19.8823529 -13.8823529 -11.8823529 -20.8823529
##  [19]  -9.8823529 -15.8823529 -18.8823529  -4.8823529 -16.8823529 -16.8823529
##  [25] -20.8823529 -19.8823529 -20.8823529 -10.8823529   3.1176471   1.1176471
##  [31]  -1.8823529   0.1176471  -3.8823529 -10.8823529   6.1176471   7.1176471
```

```
##  [37]   1.1176471   4.1176471   9.1176471  12.1176471   9.1176471  15.1176471
##  [43]  14.1176471   4.1176471   2.1176471   1.1176471  -0.8823529  -5.8823529
##  [49] -12.8823529  -4.8823529  -1.8823529  -0.8823529  -1.8823529  -1.8823529
##  [55]  -1.8823529  -2.8823529   0.1176471  -4.8823529   2.1176471  -0.8823529
##  [61]   5.1176471   6.1176471   7.1176471   3.1176471   6.1176471   5.1176471
##  [67]   5.1176471  10.1176471  14.1176471  14.1176471  11.1176471   4.1176471
##  [73]  -4.8823529   3.1176471  13.1176471   2.1176471   3.1176471   4.1176471
##  [79]   6.1176471   9.1176471   7.1176471  -3.8823529   3.1176471   4.1176471
##  [85]   8.1176471   7.1176471   4.1176471   8.1176471  10.1176471   8.1176471
##  [91]   5.1176471   3.1176471   3.1176471   3.1176471   4.1176471   8.1176471
##  [97]   7.1176471   9.1176471  11.1176471  12.1176471  12.1176471  14.1176471
## [103]   8.1176471   8.1176471   4.1176471   2.1176471   1.1176471  -0.8823529
## [109]   1.1176471  -1.8823529   0.1176471   0.1176471  -0.8823529  -5.8823529
## [115]  -2.8823529   1.1176471   3.1176471   8.1176471  10.1176471  19.1176471
## [121]  16.1176471  18.1176471  16.1176471  13.1176471  14.1176471  15.1176471
## [127]  15.1176471   9.1176471   6.1176471   2.1176471   0.1176471  -2.8823529
## [133]  -4.8823529   3.1176471  -1.8823529  -0.8823529  -6.8823529  -6.8823529
## [139]   0.1176471 -10.8823529  -1.8823529  -9.8823529   4.1176471 -13.8823529
## [145]  -6.8823529   3.1176471  -8.8823529 -14.8823529  -7.8823529  -0.8823529
## [151]  -2.8823529  -1.8823529  -9.8823529
```

But if we also divide by the standard deviation, we get a standardized variable (or a set of z-scores). Note the extra parentheses to make sure we get the order of operations right. We have to subtract first, but then divide that whole mean-centered quantity by the standard deviation.

```
(airquality$Temp - mean(airquality$Temp))/sd(airquality$Temp)
```

```
##   [1] -1.14971398 -0.62146702 -0.41016823 -1.67796094 -2.31185730 -1.25536337
##   [7] -1.36101276 -1.99490912 -1.78361034 -0.93841519 -0.41016823 -0.93841519
##  [13] -1.25536337 -1.04406459 -2.10055851 -1.46666216 -1.25536337 -2.20620791
##  [19] -1.04406459 -1.67796094 -1.99490912 -0.51581762 -1.78361034 -1.78361034
##  [25] -2.20620791 -2.10055851 -2.20620791 -1.14971398  0.32937752  0.11807873
##  [31] -0.19886945  0.01242934 -0.41016823 -1.14971398  0.64632570  0.75197509
##  [37]  0.11807873  0.43502691  0.96327387  1.28022205  0.96327387  1.59717023
##  [43]  1.49152084  0.43502691  0.22372813  0.11807873 -0.09322005 -0.62146702
##  [49] -1.36101276 -0.51581762 -0.19886945 -0.09322005 -0.19886945 -0.19886945
##  [55] -0.19886945 -0.30451884  0.01242934 -0.51581762  0.22372813 -0.09322005
##  [61]  0.54067630  0.64632570  0.75197509  0.32937752  0.64632570  0.54067630
##  [67]  0.54067630  1.06892327  1.49152084  1.49152084  1.17457266  0.43502691
##  [73] -0.51581762  0.32937752  1.38587145  0.22372813  0.32937752  0.43502691
##  [79]  0.64632570  0.96327387  0.75197509 -0.41016823  0.32937752  0.43502691
##  [85]  0.85762448  0.75197509  0.43502691  0.85762448  1.06892327  0.85762448
##  [91]  0.54067630  0.32937752  0.32937752  0.32937752  0.43502691  0.85762448
##  [97]  0.75197509  0.96327387  1.17457266  1.28022205  1.28022205  1.49152084
```

```
## [103]  0.85762448  0.85762448  0.43502691  0.22372813  0.11807873 -0.09322005
## [109]  0.11807873 -0.19886945  0.01242934  0.01242934 -0.09322005 -0.62146702
## [115] -0.30451884  0.11807873  0.32937752  0.85762448  1.06892327  2.01976780
## [121]  1.70281962  1.91411841  1.70281962  1.38587145  1.49152084  1.59717023
## [127]  1.59717023  0.96327387  0.64632570  0.22372813  0.01242934 -0.30451884
## [133] -0.51581762  0.32937752 -0.19886945 -0.09322005 -0.72711641 -0.72711641
## [139]  0.01242934 -1.14971398 -0.19886945 -1.04406459  0.43502691 -1.46666216
## [145] -0.72711641  0.32937752 -0.93841519 -1.57231155 -0.83276580 -0.09322005
## [151] -0.30451884 -0.19886945 -1.04406459
```

The easier way to do this in R is to use the `scale` command. (Sorry, the output is a little long. Keep scrolling below.)

```
scale(airquality$Temp)
```

```
##               [,1]
##   [1,] -1.14971398
##   [2,] -0.62146702
##   [3,] -0.41016823
##   [4,] -1.67796094
##   [5,] -2.31185730
##   [6,] -1.25536337
##   [7,] -1.36101276
##   [8,] -1.99490912
##   [9,] -1.78361034
##  [10,] -0.93841519
##  [11,] -0.41016823
##  [12,] -0.93841519
##  [13,] -1.25536337
##  [14,] -1.04406459
##  [15,] -2.10055851
##  [16,] -1.46666216
##  [17,] -1.25536337
##  [18,] -2.20620791
##  [19,] -1.04406459
##  [20,] -1.67796094
##  [21,] -1.99490912
##  [22,] -0.51581762
##  [23,] -1.78361034
##  [24,] -1.78361034
##  [25,] -2.20620791
##  [26,] -2.10055851
##  [27,] -2.20620791
##  [28,] -1.14971398
##  [29,]  0.32937752
```

```
##  [30,]  0.11807873
##  [31,] -0.19886945
##  [32,]  0.01242934
##  [33,] -0.41016823
##  [34,] -1.14971398
##  [35,]  0.64632570
##  [36,]  0.75197509
##  [37,]  0.11807873
##  [38,]  0.43502691
##  [39,]  0.96327387
##  [40,]  1.28022205
##  [41,]  0.96327387
##  [42,]  1.59717023
##  [43,]  1.49152084
##  [44,]  0.43502691
##  [45,]  0.22372813
##  [46,]  0.11807873
##  [47,] -0.09322005
##  [48,] -0.62146702
##  [49,] -1.36101276
##  [50,] -0.51581762
##  [51,] -0.19886945
##  [52,] -0.09322005
##  [53,] -0.19886945
##  [54,] -0.19886945
##  [55,] -0.19886945
##  [56,] -0.30451884
##  [57,]  0.01242934
##  [58,] -0.51581762
##  [59,]  0.22372813
##  [60,] -0.09322005
##  [61,]  0.54067630
##  [62,]  0.64632570
##  [63,]  0.75197509
##  [64,]  0.32937752
##  [65,]  0.64632570
##  [66,]  0.54067630
##  [67,]  0.54067630
##  [68,]  1.06892327
##  [69,]  1.49152084
##  [70,]  1.49152084
##  [71,]  1.17457266
##  [72,]  0.43502691
##  [73,] -0.51581762
##  [74,]  0.32937752
##  [75,]  1.38587145
```

```
##  [76,]  0.22372813
##  [77,]  0.32937752
##  [78,]  0.43502691
##  [79,]  0.64632570
##  [80,]  0.96327387
##  [81,]  0.75197509
##  [82,] -0.41016823
##  [83,]  0.32937752
##  [84,]  0.43502691
##  [85,]  0.85762448
##  [86,]  0.75197509
##  [87,]  0.43502691
##  [88,]  0.85762448
##  [89,]  1.06892327
##  [90,]  0.85762448
##  [91,]  0.54067630
##  [92,]  0.32937752
##  [93,]  0.32937752
##  [94,]  0.32937752
##  [95,]  0.43502691
##  [96,]  0.85762448
##  [97,]  0.75197509
##  [98,]  0.96327387
##  [99,]  1.17457266
## [100,]  1.28022205
## [101,]  1.28022205
## [102,]  1.49152084
## [103,]  0.85762448
## [104,]  0.85762448
## [105,]  0.43502691
## [106,]  0.22372813
## [107,]  0.11807873
## [108,] -0.09322005
## [109,]  0.11807873
## [110,] -0.19886945
## [111,]  0.01242934
## [112,]  0.01242934
## [113,] -0.09322005
## [114,] -0.62146702
## [115,] -0.30451884
## [116,]  0.11807873
## [117,]  0.32937752
## [118,]  0.85762448
## [119,]  1.06892327
## [120,]  2.01976780
## [121,]  1.70281962
```

```
## [122,]  1.91411841
## [123,]  1.70281962
## [124,]  1.38587145
## [125,]  1.49152084
## [126,]  1.59717023
## [127,]  1.59717023
## [128,]  0.96327387
## [129,]  0.64632570
## [130,]  0.22372813
## [131,]  0.01242934
## [132,] -0.30451884
## [133,] -0.51581762
## [134,]  0.32937752
## [135,] -0.19886945
## [136,] -0.09322005
## [137,] -0.72711641
## [138,] -0.72711641
## [139,]  0.01242934
## [140,] -1.14971398
## [141,] -0.19886945
## [142,] -1.04406459
## [143,]  0.43502691
## [144,] -1.46666216
## [145,] -0.72711641
## [146,]  0.32937752
## [147,] -0.93841519
## [148,] -1.57231155
## [149,] -0.83276580
## [150,] -0.09322005
## [151,] -0.30451884
## [152,] -0.19886945
## [153,] -1.04406459
## attr(,"scaled:center")
## [1] 77.88235
## attr(,"scaled:scale")
## [1] 9.46527
```

Although the outputs are formatted a little differently, you can go back and check that these sets of numbers match each other.

What is the mean of a standardized variable? How do you know this?

Let's calculate the variance of a standardized variable. To do so, I'll note that the mean $\overline{X}$ is just a number. Also, the standard deviation $SD(X)$ is just a number. To make the calculation easier to understand, let's just substitute letters that are easier to work with:

$M = \overline{X}$

$S = SD(X)$

Remember, M and S are *constants*.

Now we need to calculate $Var(Z)$. I'll do the first couple of steps. Then you take over and, using the variance rules from earlier in the chapter, simplify the expression until you get to a numerical answer. Be sure to justify each step by citing the rule you invoked to get there.

$$Var(Z) = Var\left(\frac{(X - \overline{X})}{SD(X)}\right) \tag{2.3}$$

$$= Var\left(\frac{(X - M)}{S}\right) \tag{2.4}$$

$$= Var\left(\frac{1}{S}(X - M)\right) \tag{2.5}$$

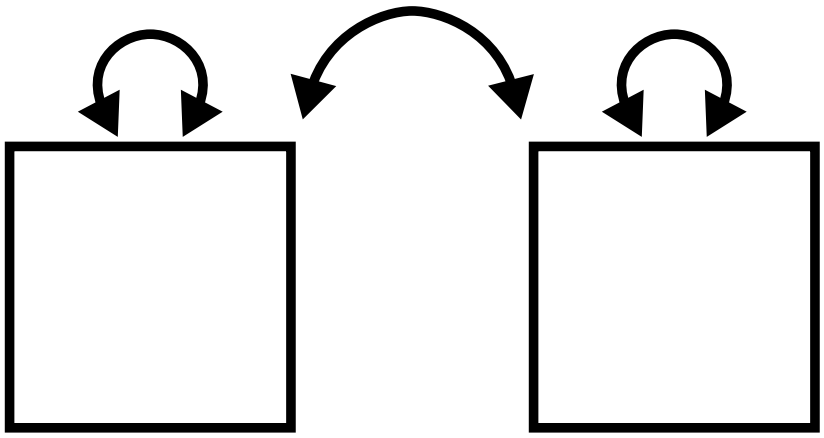
$$= \quad ??? \tag{2.6}$$

You may feel a little uncomfortable applying **Rule 3** because you might worry if X and M are independent. Since M is the mean of X, it seems like that is *very* dependent on X. This is where some of the intuition about independence breaks down and we have to rely on mathematical rules that we haven't really gotten into. *All* constants are independent of any other variable.

You should get the answer 1. A standardized variable always has variance 1. This will be an important fact in future chapters.

Chapter 3

Covariance



3.1 Calculating covariance

The last chapter was about variance, which measures the spread of a single variable. Now we extend this idea to pairs of variables.

We say that two variables "co-vary" when the spread of one variable is related to the spread of another variable. This relationship represents an *association* between the two variables.

We'll call our two variables X_1 and X_2. To keep things simple, let's assume that we have already mean centered our variables.

If X_1 and X_2 are already mean centered, then what are $\overline{X_1}$ and $\overline{X_2}$?

As we did in the last chapter with variance, we'll build up the calculation of

covariance step-by-step using a table to keep track of intermediate quantities we need.

Here are two variables (with $n = 7$) that have been mean centered:

X_1	X_2
-1	-2
-2	2
2	-2
-3	-1
4	2
-1	-2
1	3

Check that the mean of both columns is truly zero.

Something interesting happens when we look at the product X_1X_2.

If X_1 and X_2 both lie above their means, they are both positive numbers. Therefore, their product is positive.

What if both X_1 and X_2 lie below their means? What do we know about their values individually and what do we know about their product?

Here is the chart again, but with the products listed in a new column:

X_1	X_2	X_1X_2
-1	-2	2
-2	2	-4
2	-2	-4
-3	-1	3
4	2	8
-1	-2	2
1	3	3

Now we add up the products across all seven data pairs:

X_1	X_2	X_1X_2
-1	-2	2
-2	2	-4
2	-2	-4
-3	-1	3
4	2	8
-1	-2	2

X_1	X_2	X_1X_2
1	3	3
		Sum: 10

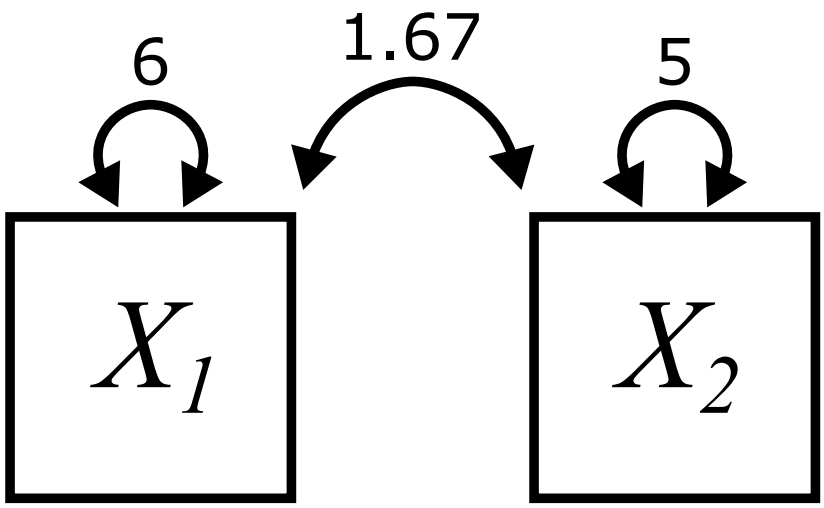
So when X_1 and X_2 tend to have similar values (both positive or both negative), their product is usually positive. It's not true of every pair of values in the table above; some products are negative. But the majority are positive. Therefore, the sum of all such products will be positive.

We're almost there. Just like we wanted the average squared deviation to calculate the variance, here we want the average of the products from the third column above. And just like in the case of variance, it's not *quite* the average we calculate. Instead of dividing by n, we divide by $n-1$ for exactly the same esoteric reason. In our example, there are 7 data points (in other words, 7 rows of data), so we divide by 6.

Putting this all together:

X_1	X_2	X_1X_2
-1	-2	2
-2	2	-4
2	-2	-4
-3	-1	3
4	2	8
-1	-2	2
1	3	3
		Sum: 10
	Covariance: 10/6 =	$\boxed{1.67}$

In our diagrams, the covariance of two variables is indicated by a curved, double-headed arrow pointing at both boxes and labeled with the value of the covariance, like this:



Note that we still include the variances of each of the individual variables. They are still important to us. We just have one new type of arrow now.

Verify that the variances in the diagram are correct for our example. You can do it by hand if you want, but using R is fine too.

Here is the final formula for covariance, written as $Cov(X_1, X_2)$. This works for all pairs of variables, even if they aren't mean centered. The terms $(X_1 - \overline{X_1})$ and $(X_2 - \overline{X_2})$ do the mean centering:

$$Cov(X_1, X_2) = \frac{\sum (X_1 - \overline{X_1})(X_2 - \overline{X_2})}{n-1}$$

Suppose X_1 tends to be above its mean when X_2 is below its mean and X_1 tends to be below its mean when X_2 is above its mean. What will the product $(X_1 - \overline{X_1})(X_2 - \overline{X_2})$ usually be? Therefore, what will the sum of all such products likely be?

For general variables (not necessarily mean centered), the table will actually look like this:

X_1	X_2	$(X_1 - \overline{X_1})$	$(X_2 - \overline{X_2})$	$(X_1 - \overline{X_1})(X_2 - \overline{X_2})$
17	23	-2	11	-22
25	15	6	3	18
...	...	...	...	...

Calculate the covariance by hand by making a table like the one above. (These variables are *not* mean centered, so you'll have to calculate the mean of each variable in order to fill out the third and fourth columns.)

X_3: 8, 10, 16, 7, 4, 3

X_4: 6, 5, 4, 9, 11, 7

Explain intuitively why the covariance is negative for these two variables.

When calculating variance, the order of the data points does not matter. Why?

When calculating covariance, the order of the data points *does* matter. Why?

What if you keep pairs together, but rearrange the rows of the table. How does that affect the covariance?

3.2 Calculating covariance in R

Once we've done it by hand a few times to make sure we understand how the formula works, from here on out we can let R do the work for us:

```
X1 <- c(-1,-2, 2, -3, 4, -1, 1)
X2 <- c(-2, 2, -2, -1, 2, -2, 3)
cov(X1, X2)
```

```
## [1] 1.666667
```

```
X3 <- c(8, 10, 16, 7, 4, 3)
X4 <- c(6, 5, 4, 9, 11, 7)
cov(X3, X4)
```

```
## [1] -9.2
```

And here's some real world data. In addition to temperature (which we've already seen), we can use wind speed and see if there is an association between them:

```
cov(airquality$Temp, airquality$Wind)
```

```
## [1] -15.27214
```

3.3 Covariance rules

We'll think of the variance and covariance rules as one big list. We left off on **Rule 4**, so now we'll introduce **Rule 5**.

- **Rule 5**

$$Cov(X, X) = Var(X)$$

In words, **Rule 5** states that the covariance of a variable *with itself* is just the same thing as the variance of that variable. This is quite remarkable! It means that variance is really just a special case of covariance.

Explain why **Rule 5** is true. (Hint: think about how you would calculate $Cov(X, X)$ using either the formula or the table—or both!)

- **Rule 6**

$$Cov\left(X_1, X_2\right) = Cov\left(X_2, X_1\right)$$

In words, we would say that covariance is *symmetric*.

Explain why **Rule 6** is true. (Again, think about the formula or the table—or both!)

The next four rules are analogous to similar rules for variance (**Rule 1**, **Rule 2**, **Rule 3**, and **Rule 4**).

- **Rule 7**

Suppose that C is a "constant" variable, meaning that it always has the same value (rather than being a variable that could contain lots of different numbers). Then,

$$Cov\left(X, C\right) = 0$$

As always, try to explain this rule. Give an intuitive explanation of why this rule "should" be true. Then think about it computationally, thinking of either the formula or the table—or both!

- **Rule 8**

$$Cov\left(X_1 + X_2, X_3\right) = Cov\left(X_1, X_3\right) + Cov\left(X_2, X_3\right)$$

What you should appreciate here is that there is no longer any restriction on the relationships among the variables involved. **Rule 2** only worked when the two variables were independent. On the other hand, **Rule 8** works for any combination of variables, no matter their relation.

Even more satisfying is this next rule:

- **Rule 9**

$$Cov\left(X_1 - X_2, X_3\right) = Cov\left(X_1, X_3\right) - Cov\left(X_2, X_3\right)$$

Yay! The minus sign behaves sensibly now! Of course, since covariances can be positive or negative (unlike variances which are always positive!) we can more safely subtract two of them without worry. And this rule, like **Rule 8**, does not depend on X_1 and X_2 being independent. They can be any two variables.

There are versions of these rules with the addition or subtraction on the other side, but these are just minor variations of **Rule 8** and **Rule 9**, so they're not worth mentioning as a separate rule. Remember that covariance is symmetric, so you can always swap things on the left and right of the comma.

$$Cov\left(X_1, X_2 \pm X_3\right) = Cov\left(X_1, X_2\right) \pm Cov\left(X_1, X_3\right)$$

- **Rule 10**

If a is any number,

$$Cov\left(aX_1, X_2\right) = aCov\left(X_1, X_2\right) = Cov\left(X_1, aX_2\right)$$

This rule is also very sensible. Instead of **Rule 4** that takes a number a and pulls out an a^2, **Rule 10** just pulls out a single factor of a (from either slot).

Just a couple more rules. We were talking about independence in conjunction with **Rule 8** and **Rule 9**. That leads directly to an interesting and super-important rule:

- **Rule 11**

If X_1 and X_2 are independent, then

$$Cov\left(X_1, X_2\right) = 0$$

Why is **Rule 11** true, intuitively?

It's interesting to note that this rule only works one way. In other words, if you know that two variables are independent, then you can conclude their covariance is zero. However, if you know the covariance is zero, that doesn't necessarily mean that the two variables are independent. We'll see an example of this later in the chapter.

Finally, one rule to rule them all:

- **Rule 12**

For *any* two variables X_1 and X_2:

$$Var(aX_1 + bX_2) = a^2Var(X_1) + b^2Var(X_2) + 2abCov(X_1, X_2)$$

This brings practically everything we know together into one rule!

Proving **Rule 12** will give us good practice with the type of manipulation we'll need to do in future chapters. So here goes. For the first few steps, you name what rule we're invoking. Then, you'll pick up the thread and follow it through the last few steps on your own.

$$\begin{aligned} Var(aX_1 + bX_2) &= Cov(aX_1 + bX_2, aX_1 + bX_2) && (3.1)\\ &= Cov(aX_1 + bX_2, aX_1) + Cov(aX_1 + bX_2, bX_2) && (3.2)\\ &= Cov(aX_1, aX_1) + Cov(bX_2, aX_1) + && (3.3)\\ &\quad Cov(aX_1, bX_2) + Cov(bX_2, bX_2) && (3.4)\\ &= \quad ??? && (3.5) \end{aligned}$$

You'll need these rules to do calculations in future chapters. Rather than having to search for them in Chapter 2 and this chapter, we've gathered up all the rules in one convenient place in Appendix A.

3.4 Correlation

The pros and cons for calculating covariance are similar to those for variance. The mathematics is much nicer for covariance, but we lose interpretability.

Let's suppose that X_1 measures salary in dollars and X_2 measures years of education. We would expect there to be some association between these variables, so we calculate the covariance. What is the unit of measurement of the resulting number?

The solution to the problem here is not as simple as it was for variance. Since variance had squared units, all we had to do was take the square root. Covariance has a weird product of units, so we have to do something more clever.

Following up on the activity above, let's suppose we have a covariance with units of "dollar-years". If we divide by a number expressed in dollars, we get rid of those units and we're left with years. But that seems unsatisfying; covariance should express something about both variables that went into it. Likewise, it makes no sense to divide by a number expressed in years as that would leave us just with dollars.

The solution to the dilemma is to accept that we aren't going to be able to keep any units in a meaningful way. Therefore, what we want is something *standardized*, meaning that it has no units.

If X_1 is expressed in dollars, can you think of a statistic that measures spread and is also in units of dollars?

Likewise, if X_2 is measured in years, what statistic that measures spread is also in units of years?

The previous activity gives us an idea. What if we divide the covariance by *both* the standard deviation of X_1 *and* the standard deviation of X_2?

$$\frac{Cov(X_1, X_2)}{SD(X_1)SD(X_2)}$$

Sometimes it's written like this:

$$\frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}}$$

But that's the same thing, right?

This quantity has no units. We call this the *correlation* between X_1 and X_2. We'll either write

$$Corr(X_1, X_2)$$

or, if we need to be more concise,

$$r_{X_1X_2}$$

Yes, this is the same as the correlation coefficient you learned about in your intro stats class, although it wasn't likely presented to you quite this way.[1]

One great thing about correlation is that it has no units, so it serves as a sort of "universal" measure of how two variables co-vary. But the best part is that it has a nice intuitive meaning precisely because it factors out the pieces of the covariance that are only there because of the spread of the two variables individually. In other words, the fact that X_1 and X_2 have their own variability actually *complicates* the notion of covariance. Those individual variances "corrupt" the interpretation of covariance. But after excising them, all that's left in the correlation is the "pure" part of the covariance that expresses the relationship or association between X_1 and X_2.

[1] Karl Pearson is credited with inventing the correlation coefficient. Pearson was a life-long eugenicist and a proponent of using "science" to prove that some races were superior to others. It important to disentangle the truly valuable notion of correlation from the discredited hands that may have first written it down. Therefore, we will not be referring to it in this text as the Pearson correlation coefficient.

3.5 Covariance with standardized data

In the last chapter, you showed that the variance of a standardized variable was 1. What is the covariance between two standardized variables?

Let's standardize both X_1 and X_2. To make the math a little easier, we'll use similar notation to what we used at the end of the last chapter.

$M_1 = \overline{X_1}$

$S_1 = SD(X_1)$

$M_2 = \overline{X_2}$

$S_2 = SD(X_2)$

And we'll write the z-scores in a way that is more amenable to mathematical manipulation (like before):

$$Z_1 = \frac{1}{S_1}(X_1 - M_1)$$

$$Z_2 = \frac{1}{S_2}(X_2 - M_2)$$

This looks a little more intimidating, but if you apply the rules, it works out:

$$Cov(Z_1, Z_2) = Cov\left(\frac{1}{S_1}(X_1 - M_1), \frac{1}{S_2}(X_2 - M_2)\right) \tag{3.6}$$

$$= \quad ??? \tag{3.7}$$

Work this out. Take your time. Apply the rules carefully. So that you know what you're aiming for, you should get

$$Cov(Z_1, Z_2) = \frac{Cov(X_1, X_2)}{S_1 S_2}$$

Okay, now remember that S_1 is just a convenient substitute for $SD(X_1)$ and S_2 is just a substitute for $SD(X_2)$. Wait, does that answer look familiar?

This is cool! Correlation is simply the covariance of two variables after they've been standardized.

This also reinforces the earlier comment about interpreting covariance after removing the extraneous influence of the spread of the individual variables. Standardizing variables makes the spread of all variables 1, so their covariance is now a pure representation of just the association between them.

You probably remember from intro stats that correlation takes on values between -1 and 1. That fact is not obvious from the formula we have. Why should the fraction

$$\frac{Cov(X_1, X_2)}{SD(X_1)SD(X_2)}$$

be bounded by -1 and 1?

Let's go back to standardized variable to keep things simple. The correlation is just the covariance of two standardized variables:

$$Corr(X_1, X_2) = Cov(Z_1, Z_2)$$

Use the rules to calculate this:

$$Var(Z_1 + Z_2)$$

Remember that Z_1 and Z_2 are not necessarily independent. (In fact, we hope they are not. Otherwise, why do we care about their correlation? It would be zero!) So you need **Rule 12**, not **Rule 2**. Keep manipulating until you get

$$2 + 2Corr(X_1, X_2)$$

Since variances are always non-negative, we now know that

$$0 \leq 2 + 2Corr(X_1, X_2)$$

Solve this inequality for $Corr(X_1, X_2)$.

Now follow the exact same steps for

$$Var(Z_1 - Z_2)$$

Very little should change in your answer, but there is one small change. Again, solve the resulting inequality. (Don't forget the key rule when working with inequalities that multiplying or dividing by a negative number changes the direction of the inequality.)

Here is a fact we will state without proof:

Correlations are only interpretable as the strength of **linear** associations.

Why is this? Basically, it boils down to the fact that a "perfect" correlation of 1 or -1 is only achievable when data points lie on a perfectly straight line. Therefore, thinking of correlation as lying between 0 and 1 (or 0 and -1) is only sensible if you are judging how close points are to lying on a straight line. We'll see examples of this in the next section when we plot some data.

To calculation correlation in R, use the `cor` command:

```
cor(airquality$Temp, airquality$Wind)
```

```
## [1] -0.4579879
```

Use R to confirm that the number above is the covariance divided by the product of the standard deviations.

3.6 Visualizing correlation

Covariance is hard to interpret, so when we're visualizing data and we want to understand any association that might exist between two variables, correlation is a much better statistic to calculate. Let's see how correlation relates to the graph of two variables.

Before getting into the graphing, we will need to load some packages. The `tidyverse` is a whole set of commonly used packages that will allow us to work with data frames (or "tibbles" as the cool kids are calling them) and make graphs. Be sure to load the package by typing the following in R before going any further:

```
library(tidyverse)
```

```
## -- Attaching packages --------------------------------------- tidyverse 1.3.1 --
```

```
## v ggplot2 3.3.6     v purrr   0.3.4
## v tibble  3.1.7     v dplyr   1.0.9
## v tidyr   1.2.0     v stringr 1.4.0
## v readr   2.1.2     v forcats 0.5.1
```

```
## -- Conflicts ------------------------------------------ tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
```

In fact, from here on out, we'll start each chapter by loading any necessary libraries in R that we'll need.

The standard graph of two numerical variables is a scatterplot. Let's start with a straight line relationship. First, we define two variables. We'll use some shortcuts here to make our lives a little easier. The `seq` command just generates a sequence of numbers.

```
X5 <- seq(1, 9)
X5
```

```
## [1] 1 2 3 4 5 6 7 8 9
```

Then we can establish a linear relationship just by declaring one in a formula:

```
X6 <- 3 + 0.5 * X5
X6
```

```
## [1] 3.5 4.0 4.5 5.0 5.5 6.0 6.5 7.0 7.5
```

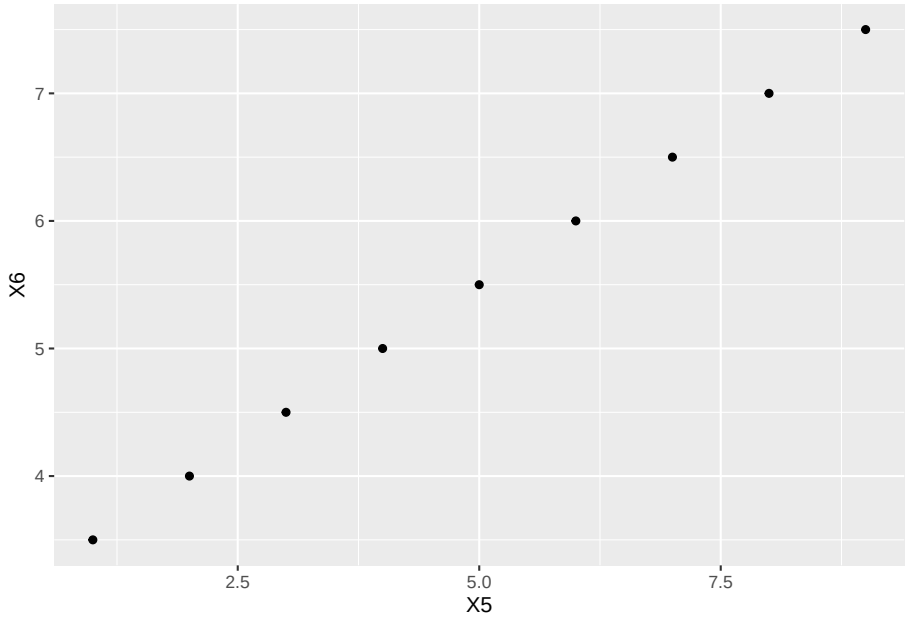
To put both variables into the same graph, it helps to make them both columns in a single tibble.

```
linear_data <- tibble(X5, X6)
linear_data
```

```
## # A tibble: 9 x 2
##      X5    X6
##   <int> <dbl>
## 1     1   3.5
## 2     2   4
## 3     3   4.5
## 4     4   5
## 5     5   5.5
## 6     6   6
## 7     7   6.5
## 8     8   7
## 9     9   7.5
```

And here is the graph:

```
ggplot(linear_data, aes(y = X6, x = X5)) +
    geom_point()
```



Now the correlation:

```
cor(X5, X6)
```

```
## [1] 1
```

It is 1, as expected.

What about a perfectly straight line with a negative slope?

```
X7 <- 5 - 0.2 * X5
X7
```

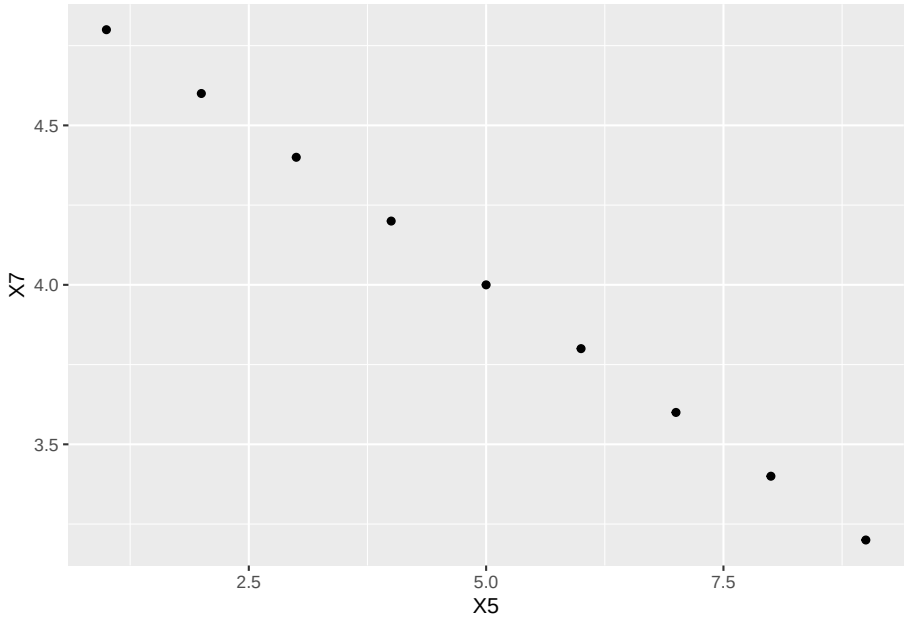
```
## [1] 4.8 4.6 4.4 4.2 4.0 3.8 3.6 3.4 3.2
```

We'll throw this new variable into the tibble we already have (for convenience). To explain the syntax below, the `%>%` symbol is called a "pipe" and it tells R to pass the `linear_data` tibble on to the next row to process it. And the processing itself is dictated by the `bind_cols` command which tells R to "bind a new column" to the tibble. The part that says `X7 = X7` may be a little confusing. It says to add the new column `X7`, but also still call it `X7`.

```
linear_data <- linear_data %>%
    bind_cols(X7 = X7)
linear_data
```

```
## # A tibble: 9 x 3
##      X5    X6    X7
##   <int> <dbl> <dbl>
## 1     1   3.5   4.8
## 2     2   4     4.6
## 3     3   4.5   4.4
## 4     4   5     4.2
## 5     5   5.5   4
## 6     6   6     3.8
## 7     7   6.5   3.6
## 8     8   7     3.4
## 9     9   7.5   3.2
```

```
ggplot(linear_data, aes(y = X7, x = X5)) +
    geom_point()
```




```
cor(X5, X7)
```

```
## [1] -1
```

Again, that is what we expected.

What happens if we plot random data? The `runif` command just chooses random numbers uniformly between 0 and 1.[2] We use `set.seed` to make our work reproducible. You will get the same set of random numbers on your machine if you use the same seed as we use here.

```
set.seed(1234)
X8 <- runif(20)
X9 <- runif(20)
```

```
X8
```

```
##  [1] 0.113703411 0.622299405 0.609274733 0.623379442 0.860915384 0.640310605
##  [7] 0.009495756 0.232550506 0.666083758 0.514251141 0.693591292 0.544974836
## [13] 0.282733584 0.923433484 0.292315840 0.837295628 0.286223285 0.266820780
## [19] 0.186722790 0.232225911
```

```
X9
```

```
##  [1] 0.31661245 0.30269337 0.15904600 0.03999592 0.21879954 0.81059855
##  [7] 0.52569755 0.91465817 0.83134505 0.04577026 0.45609148 0.26518667
## [13] 0.30467220 0.50730687 0.18109621 0.75967064 0.20124804 0.25880982
## [19] 0.99215042 0.80735234
```

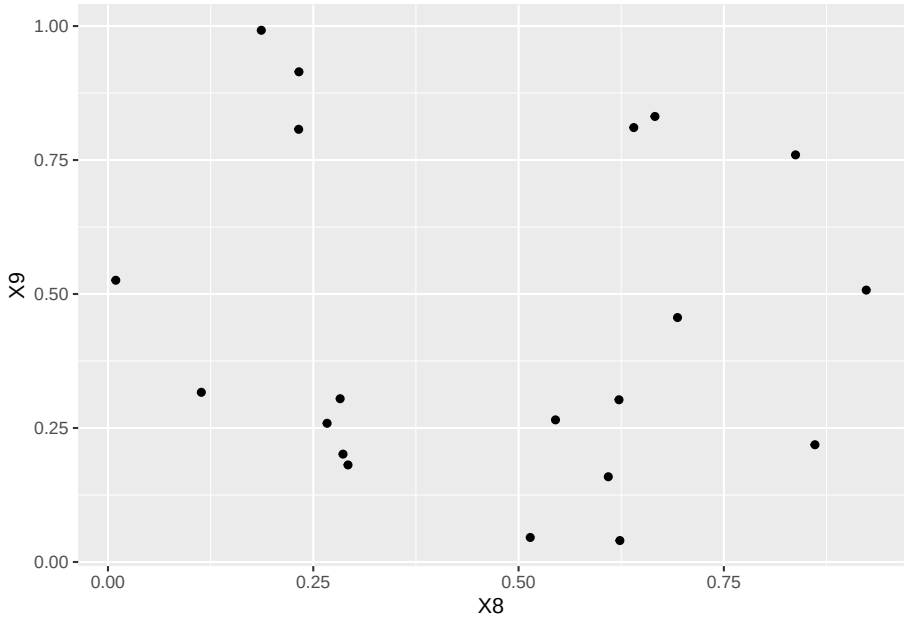
```
random_data <- tibble(X8, X9)
random_data
```

```
## # A tibble: 20 x 2
##         X8     X9
##      <dbl>  <dbl>
##  1 0.114   0.317
##  2 0.622   0.303
##  3 0.609   0.159
##  4 0.623   0.0400
##  5 0.861   0.219
##  6 0.640   0.811
##  7 0.00950 0.526
##  8 0.233   0.915
##  9 0.666   0.831
## 10 0.514   0.0458
## 11 0.694   0.456
```

[2] Sean's brain always want to parse this command as "run if". Run if what? No, no, it's "r unif".

```
## 12 0.545   0.265
## 13 0.283   0.305
## 14 0.923   0.507
## 15 0.292   0.181
## 16 0.837   0.760
## 17 0.286   0.201
## 18 0.267   0.259
## 19 0.187   0.992
## 20 0.232   0.807
```

```
ggplot(random_data, aes(y = X9, x = X8)) +
    geom_point()
```



What do you guess is the correlation between X_8 and X_9?

Now calculate it using R? Did you get the *exact* answer you guessed? If not, why not?

What about data that follows a perfect mathematical relationship that is not a straight line? For example, here is a part of a parabola.

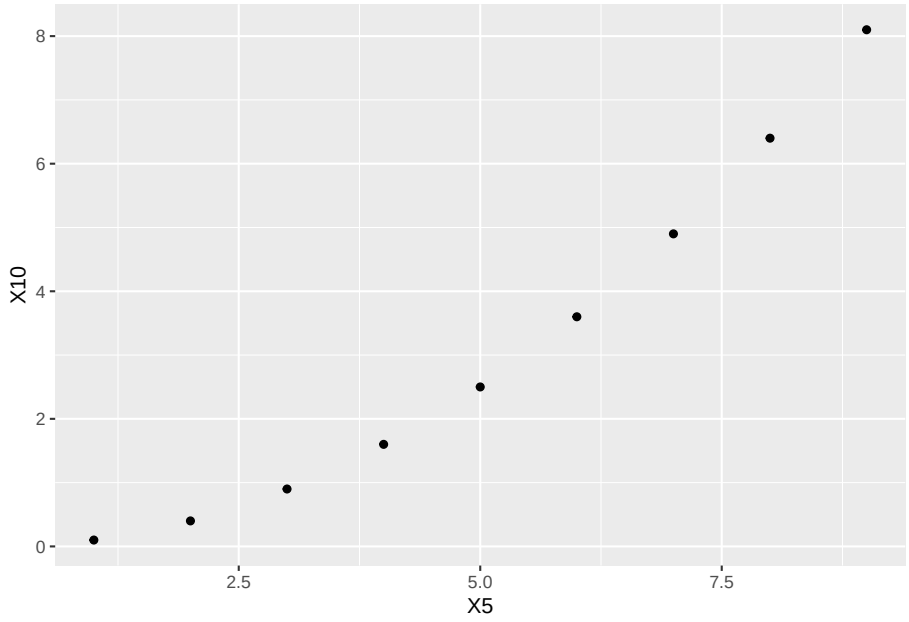
```
X10 <- 0.1 * X5^2
X10
```

```
## [1] 0.1 0.4 0.9 1.6 2.5 3.6 4.9 6.4 8.1
```

```
nonlinear_data <- tibble(X5, X10)
nonlinear_data
```

```
## # A tibble: 9 x 2
##      X5   X10
##   <int> <dbl>
## 1     1   0.1
## 2     2   0.4
## 3     3   0.9
## 4     4   1.6
## 5     5   2.5
## 6     6   3.6
## 7     7   4.9
## 8     8   6.4
## 9     9   8.1
```

```
ggplot(nonlinear_data, aes(y = X10, x = X5)) +
    geom_point()
```



Now for the correlation:

```
cor(X5, X10)
```

```
## [1] 0.975281
```

This is a large correlation, but it is not exactly 1, even though the points follow a precise mathematical relationship. That relationship is not linear.

Here's a fascinating example. For this, we'll want a parabola that goes down and then up.

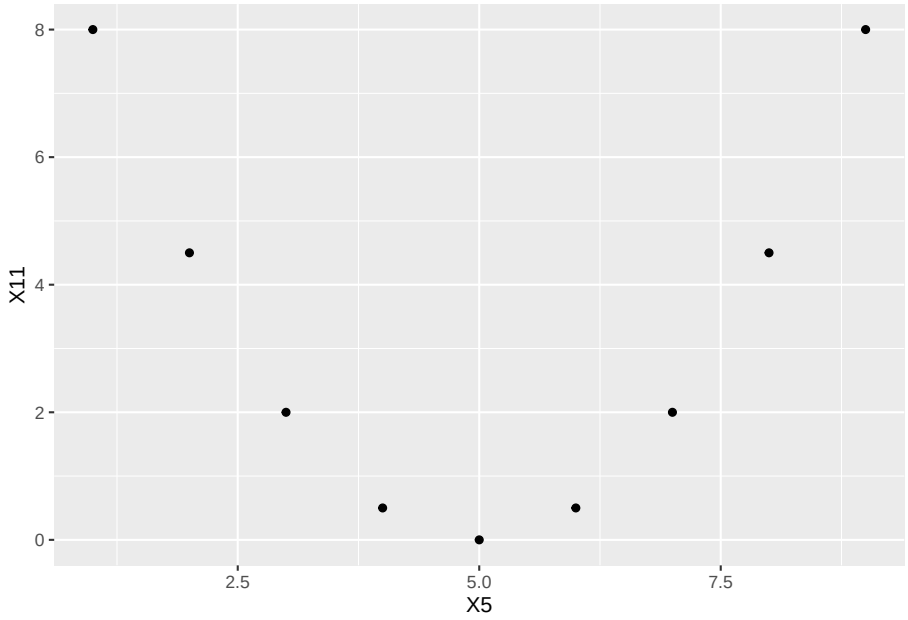
```
X11 <- 0.5 * (X5 - 5)^2
X11
```

```
## [1] 8.0 4.5 2.0 0.5 0.0 0.5 2.0 4.5 8.0
```

```
nonlinear_data <- nonlinear_data %>%
    bind_cols(X11 = X11)
nonlinear_data
```

```
## # A tibble: 9 x 3
##      X5   X10   X11
##   <int> <dbl> <dbl>
## 1     1   0.1   8
## 2     2   0.4   4.5
## 3     3   0.9   2
## 4     4   1.6   0.5
## 5     5   2.5   0
## 6     6   3.6   0.5
## 7     7   4.9   2
## 8     8   6.4   4.5
## 9     9   8.1   8
```

```
ggplot(nonlinear_data, aes(y = X11, x = X5)) +
    geom_point()
```



Before looking at the answer, what is your guess for the correlation between X_5 and X_{11}?

Now calculate the correlation in R.

Again, there's a perfect mathematical relationship between these two variables. They are most definitely associated. So why is the correlation 0?

Recall the earlier promise to discuss **Rule 11**. If two variables are independent, then their covariance is zero, and, therefore, their correlation is also zero. However, this rule doesn't work the other way around. The claim is that knowing the covariance/correlation is zero does not imply (necessarily) that the two variables are independent. Here is the promised example of that phenomenon. X_5 and X_{11} have zero correlation. And yet, X_5 and X_{11} are definitely *not* independent.

This is important enough for a fancy box:

When you see that the correlation between two variables is zero or near zero, be careful not to conclude that the variables are independent.

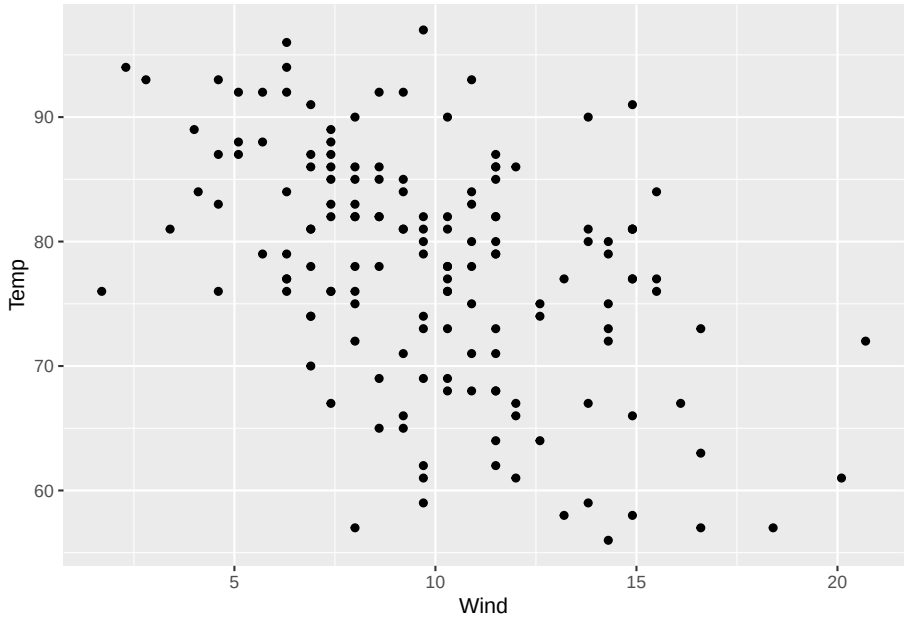
A zero or near-zero correlation indicates only the lack of a *linear* association between two variables. There may be nonlinear associations. That's why it's always a good idea to graph your data.

Real data is, of course, much messier and it's just not possible to have perfect correlations between two variables measured out there in the real world. (If you do find a perfect correlation between two columns of your data, chances are

that you either recorded the same column twice, or the second column is some simple transformation of first column, like multiplying every value by the same number or something like that.)

Here is a plot of the temperature (degrees Fahrenheit) and wind speed (mph) from the New York air quality data set.

```
ggplot(airquality, aes(y = Temp, x = Wind)) +
    geom_point()
```



Just looking at the scatterplot (without calculating anything), is the correlation between these two variables positive or negative? Try guessing the exact value of the correlation.

Now calculate the exact value of the correlation to see how close you were.

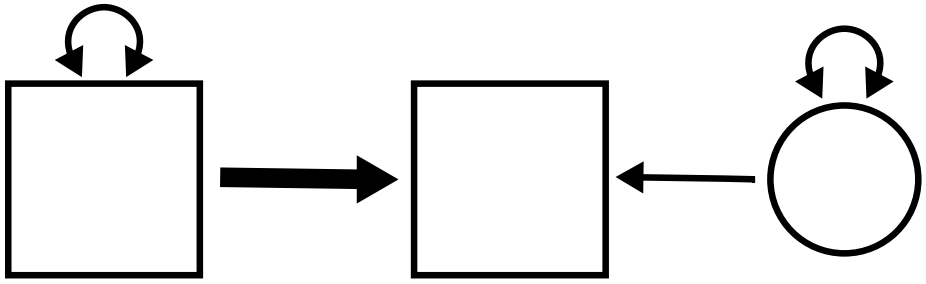
If you want some practice with looking at scatterplots and guessing the correlation, try this online game:

Guess the Correlation

Turn up the sound! If the whole class plays at the same time, your classroom will sound like an arcade. Compete with your classmates to see who can get the high score.

Chapter 4

Simple regression



Preliminaries

We need to load the packages we will use for this chapter. The `tidyverse` package has all sorts of utilities for working with tibbles (data frames). The `broom` package will be used to calculate and store the residuals of the model. This will also be our first introduction to the `lavaan` package that will be used throughout the rest of the book.

```
library(tidyverse)
library(broom)
library(lavaan)
```

```
## This is lavaan 0.6-11
## lavaan is FREE software! Please report any bugs.
```

4.1 Some friendly advice

Even if you have seen regression before reading this book, be sure to read and study the this chapter and the next chapter thoroughly. If nothing else, you need to be comfortable with the notation and terminology established here. But we will also take special care to motivate and justify all the calculations that are taken for granted in some treatments of regression. This framework will be important as we move into mediation and path analysis in the following chapters. If you are comfortable with the content of this chapter, there won't be much "new" to say about multiple regression, mediation, and path analysis more generally.

4.2 Prediction

One of the most important tasks in statistics is *prediction*. Given some data, can we predict the value of something important about a population of interest?

Suppose you have gathered some data on anxiety among Utah high school students. There are various instruments available for measuring anxiety, so say you have administered the Beck Anxiety Inventory. This instrument assigns a score from 0 to 63, with lower numbers indicating less anxiety and higher numbers indicating more.

You take care to make sure your sample is as close to a simple random sample as possible so that it's representative of the population (all high school students in the state of Utah). From your sample data, you can calculate summary statistics. For example, you might find that the mean anxiety score for Utah high school students is 7.1 with a standard deviation of 3.9.

A random Utah high school student walks through the door. You don't know anything about them. Can you say anything about their anxiety? What is your best guess as to what their score might be on the Beck Anxiety Inventory?

We can do a lot better if we have another variable we can measure. For example, let's suppose our data records not only anxiety, but also the minutes of smart phone usage per day.

In theory, why would having information about smart phone usage potentially help us make better predictions of anxiety?

Do you suspect that the association between anxiety and smart phone usage is positive or negative? (You can Google this question to check if there is any empirical evidence out there for your guess.)

Now imagine that another random Utah high school student walks through the door. This time, I tell you that their smart phone usage is average (sitting at the mean). What is your best prediction for their anxiety score? (Give an exact value.)

What if I told you that the student who walked through the door had *higher* than average smart phone usage? What would be your prediction of their anxiety score? (You can't give an exact value here, but give a qualitatively sensible answer.)

What if I told you that the student who walked through the door had *lower* than average smart phone usage? What would be your prediction of their anxiety score? (Again, just give a qualitatively sensible answer.)

4.3 Regression terminology

When we have one variable we suspect may help us predict another variable, one way to study it is using a simple regression model.

This is related to, but somewhat different from, covariance. Covariance is symmetric, so it expresses the idea that two variables are mutually related. But there is no "directionality" to that relationship. By way of contrast, a simple regression model asserts that one of the variables is a "predictor" and the other is "response". In other words, we start with the values or properties of the predictor variable and try to deduce what we can about the values or properties of the response variable.

Keep in mind that "directionality" is not the same as "causality". While it's possible that one variable causes another, there needs to be a data collection process (often a carefully controlled experiment) and a clear scientific rationale that justifies a causal relationship between variables before we can start thinking about inferring causality. For purposes of much of this book, directionality just means that we wish to establish a predictive relationship wherein we start with the properties of one variable and try to predict the properties of another variable. There is often a "sensible" order in which to do this based on the research questions asked or the hypotheses posed.

There are many different terminological conventions in statistics, so be aware that "predictor" variables are also called—often depending on the discipline and the context—features, covariates, controls, regressors, inputs, explanatory variables, or independent variables. In fact, in the context of structural equation modeling, we will use the term "exogenous" to refer to variables that play this role. (That term has a much more precise definition that we'll discuss in future chapters.) And "response" variables might be called outcomes, outputs, targets, criteria, predicted variables, explained variables, or dependent variables, among others. In this book, we will often use the term "endogenous" (again, in a very specific way yet to be explained). If there is a data collection process and a clear scientific rationale that justifies a causal relationship between variables, then we might be able to refer to variables as either "cause" or "effect".

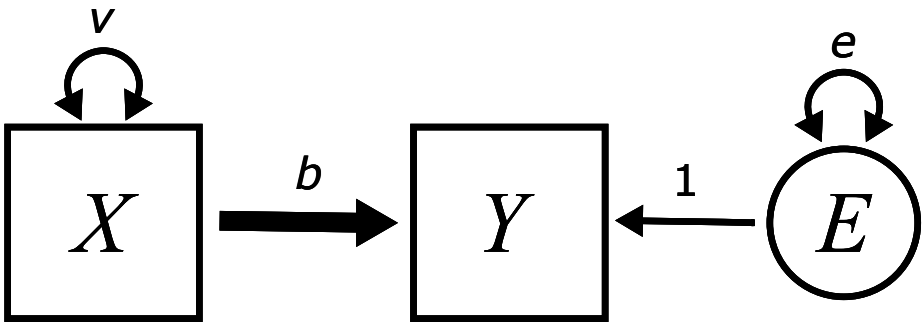
Keep in mind that it's the scientific question we want to ask that determines

the predictor/response relationship. A different researcher with a different hypothesis might use the same two variables but with the roles reversed.

In the anxiety/smart-phone example above, which variable is predictor and which is response, at least according to the way we stated the scenario?

4.4 The simple regression model

Here is the figure from the top of the chapter, only now we have decorated it with some letters (and a number):



The goal of this section is to explain all these.

The variable names are X and Y. X is the exogenous variable and Y is the endogenous variable. For example, X might be smart phone usage and Y might be the anxiety score from the example above. In this section, we're going to do some concrete calculations using the example from the last chapter about wind speed and temperature from the `airquality` data set. In the last chapter, we simply calculated the (symmetric) correlation between wind speed and temperature. Here, we will consider wind speed as exogenous and temperature as endogenous. In other words, our goal is to use the wind speed as a predictor of temperature.

The letter v requires no further explanation. This is the variance of the variable X, so we already know about it.

The parameter b is supposed to measure something about the predictive relationship between X and Y. It is attached to an arrow that is drawn a little thicker than the other arrows in the diagram. More to say about that in a moment.

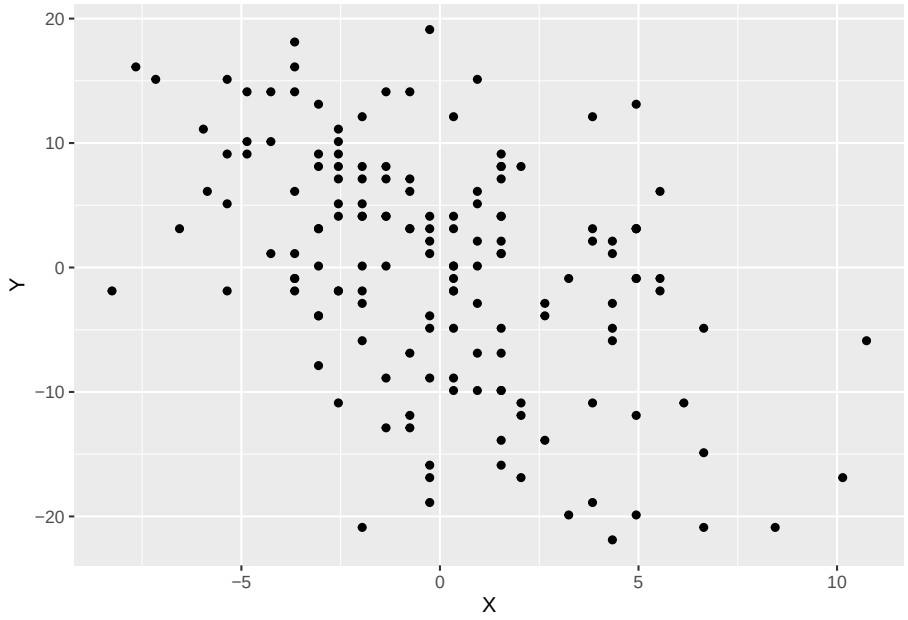
The really weird, new part is the circle on the right. This will be the "error" term.

What is "error" and why is it here? To illustrate, let's plot wind speed against temperature. Before plotting and analyzing these variables, we are going to mean-center them and put them in a tibble.

```
X <- airquality$Wind - mean(airquality$Wind)
Y <- airquality$Temp - mean(airquality$Temp)
airquality_mc <- tibble(X, Y)
airquality_mc
```

```
## # A tibble: 153 x 2
##        X      Y
##    <dbl>  <dbl>
##  1 -2.56 -10.9
##  2 -1.96  -5.88
##  3  2.64  -3.88
##  4  1.54 -15.9
##  5  4.34 -21.9
##  6  4.94 -11.9
##  7 -1.36 -12.9
##  8  3.84 -18.9
##  9 10.1  -16.9
## 10 -1.36  -8.88
## # ... with 143 more rows
```

```
ggplot(airquality_mc, aes(y = Y, x = X)) +
    geom_point()
```

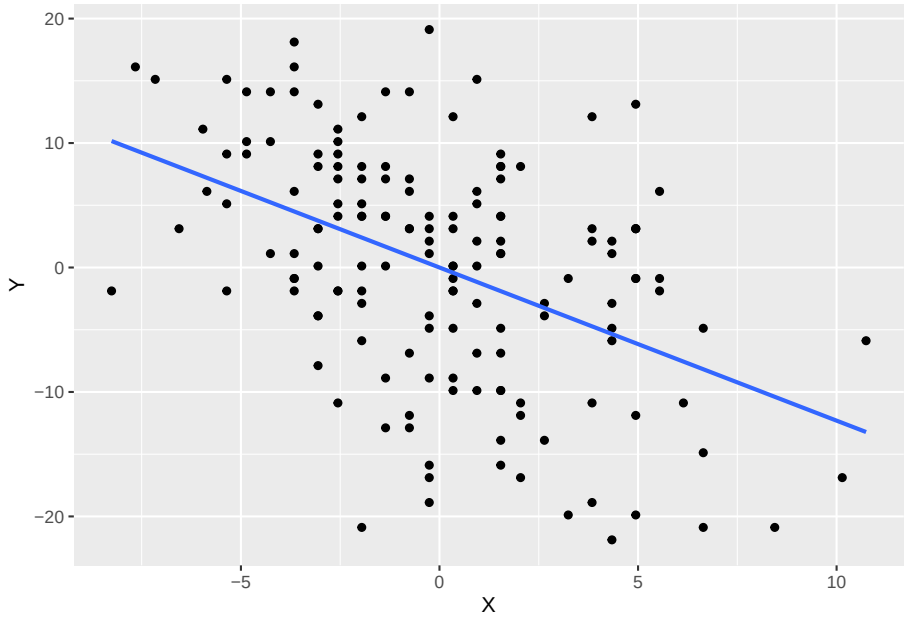


Note that the exogenous variable (wind speed) is on the x-axis and the endogenous variable (temperature) is on the y-axis.

We can see a negative and reasonably linear association between these variables, so let's add a line of best fit to the data.

```
ggplot(airquality_mc, aes(y = Y, x = X)) +
    geom_point() +
    geom_smooth(method = lm, se = FALSE)
```

```
## `geom_smooth()` using formula 'y ~ x'
```



The line passes right through the origin $(0, 0)$. Why?

The slope of this line is -1.23. We'll see how to calculate this slope in a bit. But what does this slope mean?

Look at the help file for the `airquality` data set. (Either use the Help tab in RStudio or type `?airquality` at the Console.)

What are the units of measurement of X? What are the units of measurement of Y? Since slope is "rise over run", what are the units of the slope?

So, the idea is that for every additional mile per hour of wind speed, we predict that the temperature goes down by 1.23 degrees Fahrenheit.

Why is the following sentence incorrect?

> For every additional mile per hour of wind speed, the temperature goes down by 1.23 degrees Fahrenheit.

The point is that the line is a *model* that makes predictions. As long as X and Y are both mean-centered, the equation of this line is

$$\hat{Y} = bX$$

There is a new piece of notation here: $\hat{Y}$. This symbol represents the *predicted* value of Y according to the model. We will not get the *actual* value of Y from this piece of the model because the actual values of Y differ from the model

because real-world data doesn't lie on a perfect straight line. More on that in a moment.

According to the information above, we can estimate that the value of b is -1.23:

$$\hat{Y} = -1.23X$$

This is a proportional effect. Again, as long as X and Y are both mean-centered, knowing the value of X allows us to predict the value of Y by multiplying by b.

But those predictions will almost always be wrong. On any given day, given an increase of 1 mile per hour wind in wind speed, it will very rarely happen that the temperature will drop by *exactly* 1.23 degrees. That's just a sort of "average" over time. On average, there's a slight temperature change associated with 1 mph change in wind speed, and the number -1.23 is the best estimate of that average change across our whole data set. We need to be *especially* clear that an increase in wind speed does not necessary *cause* a drop in temperature. I mean, that might be partially true, but we can't prove it from our observational data. There are all sorts of other reasons to explain both an increase in wind speed and a drop in temperature (like a cold front moving in).

Since our predictions are average effects and not specific guarantees, every prediction we make will be wrong by some amount. (We could get extremely lucky, but even then, it's difficult to imagine a situation in which our prediction is precisely correct to, say, 10 decimal places or something like that.) Therefore, there is error in our prediction. The new equation—accounting for error—is

$$Y = bX + E$$

or

$$Y = -1.23X + E$$

Now we use Y instead of $\hat{Y}$. Once we include the error, we can recover the exact value of Y, so this is no longer just a prediction from the straight-line model. Remember this:

If you write down a regression equation for an endogenous variable that includes all incoming arrows, *including the error term*, use Y.

If you write down a regression equation for an endogenous variable that includes all incoming arrows, *excluding the error term*, use $\hat{Y}$.

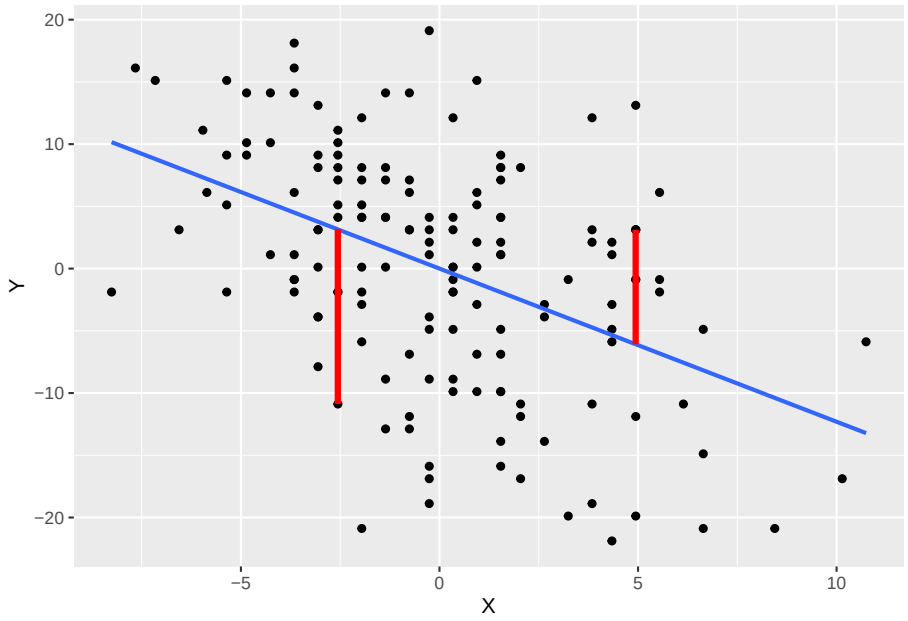
Error is a funny word because it has a negative connotation. It sounds like we made a mistake. Well, the model does make mistakes. Every model prediction is technically wrong. But this is not the kind of mistake that results from doing our arithmetic wrong or anything like that. It's simply the "natural" error that results from the messiness of the real world and the impossibility of predicting anything with certainty. For this reason, we will often prefer the term "residual".

It's what is "left over" after we have made a prediction. It's the extra change in temperature, for example, that is not accounted for by the model with wind speed alone.

The residuals are evident in the plot above. If there were no residuals, every data point would lie on a perfect straight line. But the data points are all either a little above or below the line. Those vertical distances between the data and the line are the residuals or errors. Here is an example of two residuals plotted below in red.

```
ggplot(airquality_mc, aes(y = Y, x = X)) +
    geom_point() +
    geom_smooth(method = lm, se = FALSE) +
    annotate("segment",
             x = -2.56, y = 3.15,
             xend = -2.56, yend = 3.15 - 14.03,
             color = "red", size = 1.5) +
    annotate("segment",
             x = 4.94, y = -6.08,
             xend = 4.94, yend = -6.08 + 9.20,
             color = "red", size = 1.5)
```

```
## `geom_smooth()` using formula 'y ~ x'
```



Points below the line have negative residuals and points above the line have positive residuals.

The residuals do not appear as observed or measured variables in our data. They are a consequence of a variety of *unmeasured* factors that determine temperature aside from wind speed. An unmeasured variable that appears in a model is called a *latent variable*. We will discuss latent variables in far greater detail in Chapter 8. For now, just know that latent variables are indicated by circles in the diagram. That's why there is a circle with the letter E inside.

The equation

$$Y = bX + E$$

can also be written as

$$Y = bX + 1 \cdot E$$

How is that "1" represented in the diagram?

The letters v, b, and e are called *free parameters* because they are free to vary depending on the data. The "1" is called a *fixed parameter*. It is attached to an arrow, so it's technically a parameter of the model, but it is not a parameter that we need to calculate. It is "fixed" at the value 1 because the error term in the model is represented by $+E$ with a fixed coefficient of 1. In general, throughout the book, if the word "parameter" is used without qualification, you can assume we are talking only about the free parameters, the ones we need to calculate.

Arrows that represent the coefficients of regression relationships will be drawn a little thicker than the other arrows in the diagram. This convention is, to our knowledge, unique to this book. It is not absolutely necessary, but it will be helpful later when there are more arrows floating about to distinguish between the regression relationship and other kinds of relationships (like error terms or covariances, for example).

The only thing in the diagram that hasn't been explained yet is e.

Where does e appear in the diagram? Given where it appears, what does it represent mathematically?

We know that curved arrows represent variance. But what does it mean to measure the variance of a variable we can't observe?

What would the scatterplot look like if the error variance were very small. What about if the error variance were large?

There is variability in the size of the residuals. Some are small (points that are close to the line) and some are large (points that are far from the line). This spread of residuals can be estimated from data, just like any other variance calculation.[1] It turns out to be about 70.8. We'll see how to calculate that

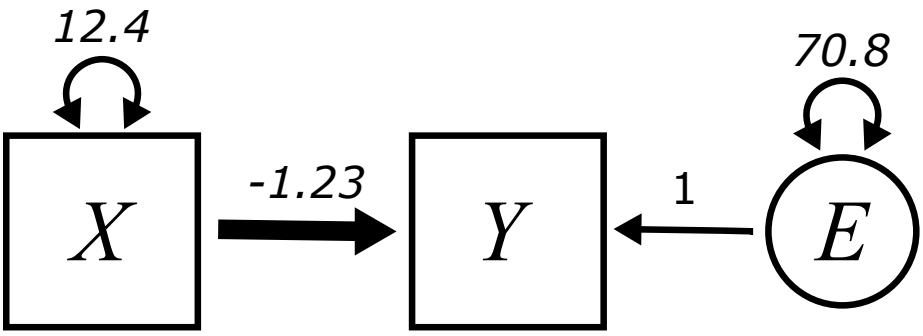
[1] We know that variance is close to the average squared deviation, except we divide by $n-1$ instead of n. Well, residuals are a little more weird still. To get an unbiased estimate, you have divide by $n-2$. However, the calculation that appears next is the one that uses $n-1$. This is for reasons that, regrettably, we'll have to sweep under the rug here.

below.

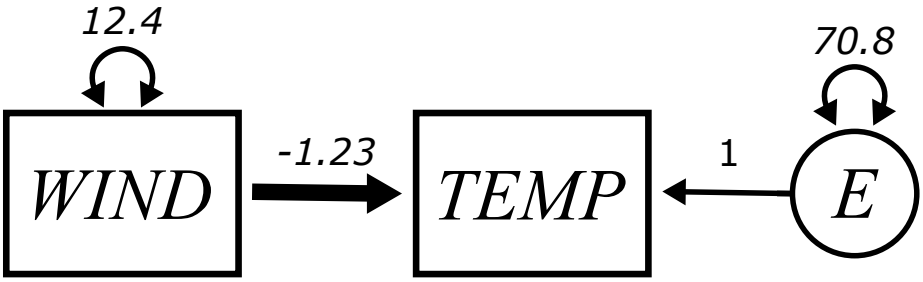
Let's put everything together into a diagram.

First, we need the variance of X (wind speed). Calculate it in R. You should get 12.4.

Does it matter if you calculate the variance of the variable `Wind` from the original `airquality` data, or the variance of the X variable from `airqualilty_mc`? Why or why not?



While it's helpful to see X and Y as generic prototypes for any simple regression model, in most applied problems from now on we'll refer to variables using contextually meaningful names. The final diagram looks like this:



Is the error variable E exogenous or endogenous?

One final note about this diagram: you may have noticed that the Y variable (or *TEMP*) does *not* have a variance term attached. There is no double-headed arrow on that box. Why not? The point here is that we are trying to understand the variance of Y using other elements of the model. In other words, Y has a variance, but that variance is partially predicted by X. And the rest of the variance not predicted by X is swept up in the error term E. So all the variance is Y is accounted for through the contribution of X and E combined.

4.5 Simple regression assumptions

All the calculations we need to do, and our ability to interpret the results, depend on certain assumptions being met.

If you look up regression assumptions, you might find a huge list of requirements. Some of these requirements relate to calculating statistics like P-values for regression parameters. For now, we are content simply to know that it makes sense to interpret the parameters in the model above.

For that, we really only need five assumptions:

1. The data should come from a "good" sample.
2. The values of the exogenous variable should be measured without error.
3. The relationship between Y and X should be approximately linear.
4. The residuals should be independent of the X values.
5. There should be no influential outliers.

Let's address these one at a time:

1. What do we mean by a "good" sample? While a simple random sample is the gold standard, it's usually not possible to obtain one in the real world. So we make our sampling process as random as possible and ensure that the resulting sample is as representative of the population we're trying to study as possible.
2. We should always strive to measure things precisely. When measuring physical phenomena with precise scientific instruments, we can usually minimize so-called "measurement error". But some measurements are a lot messier. You might ask a person a series of survey questions today and then ask them the same questions tomorrow and get somewhat different answers. Or you might have to record data that consists of "educated guess" estimates about things that are difficult to pin down precisely. Whenever you have an exogenous variable that is unreliable, that can introduce bias into your model. (Curiously, measurement error in the endogenous variable doesn't matter quite as much. It may introduce more variability in your estimates, but it will not bias the values of the parameters of the regression model.)
3. You can check linearity with a scatterplot. Just make sure the pattern of dots doesn't have strong curvature to it.
4. There should be no patterns in the residuals at all. They should be randomly scattered around the best-fit line and the average size of the residuals should not change radically from one side of the graph to the other.[2] You can check this by plotting the residuals, but that can't be done until

[2]This property of similarly-sized residuals is called *homoskesdasticity*. The violation of that condition is called *heteroskedasticity* which is one of Sean's favorite words ever!

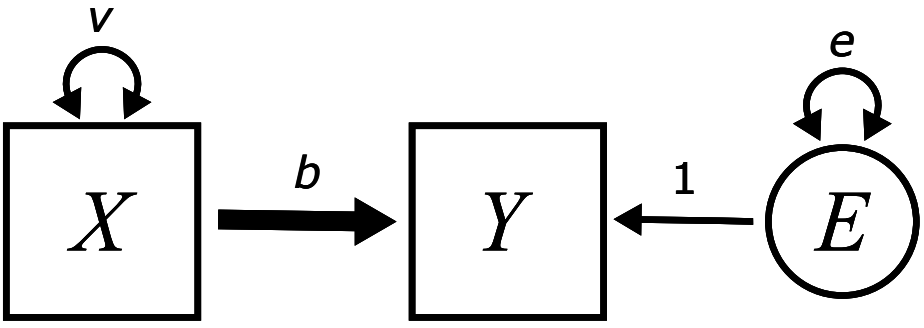
after the model is fit. (The residuals don't exist until we have a value of $\hat{Y}$ with which to compare Y.)

5. Check the scatterplot for outliers. If there are serious ones, assess them to make sure they are not data entry mistakes. If they correspond to valid data, you cannot just throw them away.[3] Often, the solution is to run the analysis both including and (temporarily) excluding the outliers to make sure their presence doesn't radically alter the parameter estimates.

4.6 Calculating regression parameters

Now we'll show one way to calculate the parameters (the numbers) in the above diagram. This isn't the only way to do it. In fact, this is not the approach that is used in most intro stats classes. But this approach will be helpful to illustrate the way we will do these calculations in future chapters.

Let's go back to the diagram without the numbers:



The letter v is easy because it's just the variance of X:

$$Var(X) = v$$

We can estimate it directly from the data. (You already did it in R above for wind speed.) Through completing the activities below, we will also calculate b and e.

To get the other parameters, we have to set up a few equations. The first observation we need to make is that, as important as the arrows are in a diagram, it's just as important where the arrows are *not*.

[3] In other words, don't follow the all-too-common "rule of thumb" for outliers, which is just covering them up with your thumb and pretending they don't exist.

Are there any arrows directly connecting X and E? What might that imply about the relationship between X and E? Which regression assumption is related to this question?

So what would that imply about the value of $Cov(E, X)$?

We have to be a little careful with the line of reasoning above. Even if there are no *direct* paths from X to E, are there any *indirect* paths from X to E? Such an indirect path might be the source of some kind of association between X and E.

The only possible path goes through Y and looks like

$$X \rightarrow Y \leftarrow E$$

For reasons that we won't explain here (but will be explained in Chapter 6), this type of path does *not* imply any kind of association between X and E. Therefore, the model *does* imply that X and E are independent, and, therefore, $Cov(E, X) = 0$.

Next, because Y is the combination of X and E, we've already seen that we can write

$$Y = bX + E$$

Therefore, we can calculate $Var(Y)$ according to this formula using the established rules. (A convenient list of all of them in one place is located in Appendix A.)

Keep simplifying the following as much as possible:

$$\begin{aligned} Var(Y) &= Var(bX + E) && (4.1)\\ &= \quad ??? && (4.2) \end{aligned}$$

You should end up with

$$b^2 v + e$$

Don't forget that $Var(X) = v$ and $Var(E) = e$ in the diagram!

We also need to use information about the covariance between Y and X. Keep simplifying the calculation below:

$$\begin{aligned} Cov(Y, X) &= Cov(bX + E, X) && (4.3)\\ &= \quad ??? && (4.4) \end{aligned}$$

You should end up with

$$bv$$

Use R to calculate $Var(Y)$ and $Cov(Y, X)$ for the `airquality` data. (You've already computed $Var(X)$. It was 12.4.)

You should get 89.6 and -15.3, respectively.

Now we can set up all the equations we need to solve for the various letters we want. Here are the three equations we have established:

$$12.4 = v \tag{4.5}$$
$$89.6 = b^2 v + e \tag{4.6}$$
$$-15.3 = bv \tag{4.7}$$

Time to do a little algebra. You know v. Using that value, solve for b first (using the last equation). Then, using both values of b and v, solve for e in the second equation.

Check that the values you got are the same as the ones from the earlier diagram (with the possibility of a little rounding error).

Now let's go through that again, but this time, in full generality:

$$Var(X) = v \tag{4.8}$$
$$Var(Y) = b^2 v + e \tag{4.9}$$
$$Cov(Y, X) = bv \tag{4.10}$$

Therefore,

$$v = Var(X)$$

$$b = \frac{Cov(Y, X)}{Var(X)}$$

$$e = Var(Y) - \left(\frac{Cov(Y, X)}{Var(X)}\right)^2 Var(X)$$

4.7 The model-implied matrix

It will be convenient in future chapters to collect up all these numbers we need in an array of terms called a *sample covariance matrix*. (Sometimes this is called a variance-covariance matrix.) The idea is to take the covariance of all possible pairs of observed variables and arrange them as follows:

$$\begin{bmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y) \end{bmatrix}$$

There are some immediate simplifications to make.

1. Since $Cov(X,Y) = Cov(Y,X)$, there is no point in writing it twice. We will just use a dot ($\bullet$) to replace $Cov(X,Y)$.
2. We can replace the upper-left and lower-right entries (the entries on the so-called "diagonal" of the matrix) with variances.

This is our final sample covariance matrix:

$$\begin{bmatrix} Var(X) & \bullet \\ Cov(Y,X) & Var(Y) \end{bmatrix}$$

Alternatively, we could also write this:

$$\begin{bmatrix} Var(X) & Cov(X,Y) \\ \bullet & Var(Y) \end{bmatrix}$$

The former is called the *lower-triangular* form, and the latter is the *upper-triangular* form. Both contain the same information, so it really doesn't matter which one we use.

With the data we have, we can calculate numbers for all these quantities.

There is another important matrix called the *model-implied matrix*. Given the model, what does the covariance matrix look like? From the calculations above, we know that the model-implied matrix is

$$\begin{bmatrix} v & \bullet \\ bv & b^2v + e \end{bmatrix}$$

The letters b, v, and e are unknowns. Okay, v is not *very* unknown. It's an unknown in the sense of being a parameter in the model, but you don't have to work very hard to find it. The point is that model parameters are estimated by equating the covariance matrix (calculated from the data) with model-implied matrix and trying to solve for all the unknown parameters.

There is no new math to do in this section. The matrices are just convenient ways to organize the work we've already done. All parameter estimation in structural equation modeling is essentially setting these two matrices (the sample covariance matrix and the model-implied matrix) equal to each other and solving:

$$\begin{bmatrix} Var(X) & \bullet \\ Cov(Y,X) & Var(Y) \end{bmatrix} = \begin{bmatrix} v & \bullet \\ bv & b^2v + e \end{bmatrix}$$

Don't forget that the matrix on the left—the sample covariance matrix—consists of numbers that we calculate from data. The matrix on the right—the model-implied matrix—contains letters, which are the unknown parameters we're trying to find.

4.8 Coefficients in terms of correlation

The formulas we derived are fine as far as they go. They allow you to take quantities calculated from data (variances and covariances of observed variables) and translate that into estimates of model parameters.

The formula for the slope parameter b is pretty simple and has some intuitive content. It's the covariance between Y and X, but dividing by the variance of X to make sure it has the right units.

$$b = \frac{Cov(Y,X)}{Var(X)}$$

Another way to look at this formula is to rearrange things a bit as follows:

The formula for b above is equivalent to

$$b = \frac{Cov(Y,X)\sqrt{Var(Y)}}{Var(X)\sqrt{Var(Y)}}$$

Why?

Now write it like this:

$$b = \frac{Cov(Y,X)\sqrt{Var(Y)}}{\sqrt{Var(X)}\sqrt{Var(X)}\sqrt{Var(Y)}}$$

What happened here?

Finally, write it like this:

$$b = \left(\frac{Cov(Y, X)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \right) \left(\frac{\sqrt{Var(Y)}}{\sqrt{Var(X)}} \right)$$

Explain why this simplifies to

$$b = Corr(Y, X) \left(\frac{SD(Y)}{SD(X)} \right)$$

This is often the formula taught in intro stats classes. In more concise notation:

$$b = r_{YX} \left(\frac{s_Y}{s_X} \right)$$

The intuition here is that b is basically just the correlation between Y and X, but it has to account for the scales and units of Y and X.

Why does the standard deviation of Y have to be in the numerator and the standard deviation of X have to be in the denominator? Think about the units b must have.

The formula for the error variance e is a little more gross. With similar trickery, though, we can simplify that formula quite a bit.

Here is the starting point:

$$e = Var(Y) - \left(\frac{Cov(Y, X)}{Var(X)} \right)^2 Var(X)$$

Explain why the right-hand side can be rewritten as

$$Var(Y) - \frac{Cov(Y, X)^2}{Var(X)}$$

Explain why the next step is valid:

$$Var(Y) - \frac{Cov(Y, X)^2 Var(Y)}{Var(X)Var(Y)}$$

What about this next one?

$$Var(Y) \left(1 - \frac{Cov(Y, X)^2}{Var(X)Var(Y)} \right)$$

Why would we do such a thing? In other words, does the new fraction on the right look familiar in any way?

We hope you recognize that the fraction on the right is just the correlation coefficient squared. The whole equation can now be written as

$$e = Var(Y)\,(1 - r_{YX}^2)$$

There is a nice consequence of this last equation. The term in parentheses $(1 - r_{YX}^2)$ is a number between 0 and 1, right? Since we are multiplying this by the variance of Y, we can think of the term in parentheses as a *proportion*. All the variance of Y is explained in our model in one of two ways. The thick arrow coming in from the left uses X to predict some of the variance of Y. All the rest of the variance of Y is left over in the error term e. Therefore, $(1 - r_{YX}^2)$ is the proportion of the variance of Y left over as error.

And if that is true, it must also be the case that r_{YX}^2 is the proportion of the variance of Y explained by X. Calculating one minus a proportion gives the complementary proportion. For example, if $(1 - r_{YX}^2) = 0.3$, then 30% of the variance of Y is left over as error. But that implies that 70% of the variance of Y is explained by X. $1 - 0.3 = 0.7$.

Most authors will write R^2 instead of r^2 for some reason. Rearranging the equation above, replacing r_{YX}^2 with R^2, and writing e as $Var(E)$ looks like

$$R^2 = 1 - \frac{Var(E)}{Var(Y)}$$

In other words, we can think of the error variance as a *proportion* of the total variance of Y, and then R^2 is the complementary proportion. Therefore, R^2 is the proportion of the variance *accounted for by the model*.

4.9 Regression with standardized variables

Recall that if we convert our variables to z-scores, variances are all 1 and covariances become correlation coefficients. In other words, the covariance matrix becomes a *correlation matrix* and looks like this:

$$\begin{bmatrix} 1 & \bullet \\ r_{YX} & 1 \end{bmatrix}$$

The model-implied matrix does not change. Solving for the parameters as before is the same, then, except we can now replace $Var(X)$ and $Var(Y)$ with 1, and $Cov(Y, X)$ with r_{YX}.

$$\begin{bmatrix} 1 & \bullet \\ r_{YX} & 1 \end{bmatrix} = \begin{bmatrix} v & \bullet \\ bv & b^2v + e \end{bmatrix}$$

$$1 = v \tag{4.11}$$
$$r_{YX} = bv \tag{4.12}$$
$$1 = b^2 v + e \tag{4.13}$$

Therefore,

$$v = 1$$

$$b = r_{YX}$$

$$e = 1 - r_{YX}^2$$

When the variables are standardized, the slope of the regression is just the correlation! And the error variance is just a proportion between 0 and 1 which is complementary to r_{YX}^2 (aka, R^2, or the variance explained by the model). Those two variances, e and R^2 now add up to 1.

We'll use the `scale` command to create standardized variables for temperature and wind speed and put them in a new tibble.

```
X_std <- scale(airquality$Wind)
Y_std <- scale(airquality$Temp)
airquality_std <- tibble(X_std, Y_std)
airquality_std
```

```
## # A tibble: 153 x 2
##    X_std[,1] Y_std[,1]
##        <dbl>     <dbl>
##  1    -0.726    -1.15
##  2    -0.556    -0.621
##  3     0.750    -0.410
##  4     0.438    -1.68
##  5     1.23     -2.31
##  6     1.40     -1.26
##  7    -0.385    -1.36
##  8     1.09     -1.99
##  9     2.88     -1.78
## 10    -0.385    -0.938
## # ... with 143 more rows
```

Modify the `ggplot` code from earlier in the chapter to create a scatterplot of the new standardized variables along with a best-fit line. What is the slope of this line? (Hint: calculate the correlation coefficient between the two standardized variables.)

4.10 Simple regression in R

4.10.1 Using `lm`

The straightforward way to run regression in R is to use the `lm` command. This stands for "linear model". It uses a special symbol, the tilde ~, to express the relationship between the endogenous variable and the exogenous variable. The endogenous (response) variable always goes on the left, before the tilde. The exogenous (predictor) variable goes on the right, after the tilde. Finally, there is a `data` argument to tell `lm` where to find the variables to model.

```
YX_lm <- lm(Y ~ X, data = airquality_mc)
YX_lm
```

```
##
## Call:
## lm(formula = Y ~ X, data = airquality_mc)
##
## Coefficients:
## (Intercept)            X
##   1.117e-14   -1.230e+00
```

Which of the two numbers above is the slope b?

We haven't talked about the intercept yet, but according to this output, what is it? (Hint: it's not literally 1.117×10^{-14}. What does that number really mean?)

Run the `lm` command, but this time using the standardized variables from the `airquality_std` tibble. The value of the slope should not surprise you. Explain why it is what it is.

Don't forget: these parameters make no sense to interpret unless the regression assumptions are met. We looked at a scatterplot already and determined that is was approximately linear. But we haven't checked the residuals.

The residuals can be obtained most easily from the model by using the `augment` command from the `broom` package in the following way:

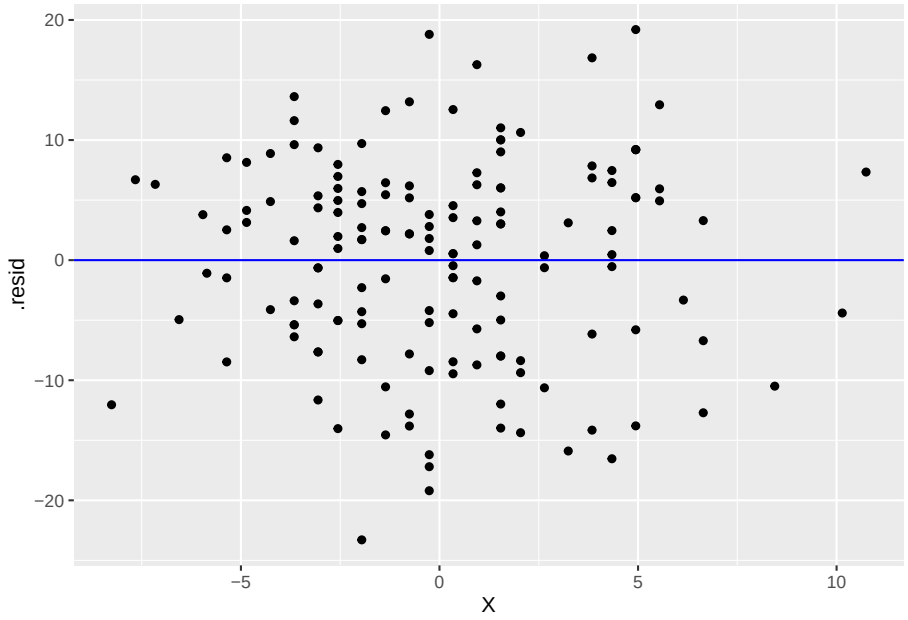
```
YX_aug <- augment(YX_lm)
YX_aug
```

```
## # A tibble: 153 x 8
##         Y     X .fitted  .resid    .hat .sigma   .cooksd .std.resid
##     <dbl> <dbl>   <dbl>   <dbl>   <dbl>  <dbl>     <dbl>      <dbl>
##  1 -10.9  -2.56    3.15 -14.0   0.0100    8.39 0.0141       -1.67
##  2  -5.88 -1.96    2.41  -8.29  0.00857   8.44 0.00420      -0.986
##  3  -3.88  2.64   -3.25  -0.631 0.0102    8.47 0.0000292    -0.0751
##  4 -15.9   1.54   -1.90 -14.0   0.00780   8.39 0.0109       -1.66
##  5 -21.9   4.34   -5.34 -16.5   0.0165    8.36 0.0328       -1.98
##  6 -11.9   4.94   -6.08  -5.80  0.0195    8.46 0.00478      -0.694
##  7 -12.9  -1.36    1.67 -14.6   0.00751   8.39 0.0113       -1.73
##  8 -18.9   3.84   -4.73 -14.2   0.0144    8.39 0.0208       -1.69
##  9 -16.9  10.1   -12.5   -4.40  0.0611    8.46 0.00942      -0.538
## 10  -8.88 -1.36    1.67 -10.6   0.00751   8.43 0.00596      -1.25
## # ... with 143 more rows
```

There are many columns here and we're not going to discuss most of them, but the residuals of the model are stored in the column called `.resid`. There are also standardized residuals stored in column `std.resid`. Both will look exactly the same in a plot except for the scale of the axes, so it doesn't much matter which we use.

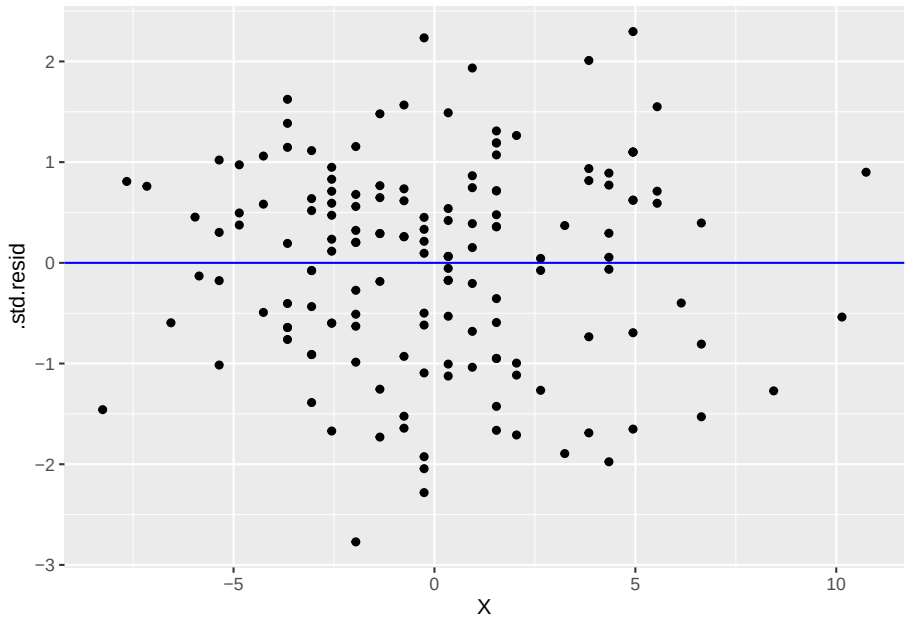
The residuals are now stored in a new tibble called `YX_aug`, so be sure to use that in the following `ggplot` command and *not* the original data. We'll put the residuals on the y-axis. Since we're interested in checking that the residuals are independent of the X variable, we will put that, unsurprisingly, on the x-axis. A reference line at $y = 0$ helps us see where the residuals are centered.

```
ggplot(YX_aug, aes(y = .resid, x = X)) +
    geom_point() +
    geom_hline(yintercept = 0, color = "blue")
```



or

```
ggplot(YX_aug, aes(y = .std.resid, x = X)) +
    geom_point() +
    geom_hline(yintercept = 0, color = "blue")
```



What do you see in the graphs above? Is this good or bad? What indicates in these graphs that the residuals are independent of X?

4.10.2 Using `lavaan`

We will also introduce you briefly to the `lavaan` package. While it's totally overkill for simple regression, getting used to the syntax now will make it easier to continue to build up your confidence in using it when it will be the only tool we use.

A `lavaan` model is built in a similar way to `lm` using the tilde ~ notation. One big difference is that the model needs to be specified inside quotation marks first and assigned to a name like this:

```
TEMP_WIND_model <- "Y ~ X"
```

Then we pass that model text to the `sem` function from `lavaan`:

```
TEMP_WIND_fit <- sem(TEMP_WIND_model, data = airquality_mc)
```

The model is now stored as `TEMP_WIND_fit`. One way to learn about the model is to use the `parameterEstimates` function.

```
parameterEstimates(TEMP_WIND_fit)
```

```
##   lhs op rhs    est    se      z pvalue ci.lower ci.upper
## 1   Y  ~   X -1.230 0.193 -6.373      0   -1.609   -0.852
## 2   Y ~~   Y 70.337 8.042  8.746      0   54.575   86.098
## 3   X ~~   X 12.330 0.000     NA     NA   12.330   12.330
```

There is a lot of output here, and we're not going to talk about most of it now. Focus on the `est` column.

You should recognize these three estimates. Explain what these numbers represent.

In particular, pay close attention to the second line. If you are hasty, you may think this is the variance of Y, but that is not correct.

We can also produce the standardized estimates.

```
standardizedSolution(TEMP_WIND_fit)
```

```
##   lhs op rhs est.std    se      z pvalue ci.lower ci.upper
## 1   Y  ~   X  -0.458 0.060 -7.577      0   -0.576   -0.340
## 2   Y ~~   Y   0.790 0.055 14.273      0    0.682    0.899
## 3   X ~~   X   1.000 0.000     NA     NA    1.000    1.000
```

Again, explain these three numbers. (They are now listed in a column called `est.std` for "standardized estimates".)

Verify that the second line is actually the error variance. (Hint: remember $1 - r^2_{YX}$.) How do we know it's not the standardized variance of Y? (In other words, what do you actually know to be the standardized variance of Y?)

One downside of using `lavaan` is that it doesn't store the residuals, so we have no way of checking that regression assumption. For more complex models in future chapters where `lavaan` (or some comparable package) is the only choice, the residual independence assumption will be just that: an assumption. We must have substantive reason to believe that's true when we specify the model.

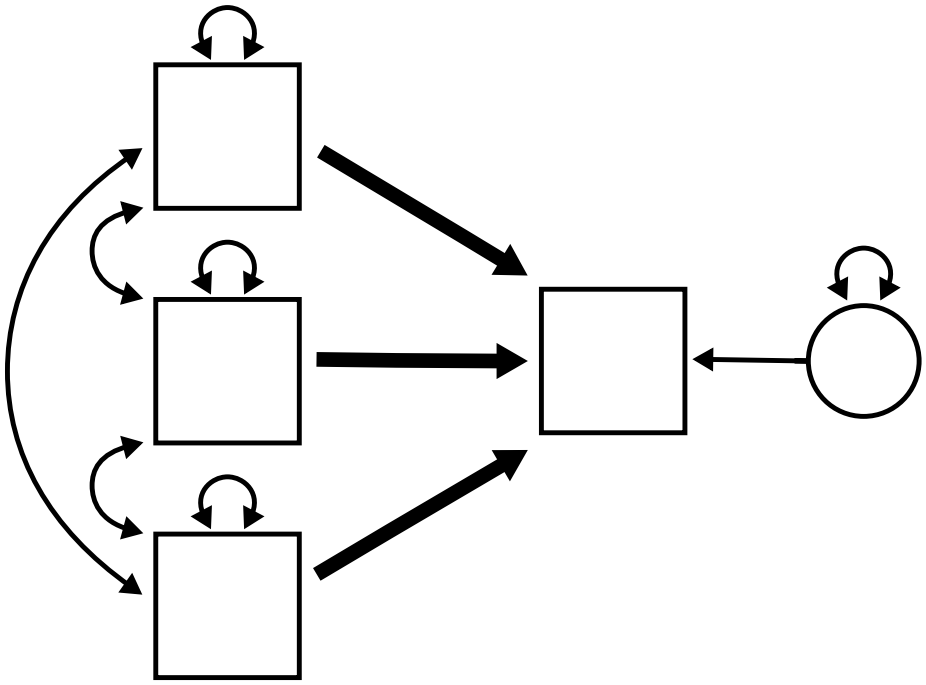
4.11 What about intercepts?

If you are familiar with regression from another course, you may be wondering where the intercepts went. Because we mean-centered and/or standardized all the data, there were no intercepts. The regression line always passes through $(0, 0)$ for mean-centered or standardized data.

[PUT A REFERENCE HERE IF WE DECIDE TO COVER MEAN STRUCTURE IN A FUTURE CHAPTER.]

Chapter 5

Multiple regression



Preliminaries

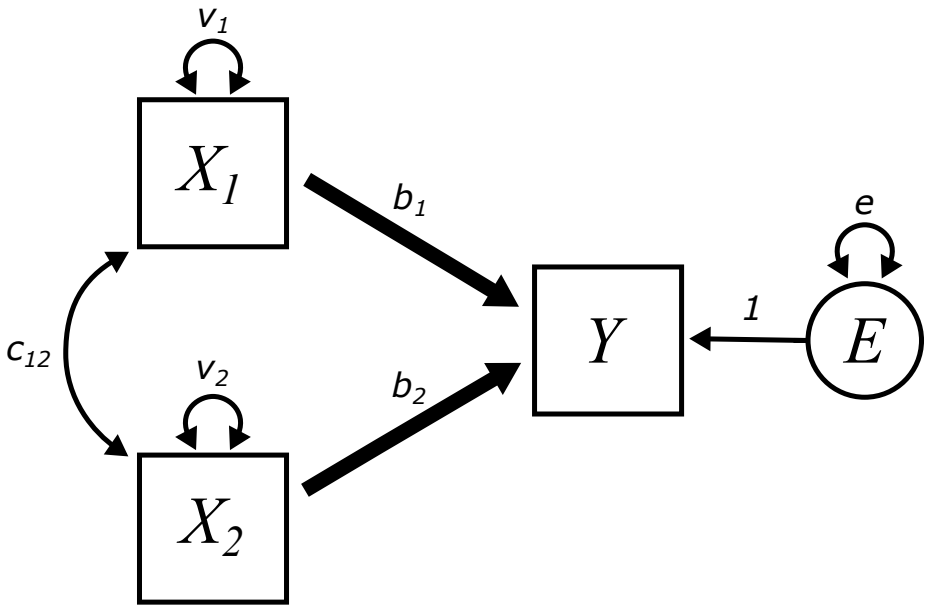
We will load the `tidyverse` package to work with tibbles, the `broom` package to calculate residuals, and `lavaan`.

```
library(tidyverse)
library(broom)
library(lavaan)
```

5.1 The multiple regression model

This chapter is an extension of all the ideas established in the last chapter. Multiple regression is like simple regression, but with more exogenous variables. There will still be only one endogenous variable. Although the archetype illustrated at the beginning of the chapter has three predictor variables, we will start with only two predictor variables to keep things simple. If you understand what happens with two variables, it's fairly straightforward to generalize that knowledge to three or more predictors. The logic is the same.

Here is a multiple regression model with two predictors and with all paths given parameter labels:



How many free parameters appear in this model?

How many fixed parameters appear in this model?

The equation describing the relationship among these variables can be written as either

$$\widehat{Y} = b_1 X_1 + b_2 X_2$$

or

$$Y = b_1 X_1 + b_2 X_2 + E$$

Why do we use $\widehat{Y}$ in the first equation and Y in the second equation?

Although we'll work through the details for only two predictors, a multiple regression model with k predictors will look like

$$\widehat{Y} = b_1 X_1 + b_2 X_2 + \cdots + b_k X_k$$

or

$$Y = b_1 X_1 + b_2 X_2 + \cdots + b_k X_k + E$$

5.2 Multiple regression assumptions

Fortunately, the assumptions for multiple regression are basically the same as they are for simple regression with a few minor modifications and one addition:

1. The data should come from a "good" sample.
2. The exogenous variables should be measured without error.
3. The relationship between $X_1, \ldots, X_k$, and Y should be approximately linear.
4. The residuals should be independent of the $X_1, \ldots, X_k$ values.
5. There should be no influential outliers.
6. The exogenous variables should not be highly correlated with one another.

We discuss these briefly:

1. Nothing has changed here. Good analysis starts with good data collection practices.
2. Nothing has changed here. It's a good idea to try to measure all our variables with as little error as possible, but in particular, measurement errors in the exogenous variables can bias our parameter estimates.
3. With only Y against X, the regression model is a line. With Y against X_1 and X_2, the regression model is a plane (a 2-dimensional plane sitting in 3-dimensional space) which is a little challenging to graph. With more predictors, the regression model lives in even higher dimensions and it's

impossible to visualize. To check this condition, the best you can usually do is to check that the scatterplots of Y against each X_i individually are approximately linear.

4. Once we fit the model, we can check the residuals. Rather than plotting the residuals against each X_i separately, we can employ a trick that we'll explain later in the chapter.
5. Nothing changes here.
6. This is the new condition. When two or more predictors variables are highly correlated with each other, this induces a condition called *multicollinearity*.

To illustrate why multicollinearity is a problem, think about the two-variable case:

$$\hat{Y} = b_1 X_1 + b_2 X_2$$

In general, we will be able to compute the values of b_1 and b_2 that best fit the model to data.

But now suppose that X_2 is just a multiple of X_1, say $X_2 = 2X_1$. Now the equation looks more like

$$\begin{aligned} \hat{Y} &= b_1 X_1 + b_2 X_2 && (5.1) \\ &= b_1 X_1 + b_2 (2X_1) && (5.2) \\ &= (b_1 + 2b_2) X_1 && (5.3) \end{aligned}$$

So even though it "looked like" there were two distinct predictors variables, this is just a simple regression in disguise. Okay, so now let's suppose we try to calculate the slope of this simple regression. Say it's 10. What are the values of b_1 and b_2? In other words, what values of b_1 and b_2 solve the following equation?

$$b_1 + 2b_2 = 10$$

Explain why it is impossible to pin down unique values for b_1 and b_2 that make the above equation true.

If you choose a large, negative value of b_1, what does that imply about the value of b_2?

If you choose a large, positive value of b_1, what does that imply about the value of b_2?

Multicollinearity works a lot like that. Even when variables are not exact multiples of each other, sets of highly correlated variables will result in equations with a large range of possible values that are consistent with the data. Even

more dangerously, your fitting algorithm may estimate values for these coefficients, but those numbers will likely be meaningless. A completely different set of numbers may also be perfectly consistent with the data.

To be clear, it's not a problem that there is covariance among our predictors. We expect that. The problem only arises when two or more predictors are *highly* correlated with each other.

5.3 Calculating regression parameters

There is nothing new here, but the calculations do start to get a little messy. Everything that follows is for two predictors only. We will not do any calculations for three or more predictors. It gets out of hand pretty quickly.

First, let's remember what we're trying to do. From the data, we can calculate the sample covariance matrix. These are all the variances and covariances among the observed variables:

$$\begin{bmatrix} Var(X_1) & \bullet & \bullet \\ Cov(X_2, X_1) & Var(X_2) & \bullet \\ Cov(Y, X_1) & Cov(Y, X_2) & Var(Y) \end{bmatrix}$$

Remember that these entries are all just numbers that we calculate directly from the data.

To get started on the model-implied matrix, let's extend **Rule 12** a little.

For any three variables X_1, X_2, and X_3:

$$\begin{aligned} Var(aX_1 + bX_2 + cX_3) = a^2Var(X_1) + b^2Var(X_2) + c^2Var(X_3) \quad (5.4)\\ + 2abCov(X_1, X_2) \quad (5.5)\\ + 2acCov(X_1, X_3) \quad (5.6)\\ + 2bcCov(X_2, X_3) \quad (5.7) \end{aligned}$$

This can be extended to any number of variables. Each variance appears with a coefficient squared and each pair of variables gets a covariance term with 2 times the product of the corresponding variable coefficients. (It's hard to describe in words, but it's still more trouble than it's worth writing it down in formal mathematical notation. Hopefully you can see how the pattern of coefficients generalizes.)

Now we can compute, for example, $Var(Y)$:

$$
\begin{aligned}
Var(Y) &= Var(b_1X_1 + b_2X_2 + E) && (5.8)\\
&= b_1^2Var(X_1) + b_2^2Var(X_2) + Var(E) && (5.9)\\
&\quad + 2b_1b_2Cov(X_1, X_2) && (5.10)\\
&\quad + 2b_1Cov(X_1, E) && (5.11)\\
&\quad + 2b_2Cov(X_2, E) && (5.12)
\end{aligned}
$$

What happens to the last two lines above? Why?

Therefore,

$$Var(Y) = b_1^2v_1 + b_2^2v_2 + 2b_1b_2c_{12} + e$$

Rule 8 and **Rule 9** extend in a similar way to sums of three or more terms. But that's even easier: just split up the covariance into as many pieces as there are terms to split.

Your turn.

Calculate $Cov(Y, X_1)$. You should get

$$b_1v_1 + b_2c_{12}$$

Calculate $Cov(Y, X_2)$. You should get

$$b_2v_2 + b_1c_{12}$$

That turns out to be all the computation we need to write down the model-implied matrix.

The first three entries are easy because they are just the parameters v_1, c_{12}, and v_2. The last column contains the entries we just calculated above.

Therefore, the model-implied matrix is

$$
\begin{bmatrix}
v_1 & \bullet & \bullet \\
c_{12} & v_2 & \bullet \\
b_1v_1 + b_2c_{12} & b_2v_2 + b_1c_{12} & b_1^2v_1 + b_2^2v_2 + 2b_1b_2c_{12} + e
\end{bmatrix}
$$

If we set these expressions equal to the numbers from the sample covariance matrix, *in theory* we could then solve for the unknown parameters in the model-implied matrix above. Three of them are basically already done since we can just read off v_1, c_{12}, and v_2. But solving for b_1, b_2, and e is no joke! And even if we did, the resulting expressions are not particularly enlightening. This is where we are quite happy turning over the computational details to a computer.

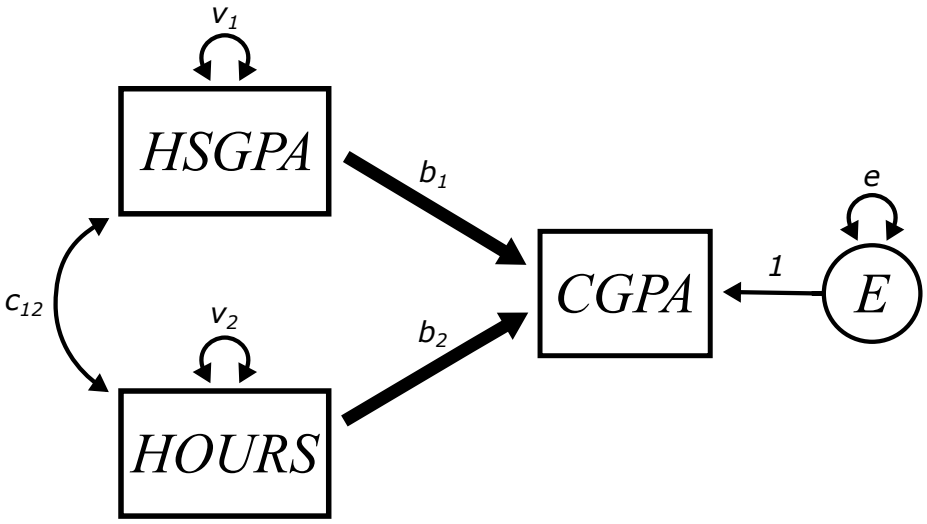
5.4 Interpreting the coefficients

Without explicit mathematical expressions for these parameters, it's a bit challenging to explain their interpretation. For now, we'll take it on faith that the following is true:

In a multiple regression model, each b_i represents the slope of the linear association between Y and X_i *while holding the value of all other predictors constant.*

What does this mean?

Let's work with a concrete example. Suppose we think that college GPA can be predicted using high school GPA along with the number of hours per week spent studying in college. Here is what such a model might look like:



If high school GPA and hours per week studying are correlated (and they likely are), they influence each other, and some of the influence has the danger of "corrupting" the estimates of the path coefficients. For example, if b_2 is positive, that would suggest that hours spent studying is associated with predicted increases in college GPA. But how do we know that's really due to the studying? Maybe students who did well in high school are just "smarter".[1] Sure, they also put in more hours studying, but maybe that doesn't matter. Maybe those students would do just as well in college even if they didn't study a whole lot. If that were the case, the coefficient b_2 would be positive just because that set of students (who happen to study more, even though it doesn't matter) also are the ones who have high college GPAs.

This is why it's important to *control* for other variables. All this means is that we need to temporarily fix the value of other variables to make the comparison

[1] For the record, we don't actually believe that is true.

fair. For example, we could look only at students with a 3.0 in high school. Among those students, there will be variability in the number of hours they study in college. If that variability is associated with variability in college GPA, we know that the hours spent studying is at least partly associated with that change. (There are lots of other factors too, but those will be swept up in the error variance.) The high school GPA can't predict that because it was fixed at 3.0, so we're comparing apples to apples. Students who got a 2.0 in high school may do more poorly overall, but the relative increase in GPA due to studying would be the same (at least if everything is linear, as is assumed).

If the parameter b_2 is estimated to be 0.13, that suggests that each additional hour of study time per week predicts an increase of 0.13 points in the college GPA, holding high school GPA constant. This means that the increase of 0.13 is only predicted within groups of students with the same high school GPA.

If the parameter b_1 is estimated to be 1.2, that suggests that college GPA is predicted to increase 1.2 points for every point increase in high school GPA. This coefficient can only be interpreted while holding hours per week studying constant. This means that this estimate only makes sense to interpret within groups of students who put in the same number of hours of studying. That takes out the hours of studying as an explanation and only accounts for changes in high school GPA to be associated with changes in college GPA.

5.5 Regression with standardized variables

Things get a little easier (although not completely trivial) with standardized variables.

First, a notational simplification. The correlations between our variables—according to our convention—would be called $r_{X_2X_1}$, r_{YX_1}, and r_{YX_2}. These are a little hard to look at in complex expressions, so we will replace them with r_{21}, r_{Y1}, and r_{Y2}. (Don't be confused by r_{21} vs r_{12} or c_{21} vs c_{12}. Since covariance and correlation are symmetric, the order of the subscripts does not matter.)

Let's look at the sample covariance matrix and the model-implied matrix for standardized variables:

$$\begin{bmatrix} 1 & \bullet & \bullet \\ r_{21} & 1 & \bullet \\ r_{Y1} & r_{Y2} & 1 \end{bmatrix} = \begin{bmatrix} v_1 & \bullet & \bullet \\ c_{12} & v_2 & \bullet \\ b_1v_1 + b_2c_{12} & b_2v_2 + b_1c_{12} & b_1^2v_1 + b_2^2v_2 + 2b_1b_2c_{12} + e \end{bmatrix}$$

The entry in the lower-left corner yields

$$r_{Y1} = b_1v_1 + b_2c_{12}$$

which simplifies to

$$r_{Y1} = b_1 + b_2 r_{21}$$

The next entry to the right of that yields

$$r_{Y2} = b_2 v_2 + b_1 c_{12}$$

which simplifies to

$$r_{Y2} = b_2 + b_1 r_{21}$$

These two equations can be solved for the two unknown parameters b_1 and b_2.

Are you feeling brave? Are your algebra skills sharp? Totally optional, but see if you can derive the final answers below:

$$b_1 = \frac{r_{Y1} - r_{Y2} r_{21}}{1 - r_{21}^2}$$

$$b_2 = \frac{r_{Y2} - r_{Y1} r_{21}}{1 - r_{21}^2}$$

These are still pretty gross, but there is some intuitive content to them. Look at the numerator of the fraction for b_1. Essentially, this is just r_{Y1} with some extra stuff. If this were simple regression, we would expect the slope b_1 to simply be the correlation between X_1 and Y. But in multiple regression, we also have to *control* for any contribution to the model coming from X_2. How do we do that? By subtracting off that contribution, which turns out to be $r_{Y2} r_{21}$. And why does the latter term appear the way it does? Because we only need to control for the effect of X_2 if X_2 is providing some of the same "information" to the regression model as X_1. Therefore, we don't need to subtract *all* of r_{Y2} to control for X_2, just a *fraction* of r_{Y2}. What fraction? r_{21}! We just need the part of X_2 that it has in common with X_1. We don't want to "double-count" the contribution to the model that is common to both X_2 and X_1.

Here's another way to think about it. What if X_1 and X_2 are independent? Calculate b_1 and b_2 from the above formulas in this much easier case. (Don't overthink this. What is r_{21} in this case?)

So if X_1 and X_2 are independent, they both offer a unique contribution to predicting Y in the model. And that contribution is just their correlation with Y (r_{Y1} and r_{Y2}, respectively). There is no overlap. But if X_1 and X_2 are correlated, then some of their "influence" is counted twice. We have to subtract out that influence so that b_1 and b_2 are only measuring the "pure" contribution of X_1 and X_2, controlling for the other one.

What about the $1 - r_{21}^2$ in the denominator? There's less of a good intuitive explanation here. It's there because—mathematically speaking—it has to be there. It rescales the slope coefficients to make everything work out the way it has to.

The final equation is the one for $Var(Y)$ in the lower-right corner of the matrix. It says

$$1 = b_1^2 v_1 + b_2^2 v_2 + 2 b_1 b_2 c_{12} + e$$

which simplifies to

$$1 = b_1^2 + b_2^2 + 2 b_1 b_2 r_{21} + e$$

Rearranging to solve for e,

$$e = 1 - (b_1^2 + b_2^2 + 2 b_1 b_2 r_{21})$$

It is *not* enlightening in any way to replace b_1 and b_2 here with the earlier fractions. We can leave e like this.

Since the standardized variance of Y is 1, the stuff inside the parentheses above represents the variance *accounted for by the model*. (That is subtracted from 1, then, to be left with e, the error variance.) This is analogous to the R^2 term described in the last chapter.

This makes some conceptual sense too. All the pieces of $(b_1^2 + b_2^2 + 2 b_1 b_2 r_{21})$ correspond to various pieces of the model. The first two relate to the direct effects of X_1 and X_2 and the third piece relates to an "indirect" effect shared between them.

5.6 Multiple regression in R

Let's fit a multiple regression model on some data about music. The data is a sample of 10,000 songs from the Million Song Dataset, a collection of metrics about the audio for a million contemporary popular music tracks.

This data set was downloaded from the CORGIS Dataset Project and more information about the variables in this data set can be found here.

```
music <- read_csv("https://raw.githubusercontent.com/VectorPosse/sem_book/main/data/mus
```

```
## Rows: 10000 Columns: 35
## -- Column specification --------------------------------------------------------
## Delimiter: ","
## chr  (4): artist.id, artist.name, artist.terms, song.id
## dbl (31): artist.familiarity, artist.hotttnesss, artist.latitude, artist.loc...
##
## i Use `spec()` to retrieve the full column specification for this data.
## i Specify the column types or set `show_col_types = FALSE` to quiet this message.
```

```
music
```

```
## # A tibble: 10,000 x 35
##    artist.familiarity artist.hotttnes~ artist.id artist.latitude artist.location
##                 <dbl>            <dbl> <chr>               <dbl>           <dbl>
##  1              0.582            0.402 ARD7TVE1~             0                 0
##  2              0.631            0.417 ARMJAGH1~            35.1               0
##  3              0.487            0.343 ARKRRTF1~             0                 0
##  4              0.630            0.454 AR7G5I41~             0                 0
##  5              0.651            0.402 ARXR32B1~             0                 0
##  6              0.535            0.385 ARKFYS91~             0                 0
##  7              0.556            0.262 ARDOS291~             0                 0
##  8              0.801            0.606 AR10USD1~             0                 0
##  9              0.427            0.332 AR8ZCNI1~             0                 0
## 10              0.551            0.423 ARNTLGG1~             0                 0
## # ... with 9,990 more rows, and 30 more variables: artist.longitude <dbl>,
## #   artist.name <chr>, artist.similar <dbl>, artist.terms <chr>,
## #   artist.terms_freq <dbl>, release.id <dbl>, release.name <dbl>,
## #   song.artist_mbtags <dbl>, song.artist_mbtags_count <dbl>,
## #   song.bars_confidence <dbl>, song.bars_start <dbl>,
## #   song.beats_confidence <dbl>, song.beats_start <dbl>, song.duration <dbl>,
## #   song.end_of_fade_in <dbl>, song.hotttnesss <dbl>, song.id <chr>, ...
```

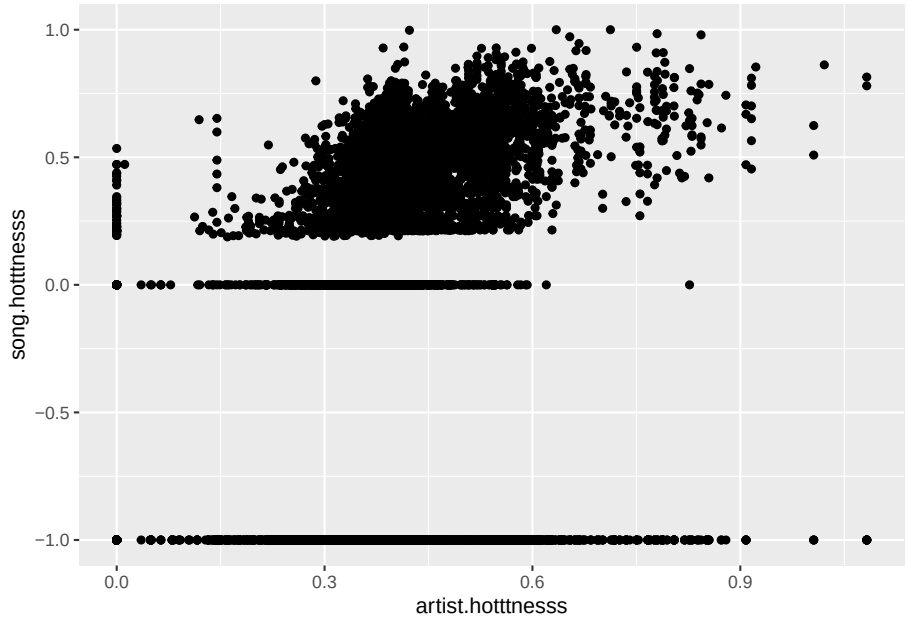
The endogenous variable of interest to us will be the measure of the song's popularity, called `song.hotttnesss` (on a scale from 0 to 1).[2] There are many possible exogenous predictors, but let's focus on three:

- `artist.hotttnesss`
  - This is the popularity of the artist (on a scale from 0 to 1).
- `song.loudness`
  - Not clear from the website what this is exactly, but it appears to be some kind of average dBFS (decibels relative to full scale). Numbers close to zero are actually as loud as recordings reasonably go and increasingly negative numbers represent softer volumes.
- `song.tempo`
  - This is measured in beats per minute (BPM).

Let's plot `song.hotttnesss` against each of the three proposed predictors to test the linearity assumption, starting with `artist.hotttnesss`:

[2] Very important that there are three t's and three s's!

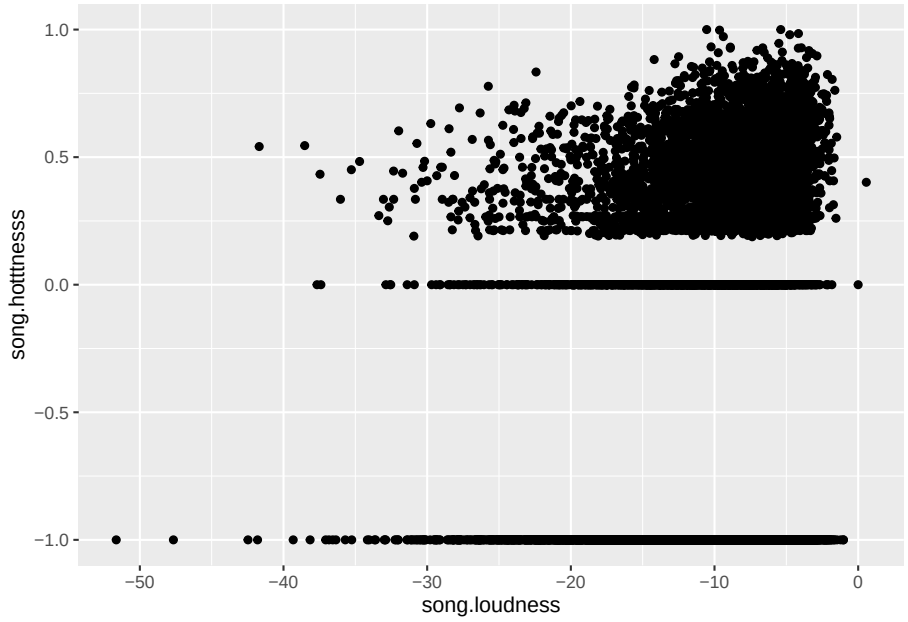
```
ggplot(music, aes(y = song.hotttnesss,
                  x = artist.hotttnesss)) +
    geom_point()
```



Uh, we've got some issues here to deal with. Since `song.hotttness` is supposed to be from 0 to 1, we can guess that the -1 values are likely coded to represent "missing" data. Even the values of 0 don't seem valid given that there is a big gap between the row of zeros and any of the rest of the cluster of actual data. The `artist.hotttness` variable also seems to have some zeros that are disconnected from the rest of the data. These may be genuine outliers, but it's more likely that these were artists for whom no data was collected.

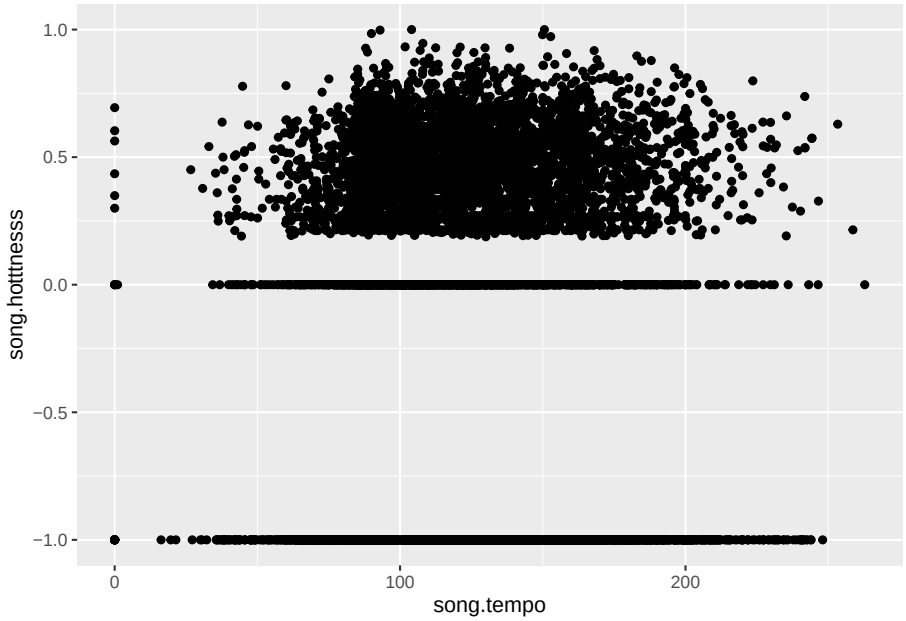
While we're suspecting issues, let's also check `song.loudness` and `song.tempo`.

```
ggplot(music, aes(y = song.hotttnesss,
                  x = song.loudness)) +
    geom_point()
```



The `song.loudness` distribution looks reasonable. It's definitely skewed to the left, but there are no strict requirements about the predictor variables having any particular type of distribution.

```
ggplot(music, aes(y = song.hotttnesss,
                  x = song.tempo)) +
    geom_point()
```



Is it possible for a song tempo to be 0 BPM?

To make it a little cleaner, the following code will `select` only the variables in which we're interested. Then it will `filter` out the values we want to keep (discarding the ones that represent missing/invalid data). We'll put this into a new tibble called `music_clean`.

```
music_clean <- music %>%
    select(song.hotttnesss, artist.hotttnesss,
           song.loudness, song.tempo) %>%
    filter(song.hotttnesss > 0,
           artist.hotttnesss > 0,
           song.tempo > 0)
music_clean
```

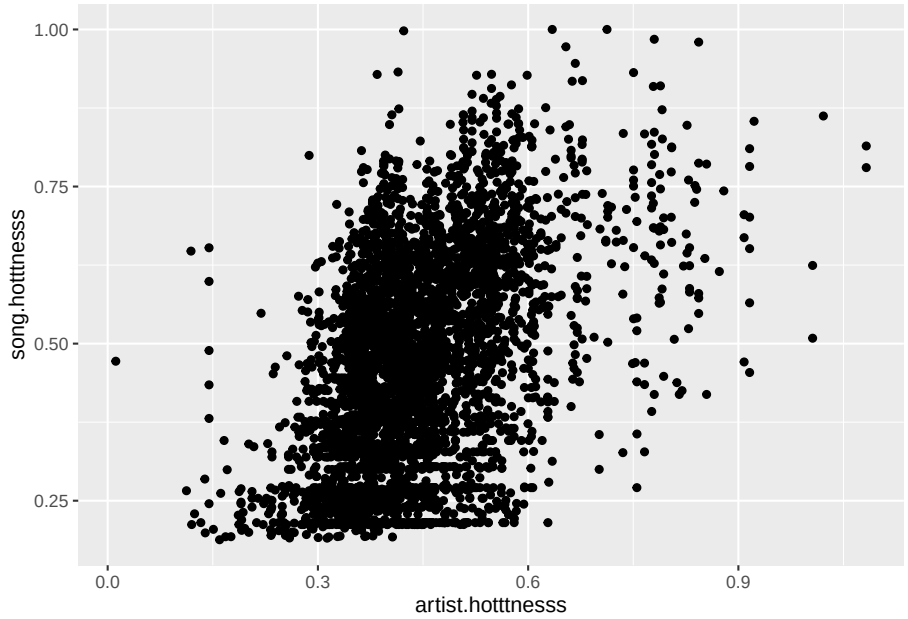
```
## # A tibble: 4,157 x 4
##    song.hotttnesss artist.hotttnesss song.loudness song.tempo
##              <dbl>             <dbl>         <dbl>      <dbl>
##  1           0.602             0.402        -11.2        92.2
##  2           0.605             0.402         -4.50      130.
##  3           0.266             0.332        -13.5        86.6
##  4           0.266             0.352         -7.54      118.
##  5           0.405             0.448         -8.58      120.
##  6           0.335             0.331        -16.1       128.
##  7           0.684             0.513         -5.27      150.
```

```
##  8           0.314            0.378         -8.05      112.
##  9           0.667            0.542         -4.26      167.
## 10           0.495            0.306        -12.3       138.
## # ... with 4,147 more rows
```

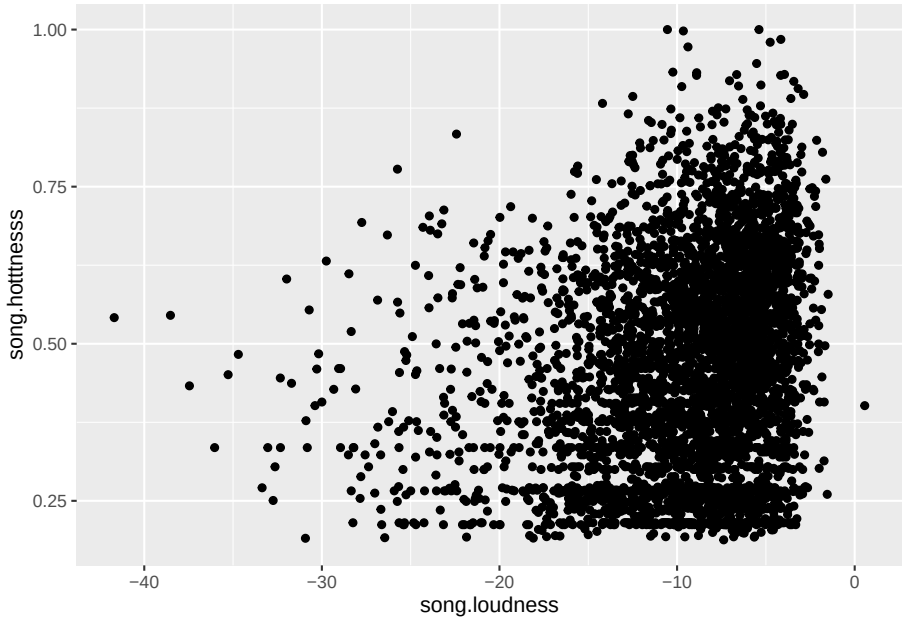
This has reduced the number of rows to 4,157, but that is still a huge sample size.

Let's check the scatterplots once more, now with the `music_clean` data.

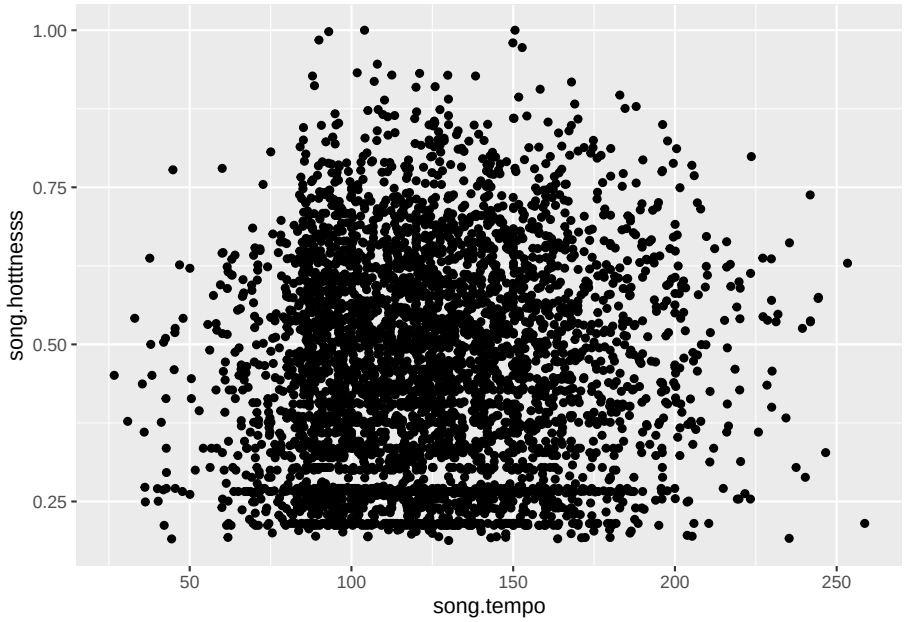
```
ggplot(music_clean, aes(y = song.hotttnesss,
                        x = artist.hotttnesss)) +
    geom_point()
```




```
ggplot(music_clean, aes(y = song.hotttnesss,
                        x = song.loudness)) +
    geom_point()
```




```
ggplot(music_clean, aes(y = song.hotttnesss,
                        x = song.tempo)) +
    geom_point()
```



There doesn't appear to be much of an association with loudness or tempo. But that doesn't violate any assumptions. (A violation of the assumptions would be a decidedly non-linear association, not just a near-zero association.) Given these graphs, we will expect the model to tell us that song popularity is maybe somewhat associated with artist popularity, but not much with loudness or tempo.

5.6.1 Using `lm`

The `lm` model specification is a minor extension of what you learned for simple regression. Just use plus signs on the right side of the tilde ~ to add more predictors. Be sure to use `music_clean` and not `music`!

```
SONG_lm <- lm(song.hotttnesss ~ artist.hotttnesss +
                  song.loudness +
                  song.tempo,
              data = music_clean)
SONG_lm
```

```
##
## Call:
## lm(formula = song.hotttnesss ~ artist.hotttnesss + song.loudness +
##     song.tempo, data = music_clean)
##
## Coefficients:
##       (Intercept)  artist.hotttnesss      song.loudness         song.tempo
##         0.1636446          0.7003692          0.0036969          0.0002057
```

We didn't go to the trouble of mean-centering the data this time, so the intercept is no longer 0. But we will not attempt to interpret the intercept anyway. The other three coefficients are b_1, b_2 and b_3, the path coefficients of the model. These are interpreted as follows:

- b_1:
  - Song popularity is predicted to increase 0.7 points for every point increase in artist popularity.

While this is mathematically true, it's kind of nonsensical to report using numbers of that magnitude. Both scales only go from 0 to 1, so an increase in 1 point would be measuring the difference between an artist with 0 popularity (the lowest possible value of popularity) to an artist with 1 popularity (the highest possible value of popularity).

A better way to report this would be to scale everything down by a factor of 10:

- Song popularity is predicted to increase 0.07 points for every 0.1 increase in artist popularity.

- b_2:

  - Song popularity is predicted to increase 0.004 points for every increase of 1 dB of loudness.

An increase of 1 dB is not very much, so again, we can scale the result to make it more meaningful. This time we'll multiply by a factor of 10:

- Song popularity is predicted to increase 0.04 points for every increase of 10 dB of loudness.

- b_3:

  - Song popularity is predicted to increase 0.0002 points for every increase of 1 BPM in the tempo.

Restate the interpretation of b_3 on a scale that makes sense. If you're not familiar with BPM, Google it to get a sense of what a reasonable jump in tempo might be.

Now that we have the model fit, we can use `broom` to capture the residuals.

```
SONG_aug <- augment(SONG_lm)
SONG_aug
```

```
## # A tibble: 4,157 x 10
##    song.hotttnesss artist.hotttnesss song.loudness song.tempo .fitted  .resid
##              <dbl>             <dbl>         <dbl>      <dbl>   <dbl>   <dbl>
##  1           0.602             0.402        -11.2        92.2   0.423  0.179
##  2           0.605             0.402         -4.50      130.    0.455  0.149
##  3           0.266             0.332        -13.5        86.6   0.364 -0.0984
##  4           0.266             0.352         -7.54      118.    0.406 -0.140
##  5           0.405             0.448         -8.58      120.    0.470 -0.0652
##  6           0.335             0.331        -16.1       128.    0.362 -0.0273
##  7           0.684             0.513         -5.27      150.    0.535  0.150
##  8           0.314             0.378         -8.05      112.    0.422 -0.108
##  9           0.667             0.542         -4.26      167.    0.562  0.105
## 10           0.495             0.306        -12.3       138.    0.361  0.134
## # ... with 4,147 more rows, and 4 more variables: .hat <dbl>, .sigma <dbl>,
## #   .cooksd <dbl>, .std.resid <dbl>
```

But how do we graph them now that there are three predictor variables? We could graph the residuals against all three predictors separately, but there's a more efficient method.

Calculate

$$Cov(E, \hat{Y})$$

by substituting

$$\hat{Y} = b_1 X_1 + b_2 X_2 + b_3 X_3$$

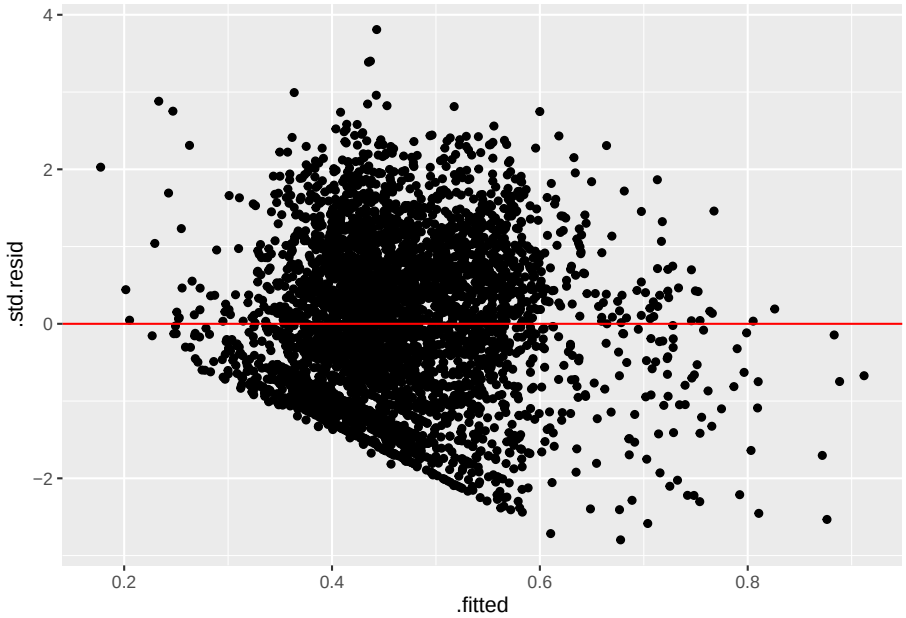
If we assume that E is independent of all the predictors, what is the value of $Cov(E, \hat{Y})$?

Of course, if $Cov(E, \hat{Y}) = 0$, that does not necessarily imply that all the $Cov(E, X_i)$ must be zero. And even if those are zero, that doesn't imply independence. But if $Cov(E, \hat{Y}) \neq 0$, then we know at least one of the $Cov(E, X_i)$ also must be non-zero. **Therefore, we can plot the residuals against the fitted values and this will serve as a *disqualifying* condition. A problem in the plot of residuals against fitted values serves as evidence of a problem with the model.**

One of the nice features of the `augment` output is that it also has a column called `.fitted` that stores the $\hat{Y}$ values.

Here are the (standardized) residuals graphed against the fitted values:

```
ggplot(SONG_aug, aes(y = .std.resid, x = .fitted)) +
    geom_point() +
    geom_hline(yintercept = 0, color = "red")
```



The weirdness in the residuals is not ideal. It doesn't prevent us from fitting the model, but we will state our results cautiously knowing that variance toward the left half of the graph is compressed relative to the right side of the graph. Therefore, the error variance is not "acting" in the model the same way across all combinations of the predictor variables.

Go back and look at the original scatterplots of the data (from `music_clean` and see if you can figure out why the residuals are cut off funny like that in the lower left quadrant.)

5.6.2 Using `lavaan`

Model specification in `lavaan` happens in a separate step with the model in quotes:

```
SONG_model <- "song.hotttnesss ~ artist.hotttnesss +
    song.loudness +
    song.tempo"
```

Then the model is fit with the `sem` function.

```
SONG_fit <- sem(SONG_model, data = music_clean)
```

Here are the unstandardized parameter estimates:

```
parameterEstimates(SONG_fit)
```

```
##                  lhs op               rhs      est    se      z pvalue ci.lower
## 1    song.hotttnesss  ~ artist.hotttnesss    0.700 0.021 33.427  0.000    0.659
## 2    song.hotttnesss  ~     song.loudness    0.004 0.000  7.987  0.000    0.003
## 3    song.hotttnesss  ~        song.tempo    0.000 0.000  3.096  0.002    0.000
## 4    song.hotttnesss ~~   song.hotttnesss    0.021 0.000 45.591  0.000    0.020
## 5  artist.hotttnesss ~~ artist.hotttnesss    0.012 0.000     NA     NA    0.012
## 6  artist.hotttnesss ~~     song.loudness    0.112 0.000     NA     NA    0.112
## 7  artist.hotttnesss ~~        song.tempo    0.091 0.000     NA     NA    0.091
## 8      song.loudness ~~     song.loudness   25.426 0.000     NA     NA   25.426
## 9      song.loudness ~~        song.tempo   27.118 0.000     NA     NA   27.118
## 10        song.tempo ~~        song.tempo 1185.061 0.000     NA     NA 1185.061
##    ci.upper
## 1     0.741
## 2     0.005
## 3     0.000
## 4     0.022
## 5     0.012
## 6     0.112
## 7     0.091
## 8    25.426
## 9    27.118
## 10 1185.061
```

Focus on the estimate column (`est`).

Do you recognize the values from lines 1 through 3?

What does line 4 mean? (Hint: it's not the variance of `song.hotttnesss` even though the notation makes it look like that.)

What's going on in lines 5 through 10?

Here are the standardized parameter estimates:

```
standardizedSolution(SONG_fit)
```

```
##                  lhs op               rhs est.std    se      z pvalue ci.lower
## 1    song.hotttnesss  ~ artist.hotttnesss   0.459 0.012 39.345  0.000    0.436
## 2    song.hotttnesss  ~     song.loudness   0.111 0.014  8.051  0.000    0.084
## 3    song.hotttnesss  ~        song.tempo   0.042 0.014  3.100  0.002    0.016
## 4    song.hotttnesss ~~   song.hotttnesss   0.752 0.011 69.186  0.000    0.731
## 5  artist.hotttnesss ~~ artist.hotttnesss   1.000 0.000     NA     NA    1.000
## 6  artist.hotttnesss ~~     song.loudness   0.202 0.000     NA     NA    0.202
## 7  artist.hotttnesss ~~        song.tempo   0.024 0.000     NA     NA    0.024
```

```
## 8      song.loudness ~~     song.loudness   1.000 0.000     NA     NA    1.000
## 9      song.loudness ~~        song.tempo   0.156 0.000     NA     NA    0.156
## 10        song.tempo ~~        song.tempo   1.000 0.000     NA     NA    1.000
##    ci.upper
## 1     0.482
## 2     0.138
## 3     0.069
## 4     0.773
## 5     1.000
## 6     0.202
## 7     0.024
## 8     1.000
## 9     0.156
## 10    1.000
```

Focus on the estimates again (`est.std`).

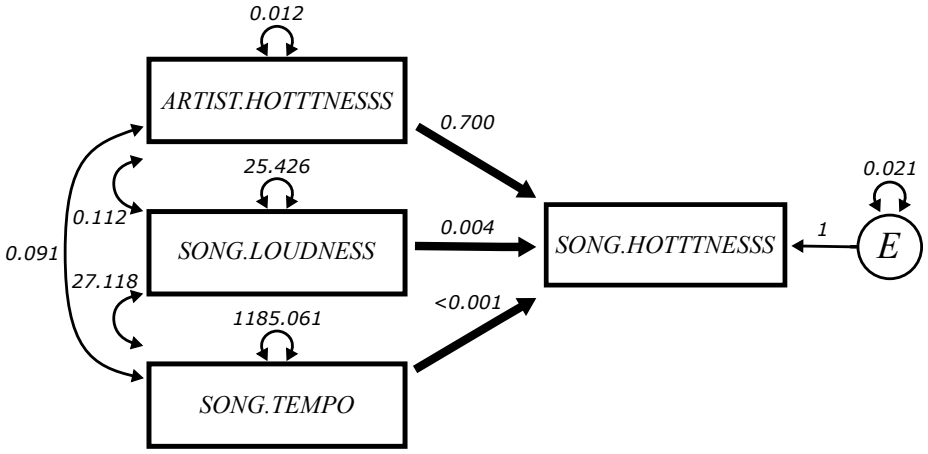
Why is it easier to compare the values in lines 1 though 3 in this output than it was in the unstandardized table? (Hint: think about units of measurement or lack thereof.)

What does the value in line 4 tell you? (Hint: it's closer to 1 than to 0.)

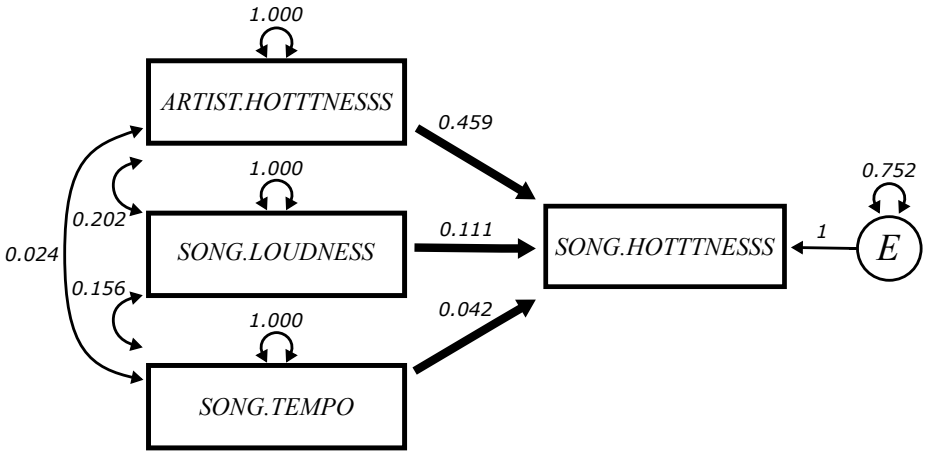
Why are lines 5, 8, and 10 equal to 1?

How do you interpret lines 6, 7, and 9?

This is the final model with all variables labeled and all unstandardized parameter estimates identified:

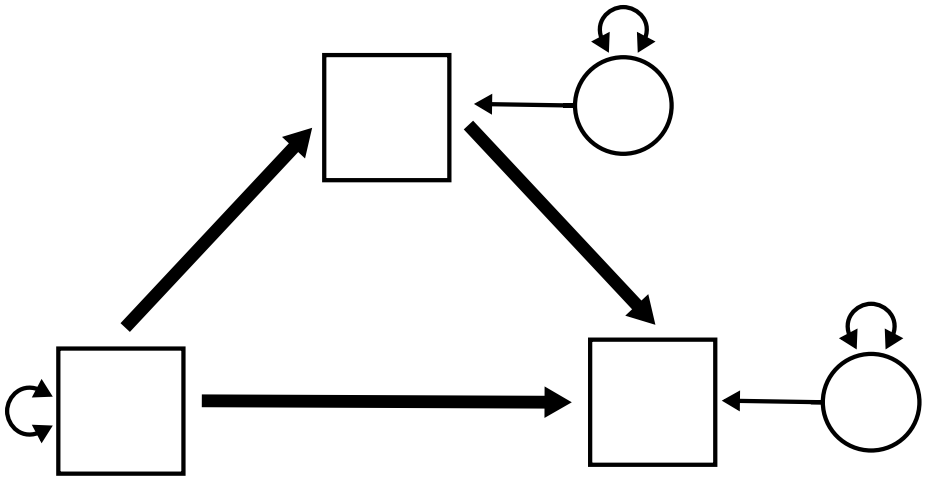


This is the same thing, but with standardized parameter estimates:


1.000
ARTIST.HOTTTNESSS
0.459
1.000
0.752
0.202
SONG.LOUDNESS
0.111
SONG.HOTTTNESSS
1
E
0.024
0.156
1.000
0.042
SONG.TEMPO

Chapter 6

Mediation



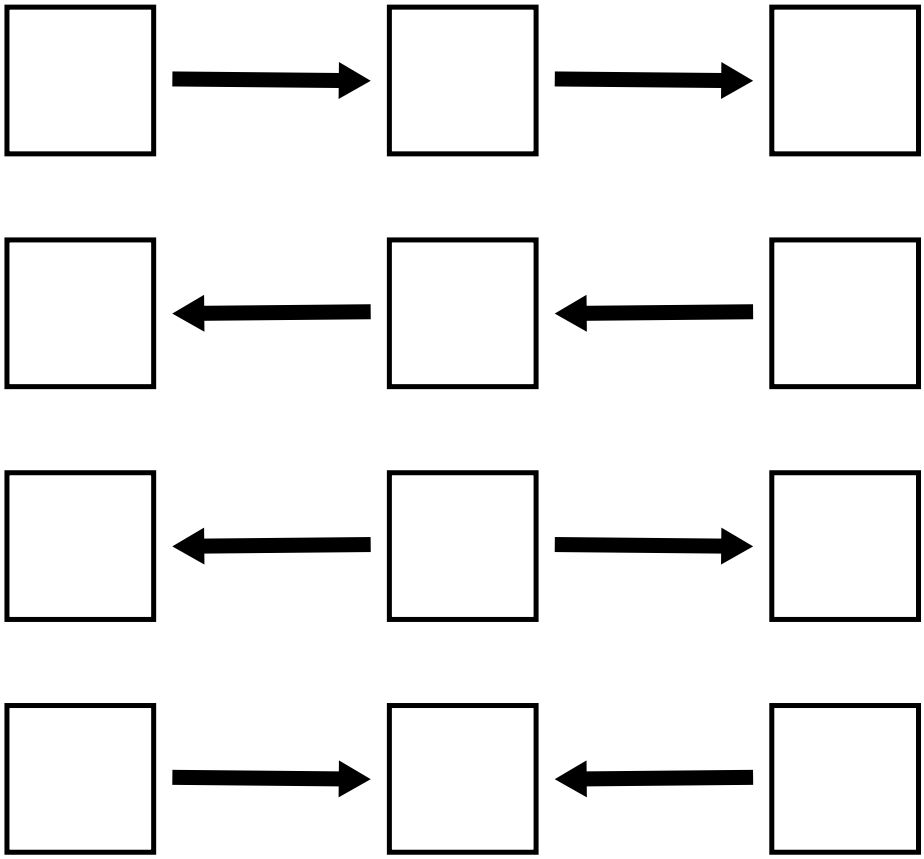
Preliminaries

We will load the `tidyverse` package to work with tibbles and `lavaan`.

```
library(tidyverse)
library(lavaan)
```

6.1 Arrows going everywhere!

To start off, let's look at all possible paths that connect three variables with two arrows. (For the moment, we'll leave out variances, covariances, and error terms.)



The second model is just a copy of the first model reversed, so we can disregard it. The other three models are genuinely distinct models with somewhat different consequences for the relationship among the three variables:

- The first model represents a "mediator".
- The third model represents a "confounder".
- The fourth model represents a "collider".

The first part of this chapter will make these distinctions clear.

6.2 Exogenous and endogenous variables

Look at the first model above. It's clear that the variable on the left is exogenous and the variable on the right is endogenous. The middle variable is called a *mediator*. Is it exogenous or endogenous?

Here we give a more specific definition of these two terms:

An *exogenous* variable is one that has no unidirectional arrows (so not counting double-headed arrows) entering it in the model diagram. It has only unidirectional arrows leaving it.

An *endogenous* variable is one that has at least one unidirectional arrow entering it (again, not counting double-headed arrows). It may have other unidirectional arrows both entering and/or leaving.

- The prefix *exo-* means "outside". So whatever variability there is in an exogenous variable must come from "outside" the model. There are no unidirectional arrows coming in, so there is nothing in the model to account for its variance, or, for that matter, its covariance with other exogenous variables.
- The prefix *endo-* means "within". The variability of endogenous variables is accounted for by other variables (including error terms) inside the model. The fact that there might be arrows leaving endogenous variables is irrelevant for this definition. It's only about arrows coming in.

According to the definition above, is a mediator exogenous or endogenous?

Here are the Really Important Rules (RIR™) for working with exogenous and endogenous variables in models. They come in three pairs:

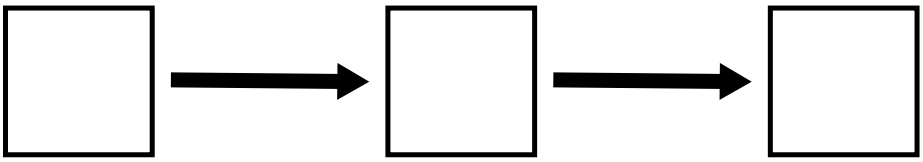
- **Rule 1:**
  - Every exogenous variable in a model requires a double-headed arrow pointing to itself, representing its variance.
  - No endogenous variable should have a double-headed arrow pointing to itself.
- **Rule 2:**
  - Every pair of exogenous variables in a model—except error terms—requires a double-headed arrow joining them, representing their covariance.
  - No other pair of variables in a model (between exogenous and endogenous, or between endogenous) should have a double-headed arrow joining them.
- **Rule 3:**

  - Every endogenous variable in a model requires an error term.
  - No exogenous variable in a model should have an error term.

These rules have important justifications. Don't just memorize the rules blindly. Understand why they are imperative.

- **Rule 1:**
  - Exogenous variables vary, but the source of their variance is not in the model. (That's what makes them exogenous.) Therefore, we have to represent their variance "manually" in the model by indicating it with a double-headed arrow.
  - On the other hand, the variance of endogenous variables is accounted for by other variables in the model already, so it doesn't need a separate parameter representing its variance.
- **Rule 2:**
  - Pairs of exogenous variables co-vary. The source of that covariance is not in the model, so we have to represent it "manually" by indicating it with a double-headed arrow. Error terms are the exception to this rule. While it's possible that error terms can co-vary, that usually isn't sensible for most models. A future chapter [LINK] will cover how and when error terms can be correlated, but it should never be the default assumption of the model.
  - Covariances between other types of variables (exogenous to endogenous, or endogenous to endogenous) are consequences of the other arrows in the diagram that create direct and indirect paths among the variables, so their covariance is not separately drawn as a double-headed arrow.
- **Rule 3:**
  - While the model is supposed to account for the variance of endogenous variables through incoming arrows, it will never be able to explain 100% of that variance just using other variables in the model. There will always be residuals, so these residuals have to be represented "manually" in the model using error terms.
  - Exogenous variables are assumed to be measured without error. While that assumption is not always very realistic in the real world, we don't have much of a choice. By their very definition, the variance of exogenous variable isn't accounted for by anything else in the model, so error terms just don't make any sense for them.

Here's the first model of the four shown earlier:



Draw this model on your own piece of paper. Following the rules above, draw in all variances, covariances, or error terms that should be present in the diagram. (Don't worry about labeling anything with letters yet. Just draw the arrows and the circles.)

6.3 Naming conventions

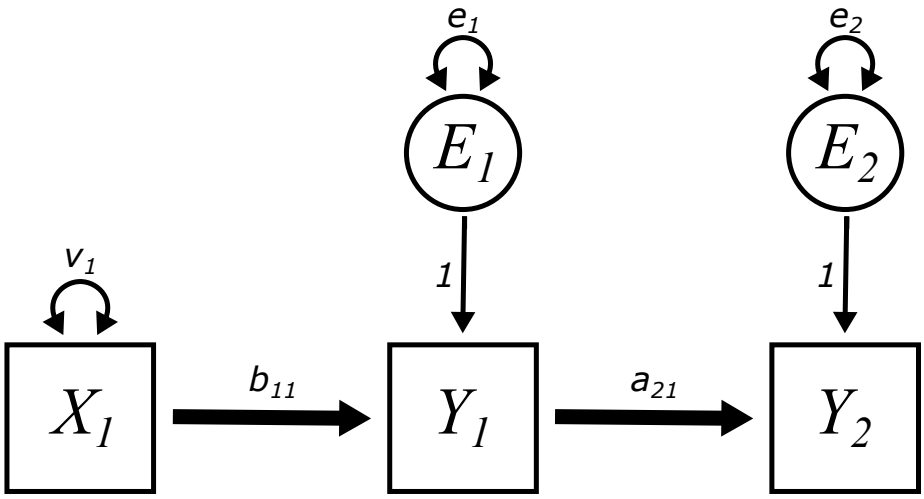
We need to establish some conventions for naming things.

- We need to name our variables. When we model real-world data, we'll use contextually meaningful names, but for abstract models we draw, we need a consistent way of labeling them.
  - Exogenous variables will be called X_i (using numbers as subscripts).
  - Endogenous variables will be called Y_i (also using numbers as subscripts).
  - Error terms will be called E_i with subscripts matching the ones on the endogenous variables Y_i to which they're attached.
- We need to label the parameters along the various paths of the model:
  - Variances will be called v_i with subscripts matching the exogenous variables X_i to which they're attached.
  - Error variances will be called e_i with subscripts matching the error terms E_i to which they're attached.
  - Covariances will be called c_{ij} connecting exogenous variables X_i and X_j. (Since covariance is symmetric, it could also be called c_{ji}.)
  - Unidirectional arrows from error terms to their corresponding endogenous variables will always be fixed parameters labeled with "1".
  - Thick, unidirectional arrows between an exogenous variable X_i and an endogenous variable Y_j will be called b_{ji}. Note the order of the subscripts: we always start with the subscript of the target variable and end with the subscript of the predictor.
  - Thick, unidirectional arrows between an endogenous variable Y_i and another endogenous variable Y_j will be called a_{ji}.

Why do we not need a naming convention for thick arrows between two exogenous variables?

6.4 Mediators

With all the rules in place for our diagrams, we can now revisit the model from above, but now, let's include all the extra bits of the model required by the aforementioned rules: a variance term for the exogenous variable X_1, error terms for the two endogenous variables Y_1 and Y_2, and parameter labels for everything.



Why did we not include any covariances in the model above?

As a concrete example to illustrate this phenomenon, imagine that the variables measure the following:

- X_1 is smoking.
- Y_1 is tar deposits in the lungs.
- Y_2 is lung cancer.

The idea is that smoking is associated with lung cancer. But what smoking *really* does is cause specific chemical processes in the lungs (including the deposition of tar), and that—among other factors—is what contributes to lung cancer. Tar serves as a "mediator" for the process that connects smoking to lung cancer.

Since there are two endogenous variables present in this model, there are two regression equations we have to write down:

$$Y_1 = b_{11}X_1 + E_1 \tag{6.1}$$
$$Y_2 = a_{21}Y_1 + E_2 \tag{6.2}$$

The sample covariance matrix will look like

$$\begin{bmatrix} Var(X_1) & \bullet & \bullet \\ Cov(Y_1, X_1) & Var(Y_1) & \bullet \\ Cov(Y_2, X_1) & Cov(Y_2, Y_1) & Var(Y_2) \end{bmatrix}$$

When working through covariance calculations in the past chapters, we've seen lots of terms pop out of the form $Cov(E, X)$. We've gotten used to canceling these terms because they are zero. (Why must they be zero?)

In this model, some of the covariance calculations will result in terms of the form $Cov(E, Y)$. These will not necessarily cancel, so we need to be more cautious.

Calculate $Cov(E_1, Y_1)$ for the model above by substituting the regression equation $Y_1 = b_{11}X_1 + E_1$. You should get e_1 (and *not* zero).

Without doing any calculations, why would we also expect $Cov(E_1, Y_2)$ to be non-zero? (Hint: how are E_1 and Y_2 connected in the diagram?)

On the other hand, why would we expect $Cov(E_1, E_2)$ to be zero in general? (Hint: look back to Really Important Rule 2 above.)

Finally, we *will* expect $Cov(E_2, Y_1)$ to be zero. Why? If you're stuck, go ahead and do the calculation to confirm.

Calculate the full model-implied matrix. You should get the following:

$$\begin{bmatrix} v_1 & \bullet & \bullet \\ b_{11}v_1 & b_{11}^2 v_1 + e_1 & \bullet \\ a_{21}b_{11}v_1 & a_{21}b_{11}^2 v_1 + a_{21}e_1 & a_{21}^2 b_{11}^2 v_1 + a_{21}^2 e_1 + e_2 \end{bmatrix}$$

If this is too tedious and time-consuming, just pick one or two of these entries to compute.

If we standardize our variables, the sample covariance matrix (which is now a correlation matrix) is

$$\begin{bmatrix} 1 & \bullet & \bullet \\ Corr(Y_1, X_1) & 1 & \bullet \\ Corr(Y_2, X_1) & Corr(Y_2, Y_1) & 1 \end{bmatrix}$$

We've switched to using $Corr$ instead of using the letter r for this exercise. That's because $r_{Y_1X_1}$, $r_{Y_2X_1}$, and $r_{Y_2Y_1}$ have subscripts inside of subscripts and are hard to read and process.

Setting the correlation matrix equal to the model-implied matrix above, we get

$$v_1 = 1$$

pretty much for free.

Now solve for b_{11} and e_1 next using the two terms in the second row of the matrix. You should get:

$$b_{11} = Corr(Y_1, X_1) \tag{6.3}$$
$$e_1 = 1 - Corr(Y_1, X_1)^2 \tag{6.4}$$

Why is this not surprising? (Hint: if you ignore Y_2 altogether and only pay attention to relationships between X_1, Y_1, and E_1, what kind of model is this?)

The parameter a_{21} is interesting. The equation implied by the lower-left element of the matrix—corresponding to $Corr(Y_2, X_1)$—is

$$Corr(Y_2, X_1) = a_{21} b_{11} v_1 \tag{6.5}$$
$$= a_{21} Corr(Y_1, X_1) \tag{6.6}$$

Solving for a_{21}:

$$a_{21} = \frac{Corr(Y_2, X_1)}{Corr(Y_1, X_1)}$$

On the other hand, the equation implied by the center element on the bottom row of the matrix—corresponding to $Corr(Y_2, Y_1)$—is

$$Corr(Y_2, Y_1) = a_{21} b_{11}^2 v_1 + a_{21} e_1 \tag{6.7}$$
$$= a_{21} Corr(Y_1, X_1)^2 + a_{21} \left(1 - Corr(Y_1, X_1)^2\right) \tag{6.8}$$
$$= a_{21} Corr(Y_1, X_1)^2 + a_{21} - a_{21} Corr(Y_1, X_1)^2 \tag{6.9}$$
$$= a_{21} \tag{6.10}$$

So we get two different answers for a_{21}!

Use $a_{21} = Corr(Y_2, Y_1)$ along with everything else you've learned to solve for e_2 using the equation in the lower-right corner of the matrix. (This is the only one we can use because it's the only term involving e_2!)

It may look ugly, but you might be surprised at the simplicity of the answer that pops out. You should get

$$e_2 = 1 - Corr(Y_2, Y_1)^2$$

Again, though, why is that not really all that surprising? (Hint: what if you ignore X_1 and treat the relationship between Y_1, Y_2, and E_2 as a simple regression?)

Since we got two different answers for a_{21}, if this model is correct, they must be equal:

$$a_{21} = Corr(Y_2, Y_1) = \frac{Corr(Y_2, X_1)}{Corr(Y_1, X_1)}$$

which, if we rearrange the fraction, implies

$$Corr(Y_1, X_1)Corr(Y_2, Y_1) = Corr(Y_2, X_1)$$

Another way to state this is that the correlation along the first path (b_{11}) and the correlation along the second path (a_{21}) multiply to give the correlation along both paths combined.

But these three correlations are numbers that are measured using the data. Is there any guarantee that the product of two of the correlations will necessarily equal the third?

No!

In fact, this will almost never be true with real data.

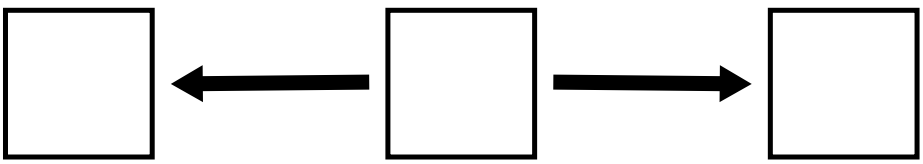
So if the model *implies* that there must be a mathematical relationship among the correlations, but the data does not support that implication, what does that say about the model?

Think about the ramifications of the above discussion for smoking and lung cancer. Smoking is correlated to tar deposits, and tar deposits are correlated with lung cancer. But if the product of those two correlations doesn't equal the overall correlation between smoking and lung cancer, what does that say about the model? What other "path" might be missing in the model that would help account for the discrepancy?

We'll return to this example in a moment. But first, let's explore the other model configurations set up at the beginning of the chapter.

6.5 Confounders

The variable in the middle of the diagram below is called a "confounder":



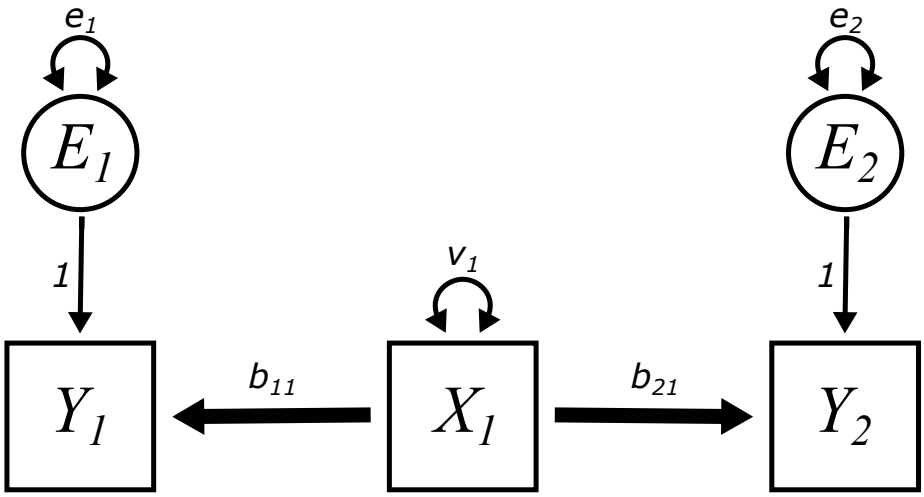
Draw this model on your own piece of paper.

Identify which variables are exogenous or endogenous.

Following the Really Important Rules, draw in all variances, covariances, or error terms that should be present in the diagram.

Finally, see if you can label all paths with letters and subscripts according to the naming conventions described earlier.

Here is the final model:



As a concrete example to illustrate this phenomenon, imagine that the variables measure the following:

- Y_1 is the presence of power lines near homes.
- Y_2 is the incidence of cancer.

For a moment, we're not going say what X_1 is.

Does a positive correlation between power lines and cancer imply that living near power lines *causes* cancer?

Does a positive correlation between power lines and cancer imply that cancer *causes* people to live near power lines? (Okay, that one is a little ridiculous, but we're making a point here.)

So if Y_1 doesn't cause Y_2 and Y_2 doesn't cause Y_1, why else might they be correlated?

There may be several plausible answers to the last question above, but here is one possibility:

- X_1 is poverty.

Give a plausible explanation for how poverty might be correlated to *both* living near power lines and cancer.

Now we turn our attention to the mathematics.

The two regression equations are

$$Y_1 = b_{11}X_1 + E_1 \tag{6.11}$$
$$Y_2 = b_{21}X_1 + E_2 \tag{6.12}$$

The sample covariance matrix will look exactly the same as it did for the mediation model above.

$$\begin{bmatrix} Var(X_1) & \bullet & \bullet \\ Cov(Y_1, X_1) & Var(Y_1) & \bullet \\ Cov(Y_2, X_1) & Cov(Y_2, Y_1) & Var(Y_2) \end{bmatrix}$$

This is because there are still three observed variables and they have the same three names, even if they are connected with arrows in a different way.

This also means the sample correlation matrix is the same:

$$\begin{bmatrix} 1 & \bullet & \bullet \\ Corr(Y_1, X_1) & 1 & \bullet \\ Corr(Y_2, X_1) & Corr(Y_2, Y_1) & 1 \end{bmatrix}$$

Calculate the full model-implied matrix. You should get the following:

$$\begin{bmatrix} v_1 & \bullet & \bullet \\ b_{11}v_1 & b_{11}^2 v_1 + e_1 & \bullet \\ b_{21}v_1 & b_{11}b_{21}v_1 & b_{21}^2 v_1 + e_2 \end{bmatrix}$$

Don't slack off on this one! Unlike the mediation example, all these terms are very straightforward to compute.

Now calculate the standardized solution. In other words, solve for all the free parameters using the sample correlation matrix.

You should get the following:

$$v_1 = 1 \tag{6.13}$$
$$b_{11} = Corr(Y_1, X_1) \tag{6.14}$$
$$e_1 = 1 - Corr(Y_1, X_1)^2 \tag{6.15}$$
$$e_2 = 1 - Corr(Y_2, X_1)^2 \tag{6.16}$$

Check that two of the equations give two different solutions for b_{21}:

$$b_{21} = Corr(Y_2, X_1) \tag{6.17}$$

$$b_{21} = \frac{Corr(Y_2, Y_1)}{Corr(Y_1, X_1)} \tag{6.18}$$

The last calculation implies that

$$Corr(Y_1, X_1)Corr(Y_2, X_1) = Corr(Y_2, Y_1)$$

Another way to state this is that the correlation along the first path (b_{11}) and the correlation along the second path (b_{21}) multiply to give the correlation along both paths combined.

But these three correlations are numbers that are measured using the data. Is there any guarantee that the product of two of the correlations will necessarily equal the third?

No!

In fact, this will almost never be true with real data.

So if the model *implies* that there must be a mathematical relationship among the correlations, but the data does not support that implication, what does that say about the model?

Does this all sound familiar? It's even more déjà vu than you think. Here are the standardized parameter solutions from the mediator example:

$$v_1 = 1 \tag{6.19}$$

$$e_1 = 1 - Corr(Y_1, X_1)^2 \tag{6.20}$$

$$e_2 = 1 - Corr(Y_2, Y_1)^2 \tag{6.21}$$

$$b_{11} = Corr(Y_1, X_1) \tag{6.22}$$

$$a_{21} = Corr(Y_2, Y_1) = \frac{Corr(Y_2, X_1)}{Corr(Y_1, X_1)} \tag{6.23}$$

And here are the standardized parameter solutions from the confounder example:

$$v_1 = 1 \tag{6.24}$$
$$e_1 = 1 - Corr(Y_1, X_1)^2 \tag{6.25}$$
$$e_2 = 1 - Corr(Y_2, X_1)^2 \tag{6.26}$$
$$b_{11} = Corr(Y_1, X_1) \tag{6.27}$$
$$b_{21} = Corr(Y_2, X_1) = \frac{Corr(Y_2, Y_1)}{Corr(Y_1, X_1)} \tag{6.28}$$

Other than just a change of notation—owing to the fact that the roles of X_1 and Y_1 are reversed in the collider example—the solutions are *identical*.

Think about the ramifications of the above discussion for living near power lines and cancer. Living near power lines is correlated to poverty, and poverty is correlated with cancer. But if the product of those two correlations doesn't equal the overall correlation between power lines and cancer, what does that say about the model? What other "path" might be missing in the model that would help account for the discrepancy?

Now suppose that scientists are able to use a carefully controlled experiment (ethical considerations aside) to prove that there is no direct effect of power lines on cancer. Note that this is *not* the same thing as saying that the correlation between power lines and cancer is zero. How does the model explain this?

When a confounder accounts for all (or nearly all) the covariance between two variables, the resulting association is called *spurious*. The association exists, but it doesn't exist due to any direct pathway.

Given that the mediator model and the confounder model are *statistically identical*, why would you use one model versus the other? Are there "philosophical" differences between mediators and confounders, even though the two models give the same results?

One of the most important takeaways from this section is the realization that the arrows along an indirect path between two variables need not go in the same direction to imply a statistical relationship between those variables.

For a mediator, the arrows do go the same way:

$$X_1 \rightarrow Y_1 \rightarrow Y_2$$

And it's no surprise to anyone that X_1 and Y_2 are associated. The association is "transmitted" from X_1 to Y_1 and then from Y_1 to Y_2 in an obvious way.

But for a collider, the arrows don't go the same way:

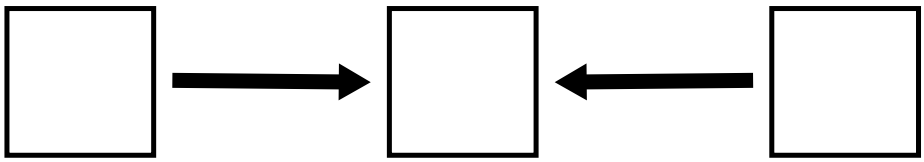
$$Y_1 \leftarrow X_1 \rightarrow Y_2$$

And, yet, there is still an association between Y_1 and Y_2. Sometimes even "backwards" arrows can "transmit" an association through indirect pathways. This is often called a "backdoor path".

But there are limits to that logic. The next example will illustrate.

6.6 Colliders

The variable in the middle of the diagram below is called a "collider":



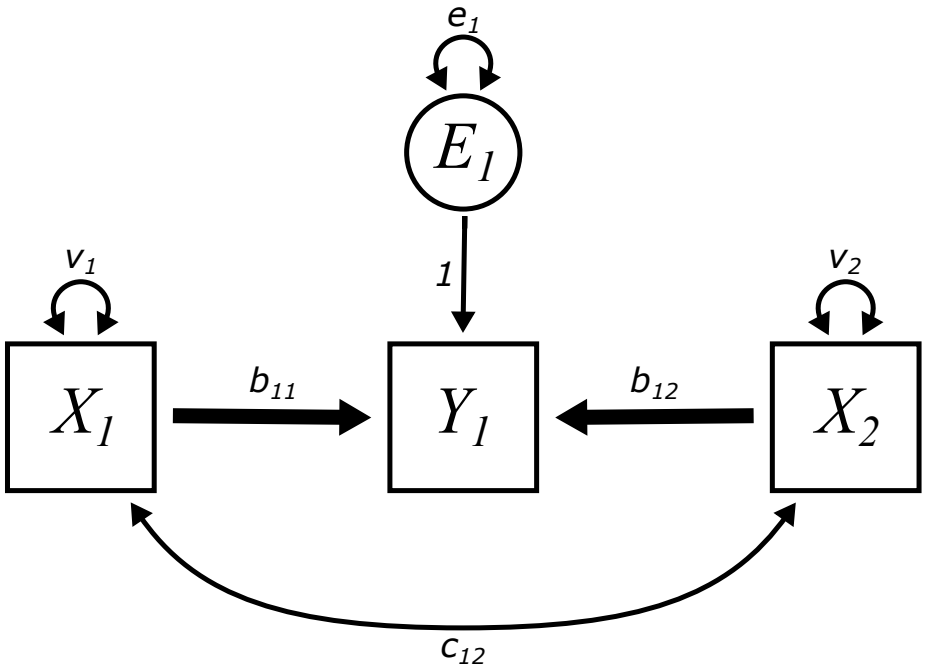
Draw this model on your own piece of paper.

Identify which variables are exogenous or endogenous.

Following the Really Important Rules, draw in all variances, covariances, or error terms that should be present in the diagram.

Finally, see if you can label all path with letters and subscripts according to the naming conventions described earlier.

Here is the final model:



As a concrete example to illustrate this phenomenon, imagine that the variables measure the following:

- X_1 is the height of basketball players.
- X_2 is the shooting accuracy of basketball players.
- Y_1 is the probability of being selected to play in a professional league.

The paths b_{11} and b_{12} make sense. Taller players and players who shoot the ball better are more likely to make it to a professional level of play. These are positive associations. Do these two paths together create an indirect path between height and shooting that accounts for covariance between them?

It turns out the answer is no!

This collider model is masquerading as another model that you have already studied in a previous chapter. It was drawn a little differently there, but the relationships among the variables and arrows are exactly the same. What is that model?

Are there any restrictions on the value of c_{12} in a multiple regression model?

While the value of c_{12} does change the interpretation of the path coefficients in a multiple regression model, analysis of the model-implied matrix always results in

$$c_{12} = Cov(X_1, X_2)$$

That value is just calculated directly from the data. It does not depend on any other parameter of the model. Since X_1 and X_2 are exogenous, the source of this covariance is independent of anything else in the model. In particular, it's possible that $c_{12} = 0$.

For example, in the basketball scenario, there's no reason to believe that height and shooting ability are correlated in the general population. The fact that they are both correlated with a higher probability of being in a professional league is irrelevant to their correlation in the population. Even if they were correlated in the population ($c_{12} \neq 0$), this would have nothing to do with the collider variable.

There's no math to do in this section. All the math we need was already done in the previous chapter.

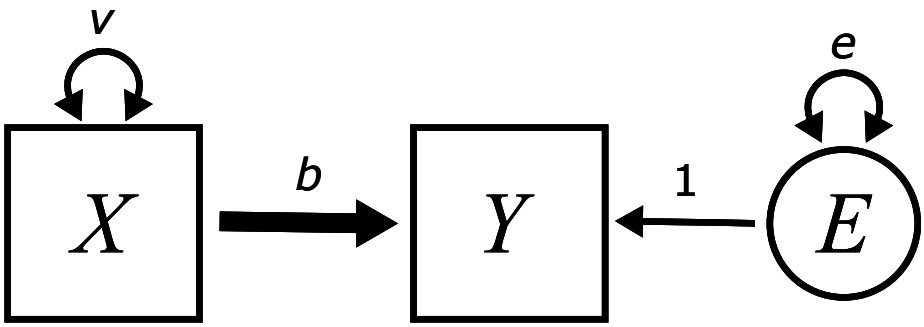
The takeaway message here is that colliders do not transmit association through them. Pathways like the following do not imply anything about the association between X_1 and X_2:

$$X_1 \rightarrow Y_1 \leftarrow X_2$$

This does not mean that X_1 and X_2 are uncorrelated. They may be correlated, but this correlation must arise from some other source—either "nature", external to the model (exogenous covariance), or some other path in the model (perhaps through a mediator or confounder).

It is not a problem to have colliders in a model. In fact, as we'll see below, every model we have analyzed in this course so far contains collider variables! The goal here is just to understand that they do not provide indirect paths for associations to be transmitted from one variable to another. Any such association must be accounted for some other way.

Consider a simple regression model:

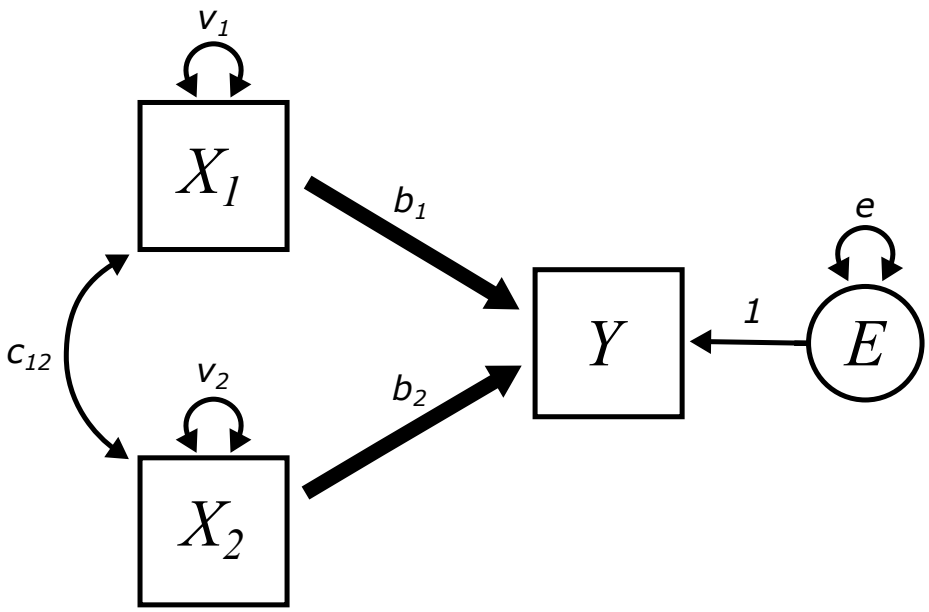


In the simple regression chapter, we explained that the error term E is uncorrelated with the exogenous variable X because there is no arrow connecting X and E. We can now admit that, while the fact about lack of correlation is true, the explanation we gave was a little misleading. Correlations can be created indirectly through sequences of paths. X and and E are connected through Y as follows:

$$X \rightarrow Y \leftarrow E$$

But why does this path not imply a correlation between X and E?

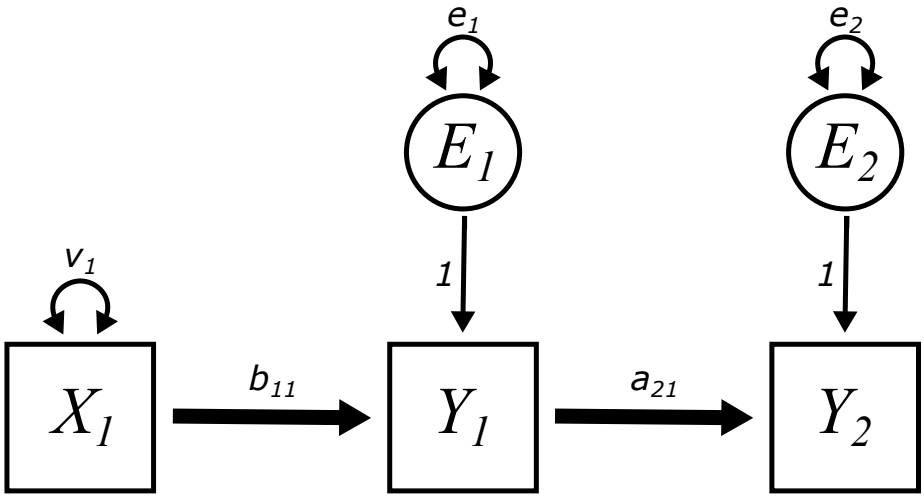
Consider the multiple regression model:



How do we know that the only covariance between X_1 and X_2 is captured by c_{12}? In other words, why is no additional covariance explained by the following path?

$$X_1 \rightarrow Y \leftarrow X_2$$

Consider the mediator example again:



One of the Really Important Rules was that the error terms should not (at least not by default) be correlated.

It's true that we haven't specified a double-headed arrow between E_1 and E_2, but how do we know there isn't an indirect path accounting for some covariance between them?

Hint: the only possible path would be

$$E_1 \rightarrow Y_1 \rightarrow Y_2 \leftarrow E_2$$

Why is that path not a problem?

Why do we have to be more careful about making assumptions about possible covariances between E_1 and Y_2?

6.7 The simple mediation model

We learned above that an indirect path through a mediator

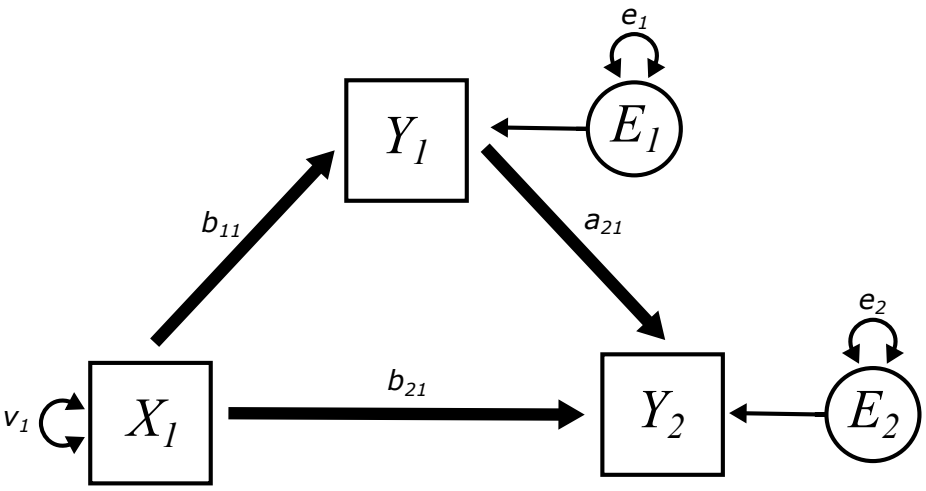
$$X_1 \rightarrow Y_1 \rightarrow Y_2$$

implies a mathematical relationship among the correlations:

$$Corr(Y_1, X_1)Corr(Y_2, Y_1) = Corr(Y_2, X_1)$$

If the correlations among these variables in the data do *not* satisfy this equation, then there is a problem with the model. There is some "left-over" association that isn't explain by this pathway.

To accommodate this possibility (which, for real-world data, is almost always the case), we can simply add a direct path between X_1 and Y_2 that will "soak up" any remaining association. The following model will be called the "simple mediation" model:



Another way to look at this model—maybe one that is more in line with typical scientific hypotheses—is to focus on the relationship between two variables, X_1 and Y_2. A simple regression will produce an estimate for the slope b. But is that path coefficient meaningful?

Yes, it represents the "total effect" of X_1 on Y_2. (If we're being careful about causal language, however, we might simply say that all covariance between X_1 and Y_2 is accounted for by b.)

But does that one path coefficient tell the whole story? Maybe not. There could be other variables that account for some of that covariance. Controlling for those variables will tell a richer story about the sources of covariation in our response variable.

To reprise the example from earlier, a scientist may want to know if smoking is associated with lung cancer. (Actually, that scientist probably wants to know if smoking *causes* lung cancer, but let's set aside causal questions for now.) A study shows a strong association. But by what mechanism is that association created? What aspect of smoking is associated with lung cancer?

Someone posits that smoking leaves tar deposits in the lungs. So more data is collected and analyzed. The model above can now tell us how much of the association between smoking and lung cancer might be accounted for through an indirect pathway that passes through Y_1, tar deposits in the lungs.

That's not the end of the story, either, but to keep things simple, we'll work with this simple mediation model with only three variables.

Here comes the math.

The regression equations are

$$Y_1 = b_{11}X_1 + E_1 \tag{6.29}$$
$$Y_2 = b_{21}X_1 + a_{21}Y_1 + E_2 \tag{6.30}$$

The sample correlation matrix is the same as for any three variables:

$$\begin{bmatrix} 1 & \bullet & \bullet \\ Corr(Y_1, X_1) & 1 & \bullet \\ Corr(Y_2, X_1) & Corr(Y_2, Y_1) & 1 \end{bmatrix}$$

But the model-implied matrix is involved enough that it doesn't even fit on the screen (nor are we making you compute it by hand). Here are the six equations separately:

$$1 = v_1 \tag{6.31}$$
$$Corr(Y_1, X_1) = b_{11}v_1 \tag{6.32}$$
$$1 = b_{11}^2 v_1 + e_1 \tag{6.33}$$
$$Corr(Y_2, X_1) = b_{21}v_1 + a_{21}b_{11}v_1 \tag{6.34}$$
$$Corr(Y_2, Y_1) = b_{11}b_{21}v_1 + a_{21}b_{11}^2 v_1 + a_{21}e_1 \tag{6.35}$$
$$1 = b_{21}^2 v_1 + a_{21}^2 b_{11}^2 v_1 + a_{21}^2 e_1 + 2a_{21}b_{11}b_{21}v_1 + e_2 \tag{6.36}$$

Skipping some algebra, we get the following (standardized) solution:

$$v_1 = 1 \tag{6.37}$$
$$b_{11} = Corr(Y_1, X_1) \tag{6.38}$$
$$e_1 = 1 - Corr(Y_1, X_1)^2 \tag{6.39}$$
$$e_2 = 1 - (b_{21}^2 + a_{21}^2 b_{11}^2 + a_{21}^2 e_1 + 2a_{21}b_{11}b_{21}) \tag{6.40}$$
$$a_{21} = \frac{Corr(Y_2, Y_1) - Corr(Y_2, X_1)Corr(Y_1, X_1)}{1 - Corr(Y_1, X_1)^2} \tag{6.41}$$
$$b_{21} = \frac{Corr(Y_2, X_1) - Corr(Y_2, Y_1)Corr(Y_1, X_1)}{1 - Corr(Y_1, X_1)^2} \tag{6.42}$$

A few observations.

The expressions for v_1, b_{11}, and e_1 are totally expected. Explain why.

The expression for e_2 is the only one not expressed in terms of all correlations. But substituting in the values of a_{21}, b_{11}, and b_{21} would not be instructive in the slightest.

The expressions for a_{21} and b_{21} have some intuitive content.

Review the content from last chapter called Regression with standardized variables, in particular the formulas given for b_1 and b_2.

Even though we had to make the notation a little more complicated, do you see any similarities between those formulas and the ones shown above for a_{21} and b_{21}?

Why might that be? Compare the diagrams for the simple mediation model in this chapter and the multiple regression model from the last chapter. What are the similarities and differences?

Hopefully you could see past the notation to realize that the formulas for b_1 and b_2 in a multiple regression model are *identical* to the formulas for a_{21} and b_{21} in our simple mediation model.

This is useful because it helps us interpret these path coefficients. As they were for multiple regression, they are simply that part of the correlation due to a direct path, controlling for the correlation along the indirect path.

If the main path of interest to our scientific hypothesis is

$$X_1 \rightarrow Y_2$$

the path coefficient b_{21} is the estimate of interest. What other indirect path is being "controlled for" in that estimate?

For a mediation model, the activity above tells us the right way to think about the path coefficient b_{21}. But it's also instructive to consider a_{21}.

Suppose the substantive path of scientific interest was

$$Y_1 \rightarrow Y_2$$

In that case, how do we interpret its path coefficient a_{21}? It is the strength of the association between Y_1 and Y_2 controlling for the indirect path where?

What kind of indirect path is that? In other words, what role does X_1 play along the indirect path from Y_1 to Y_2? (Go back and look at the diagram and the arrows.)

Suppose the substantive path of scientific interest was

$$X_1 \rightarrow Y_1$$

In that case, how do we interpret its path coefficient b_{11}? It is the strength of the association between X_1 and Y_2, but does it account for any indirect paths?

The answer is "no", but why not? What role does Y_2 play along an indirect path from X_1 to Y_1 and why does that not introduce any additional association between them?

The three activities above illustrate the remarkable fact that the simple mediation model is actually an example of all three models we've studied in this chapter: there's a mediator, a confounder, and a collider! (Each variable plays one of those roles with respect to the relationship between the other two.) So while we call it the "simple mediation model", we can actually use the implications of the model for any of the three.

6.8 Simple mediation in R

Chapter 7

Path analysis

Chapter 8

Latent variables

Chapter 9

Confirmatory factor analysis

Chapter 10

Structural equation models

Chapter 11

Structural causal models

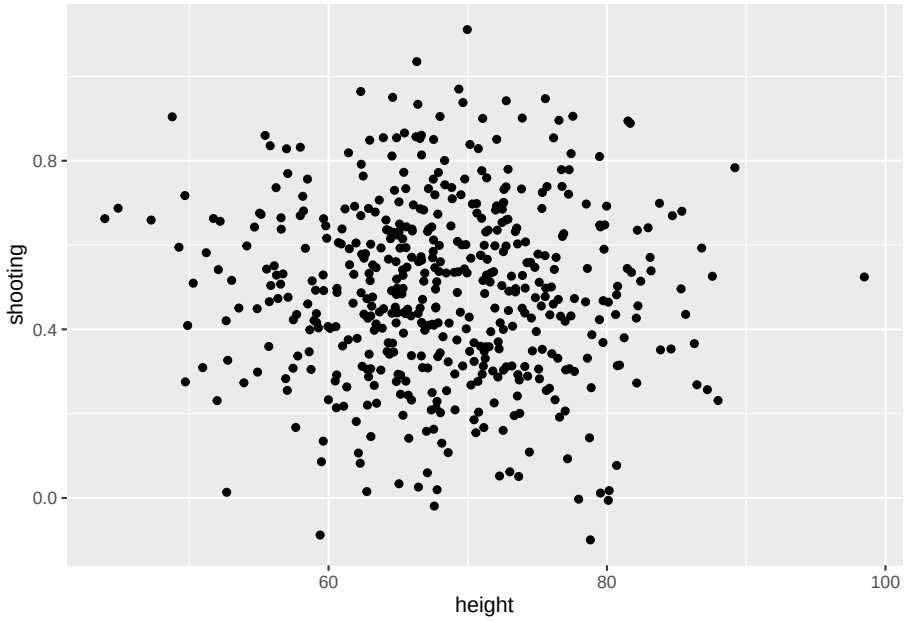
[COLLIDER BIAS EXAMPLE]

```
library(tidyverse)
```

Colliders are particularly dangerous due to a phenomenon called *collider bias*. We return to the basketball example to illustrate.

Suppose there is no association in the population between the height and the shooting accuracy of basketball players. (That may or not be true, but let's assume it for the time being.) Here is some simulated data of shooting percentages and height in inches:

```
set.seed(1)
height <- rnorm(500, mean = 68, sd = 8)
shooting <- rnorm(500, mean = 0.5, sd = 0.2)
fake_basketball_data <- tibble(height, shooting)
ggplot(fake_basketball_data, aes(y = shooting, x = height)) +
    geom_point()
```



There is no correlation between these two variables:

```
cor(fake_basketball_data$height, fake_basketball_data$shooting)
```

```
## [1] -0.04122944
```

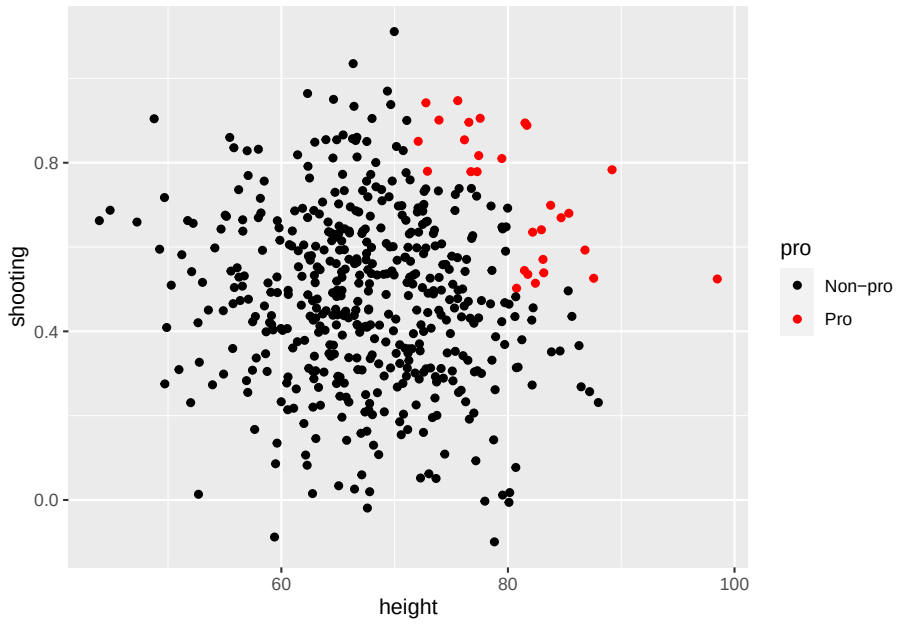
But now we'll figure out who makes it into the professional league:

```
fake_basketball_data <- fake_basketball_data %>%
    mutate(pro = ifelse((shooting > 0.75 & height > 72) |
                        (shooting > 0.50 & height > 80),
                        "Pro", "Non-pro"))
```

The condition here for making it into the pros is that *either* the player is a very good shooter (and is reasonably tall), or is very tall (and is a reasonably good shooter).

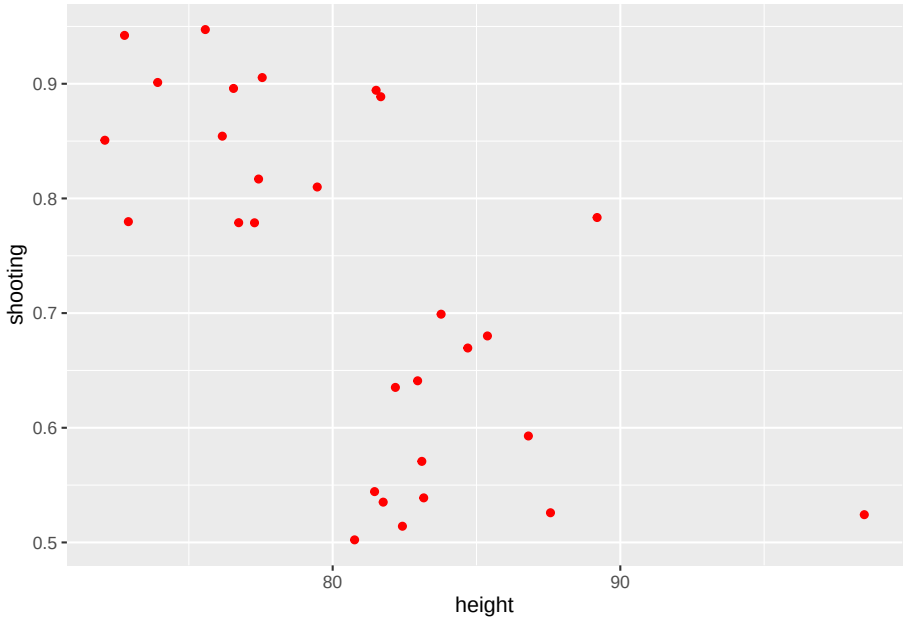
Here is the plot, with the pros colored with red points.

```
ggplot(fake_basketball_data,
       aes(y = shooting, x = height, color = pro)) +
    geom_point() +
    scale_color_manual(values = c("black", "red"))
```



We'll isolate those pro players in the graph:

```
fake_basketball_data_pros <- fake_basketball_data %>%
    filter(pro == "Pro")
ggplot(fake_basketball_data_pros,
       aes(y = shooting, x = height)) +
    geom_point(color = "red")
```



Now there is a pretty large negative correlation.

```
cor(fake_basketball_data_pros$shooting,
    fake_basketball_data_pros$height)
```

```
## [1] -0.6451966
```

So incorporating information into the model about the collider (the probability of going pro) induces an association that wasn't present in the population.

Here's another way to think about this. If you know someone's height, do you know anything about their shooting ability? Or vice versa? In the population at large, probably not much.

But what if I tell you the person is a professional basketball player? And what if I tell you this person is quite short (at least for a pro player). What information does that provide about their shooting ability?

And what if I tell you that a professional basketball player is not a very good shooter. What information does that provide about their height?

Collider bias is often called *selection bias* because it happens most often when you control for a variable that ends up selecting for certain traits in the population. In this example, if you try to "control" for the pro league variable Y_1, you end up looking at a set of players who have the same probability of going pro. In that set, players will vary in their height and shooting ability, but shorter players are forced to be better shooters, and players who don't shoot well may fare better if they are tall.

Appendix A

Variance/covariance rules

- **Rule 1**

If C is constant, then

$$Var\left(C\right) = 0$$

- **Rule 2**

If X_1 and X_2 are independent, then

$$Var\left(X_1 + X_2\right) = Var\left(X_1\right) + Var\left(X_2\right)$$

Consequence of **Rule 1** and **Rule 2**:

$$Var\left(X + C\right) = Var\left(X\right)$$

- **Rule 3**

If X_1 and X_2 are independent, then

$$Var\left(X_1 - X_2\right) = Var\left(X_1\right) + Var\left(X_2\right)$$

- **Rule 4**

If a is any number,

$$Var\left(aX\right) = a^2 Var\left(X\right)$$

Related to this rule is the corresponding one for standard deviations:

$$SD\left(aX\right) = \left|a\right| SD\left(X\right)$$

- **Rule 5**

$$Cov(X, X) = Var(X)$$

- **Rule 6**

$$Cov\left(X_1, X_2\right) = Cov\left(X_2, X_1\right)$$

- **Rule 7**

If C is constant, then

$$Cov\left(X, C\right) = 0$$

- **Rule 8**

$$Cov\left(X_1 + X_2, X_3\right) = Cov\left(X_1, X_3\right) + Cov\left(X_2, X_3\right)$$

- **Rule 9**

$$Cov\left(X_1 - X_2, X_3\right) = Cov\left(X_1, X_3\right) - Cov\left(X_2, X_3\right)$$

Consequence of **Rule 6**, **Rule 8**, and **Rule 9**:

$$Cov\left(X_1, X_2 \pm X_3\right) = Cov\left(X_1, X_2\right) \pm Cov\left(X_1, X_3\right)$$

- **Rule 10**

If a is any number,

$$Cov\left(aX_1, X_2\right) = aCov\left(X_1, X_2\right) = Cov\left(X_1, aX_2\right)$$

- **Rule 11**

If X_1 and X_2 are independent, then

$$Cov\left(X_1, X_2\right) = 0$$

- **Rule 12**

For *any* two variables X_1 and X_2:

$$Var(aX_1 + bX_2) = a^2Var(X_1) + b^2Var(X_2) + 2abCov(X_1, X_2)$$

For any three variables X_1, X_2, and X_3:

$$\begin{aligned} Var(aX_1 + bX_2 + cX_3) = {} & a^2Var(X_1) + b^2Var(X_2) + c^2Var(X_3) && \text{(A.1)} \\ & + 2abCov(X_1, X_2) && \text{(A.2)} \\ & + 2acCov(X_1, X_3) && \text{(A.3)} \\ & + 2bcCov(X_2, X_3) && \text{(A.4)} \end{aligned}$$

This can be extended to any number of variables. Each variance appears with a coefficient squared and each pair of variables gets a covariance term with 2 times the product of the corresponding variable coefficients.

Appendix B

LISREL notation