# RESURGENCE AND BOREL REGULARITY FOR ODES SEMINAR AT MATAMZEE

#### VERONICA FANTINI

#### 1. What is resurgence

Theory of resurgence was introduced by Ecalle in the '80 and it deals with divergent power series

(1.1) 
$$\tilde{\Phi}(z) = \sum_{n \ge 0} a_n z^{-n-1} \in \mathbb{C}[[z^{-1}]], \quad \text{with } a_n \sim n!$$

they have zero radius of convergence.

#### 1.1. Paradigma of Borel-Laplace sum.

$$\mathbb{C}[\![z^{-1}]\!]\ni \tilde{\Phi}:=\sum_{n\geq 0}a_nz^{-n-1} \xrightarrow{\mathcal{B}} \tilde{\phi}(\zeta)=\sum_{n\geq 0}\frac{a_n}{n!}\zeta^n\in \mathbb{C}\{\zeta\}$$

#### [make drawings of the Borel plane]

Study the analytic continuation of  $\tilde{\phi}(\zeta)$  and if  $\hat{\phi}(\zeta)$  behaves well you can go back to the z-plane via Laplace transform

$$\mathbb{C}[\![z^{-1}]\!]\ni \tilde{\Phi} \coloneqq \sum_{n\geq 0} a_n z^{-n-1} \xrightarrow{\mathcal{B}} \tilde{\phi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!} \zeta^n \in \mathbb{C}\{\zeta\}$$

$$\Phi \in \mathcal{O}(H_\theta)$$

## [make drawings of the z-plane]

The relation with  $\tilde{F}$  is via asymptotic expansion as  $\Re z e^{i\theta} \to \infty$ 

$$\mathbb{C}[\![z^{-1}]\!]\ni \tilde{\Phi} := \sum_{n\geq 0} a_n z^{-n-1} \xrightarrow{\mathcal{B}} \tilde{\phi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!} \zeta^n \in \mathbb{C}\{\zeta\}$$
 asymptotics 
$$\Phi \in \mathcal{O}(H_\theta)$$

The Laplace transform is defined

(1.2) 
$$F(z) \cong \mathcal{L}^{\theta} f(z) = \int_{0}^{+\infty} e^{-z\zeta} f(\zeta) d\zeta$$

for  $\theta \in [0, 2\pi)$  and  $F \in \mathcal{O}(H_{\theta})$  if and only if  $|f| \le Ae^{-c|z|}$  for every z in a tubular neighbourhood. In general one has to check

- $\hat{f}$  can be defined through infinity;
- $\hat{f}$  has the right decaying at  $\infty$ . HARDER

For the first one, we have to study the singularities of  $\hat{f}$ .

1.1.1. *Ecalle's resurgent function.*  $f \in \mathbb{C}\{\zeta\}$  is resurgent if it endlessly analytically continuable, i.e.

#### [draw picture]

He defined the **theory of singularities** and **relaxed the definition of Laplace transform** to deal with log- singularities, square root, poles, etc.

Examples of (minor of) singularities

$$\begin{split} \stackrel{\vee}{I}_c(\zeta) &\coloneqq \frac{\zeta^{c-1}}{(1-e^{-2\pi i c})\Gamma(c)} \quad c \in \mathbb{C} \setminus \mathbb{Z}_{>0} \\ \stackrel{\vee}{I}_c(\zeta) &\coloneqq \frac{\zeta^{c-1}}{2\pi i \Gamma(c)} \log \zeta \quad c \in \mathbb{Z}_{>0} \end{split}$$

draw singularity in a row (like Painlevé)

1.1.2. The role of  $\theta$ . Varying  $\theta$ , as long as  $|\theta - \theta'| < \pi$ ,  $\mathcal{L}^{\theta} f = \mathcal{L}^{\theta'}$  on  $H_{\theta} \cap H_{\theta'}$ . More generally, they disagree

$$\mathcal{L}^{\theta} - \mathcal{L}^{\theta'} = S_{\theta \, \theta'} \mathcal{L}^{\theta'} \quad S_{\theta \, \theta'} \in \mathbb{C}$$

These phenomena is called the Stokes phenomena. Computing the Stokes constant is crucial to understand the structure of F, and Ecalle developed the *Alien calculus* to study the Stokes phenomena.

Summarizing: the core of resurgence theory is

- (1) study the Borel plane;
- (2) compute the Stokes constants.

stress that resurgence is about Borel plane, but knows about the z-plane.

#### 1.2. Motivation: why divergent series?

#### • thimbles integrals:

COOL: if f is algebraic, I(z) is a period, i.e. it is a geometric object studied (Deligne, Malgrange, Pham, Kontsevich–Soibelman  $\sim$  generalization when f is local coordinate for 1-form  $f=\int \alpha$ ) and the resurgence of  $\tilde{I}$  has a geometric nature

– critical values of  $f \sim \text{singularity}$  in the Borel plane

- Picard-Lefschetz theory ∼ Stokes phenomena
- gradient lines ∼ Stokes indexes
- **ODEs** with irregular singularity at ∞: they admits formal solutions, and if the ODE is regular enough M.A.E.T. assures the existence of a holomorphic solution asymptotic to the formal one.
  - non linear ODEs have interesting behaviours (maybe be resonant)
- q-difference equation: like ODEs,

$$(1.3) n! \leadsto q^{n(n+1)/2} |q| > 1$$

there is a dictionary between ODEs and q-difference equation.

#### 2. ODEs

2.1. Which class of ODEs we consider.

$$(2.1) \qquad \qquad [P(\partial/\partial_z) + \frac{1}{z}Q(\partial/\partial_z) + \sum_{j=2}^d z^{-j}R_j(\partial/\partial_z)]\Phi(z) = 0$$

with

- $P(\lambda)$  a degree d polynomial
- $Q(\lambda)$  a degree d-1 polynomial
- $R_i(\lambda)$  are degree d-j polynomials

they are defined by Poincaré as *series normal de Ier ordre*. As a general fact, if  $P(-\lambda)$  has simple roots  $\alpha_1,...,\alpha_d$  then (2.1) admits d formal solution of the form

(2.2) 
$$\tilde{\Phi}_{j}(z) = e^{-\alpha_{j}z} z^{-\tau_{j}} \tilde{\phi}_{j}(z) \in e^{-\alpha_{j}z} z^{-\tau_{j}} \mathbb{C}[[z^{-1}]][\log(z)]$$

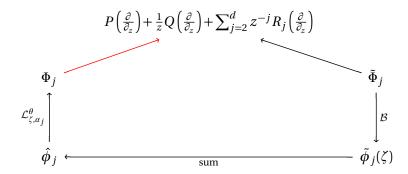
where  $\tau_i = -Q(\alpha_i)/P'(\alpha_i)$ .

The M.A.E.T theorem, guarantees under suitable assumptions on the ODE, the existence of an holomorphic solution  $\Phi_i(z)$  asymptotic to  $\tilde{\Phi}_j$  in a suitable sector.

Our goal is to prove that  $\Phi_j(z) \propto \mathcal{L}_{\zeta_j}^{\theta} \mathcal{B} \tilde{\Phi}_j$  for some angle  $\theta$ .

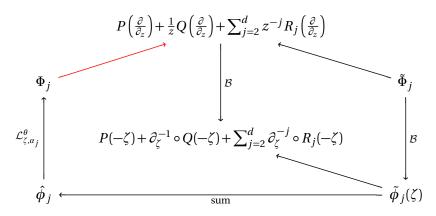
Equivalently said, Borel-Laplace summability picks an actual solution of the ODE.

#### 2.2. Borel regularity for ODEs.

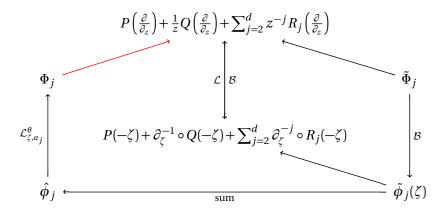


a priori it is not guarantee that  $\mathsf{L}^{\theta}_{\zeta,\alpha_j}\mathcal{B}\tilde{\Phi}_j$  is a solution of (2.1), because different functions may have the same asymptotic.

Idea of the proof is based on the following diagram:



there exists a solution  $\tilde{\phi}_j(\zeta)$  which is also slight, and in suitable coordinates it is a convergent series in  $\overset{\lor}{I}_{n+\tau_j}(\zeta_j)$ .



the Laplace transform along a Hankel contour gives an inverse for  $\mathcal{B}$ .

[repeat again the idea: a solution of (2.1) is Borel regular, because its Borel–Laplace sum gives an actual solution.]

#### Remark 2.1. [Extend the definition of $\mathcal{B}$ and $\mathcal{L}$ ]

- As far as  $\Gamma(\tau)$  is well defined, we can allow  $\tau \in \mathbb{R}$  and extend the definition of  $\mathcal{B}$ .
- Ecalle's theory of singularity introduces a generalized Laplace transform which take as contour a Hankel contour rather than a straight line. This is not the only generalization of Laplace transform Ecalle introduced.

#### 2.3. Proof of Borel regularity.

- Borel transform the  ${\rm ODE}_z$  and we get  ${\rm IE}_\zeta$ .
  - IE are not easy to be solved so usually in the application we differentiate them to get an ODE $_{\zeta}$ . If  $\hat{\phi}$  is slight we don't loose informations by differentiating.
- by Prop 1, there exists a solution  $\hat{\phi}(\zeta_i)$

$$\begin{split} \hat{\phi}(\zeta_j) &= \zeta_j^{\tau_j - 1} + \tilde{g}_j & \tilde{g}_j \in \mathcal{HL}^{\infty, 1 - \tau_j - \epsilon} \\ &= \sum_{k \geq 0} a_k \zeta_j^{\tau_j - 1 + k} + \text{h.f.} \\ &= (1 - e^{-2\pi i \tau_j}) \sum_{n \geq 0} \tilde{a}_n \overset{\vee}{I}_{\tau_j + n}(\zeta_j) & \tilde{a}_n &= a_n \Gamma(\tau_j + n), \\ & \overset{\vee}{I}_c(\xi) &= \frac{\xi^{c - 1}}{(1 - e^{-2\pi i c})\Gamma(c)} \end{split}$$

•  $\hat{\phi}(\zeta_j)$  is a germ of meromorphic function

$$\limsup_{n\to\infty} \sqrt[n]{\frac{|\tilde{a}_n|}{\Gamma(\tau_j+n)(1-e^{-2\pi i(\tau_j+n)})}} = \limsup_{n\to\infty} \sqrt[n]{\frac{a_n\Gamma(\tau_j+n)}{\Gamma(\tau_j+n)(1-e^{-2\pi i(\tau_j+n)})}} = \lim\sup_{n\to\infty} \sqrt[n]{\frac{a_n}{\Gamma(\tau_j+n)(1-e^{-2\pi i(\tau_j+n)})}} < +\infty$$

where in the last step we use

$$(2.3) \quad \infty + > \|\tilde{g}_{j}\zeta_{j}^{\tau_{j}+\epsilon-1}\|_{\infty} = \|\sum_{n\geq 1}\zeta_{j}^{\tau_{j}-1+n}\zeta_{j}^{-(\tau_{j}+\epsilon)+1}\|_{\infty} = \|\sum_{n\geq 1}a_{n}\zeta_{j}^{n-\epsilon}\|_{\infty}$$

• by Ecalle definition of Laplace transform [see Sauzin],  $\hat{\phi}(\zeta_j)$  has a well defined Laplace transform which is asymptotic to  $\sum_{n\geq 0} \tilde{a}_n z^{-\tau_j-n} (1-e^{-2\pi i \tau_j}) = (1-e^{-2\pi i \tau_j}) \sum_{n\geq 0} a_n \Gamma(\tau_j+n) z^{-\tau_j-n}$ 

$$(1-e^{-2\pi i\tau_j})\sum_{n>0}a_n\Gamma(\tau_j+n)z^{-\tau_j-n}\sim \mathcal{L}^{\theta}_{\zeta_j}\hat{\phi}_j=e^{\alpha_jz}\mathcal{L}^{\theta}_{\zeta,\alpha_j}\hat{\phi}_j$$

hence

(2.4) 
$$\mathcal{L}_{\zeta,\alpha_{j}}^{\theta}\hat{\phi} \sim e^{-\alpha_{j}z}z^{-\tau_{j}}\sum_{n\geq 0}a_{n}\Gamma(\tau_{j}+n)z^{-n} \propto \tilde{\Phi}_{j}(z)$$

### 2.4. Corollary: construct an explicit holomorphic solution predicted by M.A.E.T..

**Corollary 2.2.** The holomorphic solution which exists by M.A.E.T. can be characterized as the Borel–Laplace sum of  $\tilde{\Phi}_j$ .

#### 3. Q-DIFFERENCE EQUATIONS

# 3.1. Formal solutions vs actual solutions: paradigma of q-Borel Laplace summability.

#### 3.1.1. From ODEs to q - difference equations.

ODE	q-difference, $ q  > 1$
$\sum_{n\geq 0}^{d} a_n(z) \frac{\partial^n}{\partial z^n} \Phi = a \in \mathbb{C}(z) \text{ and } a_n \in \mathbb{C}(z)$	$\sum_{n\geq 0} a_{q,n} \sigma_q^n f = a_q \in \mathbb{C}(x) \text{ and } a_{q,n} \in \mathbb{C}(x)$
-24/(-) + 4(-) -	
$z^2\Phi'(z) + \Phi(z) = z$	$x\sigma_q f(x) + f(x) = x$
$\tilde{\Phi}(z) = \sum_{n \ge 0} (-1)^{n+1} n! z^{-n-1}$	$\hat{f}(x) = \sum_{n \ge 0} (-1)^n q^{n(n+1)/2} x^{n+1}$
General notation	
$z rac{d}{dz}$	$\sigma_q f(x) = f(qx)$
$rac{d}{dz}$	$\delta_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$
n!	$q^{n(n+1)/2}$
$e^z$	$e_q(x) = \log(1/q)\Theta_q(x)$

**Remark 3.1.** Why n! corresponds to  $q^{-n(n+1)/2}$ 

(3.1) 
$$\Gamma(n) = \int_0^{+\infty} e^{-t} t^n \frac{dt}{t}$$

(3.2) 
$$q^{-n(n-1)/2} = \int_0^{+\infty} \frac{t^n}{e_q(t)} \frac{dt}{t}$$

There is also another interesting relation between q- factorial and n!

(3.3) 
$$[n]_q! = \frac{(q;q)_n}{(1-q)^n} \to_{q\to 1} n!$$

$$\hat{f}(x) = \sum_{n \ge 0} a_n x^n \in \mathbb{C}[\![x]\!]_{(q;1)} \xrightarrow{\mathcal{B}_{(q;1)}} \varphi(\xi) = \sum_{n \ge 0} a_n q^{-n(n-1)/2} \xi^n$$
asymptotics
$$f(x)$$

3.2. **Borel regularity: Dreyfus's theorem.** Let  $\mathcal{B}_{\mu}$  be the q- Borel transform for q-Gevrey  $\mu$  series

(3.4) 
$$\mathcal{B}_{\mu} : \sum_{n>0} a_n x^n \to \sum_{n>0} a_n q^{-n(n-1)/(2\mu)} \xi^n$$

and let  $\mathsf{L}_{\mu,\kappa}^{[\lambda]}$  be the q- Laplace transform with parameters  $\mu \in \mathbb{Q}_{>0}$ ,  $\kappa \in \mathbb{N}^*$   $(\mathsf{L}_{q;1}^{[\lambda]} := \mathsf{L}_{1,1}^{[\lambda]})$ 

(3.5) 
$$\mathsf{L}_{\mu,\kappa}^{[\lambda]}\varphi(x) = \frac{\mu}{\kappa} \sum_{l \in \kappa^{-1}\mathbb{Z}} \frac{\varphi(q^l \lambda)}{\Theta_{q^{1/\mu}}(\frac{q^{1/\mu+l} \lambda}{x})} \quad \lambda \in \mathbb{C}^*/q^{\kappa^{-1}\mathbb{Z}}$$

Let  $\mathbb{H}^{[\lambda]}_{\mu,\kappa}$  be the space of functions  $\varphi \in \mathcal{M}(\mathbb{C}^*)$ , such that there exists  $\varepsilon > 0$ ,  $\Omega \subset \mathbb{C}$  connected

- $\bigcup_{l \in \kappa 1\mathbb{Z}} \{x \in \mathbb{C}^* | |x \lambda q^l| < \varepsilon |\lambda q^l| \} \subset \Omega$
- $\varphi$  can be continued analytically in  $\Omega$  with  $q^{1/\mu}$  exponential growth

$$|\varphi(\xi)| < C|\Theta_{|a|^{1/\mu}}(A|\xi|)|$$

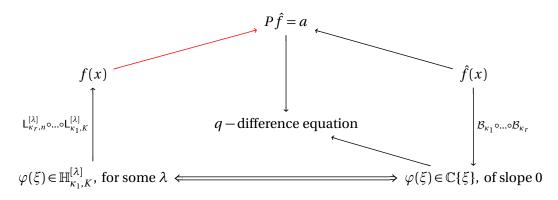
For every  $\varphi(\xi) \in \mathbb{H}^{[\lambda]}_{\mu,\kappa}$ ,  $\mathsf{L}^{[}_{\mu,\kappa}\lambda]\varphi(x) \in$ .

Under suitable assumptions on the q-difference equation, the following result holds true

**Theorem 3.2.** Let  $\hat{h}$  be a formal power series solution of a linear q-difference equation with coefficients in  $\mathbb{C}(x)$ . There exist  $\kappa_1,...,\kappa_r \in \mathbb{Q}_{>0}$ ,  $n,K \in \mathbb{N}^*$  and a finite set  $\Sigma \subset \mathbb{C}^*/q^{n^{-1}\mathbb{Z}}$ , we may compute from the q-difference equation, such that for all  $\lambda \in (\mathbb{C}^*/q^{n^{-1}\mathbb{Z}}) \setminus \mathbb{Z}$ ,

$$\mathsf{S}^{[\lambda]}(\hat{h}) := \mathsf{L}_{\kappa_r,n}^{[\lambda]} \circ \mathsf{L}_{\kappa_{r-1},K}^{[\lambda]} \circ \ldots \circ \mathsf{L}_{\kappa_1,K}^{[\lambda]} \circ \mathcal{B}_{\kappa_1} \circ \ldots \circ \mathcal{B}_{\kappa_r} \hat{h}$$

is meromorphic on  $\mathbb{C}^*$ , and is solution of the same equation as  $\hat{h}$ . Moreover,  $\mathsf{S}^{[\lambda]}(\hat{h})$  is asymptotic to  $\hat{h}$  and, for |x| close to 0 it has poles of order at most 1 that are contained in  $\lambda q^{n-1}$ ,  $n \in \mathbb{Z}$ .



# 3.3. **Summary.**

Formal series	
$\overline{\tilde{\Phi}(z) = \sum_{n \ge 0} a_n z^{-n-1} \in \mathbb{C}[\![z^{-1}]\!]_1}$	$\hat{f}(x) = \sum_{n \ge 0} a_n x^n \in \mathbb{C}[\![x]\!]_{(q;1)}$
Borel transform	
$\widetilde{\phi}(\zeta) = \sum_{n \ge 0} a_n \frac{\zeta^n}{n!} \in \mathbb{C}\{\zeta\}$	$\varphi(\xi) = \sum_{n \ge 0} a_n q^{n(n-1)/2} \xi^n \in \mathbb{C}(\!(\xi)\!)$
Laplace transform	
$\mathcal{L}_{\zeta}^{\theta} \hat{\phi}(z) = \int_{0}^{+e^{i\theta}} e^{-z\zeta} \hat{\phi}(\zeta) d\zeta \in \mathcal{O}(H_{\theta})$	$\mathcal{L}_{q;1}^{\theta}\varphi(x) = \int_{0}^{e^{i\theta}} \frac{\varphi(\xi)}{e_{q}(\xi)} d\xi$
	$\Theta$ function: $\Theta_q(x) = \sum_{n \in \mathbb{Z}} q^{-n(n+1)/2} x^n$
	$L_{q;1}^{[\lambda]}\varphi(x) = \sum_{n\geq 0} \frac{\varphi(\lambda q^n)}{\Theta_{\alpha}(\frac{\lambda q^n}{q})} \in \mathcal{O}(H^{\lambda}), \ \lambda \in \mathbb{C}^*$
	$H^{\lambda} = D_r^{\lambda} \setminus \{-\lambda q^n, n \in \mathbb{Z}\}$
	·
Domain of definition	
$\hat{\phi}(\zeta)$ s.t. $\exists C, a > 0$	$\varphi(\xi)$ s.t. $\exists C, a > 0$
$ \hat{\phi}(\zeta)  < C e^{a \zeta },  \zeta \in S_{\delta}$	$ \varphi(\xi)  < C \xi ^a q^{\frac{1}{2}\left(\frac{\log \xi }{\log q}\right)^2},  \xi \in S_\delta \cap \mathbb{C}^*$
$S_{\delta}$ is an half-strip	
	$\varphi(\xi) \in \mathbb{H}_{\mu,\kappa}^{[\lambda]}$
Gevrey asymptotics	
$\mathcal{L}_{\zeta}^{ heta}\hat{\phi}(z)\!\sim_{1}\! ilde{\Phi}(z)$	$L_{q;1}^{[\lambda]}\varphi(x)\!\sim_{q;1}\hat{f}(x)$
Stokes phenomena	
varying $\theta$ , $\mathcal{L}^{\theta}$ jumps	varying $\mu \not\equiv_q \lambda$ , $L_{q;1}^{[\mu]}$ jumps
Borel regularity	
M.A.E.T: existence of hol. solutions	[Praagman, 86]: existence of merom. solutions
F., Feynes: BL sum gives an actual solution	[Dreyfus, 14]: q–BL gives an actual solution
slight functions	slope 0 operators