

REGULAR SINGULAR VOLTERRA EQUATIONS ON COMPLEX DOMAINS

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Abstract. The inverse Laplace transform can turn a linear differential equation on a complex domain into an equivalent Volterra integral equation on a real domain. This can make things simpler: for example, a differential equation with irregular singularities can become a Volterra equation with regular singularities. It can also reveal hidden structure, especially when the Volterra equation extends to a complex domain. Our main result is to show that for a certain kind of regular singular Volterra equation on a complex domain, there is always a unique solution of a certain form. As a motivating example, this kind of Volterra equation arises when using Laplace transform methods to solve a *level 1* differential equation.

1. Introduction.

1.1. Motivation. In its most basic form, the Laplace transform \mathcal{L} turns exponential-type functions of a real “position” variable ζ into holomorphic functions of a complex “frequency” variable z . Through identities like

$$\begin{aligned}\frac{\partial}{\partial z}\mathcal{L}\varphi &= \mathcal{L}(-\zeta\varphi) \\ \mathcal{L}k\mathcal{L}\varphi &= \mathcal{L}(k * \varphi) \\ z^{-\nu}\mathcal{L}\varphi &= \mathcal{L}\partial^{-\nu}\varphi,\end{aligned}$$

where $\partial^{-\nu}$ is the Riemann-Liouville fractional integral of order $\nu \in (0, \infty)$, the Laplace transform pulls differential operators on the frequency domain back to Volterra integral operators on the position domain. The favorable regularity properties and comprehensive theory of Volterra equations can thus be brought to bear on linear differential equations.

Some differential equations pull back to Volterra equations with real-analytic kernels, which extend to holomorphic Volterra equations on complex extensions of the position domain. Solutions that look unrelated in the frequency domain may turn out to be linked by analytic continuation along the complex position domain, as seen in the phenomenon of *resurgence* [?][?][?, Section 2.4].

Differential equations with irregular singularities in the frequency domain can pull back to Volterra equations with regular singularities in the position domain. In Section 4, we’ll see this helpful behavior in a certain subclass of what Ecalle calls *level 1* differential equations, which includes classical examples like the modified Bessel equation, the equation describing the vibration modes of a solid triangular cantilever [?, from Equation 12.58], and the Airy equation after a change of coordinate. This last example generalizes to the Airy-Lucas equations, a family of level 1 equations that feature in current research [?, Equations 3.2].

From our experience with solving level 1 differential equations by Borel summation, we expect each of the corresponding integral equations to have a special kind of solution when the integration base point coincides with a regular singularity. Let’s say we’ve chosen the position variable ζ so that the integration base point is at $\zeta = 0$, and our position-domain Volterra operator has a holomorphic kernel $k(a, a')$ with a $\tau/\zeta(a)$ singularity. Assuming the residue τ is real and positive, we expect to find a solution with the following features:

- It has an $O(\zeta^{\tau-1})$ singularity at $\zeta = 0$.
- It’s of exponential type, meaning that it’s $O(e^{\Lambda|\zeta|})$ for some $\Lambda \in \mathbb{R}$ as ζ grows.

43 These conditions are just strong enough to ensure that the solution has a well-defined Laplace
 44 transform, which by construction will satisfy the differential equation we started with in the
 45 frequency domain. The first condition also tells us that the Laplace-transformed solution is
 46 $O(z^{-\tau})$ as the frequency variable z grows.

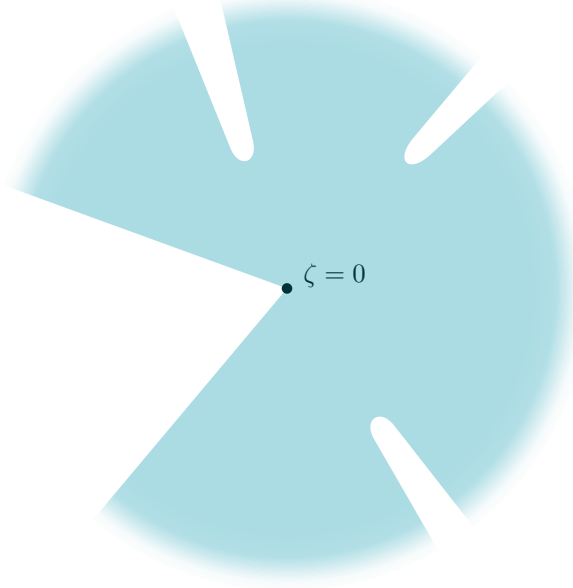
47 Our goal is to justify the expectations above with an existence and uniqueness result,
 48 stated in Section 1.4. We'll achieve it by embodying our expectations in a Banach space of
 49 holomorphic functions—the space $\mathcal{H}L_{\tau-1,\Lambda}^\infty(\Omega)$ defined in Section 2. We'll apply our results
 50 to level 1 differential equations in Section 4.

51 **1.2. Notation for uniform bounds.** Given a complex-valued function φ and a non-
 52 negative, real-valued function ω , we'll write $\varphi \lesssim \omega$ to say that $|\varphi|$ is bounded by a constant
 53 multiple of ω . Unless we say otherwise, the bound holds throughout the domain where both
 54 functions are defined.

55 **1.3. Setting.**

56 **1.3.1. The domain.** Throughout this paper, as described in Section 1.1, the position
 57 variable ζ will be the standard coordinate on \mathbb{C} . Take a simply connected open set $\Omega \subset \mathbb{C}$
 58 that touches but doesn't contain $\zeta = 0$, and satisfies the following condition.

59 (STAR) The set Ω is star-shaped around $\zeta = 0$. In other words, for any
 60 $a \in \Omega$, a straight path from $\zeta = 0$ to a stays in Ω . Since Ω doesn't
 61 contain $\zeta = 0$, we'll always omit that starting point from the path.
 62 For the applications we have in mind, Ω might look something like the set pictured below.



63

64 **1.3.2. The prototype operator.** The prototypical example of the kind of operator
 65 we'll be working with is a holomorphic Volterra operator \mathcal{V}_0 with a separable kernel and a
 66 regular singularity at $\zeta = 0$.

67 Being a holomorphic Volterra operator means that \mathcal{V}_0 sends each holomorphic function

68 φ on Ω to a new holomorphic function

$$69 \quad [\mathcal{V}_0 \varphi](a) = \int_{\zeta=0}^a k_0(a, \cdot) \varphi d\zeta.$$

70 Being separable means that the kernel $k_0(a, a')$ factors into a function of a times a function
71 of a' . We'll suppose this product can be written as a ratio

$$72 \quad k_0(a, a') = -\frac{q(a')}{p(a)},$$

73 where p and q are holomorphic functions on Ω . Having a regular singularity at $\zeta = 0$ means
74 the following:

75 (SING | τ) For some non-zero constant τ —which we'll require, for simplicity,
76 to be real and positive—the difference

$$77 \quad k_0(a, a) - \frac{\tau}{\zeta(a)}$$

78 is bounded on a neighborhood of $\zeta(a) = 0$ in Ω . In addition, for
79 each $\sigma > \tau$, the bound

$$80 \quad |k_0(a, a')| < \frac{\sigma}{|\zeta(a)|}$$

81 holds over some neighborhood of $(\zeta(a), \zeta(a')) = (0, 0)$ in Ω^2 .

82 For most of our results, we'll need to make sure that Ω doesn't touch any singularities of \mathcal{V}_0
83 other than the one at $\zeta = 0$. We'll also need to control $k_0(a, a')$ when a is away from $\zeta = 0$,
84 requiring k_0 to be bounded on the diagonal in Ω^2 and to grow at most exponentially as we
85 move away from the diagonal. These requirements can be combined into one condition.

86 (DIAG₀ | λ_Δ) For some constant λ_Δ , we have

$$87 \quad |k_0(a, a')| \lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

88 over all $a, a' \in \Omega$.

89 This condition explains why Ω might have the sort of shape illustrated in Section 1.3.1. As
90 Ω stretches out toward infinity, it has to part around the zeros of p , keeping well away from
91 every zero except the one at $\zeta = 0$. We'll occasionally mention an optional condition on p
92 that allows us to state our main results more explicitly.

93 (REG-P | B, ϵ) For some non-zero constant B and some $\epsilon > 0$,

$$94 \quad p \in B\zeta + O(|\zeta|^{1+\epsilon})$$

95 at $\zeta = 0$.

96 **1.3.3. The perturbed operator.** Now, let's perturb \mathcal{V}_0 to a more general operator
97 $\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_*$. The perturbation \mathcal{V}_* doesn't have a separable kernel,¹ but does have a

¹Unless \mathcal{V}_* is zero, of course.

98 smoothing effect that counteracts the singularity of \mathcal{V}_0 (as we'll show in Proposition 3.4).
 99 To get the smoothing effect, we'll require the kernel k_\star of \mathcal{V}_\star to vanish to some order $\gamma > 0$
 100 on the diagonal in Ω^2 . This requirement, combined with two others, will be made precise
 101 in Condition (DIAG $_\star$ | γ, λ_Δ).

102 Since \mathcal{V}_\star is a holomorphic Volterra operator, k_\star is a holomorphic function on Ω^2 . We'll
 103 allow $k_\star(a, a')$ to have a simple pole at $\zeta(a) = 0$, like $k_0(a, a')$ does, but we won't allow any
 104 sharper singularity. We'll also put an exponential bound on how fast k_\star grows away from
 105 the diagonal, mimicking Condition (DIAG $_0$ | λ_Δ) on k_0 . Altogether, we'll require:
 106 (DIAG $_\star$ | γ, λ_Δ) For some constant $\gamma > 0$, we have

$$107 \quad |k_\star(a, a')| \lesssim \frac{|\zeta(a) - \zeta(a')|^\gamma}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

108 over all $a, a' \in \Omega$.

109 Notice that this condition prevents k_\star from being separable—unless it's zero, of course.

110 Like k_0 , the combined kernel $k = k_0 + k_\star$ of \mathcal{V} grows in a controlled way when its
 111 arguments are near $\zeta = 0$, and when the difference between its arguments grows. We'll
 112 provide specific bounds in Section 3.4.2.

113 **1.4. Main results.** We want to show that when \mathcal{V} is a regular singular Volterra oper-
 114 ator of the kind described in Section 1.3.3, the equation $f = \mathcal{V}f$ has a unique solution of a
 115 certain form. For the prototypical operator \mathcal{V}_0 described in Section 1.3.2, this solution can
 116 be written explicitly.

117 **THEOREM 1.1.** *Suppose \mathcal{V}_0 satisfies Condition (SING | τ). Then the equation*

$$118 \quad f = \mathcal{V}_0 f$$

119 *has the prototype solution*

$$120 \quad (1.1) \quad f_0(a) = \frac{1}{p(a)} \exp \left(- \int_b^a \frac{q}{p} d\zeta \right).$$

121 *Changing the base point $b \in \Omega$ just multiplies f_0 by a non-zero constant.*

122 We'll prove this result in Section 3.2.1.

123 The solution f_0 from Theorem 1.1 has, at worst, a mild power-law singularity at $\zeta = 0$.
 124 With stronger constraints on \mathcal{V}_0 , we can also ensure that f_0 grows at most exponentially
 125 as $|\zeta| \rightarrow \infty$. The function spaces defined in Section 2 express both of these regularity
 126 properties.

127 **THEOREM 1.2.** *Suppose \mathcal{V}_0 satisfies Condition (SING | τ). Then, on a small enough*
 128 *neighborhood of $\zeta = 0$, we have $|f_0| \lesssim |\zeta|^{\tau-1}$.*

129 *Suppose \mathcal{V}_0 also satisfies Condition (DIAG $_0$ | λ_Δ). Then f_0 belongs to the space $\mathcal{H}L_{\tau-1, \bullet}^\infty(\Omega)$*
 130 *defined in Section 2.*

131 We'll prove this result in Section 3.2.2.

132 **Remark 1.3.** When \mathcal{V}_0 also satisfies Condition (REG-P | B, ϵ), we can get a better esti-
 133 mate of the prototype solution near $\zeta = 0$, as described in Proposition 3.3 from Section 3.2.2.

134 When we perturb \mathcal{V}_0 to the more general operator $\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_\star$ from Section 1.3.3, the
 135 equation we're trying to solve gets more complicated, but its regular singularity at $\zeta = 0$

136 stays essentially the same. We might therefore expect to find a solution that looks like f_0
 137 near the singularity, differing only by a less singular perturbation. If we strengthen the
 138 constraints on \mathcal{V}_0 a little more, this expectation is fulfilled.

139 **THEOREM 1.4.** *Suppose \mathcal{V}_0 satisfies Conditions (SING | τ) and (DIAG₀ | λ_Δ), and \mathcal{V}_\star
 140 satisfies Condition (DIAG_★ | γ, λ_Δ). Then the equation*

$$141 \quad f = \mathcal{V}f$$

142 *has a unique solution f in the affine subspace*

$$143 \quad f_0 + \mathcal{H}L_{\tau-1+\gamma, \bullet}^\infty(\Omega)$$

144 *of the space $\mathcal{H}L_{\tau-1, \bullet}^\infty(\Omega)$ defined in Section 2. Here, f_0 is the prototype solution (1.1) from
 145 Theorems 1.1–1.2, which belongs to the space $\mathcal{H}L_{\tau-1, \bullet}^\infty(\Omega)$.*

146 *For any $\rho > \tau$, the uniqueness of the solution still holds in $f_0 + \mathcal{H}L_{\rho-1, \bullet}^\infty(\Omega)$. In
 147 other words, lowering ρ into $(\tau, \tau + \gamma)$ to allow a sharper singularity won't reveal any more
 148 solutions, and raising ρ too high to admit the solution found in $f_0 + \mathcal{H}L_{\tau-1+\gamma, \bullet}^\infty(\Omega)$ will leave
 149 no solution at all.*

150 This result will follow from a more general result about inhomogeneous equations.

151 **LEMMA 1.5.** *Suppose \mathcal{V}_0 satisfies Conditions (SING | τ) and (DIAG₀ | λ_Δ), and \mathcal{V}_\star satis-
 152 fies Condition (DIAG_★ | γ, λ_Δ). Suppose we're also given a function g , which for some $\rho > \tau$
 153 belongs to the space $\mathcal{H}L_{\rho-1, \bullet}^\infty(\Omega)$ defined in Section 2. Then the inhomogeneous equation*

$$154 \quad f = \mathcal{V}f + g,$$

155 *has a unique solution f in the space $\mathcal{H}L_{\rho-1, \bullet}^\infty(\Omega)$.*

156 We'll prove Lemma 1.5 in Sections 3.4–3.5, using the contraction mapping theorem. The
 157 heart of the argument is Proposition 3.7, which shows us how to find relevant subspaces
 158 where \mathcal{V} is a contraction.

159 We'll reduce Theorem 1.4 to Lemma 1.5 in Section 3.5, by rewriting the homogeneous
 160 equation we want to solve as an inhomogeneous equation in a more regular space. To
 161 show that the inhomogeneous term, $\mathcal{V}_\star f_0$, is regular enough, we'll use Proposition 3.4 from
 162 Section 3.3 proving that \mathcal{V}_\star improves on the regularity of f_0 described by Theorem 1.2.

163 **Remark 1.6.** When \mathcal{V}_0 also satisfies Condition (REG-P | B, ϵ), Theorem 1.4 can be re-
 164 stated to give a unique solution in the affine subspace

$$165 \quad \zeta^{\tau-1} + \mathcal{H}L_{\rho-1, \bullet}^\infty(\Omega)$$

166 when $\rho > \tau$ is low enough, as described in Proposition 3.11 from Section 3.5.

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2. Function spaces for holomorphic Volterra operators.

2.1. Weighted holomorphic L^∞ spaces.

Throughout this paper, as described in Section 1.1, the “position” variable ζ will be the standard coordinate on \mathbb{C} . Take a simply connected open set $\Omega \subset \mathbb{C}$ that touches but doesn’t contain $\zeta = 0$. Let $\mathcal{C}(\Omega)$ be the space of continuous complex-valued functions on Ω . Give $\mathcal{C}(\Omega)$ the compact-open topology, recalling that this is the coarsest topology in which the seminorm $f \mapsto \sup_K |f|$ is continuous for every compact subset $K \subset \Omega$ [?, Example 2.6 and Section 4 notes]. The holomorphic functions form a closed subspace $\mathcal{H}(\Omega) \subset \mathcal{C}(\Omega)$ [?, Proposition 3.14].

Fixing a real constant Λ , let’s restrict our attention to holomorphic functions on Ω which are bounded by constant multiples of $e^{\Lambda|\zeta|}$. One might describe these functions as being uniformly of exponential type Λ .² They form a space $\mathcal{H}L_{0,\Lambda}^\infty(\Omega)$, which we’ll equip with the norm $\|f\|_{0,\Lambda} = \sup_\Omega e^{-\Lambda|\zeta|} |f|$. With respect to the seminorm on $\mathcal{H}(\Omega)$ given by a compact set $K \subset \Omega$, the inclusion map $\mathcal{H}L_{0,\Lambda}^\infty(\Omega) \hookrightarrow \mathcal{H}(\Omega)$ has norm $\sup_K e^{\Lambda|\zeta|}$. That means the inclusion is continuous.

PROPOSITION 2.1. *The space $\mathcal{H}L_{0,\Lambda}^\infty(\Omega)$ is complete.*

Proof. Take a Cauchy sequence $f_1, f_2, f_3, \dots \in \mathcal{H}L_{0,\Lambda}^\infty(\Omega)$. The inclusion map $\mathcal{H}L_{0,\Lambda}^\infty(\Omega) \hookrightarrow \mathcal{H}(\Omega)$ is bounded with respect to each of the seminorms on $\mathcal{H}(\Omega)$ given by $|f| \mapsto \sup_K |f|$ for compact subsets $K \subset \Omega$, so our sequence is Cauchy in $\mathcal{H}(\Omega)$ too. Since $\mathcal{H}(\Omega)$ is complete [?, Proposition 3.5],³ our sequence converges to a function f there.

The Cauchy property in $\mathcal{H}L_{0,\Lambda}^\infty(\Omega)$ tells us that for any $r > 0$, we can find some n for which $e^{-\Lambda|\zeta|} |f_k - f_n| \leq r$ whenever $k \geq n$. Since convergence in $\mathcal{H}(\Omega)$ implies pointwise convergence, we can see as k grows that $e^{-\Lambda|\zeta|} |f - f_n| \leq r$. This shows that our sequence converges to f in the norm $\|\cdot\|_{0,\Lambda}$. We can also see from this argument that f is in $\mathcal{H}L_{0,\Lambda}^\infty(\Omega)$: picking some $r > 0$, we observe that

$$\begin{aligned} e^{-\Lambda|\zeta|} |f| &\leq e^{-\Lambda|\zeta|} |f - f_n| + e^{-\Lambda|\zeta|} |f_n| \\ &\leq r + \|f_n\|_{0,\Lambda} \end{aligned}$$

for the corresponding n , showing that $e^{-\Lambda|\zeta|} |f|$ is bounded. \square

Now, let’s relax our norm to allow both exponential growth at infinity and a power-law singularity at $\zeta = 0$. For any $\sigma \in \mathbb{R}$, let $\mathcal{H}L_{\sigma,\Lambda}^\infty(\Omega)$ be the space of holomorphic functions on Ω which are bounded by constant multiples of $|\zeta|^\sigma e^{\Lambda|\zeta|}$. Give it the norm $\|f\|_{\sigma,\Lambda} = \sup_\Omega |\zeta|^{-\sigma} e^{-\Lambda|\zeta|} |f|$. Reprising the arguments from above, we can show that the inclusion $\mathcal{H}L_{\sigma,\Lambda}^\infty(\Omega) \hookrightarrow \mathcal{H}(\Omega)$ is continuous, and we can generalize Proposition 2.1:

PROPOSITION 2.2. *The space $\mathcal{H}L_{\sigma,\Lambda}^\infty(\Omega)$ is complete.*

2.2. Continuous inclusions between different $\mathcal{H}L_{\sigma,\Lambda}^\infty(\Omega)$.

PROPOSITION 2.3. *If $\Lambda' \leq \Lambda$, the inclusion map $\mathcal{H}L_{\sigma,\Lambda'}^\infty(\Omega) \hookrightarrow \mathcal{H}L_{\sigma,\Lambda}^\infty(\Omega)$ is continuous.*

Proof. By definition,

$$\|f\|_{\sigma,\Lambda} = \sup_\Omega |\zeta|^{-\sigma} e^{-\Lambda|\zeta|} |f|.$$

²Recall that a function f is of exponential type Λ if for every $\varepsilon > 0$, there’s a constant A_ε (which may depend on ε) such that $|f| \leq A_\varepsilon e^{(\Lambda+\varepsilon)|\zeta|}$. We instead require a uniform constant A such that $|f| \leq A e^{\Lambda|\zeta|}$.

³That is, a sequence which is Cauchy in each of the seminorms on $\mathcal{H}(\Omega)$ will always converge in the topology of $\mathcal{H}(\Omega)$, which is the coarsest topology in which all of the seminorms are continuous.

213 The norm $\|\cdot\|_{\sigma,\Lambda'}$ is designed to give $|f| \leq |\zeta|^\sigma e^{\Lambda'|\zeta|} \|f\|_{\sigma,\Lambda'}$, which tells us that

$$\begin{aligned}
214 \quad \|f\|_{\sigma,\Lambda} &\leq \sup_{\Omega} |\zeta|^{-\sigma} e^{-\Lambda|\zeta|} |\zeta|^\sigma e^{\Lambda'|\zeta|} \|f\|_{\sigma,\Lambda'} \\
215 \quad &= \sup_{\Omega} e^{-(\Lambda-\Lambda')|\zeta|} \|f\|_{\sigma,\Lambda'} \\
216 \quad &\leq \|f\|_{\sigma,\Lambda'}.
\end{aligned}$$

218 In the last step, we use the assumption that $\Lambda' \leq \Lambda$. □

219 As we mentioned in Section 2.1, one might describe $\mathcal{HL}_{0,\Lambda}^\infty(\Omega)$ as the space of functions on
220 Ω which are uniformly of exponential type Λ . Taking the union of these spaces over all
221 $\Lambda \in \mathbb{R}$, we get the space $\mathcal{HL}_{0,\bullet}^\infty(\Omega)$ that contains all functions of exponential type. Having
222 continuous inclusions between the subspaces $\mathcal{HL}_{0,\Lambda}^\infty(\Omega)$ as Λ increases, we can give $\mathcal{HL}_{0,\bullet}^\infty(\Omega)$
223 a meaningful topology: the finest topology that makes all the inclusions $\mathcal{HL}_{0,\Lambda}^\infty(\Omega) \hookrightarrow$
224 $\mathcal{HL}_{0,\bullet}^\infty(\Omega)$ continuous.⁴ For any $\sigma \in \mathbb{R}$, we define $\mathcal{HL}_{\sigma,\bullet}^\infty(\Omega)$ similarly.

225 **PROPOSITION 2.4.** *If $\sigma' > \sigma$ and $\Lambda' < \Lambda$, the inclusion map $\mathcal{HL}_{\sigma',\Lambda'}^\infty(\Omega) \hookrightarrow \mathcal{HL}_{\sigma,\Lambda}^\infty(\Omega)$*
226 *is continuous.*

227 *Proof.* By definition,

$$\begin{aligned}
228 \quad \|f\|_{\sigma,\Lambda} &= \sup_{\Omega} |\zeta|^{-\sigma} e^{-\Lambda|\zeta|} |f| \\
229 \quad &= \sup_{\Omega} |\zeta|^{\sigma'-\sigma} |\zeta|^{-\sigma'} e^{-\Lambda'|\zeta|} e^{-(\Lambda-\Lambda')|\zeta|} |f|.
\end{aligned}$$

231 The function $|\zeta|^{\sigma'-\sigma} e^{-(\Lambda-\Lambda')|\zeta|}$ is bounded near $\zeta = 0$ because the power of $|\zeta|$ is positive,
232 and it's bounded far from $\zeta = 0$ thanks to the decaying exponential. Hence,

$$\begin{aligned}
233 \quad \|f\|_{\sigma,\Lambda} &\leq C \sup_{\Omega} |\zeta|^{-\sigma'} e^{-\Lambda'|\zeta|} |f| \\
234 \quad &= C \|f\|_{\sigma',\Lambda'}
\end{aligned}$$

236 for $C = \sup_{\Omega} |\zeta|^{\sigma'-\sigma} e^{-(\Lambda-\Lambda')|\zeta|}$. □

237 **PROPOSITION 2.5.** *When $\sigma' > \sigma$, there's a continuous inclusion $\mathcal{HL}_{\sigma',\bullet}^\infty(\Omega) \hookrightarrow \mathcal{HL}_{\sigma,\bullet}^\infty(\Omega)$.* ■

238 *Proof.* For each Λ' , we can get a continuous inclusion $\mathcal{HL}_{\sigma',\Lambda'}^\infty(\Omega) \hookrightarrow \mathcal{HL}_{\sigma,\bullet}^\infty(\Omega)$ by choos-
239 ing some $\Lambda > \Lambda'$ and composing the continuous inclusion $\mathcal{HL}_{\sigma',\Lambda'}^\infty(\Omega) \hookrightarrow \mathcal{HL}_{\sigma,\Lambda}^\infty(\Omega)$ given by
240 Proposition 2.4 with the continuous inclusion $\mathcal{HL}_{\sigma,\Lambda}^\infty(\Omega) \hookrightarrow \mathcal{HL}_{\sigma,\bullet}^\infty(\Omega)$ that we get by defini-
241 tion. For any $\Lambda'' < \Lambda'$, the inclusions $\mathcal{HL}_{\sigma',\Lambda''}^\infty(\Omega) \hookrightarrow \mathcal{HL}_{\sigma,\bullet}^\infty(\Omega)$ and $\mathcal{HL}_{\sigma',\Lambda'}^\infty(\Omega) \hookrightarrow \mathcal{HL}_{\sigma,\bullet}^\infty(\Omega)$
242 constructed above automatically commute with the inclusion $\mathcal{HL}_{\sigma',\Lambda''}^\infty(\Omega) \hookrightarrow \mathcal{HL}_{\sigma',\Lambda'}^\infty(\Omega)$
243 given by Proposition 2.3, because we're ultimately working in the vector space of holomor-
244 phic functions on Ω . Because of how the topology on $\mathcal{HL}_{\sigma',\bullet}^\infty(\Omega)$ is defined, this gives us the
245 desired continuous inclusion $\mathcal{HL}_{\sigma',\bullet}^\infty(\Omega) \hookrightarrow \mathcal{HL}_{\sigma,\bullet}^\infty(\Omega)$. □

246 3. Solving holomorphic Volterra equations.

⁴In category-theoretic language, $\mathcal{HL}_{0,\bullet}^\infty(\Omega)$ is the limit of the family $\mathcal{HL}_{0,\Lambda}^\infty(\Omega)$.

247 **3.1. Overview.** The results stated in Section 1.4 lay out a method for solving the
 248 regular singular Volterra equation $f = \mathcal{V}f$. We'll now show that the method works by
 249 proving those results.

250 We start by constructing the prototype solution f_0 from the kernel of \mathcal{V}_0 , which is the
 251 separable operator that gives $\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_\star$ its regular singularity. We'll show in Section 3.2
 252 that f_0 satisfies the equation $f_0 = \mathcal{V}_0 f_0$ and belongs to the space $\mathcal{H}L_{\tau-1, \lambda_0}^\infty(\Omega)$.

253 To see what the perturbation \mathcal{V}_\star does to f_0 , we'll show in Section 3.3 that \mathcal{V}_\star has a
 254 smoothing effect, reducing the sharpness of any power-law singularity at $\zeta = 0$. In particular,
 255 it sends f_0 into $\mathcal{H}L_{\tau-1+\gamma, \bullet}^\infty(\Omega)$.

256 Now we know that \mathcal{V} sends f_0 into the affine subspace $f_0 + \mathcal{H}L_{\tau-1+\gamma, \bullet}^\infty(\Omega)$, suggesting
 257 that the equation $f = \mathcal{V}f$ has a solution there. To confirm this, we'll show in Section 3.4
 258 that \mathcal{V} is a contraction of $\mathcal{H}L_{\tau-1+\gamma, \Lambda}^\infty(\Omega)$ when Λ is large enough. This tells us that \mathcal{V} has
 259 a unique fixed point in $f_0 + \mathcal{H}L_{\tau-1+\gamma, \bullet}^\infty(\Omega)$, as we'll see in Section 3.5.

260 **3.2. Construction and regularity of the prototype solution** 261 *(proof of Theorems 1.1–1.2).*

262 **3.2.1. Construction.**

263 *Proof of Theorem 1.1.* Rewrite f_0 as $(1/p)\chi$, where

$$264 \quad \chi(a) = \exp\left(-\int_b^a \frac{q}{p} d\zeta\right).$$

265 Observing that $d\chi = -(q/p)\chi d\zeta$ greatly simplifies the calculation of $\mathcal{V}_0 f_0$. For each $a \in \Omega$,

$$\begin{aligned} 266 \quad [\mathcal{V}_0 f_0](a) &= -\int_{\zeta=0}^a \frac{q}{p(a)} f_0 d\zeta \\ 267 \quad &= -\int_{\zeta=0}^a \frac{q}{p(a)} \frac{1}{p} \chi d\zeta \\ 268 \quad &= -\frac{1}{p(a)} \int_{\zeta=0}^a \frac{q}{p} \chi d\zeta \\ 269 \quad &= \frac{1}{p(a)} \int_{\zeta=0}^a d\chi \\ 270 \quad &= \frac{1}{p(a)} \left[\chi(a) - \lim_{\zeta \rightarrow 0} \chi \right] \\ 271 \quad &= f_0(a) - \frac{1}{p(a)} \lim_{\zeta \rightarrow 0} \chi. \end{aligned}$$

273 Now, to prove that $\mathcal{V}_0 f_0 = f_0$, we just need to show that $\lim_{\zeta \rightarrow 0} \chi = 0$.

274 By Condition (SING | 7), we can find a radius $\delta > 0$ and a constant C such that

$$275 \quad (3.1) \quad \left| \frac{q}{p} - \frac{\tau}{\zeta} \right| < C$$

276 whenever $|\zeta| < \delta$. We can take advantage of this bound by rewriting the integral in the

277 definition of χ :

$$\begin{aligned}
278 \quad - \int_b^a \frac{q}{p} d\zeta &= \int_b^a \frac{\tau}{\zeta} d\zeta - \int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta} \right) d\zeta \\
279 \quad &= \tau \log \left(\frac{\zeta(a)}{\zeta(b)} \right) - \int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta} \right) d\zeta \\
280
\end{aligned}$$

281 Exponentiating both sides, we see that

$$282 \quad \chi(a) = \left(\frac{\zeta(a)}{\zeta(b)} \right)^\tau \exp \left[- \int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta} \right) d\zeta \right].$$

283 Recalling that a change of base point just multiplies f_0 by a non-zero constant, choose the
284 base point $b \in \Omega$ so that $|\zeta(b)| = \delta/2$. The exponential factor in the formula above is then
285 bounded between $\exp(-\frac{3}{2}C\delta)$ and $\exp(\frac{3}{2}C\delta)$ whenever $|\zeta(a)| < \delta$. Since τ is positive, this
286 is enough to show that $\lim_{\zeta \rightarrow 0} \chi = 0$. It follows, as discussed above, that $\mathcal{V}_0 f_0 = f_0$. \square

287 **3.2.2. Regularity.** Theorem 1.2 comprises two results with different conditions. We'll
288 prove them separately as Propositions 3.1 and 3.2.

289 PROPOSITION 3.1. Suppose \mathcal{V}_0 satisfies Condition (SING | τ). Then, on a small enough
290 neighborhood of $\zeta = 0$, we have $|f_0| \lesssim |\zeta|^{\tau-1}$.

291 *Proof.* Go back to the proof of Theorem 1.1, where we found a radius $\delta > 0$ and a
292 constant C such that Inequality (3.1) holds whenever $|\zeta| < \delta$. The subsequent argument,
293 which we used to show that $\lim_{\zeta \rightarrow 0} \chi = 0$, actually supports a stronger conclusion: it shows
294 that $|\chi| \lesssim |\zeta|^\tau$ on the region $|\zeta| < \delta$. This tells us that

$$295 \quad |f_0| \lesssim \left| \frac{1}{p} \right| |\zeta|^\tau$$

296 on the region $|\zeta| < \delta$.

297 Now, using Condition (SING | τ) again, choose some $\sigma > \tau$ and find a radius $r < \delta$ for
298 which

$$299 \quad \left| \frac{q(a')}{p(a)} \right| < \frac{\sigma}{|\zeta(a)|}$$

300 whenever $|\zeta(a)| < r$ and $|\zeta(a')| < r$. Choosing a point b' with $|\zeta(b')| < r$ and $q(b') \neq 0$, we
301 can deduce that

$$\begin{aligned}
302 \quad |f_0| &\lesssim \left| \frac{1}{q(b')} \right| \left| \frac{q(b')}{p} \right| |\zeta|^\tau \\
303 \quad &\lesssim \left| \frac{1}{q(b')} \right| \frac{\sigma}{|\zeta|} |\zeta|^\tau \\
304 \quad &\lesssim |\zeta|^{\tau-1}
\end{aligned}$$

306 on the region $|\zeta| < r$, as desired. \square

307 PROPOSITION 3.2. Suppose \mathcal{V}_0 satisfies Conditions (SING | τ) and (DIAG₀ | λ_Δ). Then
308 f_0 belongs to the space $\mathcal{H}L_{\tau-1,\bullet}^\infty(\Omega)$.

309 *Proof.* We want to show that $|f_0| \lesssim |\zeta|^{\tau-1} e^{\lambda_0 |\zeta|}$ for some real constant λ_0 .

310 By Proposition 3.1, we can find a radius $\delta > 0$ for which $|f_0| \lesssim |\zeta|^{\tau-1}$ over the region
 311 $|\zeta| < \delta$. No matter which value of λ_0 we pick, we know that $e^{\lambda_0 |\zeta|}$ can't get arbitrarily close
 312 to zero on a bounded region, so we'll have $|f_0| \lesssim |\zeta|^{\tau-1} e^{\lambda_0 |\zeta|}$ over the region $|\zeta| < \delta$.

313 By Condition (DIAG₀ | λ_Δ), we have

$$\begin{aligned} 314 \quad |k_0(a, a')| &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|} \\ 315 &\lesssim \delta^{-1} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|} \\ 316 &\lesssim e^{\lambda_\Delta |\zeta(a) - \zeta(a')|} \end{aligned}$$

318 over all $a, a' \in \Omega$ with $|\zeta(a)| \geq \delta$. One consequence is that, for some $c_\Delta > 0$, we have

$$319 \quad |k_0(a, a')| \leq c_\Delta e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

320 whenever $|\zeta(a)| \geq \delta$. Applying this bound along the diagonal in Ω^2 , we learn that $|q/p| \leq c_\Delta$
 321 on the region $|\zeta| \geq \delta$. On the other hand, by fixing some arbitrary $b' \in \Omega$, we see that

$$\begin{aligned} 322 \quad \left| \frac{q(b')}{p(a)} \right| &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(b')|} \\ 323 &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta (|\zeta(a)| + |\zeta(b')|)} \\ 324 &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a)|} e^{\lambda_\Delta |\zeta(b')|} \\ 325 &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a)|} \\ 326 &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a)|} \end{aligned}$$

327 over all $a \in \Omega$ with $|\zeta(a)| \geq \delta$. Using both of the bounds we just found, we reason that

$$\begin{aligned} 328 \quad |f_0(a)| &= \left| \frac{1}{p(a)} \exp \left(- \int_b^a \frac{q}{p} d\zeta \right) \right| \\ 329 &\leq \left| \frac{1}{p(a)} \right| \exp \left(\int_b^a \left| \frac{q}{p} \right| d\zeta \right) \\ 330 &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a)|} e^{c_\Delta (|\zeta(a)| + |\zeta(b)|)} \\ 331 &\lesssim \frac{1}{|\zeta(a)|} e^{(\lambda_\Delta + c_\Delta) |\zeta(a)|} e^{c_\Delta |\zeta(b)|} \\ 332 &\lesssim \frac{1}{|\zeta(a)|} e^{(\lambda_\Delta + c_\Delta) |\zeta(a)|} e^{c_\Delta |\zeta(b)|} \end{aligned}$$

333 over all $a \in \Omega$ with $|\zeta(a)| \geq \delta$. Setting $\lambda_0 = \lambda_\Delta + c_\Delta$, we have $|f_0| \lesssim |\zeta|^{-1} e^{\lambda_0 |\zeta|}$ over the
 334 region $|\zeta| \geq \delta$. Since we're assuming τ is real and positive, $|\zeta|^\tau$ can't get arbitrarily close
 335 to zero on the region $|\zeta| > \delta$. It follows that $|f_0| \lesssim |\zeta|^{\tau-1} e^{\lambda_0 |\zeta|}$ over the region $|\zeta| \geq \delta$.
 336 Combining this with our earlier argument on the region $|\zeta| < \delta$, we get the desired result. \square

337 PROPOSITION 3.3. Suppose \mathcal{V}_0 satisfies Conditions (REG-P | B, ϵ) and (SING | τ). Then,
 338 for some constant M ,

$$339 \quad f_0 \in M \zeta^{\tau-1} + O(|\zeta|^{\tau-1+\epsilon'})$$

340 at $\zeta = 0$, where $\epsilon' = \min\{\epsilon, 1\}$.

341 *Proof.* It's enough to show that

$$342 \quad f_0 \in M\zeta^{\tau-1} \left[1 + O(|\zeta|^{\epsilon'}) \right]$$

343 at $\zeta = 0$. Like we did in the proof of Theorem 1.1, we reason that

$$\begin{aligned} 344 \quad f_0(a) &= \frac{1}{p(a)} \exp \left(- \int_b^a \frac{q}{p} d\zeta \right) \\ 345 \quad &= \frac{1}{p(a)} \frac{\zeta(a)^\tau}{\zeta(b)^\tau} \exp \left[- \int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta} \right) d\zeta \right] \\ 346 \quad &= \zeta(b)^{-\tau} \zeta(a)^{\tau-1} \frac{\zeta(a)}{p(a)} \exp \left[- \int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta} \right) d\zeta \right]. \end{aligned}$$

348 Since $\zeta(b)^{-\tau}$ is a constant, the $\zeta(b)^{-\tau} \zeta^{\tau-1}$ factor looks like what we want.

349 With a little work, the ζ/p factor also looks like what we want. Condition (REG-P | B, ϵ)
350 implies that

$$351 \quad \frac{\zeta}{p} \in B^{-1} + O(|\zeta|^\epsilon),$$

352 at $\zeta = 0$.

353 Now, let's look at the exponential factor. By Condition (SING | τ), we can find a constant
354 C and a neighborhood Ω_{near} of $\zeta = 0$ for which

$$355 \quad \left| \frac{q}{p} + \frac{\tau}{\zeta} \right| < C$$

356 in Ω_{near} . It follows that the improper integral

$$357 \quad \eta(a) = \int_{\zeta=0}^a \left(\frac{q}{p} + \frac{\tau}{\zeta} \right) d\zeta$$

358 converges for all $a \in \Omega$, allowing us to write

$$359 \quad \exp \left[- \int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta} \right) d\zeta \right] = e^{\eta(b)} e^{-\eta(a)}.$$

360 Observing that $|\eta| \leq C|\zeta|$ in Ω_{near} , we can conclude that

$$361 \quad \exp \left[- \int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta} \right) d\zeta \right] \in e^{\eta(b)} \left[1 + O(|\zeta(a)|) \right]$$

362 at $\zeta(a) = 0$.

363 Combining the arguments above, we learn that

$$\begin{aligned} 364 \quad f_0 &\in \zeta(b)^{-\tau} \zeta^{\tau-1} \left[B^{-1} + O(|\zeta|^\epsilon) \right] e^{\eta(b)} \left[1 + O(|\zeta|) \right] \\ 365 \quad &= \zeta(b)^{-\tau} B^{-1} e^{\eta(b)} \zeta^{\tau-1} \left[1 + O(|\zeta|^\epsilon) \right] \left[1 + O(|\zeta|) \right] \\ 366 \quad &= M \zeta^{\tau-1} \left[1 + O(|\zeta|^\epsilon) + O(|\zeta|) \right] \end{aligned}$$

368 at $\zeta = 0$, with $M = \zeta(b)^{-\tau} B^{-1} e^{\eta(b)}$. Observing that $O(|\zeta|^\epsilon) + O(|\zeta|) = O(|\zeta|^{\epsilon'})$, we get
369 the desired result. \square

370 **3.3. Showing that \mathcal{V}_\star makes the prototype solution less singular**
 371 **(toward Theorem 1.4).** Now that we know the prototype solution f_0 belongs to $\mathcal{H}L_{\tau-1,\bullet}^\infty(\Omega)$,
 372 we'll show that \mathcal{V}_\star reduces the sharpness of its singularity at $\zeta = 0$, mapping it into
 373 $\mathcal{H}L_{\tau-1+\gamma,\bullet}^\infty(\Omega)$. This is a consequence of the general smoothing effect of \mathcal{V}_\star , described
 374 in the following result.

375 **PROPOSITION 3.4.** *Under Condition (DIAG \star | γ, λ_Δ), the operator \mathcal{V}_\star maps*

$$376 \quad \mathcal{H}L_{\sigma,\Lambda}^\infty(\Omega) \rightarrow \mathcal{H}L_{\sigma+\gamma,\Lambda}^\infty(\Omega)$$

377 *continuously for all $\Lambda \geq \lambda_\Delta$ and $\sigma > -1$.*

378 *Remark 3.5.* We're assuming that $\gamma > 0$, but this result holds even under the weaker
 379 assumption that $\gamma > -1$. We'll take advantage of this in Section 4.4.2

380 *Proof of Proposition 3.4.* For any function $f \in \mathcal{H}L_{\sigma,\Lambda}^\infty(\Omega)$,

$$\begin{aligned} 381 \quad |\zeta(a)|^{-(\sigma+\gamma)} e^{-\Lambda|\zeta(a)|} \left| [\mathcal{V}_\star f](a) \right| &\leq |\zeta(a)|^{-(\sigma+\gamma)} e^{-\Lambda|\zeta(a)|} \int_{\zeta=0}^a |k_\star(a, \cdot)| |f| |d\zeta| \\ 382 &\leq |\zeta(a)|^{-(\sigma+\gamma)} e^{-\Lambda|\zeta(a)|} \int_{\zeta=0}^a |k_\star(a, \cdot)| |\zeta|^\sigma e^{\Lambda|\zeta|} \|f\|_{\sigma,\Lambda} |d\zeta| \\ 383 \end{aligned}$$

384 By Condition (DIAG \star | γ, λ_Δ),

$$\begin{aligned} 385 \quad \int_{\zeta=0}^a |k_\star(a, \cdot)| |\zeta|^\sigma e^{\Lambda|\zeta|} |d\zeta| &\lesssim \int_{\zeta=0}^a \frac{|\zeta(a) - \zeta|^\gamma}{|\zeta(a)|} e^{\lambda_\Delta|\zeta(a) - \zeta|} |\zeta|^\sigma e^{\Lambda|\zeta|} |d\zeta| \\ 386 &\lesssim |\zeta(a)|^{\gamma+\sigma} \int_0^1 (1-t)^\gamma e^{\lambda_\Delta|\zeta(a)|(1-t)} t^\sigma e^{\Lambda|\zeta(a)|t} dt \\ 387 &\lesssim |\zeta(a)|^{\gamma+\sigma} e^{\lambda_\Delta|\zeta(a)|} \int_0^1 (1-t)^\gamma t^\sigma e^{(\Lambda-\lambda_\Delta)|\zeta(a)|t} dt \\ 388 &\lesssim |\zeta(a)|^{\gamma+\sigma} e^{\lambda_\Delta|\zeta(a)|} e^{(\Lambda-\lambda_\Delta)|\zeta(a)|} \int_0^1 (1-t)^\gamma t^\sigma dt. \\ 389 \end{aligned}$$

390 The last step takes advantage of the assumption that $\Lambda \geq \lambda_\Delta$. Recognizing the integral as
 391 an evaluation of the beta function, we can rewrite the bound as

$$392 \quad \int_{\zeta=0}^a |k_\star(a, \cdot)| |\zeta|^\sigma e^{\Lambda|\zeta|} |d\zeta| \lesssim |\zeta(a)|^{\gamma+\sigma} e^{\Lambda|\zeta(a)|} \frac{\Gamma(\gamma+1)\Gamma(\sigma+1)}{\Gamma(\sigma+\gamma+2)}.$$

393 Our assumptions that $\gamma > 0$ and $\sigma > -1$ ensure that the gamma functions are well-defined.⁵
 394 Rearranging to get

$$395 \quad |\zeta(a)|^{-(\gamma+\sigma)} e^{-\Lambda|\zeta(a)|} \int_{\zeta=0}^a |k_\star(a, \cdot)| |\zeta|^\sigma e^{\Lambda|\zeta|} |d\zeta| \lesssim \frac{\Gamma(\gamma+1)\Gamma(\sigma+1)}{\Gamma(\sigma+\gamma+2)},$$

396 we conclude that $|\zeta(a)|^{-\sigma-\gamma} e^{-\Lambda|\zeta(a)|} \left| [\mathcal{V}_\star f](a) \right|$ is uniformly bounded in Ω . \square

⁵We could even weaken the constraint on γ , allowing any $\gamma > -1$.

397 COROLLARY 3.6. Consider the prototype solution f_0 from Equation (1.1). If \mathcal{V}_\star satisfies
 398 Condition (DIAG \star | γ, λ_Δ), then $\mathcal{V}_\star f_0$ belongs to $\mathcal{HL}_{\tau-1+\gamma, \bullet}^\infty(\Omega)$.

399 *Proof.* We know from Theorem 1.2 that f_0 belongs to $\mathcal{HL}_{\tau-1, \bullet}^\infty(\Omega)$. Choose a constant
 400 $\Lambda \geq \lambda_\Delta$ big enough that f_0 is in $\mathcal{HL}_{\tau-1, \Lambda}^\infty(\Omega)$. Since we're assuming τ is positive, we can
 401 apply Proposition 3.4, concluding that $\mathcal{V}_\star f_0$ belongs to $\mathcal{HL}_{\tau-1+\gamma, \Lambda}^\infty(\Omega)$. \square

402 3.4. Showing that \mathcal{V} shrinks less singular functions 403 (toward Lemma 1.5).

404 **3.4.1. Overview.** In this section, we'll prove the following proposition.

405 PROPOSITION 3.7. Suppose that \mathcal{V}_0 satisfies Conditions (SING | τ) and (DIAG $_0$ | λ_Δ),
 406 and \mathcal{V}_\star satisfies Condition (DIAG \star | γ, λ_Δ). Then, for each $\rho > \tau$, we can ensure that \mathcal{V} is a
 407 contraction of $\mathcal{HL}_{\rho-1, \Lambda}^\infty(\Omega)$ by making Λ big enough.

408 *Remark 3.8.* For the argument, we'll use, "big enough" always requires $\Lambda > \lambda_\Delta$, and
 409 may require $\Lambda \gg \lambda_\Delta$.

410 First, pick some $\sigma \in (\tau, \rho)$.⁶ By Proposition 3.9, which we'll state and prove in Section 3.4.2,
 411 we can find a neighborhood $\Omega_{\text{near}} \subset \Omega$ of $\zeta = 0$ with the property that

$$412 \quad (3.2) \quad |k(a, a')| \leq \frac{\sigma}{|\zeta(a)|}$$

413 for all $a, a' \in \Omega_{\text{near}}$. By choosing a small enough positive radius δ , we can take Ω_{near} to be
 414 the part of Ω where $|\zeta| < \delta$. Complementarily, let Ω_{far} be the part of Ω where $\delta \leq |\zeta|$.

415 Take any function $\varphi \in \mathcal{HL}_{\rho, \Lambda}^\infty(\Omega)$. In Section 3.4.4, we'll bound $|\zeta|^{-(\rho-1)} e^{-\Lambda|\zeta|} |\mathcal{V}\varphi|$
 416 by $\frac{\sigma}{\rho} \|\varphi\|_{\rho-1, \Lambda}$ on Ω_{near} . In Section 3.4.5, we'll see that by making Λ big enough, we can
 417 bound $|\zeta|^{-(\rho-1)} e^{-\Lambda|\zeta|} |\mathcal{V}\varphi|$ by an arbitrarily small constant multiple of $\|\varphi\|_{\rho-1, \Lambda}$ on Ω_{far} .
 418 Together, these results show that $\|\mathcal{V}\varphi\|_{\rho-1, \Lambda} \leq \frac{\sigma}{\rho} \|\varphi\|_{\rho-1, \Lambda}$ when Λ is large enough. Since
 419 we set $\sigma < \rho$, this proves Proposition 3.7.

420 **3.4.2. Bounds on the perturbed kernel.** The conditions on k_0 and k_\star described in
 421 Sections 1.3.2 and 1.3.3 can be combined into convenient bounds on the kernel $k = k_0 + k_\star$
 422 of \mathcal{V} . One bound, which works when both arguments of k are close to $\zeta = 0$, will be used in
 423 Section 3.4.4.

424 PROPOSITION 3.9. Suppose \mathcal{V}_0 satisfies Condition (SING | τ), and \mathcal{V}_\star satisfies Condi-
 425 tion (DIAG \star | γ, λ_Δ). Then, for any $\sigma > \tau$, we can ensure that

$$426 \quad |k(a, a')| \leq \frac{\sigma}{|\zeta(a)|}$$

427 by keeping a and a' close enough to $\zeta = 0$.

428 *Proof.* First, choose some $\sigma_0 \in (\tau, \sigma)$. Condition (SING | τ) tells us that by making
 429 $\delta > 0$ small enough, we can ensure that

$$430 \quad |k_0(a, a')| \leq \frac{\sigma_0}{|\zeta(a)|}$$

⁶If you'd like, you can fix a choice of σ —for example, taking $\sigma = \frac{1}{2}(\tau + \rho)$.

whenever $|\zeta(a)| < \delta$ and $|\zeta(a')| < \delta$. Now, we'll show that under Condition $(\text{DIAG}_\star \mid \gamma, \lambda_\Delta)$, we can make $|k_\star(a, a')|$ as small as we want by keeping a and a' close to $\zeta = 0$. The condition tells us that

$$|k_\star(a, a')| \lesssim \frac{|\zeta(a) - \zeta(a')|^\gamma}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega$, which means that

$$|k_\star(a, a')| \lesssim \frac{(2\delta)^\gamma}{|\zeta(a)|} e^{\lambda_\Delta (2\delta)}$$

over all $a, a' \in \Omega$ with $|\zeta(a)| < \delta$ and $|\zeta(a')| < \delta$. Combining this conclusion with the previous one, we get the desired result. \square

Another bound, which works when the first argument of k is kept away from $\zeta = 0$, will be used in Section 3.4.5.

PROPOSITION 3.10. *Suppose \mathcal{V}_0 satisfies Condition $(\text{DIAG}_0 \mid \lambda_\Delta)$, and \mathcal{V}_\star satisfies Condition $(\text{DIAG}_\star \mid \gamma, \lambda_\Delta)$. Choose a subset $\Omega_{\text{far}} \subset \Omega$ that doesn't touch $\zeta = 0$.⁷ Then, for any $\lambda > \lambda_\Delta$, we have*

$$|k(a, a')| \lesssim e^{\lambda |\zeta(a) - \zeta(a')|}$$

over all $a \in \Omega_{\text{far}}$ and $a' \in \Omega$.

Proof. Find a radius $\delta > 0$ with $|\zeta| \geq \delta$ on Ω_{far} , and choose some $\lambda > \lambda_\Delta$. Conditions $(\text{DIAG}_0 \mid \lambda_\Delta)$ and $(\text{DIAG}_\star \mid \gamma, \lambda_\Delta)$ tell us that

$$\begin{aligned} |k(a, a')| &\leq |k_0(a, a')| + |k_\star(a, a')| \\ &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|} + \frac{|\zeta(a) - \zeta(a')|^\gamma}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|} \\ &\lesssim \delta^{-1} (1 + |\zeta(a) - \zeta(a')|^\gamma) e^{\lambda_\Delta |\zeta(a) - \zeta(a')|} \end{aligned}$$

over all $a \in \Omega_{\text{far}}$ and $a' \in \Omega$. Since

$$1 + |\zeta(a) - \zeta(a')|^\gamma$$

grows polynomially with respect to $|\zeta(a) - \zeta(a')|$, we can bound it with any growing exponential function of $|\zeta(a) - \zeta(a')|$. In particular,

$$1 + |\zeta(a) - \zeta(a')|^\gamma \lesssim e^{(\lambda - \lambda_\Delta) |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega$. It follows that

$$|k(a, a')| \lesssim e^{(\lambda - \lambda_\Delta) |\zeta(a) - \zeta(a')|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a \in \Omega_{\text{far}}$ and $a' \in \Omega$. This simplifies to the desired result. \square

⁷A subset of a topological space *touches* the points in its closure [?, Chapter 5, Definition 2.11].

460 **3.4.3. First steps toward showing that \mathcal{V} is a contraction.** The first steps of our
 461 calculation are the same throughout Ω . For each $a \in \Omega$, we have

$$462 \quad |\zeta(a)|^{-(\rho-1)} e^{-\Lambda|\zeta(a)|} |[\mathcal{V}\varphi](a)| \leq |\zeta(a)|^{-(\rho-1)} e^{-\Lambda|\zeta(a)|} \int_0^a |k(a, \cdot) \varphi \, d\zeta|$$

463 for any choice of integration path.⁸ The norm on $\mathcal{H}L_{\rho-1,\Lambda}^\infty(\Omega)$ is designed to give the bound
 464 $|\varphi| \leq |\zeta|^{\rho-1} e^{\Lambda|\zeta|} \|\varphi\|_{\rho-1,\Lambda}$, so

$$465 \quad |\zeta(a)|^{-(\rho-1)} e^{-\Lambda|\zeta(a)|} |[\mathcal{V}\varphi](a)| \leq |\zeta(a)|^{-(\rho-1)} e^{-\Lambda|\zeta(a)|} \int_0^a |k(a, \cdot)| |\zeta|^{\rho-1} e^{\Lambda|\zeta|} \|\varphi\|_{\rho-1,\Lambda} |d\zeta|$$

$$466 \quad = \|\varphi\|_{\rho-1,\Lambda} \int_0^a |k(a, \cdot)| \left| \frac{\zeta}{\zeta(a)} \right|^{\rho-1} e^{-\Lambda(|\zeta(a)|-|\zeta|)} |d\zeta|$$

$$467$$

468 What we do next depends on whether a is in Ω_{near} or Ω_{far} .

469 **3.4.4. Near the origin.** Suppose that $a \in \Omega_{\text{near}}$. Then inequality (3.2) tells us that

$$470 \quad |\zeta(a)|^{-(\rho-1)} e^{-\Lambda|\zeta(a)|} |[\mathcal{V}\varphi](a)| \leq \|\varphi\|_{\rho-1,\Lambda} \int_0^a \frac{\sigma}{|\zeta(a)|} \left| \frac{\zeta}{\zeta(a)} \right|^{\rho-1} e^{-\Lambda(|\zeta(a)|-|\zeta|)} |d\zeta|$$

$$471 \quad \leq \sigma \|\varphi\|_{\rho-1,\Lambda} \int_0^a \left| \frac{\zeta}{\zeta(a)} \right|^{\rho-1} e^{-\Lambda(|\zeta(a)|-|\zeta|)} \left| \frac{d\zeta}{\zeta(a)} \right|$$

$$472$$

473 for any integration path that stays within Ω_{near} . Taking advantage of the fact that Ω_{near} is
 474 a sector of a disk, let's use the straight path $\zeta = t\zeta(a)$, with $t \in (0, 1]$.

$$475 \quad |\zeta(a)|^{-(\rho-1)} e^{-\Lambda|\zeta(a)|} |[\mathcal{V}\varphi](a)| \leq \sigma \|\varphi\|_{\rho-1,\Lambda} \int_0^1 t^{\rho-1} e^{-\Lambda|\zeta(a)|(1-t)} dt$$

$$476 \quad \leq \sigma \|\varphi\|_{\rho-1,\Lambda} \int_0^1 t^{\rho-1} dt$$

$$477 \quad = \frac{\sigma}{\rho} \|\varphi\|_{\rho-1,\Lambda}.$$

$$478$$

479 Since we set $\sigma < \rho$, this brings us halfway to proving Proposition 3.7.

480 **3.4.5. Away from the origin.** Going back to the end of Section 3.4.3, suppose that
 481 $a \in \Omega_{\text{far}}$. Choose some $\lambda > \lambda_\Delta$. By Proposition 3.10 from Section 3.4.2, we can find a
 482 constant M for which

$$483 \quad |k(a, \cdot)| \leq M e^{\lambda|\zeta(a)-\zeta|}$$

484 for all $a \in \Omega_{\text{far}}$. Note that only the first argument of k has its domain restricted; this bound
 485 holds throughout Ω in the second argument. Applying this bound, we learn that

$$486 \quad |\zeta(a)|^{-(\rho-1)} e^{-\Lambda|\zeta(a)|} |[\mathcal{V}\varphi](a)| \leq \|\varphi\|_{\rho-1,\Lambda} \int_0^a M e^{\lambda|\zeta(a)-\zeta|} \left| \frac{\zeta}{\zeta(a)} \right|^{\rho-1} e^{-\Lambda(|\zeta(a)|-|\zeta|)} |d\zeta|.$$

⁸The absolute value of a 1-form, like $|k(a, \cdot) \varphi \, d\zeta|$, is a density on \mathbb{C} —a norm on tangent vectors.

Let's again use the straight integration path $\zeta = t\zeta(a)$, with $t \in (0, 1]$. Along this path, $|\zeta(a) - \zeta| = |\zeta(a)| - |\zeta|$, allowing us to combine the exponential factors in our bound:

$$\begin{aligned} |\zeta(a)|^{-(\rho-1)} e^{-\Lambda|\zeta(a)|} |[\mathcal{V}\varphi](a)| &\leq \|\varphi\|_{\rho-1,\Lambda} \int_0^1 M e^{\lambda|\zeta(a)|(1-t)} t^{\rho-1} e^{-\Lambda|\zeta(a)|(1-t)} dt \\ &\leq M \|\varphi\|_{\rho-1,\Lambda} \int_0^1 t^{\rho-1} e^{-(\Lambda-\lambda)|\zeta(a)|(1-t)} dt. \end{aligned}$$

Let's set $\Lambda > \lambda$, ensuring that the exponential factor shrinks as $|\zeta(a)|$ grows. Then we can make our bound uniform over all $a \in \Omega_{\text{far}}$, since $|\zeta(a)| \geq \delta$ for these points:

$$|\zeta(a)|^{-(\rho-1)} e^{-\Lambda|\zeta(a)|} |[\mathcal{V}\varphi](a)| \leq M \|\varphi\|_{\rho-1,\Lambda} \int_0^1 t^{\rho-1} e^{-(\Lambda-\lambda)\delta(1-t)} dt.$$

We can make this bound less than one, as required to show that \mathcal{V} is a contraction, by increasing Λ . To see this in the trickiest case, where $\rho < 1$, it helps to look at the beginning and end of the integration path separately. At the beginning of the path—for $t \in (0, \frac{1}{5}]$, say—we can use a worst-case estimate on the exponential factor:

$$\begin{aligned} \int_0^{1/5} t^{\rho-1} e^{-(\Lambda-\lambda)\delta(1-t)} dt &\leq e^{-\frac{4}{5}(\Lambda-\lambda)\delta} \int_0^{1/5} t^{\rho-1} dt \\ &= e^{-\frac{4}{5}(\Lambda-\lambda)\delta} \frac{1}{\rho} \left(\frac{1}{5}\right)^\rho. \end{aligned}$$

At the end of the path, we instead use a worst-case estimate on $t^{\rho-1}$:

$$\begin{aligned} \int_{1/5}^1 t^{\rho-1} e^{-(\Lambda-\lambda)\delta(1-t)} dt &\leq \max\{(1/5)^{\rho-1}, 1\} \int_{1/5}^1 e^{-(\Lambda-\lambda)\delta(1-t)} dt \\ &= \max\{(1/5)^{\rho-1}, 1\} \frac{1}{(\Lambda-\lambda)\delta} \left[1 - e^{-\frac{4}{5}(\Lambda-\lambda)\delta}\right] \\ &\leq \max\{(1/5)^{\rho-1}, 1\} \frac{1}{(\Lambda-\lambda)\delta}. \end{aligned}$$

In summary, the beginning of the integral is bounded by a decaying exponential function of $(\Lambda-\lambda)\delta$, and the end of the integral is bounded by a reciprocal function of $(\Lambda-\lambda)\delta$. That means we can make $|\zeta(a)|^{-(\rho-1)} e^{-\Lambda|\zeta(a)|} |[\mathcal{V}\varphi](a)|$ as small as we want over all $a \in \Omega_{\text{far}}$. This completes our proof of Proposition 3.7.

3.5. Existence and uniqueness of a fixed point (proof of Lemma 1.5 and Theorem 1.4).

Proof of Lemma 1.5. Choose Λ large enough to ensure that \mathcal{V} is a contraction of $\mathcal{H}L_{\rho-1,\Lambda}^\infty(\Omega)$ and g belongs to $\mathcal{H}L_{\rho-1,\Lambda}^\infty(\Omega)$. Proposition 3.7 guarantees that we can do the former, given our assumptions about \mathcal{V} and ρ , and the definition of $\mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega)$ guarantees that we can also do the latter. Our choice of Λ ensures that the affine map $f \mapsto \mathcal{V}f + g$ is also a contraction of $\mathcal{H}L_{\rho-1,\Lambda}^\infty(\Omega)$, and thus has a unique fixed point in $\mathcal{H}L_{\rho-1,\Lambda}^\infty(\Omega)$ by the contraction mapping theorem.

To see that the fixed point is still unique in the bigger space $\mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega)$, first recall from Proposition 2.3 that we have inclusions $\mathcal{H}L_{\rho-1,\Lambda'}^\infty(\Omega) \hookrightarrow \mathcal{H}L_{\rho-1,\Lambda}^\infty(\Omega)$ for all $\Lambda' \leq \Lambda$.

Any fixed point in $\mathcal{H}L_{\rho-1,\Lambda'}^\infty(\Omega)$ must map to the unique fixed point in $\mathcal{H}L_{\rho-1,\Lambda}^\infty(\Omega)$ under this inclusion. Next, observe that for any $\Lambda'' \geq \Lambda$, the map $f \mapsto \mathcal{V}f + g$ is also a contraction of $\mathcal{H}L_{\rho-1,\Lambda''}^\infty(\Omega)$. The inclusion $\mathcal{H}L_{\rho-1,\Lambda}^\infty(\Omega) \hookrightarrow \mathcal{H}L_{\rho-1,\Lambda''}^\infty(\Omega)$ must send the unique fixed point in the smaller space to the unique fixed point in the larger one. Together, these arguments show that the fixed point is unique in $\mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega)$. \square

Proof of Theorem 1.4. Start with the prototype solution f_0 constructed in Theorem 1.1, which satisfies the equation

$$f_0 = \mathcal{V}_0 f_0.$$

The base point for the construction can be chosen arbitrarily. Under our assumptions about \mathcal{V} , Theorem 1.2 tells us that f_0 is in $\mathcal{H}L_{\tau-1,\bullet}^\infty(\Omega)$. Our goal is to find a perturbation $f_\star \in \mathcal{H}L_{\tau-1+\gamma,\bullet}^\infty(\Omega)$ that makes $f = f_0 + f_\star$ a solution of

$$(3.3) \quad f = \mathcal{V}f.$$

Observing that

$$\begin{aligned} \mathcal{V}f_0 &= \mathcal{V}_0 f_0 + \mathcal{V}_\star f_0 \\ &= f_0 + \mathcal{V}_\star f_0, \end{aligned}$$

we can rewrite the homogeneous equation we're trying to solve as an inhomogeneous equation for f_\star :

$$(3.4) \quad \begin{aligned} f_0 + f_\star &= \mathcal{V}f_0 + \mathcal{V}f_\star \\ &= f_0 + \mathcal{V}_\star f_0 + \mathcal{V}f_\star \\ f_\star &= \mathcal{V}_\star f_0 + \mathcal{V}f_\star. \end{aligned}$$

Since we know that f_0 is in $\mathcal{H}L_{\tau-1,\bullet}^\infty(\Omega)$, Proposition 3.4 tells us that the inhomogeneous term $\mathcal{V}_\star f_0$ is in $\mathcal{H}L_{\tau-1+\gamma,\bullet}^\infty(\Omega)$. Since $\tau + \gamma > \tau$, and we've made all the necessary assumptions about \mathcal{V} , Lemma 1.5 guarantees that Equation (3.4) has a unique solution f_\star in $\mathcal{H}L_{\tau-1+\gamma,\bullet}^\infty(\Omega)$. Equivalently, Equation (3.3) has a unique solution f in $f_0 + \mathcal{H}L_{\tau-1+\gamma,\bullet}^\infty(\Omega)$.

Now, we just want to show that the uniqueness of the solution still holds in $f_0 + \mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega)$ for any $\rho > \tau$.

First, suppose $\rho \in (\tau, \tau + \gamma)$. In this case, we can use the inclusion $\mathcal{H}L_{\tau-1+\gamma,\bullet}^\infty(\Omega) \hookrightarrow \mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega)$ given by Proposition 2.5 to see that the inhomogeneous term $\mathcal{V}_\star f_0$ is in $\mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega)$. Lemma 1.5 then guarantees that Equation (3.4) has a unique solution in $\mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega)$ —which must be the solution we already found in $\mathcal{H}L_{\tau-1+\gamma,\bullet}^\infty(\Omega)$. \blacksquare

On the other hand, suppose $\rho > \tau + \gamma$. In this case, Proposition 2.5 gives an inclusion $\mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega) \hookrightarrow \mathcal{H}L_{\tau-1+\gamma,\bullet}^\infty(\Omega)$. Under this inclusion, any solution of Equation (3.4) that we might find in $\mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega)$ must match the unique solution we found in $\mathcal{H}L_{\tau-1+\gamma,\bullet}^\infty(\Omega)$. \square

PROPOSITION 3.11. *Suppose \mathcal{V} satisfies Condition (REG-P | B, ϵ) in addition to the other conditions we assume in Theorem 1.4. Then, as long as $\rho > \tau$ is low enough, the equation*

$$f = \mathcal{V}f$$

has a unique solution f in the affine subspace

$$\zeta^{\tau-1} + \mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega)$$

561 of the space $\mathcal{H}L_{\tau-1,\bullet}^\infty(\Omega)$. To be precise, ρ is low enough when it's in the interval

$$562 \quad (\tau, \tau + \min\{\gamma, \epsilon, 1\}].$$

563 *Proof.* Propositions 3.2 and 3.3 imply that for some constant M ,

$$564 \quad \zeta^{\tau-1} + \mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega) = M^{-1}f_0 + \mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega)$$

565 whenever $\rho \leq \tau + \min\{\epsilon, 1\}$. Theorem 1.4 implies that our equation has a unique solution
566 in

$$567 \quad M^{-1}f_0 + \mathcal{H}L_{\rho-1,\bullet}^\infty(\Omega)$$

568 as long as $\rho \in (\tau, \tau + \gamma]$. Putting these facts together, we get the desired result. \square

569 4. A motivating example.

570 **4.1. Overview.** We can use our understanding of regular singular Volterra equations
571 to study so-called *level 1 differential equations*, which each have an irregular singularity
572 at ∞ . In particular, we can build a set of analytic solutions in the frequency domain by
573 solving certain regular singular Volterra equations in the position domain. Our main results
574 guarantee that these position domain solutions exist, are unique, and have well-defined
575 Laplace transforms.

576 **4.2. Level 1 differential equations.** Let \mathcal{P} be a linear differential operator of the
577 form

$$578 \quad \mathcal{P} = P(\partial_z) + \frac{1}{z}Q(\partial_z) + \frac{1}{z^2}R(z^{-1}),$$

579 where

- 580 • P is a monic degree- d polynomial whose roots are all simple;
- 581 • Q is a degree- $(d-1)$ polynomial that's non-zero at every root of P ;
- 582 • $R(z^{-1})$ is holomorphic in some disk $|z| > A$ around $z = \infty$. In particular, the power
583 series

$$584 \quad R(z^{-1}) = \sum_{j=0}^{\infty} R_j z^{-j}$$

585 converges in the region $|z| > A$.

586 Equations of the form $\mathcal{P}\Phi = 0$ are a sub-class of what Ecalle calls *level 1* differential
587 equations [?, Section 2.1][?, Section 5.2.2.1]. In an upcoming paper [?], we'll study various
588 examples of such equations using Laplace transform methods, with the help of our existence
589 and uniqueness result Theorem 1.4.

590 **4.3. Notation.** So far, we've studied a Volterra equation with a regular singularity
591 at $\zeta = 0$. When we use Laplace transform methods to solve a level 1 differential equation
592 $\mathcal{P}\Phi = 0$ in the frequency domain, we'll end up with several Volterra equations $\hat{\mathcal{P}}_\alpha\varphi = 0$
593 in the position domain, each with a regular singularity at a different root $\zeta = \alpha$ of the
594 polynomial $P(-\zeta)$.

595 To adapt our previous reasoning to this more general situation, we'll reinterpret the
596 language of Sections 1–3 whenever we refer to it Section 4. The role of the coordinate
597 ζ in Sections 1–3 will now be played by $\zeta - \alpha$. This substitution leads to several other
598 reinterpretations, summarized below.

References to the point $\zeta = 0$ become references to the point $\zeta - \alpha = 0$, which we'll rewrite as $\zeta = \alpha$. For example, we'll now work on a domain that touches but doesn't contain $\zeta = \alpha$, introducing the notation Ω_α as a reminder of this change. Condition (STAR) now says that Ω_α is star-shaped around $\zeta = \alpha$. The function space $\mathcal{HL}_{\sigma,\Lambda}^\infty(\Omega_\alpha)$ becomes a space of holomorphic functions on Ω_α which have a power-law singularity at $\zeta = \alpha$ and are uniformly of exponential type Λ . Explicitly, f is in $\mathcal{HL}_{\sigma,\Lambda}^\infty(\Omega_\alpha)$ if

$$|f| \lesssim |\zeta - \alpha|^\sigma e^{\Lambda|\zeta - \alpha|}$$

over Ω_α , making the norm

$$\|f\|_{\sigma,\Lambda} = \sup_{\Omega_\alpha} |\zeta - \alpha|^{-\sigma} e^{-\Lambda|\zeta - \alpha|} |f|$$

well-defined.

Our conditions on Volterra operators change to describe a regular singularity at $\zeta = \alpha$. For example, consider a Volterra operator \mathcal{V}_0^α of the kind described in Section 1.3.2, with kernel k_0^α . This operator now satisfies Condition (SING | τ) if for some real, positive constant τ_α , the difference

$$k_0^\alpha(a, a) - \frac{\tau_\alpha}{\zeta(a) - \alpha}$$

is bounded on a neighborhood of $\zeta(a) = \alpha$ in Ω_α , and for each $\sigma > \tau_\alpha$, the bound

$$|k_0^\alpha(a, a')| < \frac{\sigma}{|\zeta(a) - \alpha|}$$

holds over some neighborhood of $(\zeta(a), \zeta(a')) = (\alpha, \alpha)$ in Ω_α^2 . Condition (DIAG₀ | λ_Δ) now requires that for some constant λ_Δ ,

$$|k_0^\alpha(a, a')| \lesssim \frac{1}{|\zeta(a) - \alpha|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega_\alpha$. Similarly, an operator \mathcal{V}_\star^α with kernel k_\star^α now satisfies Condition (DIAG_★ | γ, λ_Δ) if for some constants $\gamma > 0$ and λ_Δ , we have

$$|k_\star^\alpha(a, a')| \lesssim \frac{|\zeta(a) - \zeta(a')|^\gamma}{|\zeta(a) - \alpha|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega_\alpha$.

4.4. The Laplace transform.

4.4.1. Definition. Let Ω_α be a simply connected open subset of \mathbb{C} that touches but doesn't contain $\zeta = \alpha$, and let $\Gamma_{\zeta,\alpha}^\theta$ be the ray that leaves $\zeta = \alpha$ at angle θ . When Ω_α contains $\Gamma_{\zeta,\alpha}^\theta$, we can define the Laplace transform $\mathcal{L}_{\zeta,\alpha}^\theta$, which maps functions in $\mathcal{HL}_{\sigma,\Lambda}^\infty(\Omega_\alpha)$ with $\sigma > -1$ and $\Lambda \in \mathbb{R}$ to holomorphic functions on the half-plane $\operatorname{Re}(ze^{-i\theta}) > \Lambda$ in the frequency domain [?, Section 5.6]. The Laplace transform of φ is defined by the formula

$$(4.1) \quad \mathcal{L}_{\zeta,\alpha}^\theta \varphi := \int_{\Gamma_{\zeta,\alpha}^\theta} e^{-z\zeta} \varphi \, d\zeta.$$

It's a function on the frequency domain because it depends on the frequency variable z .

631 **4.4.2. Action on integral operators.** For each $\nu \in (0, \infty)$, the fractional integral
 632 $\partial_{\zeta, \alpha}^{-\nu}$ is the Volterra operator defined by

$$633 \quad [\partial_{\zeta, \alpha}^{-\nu} \varphi](a) := \frac{1}{\Gamma(\nu)} \int_{\zeta=\alpha}^a (\zeta(a) - \zeta)^{\nu-1} \varphi \, d\zeta$$

634 for each $a \in \Omega_\alpha$. It obeys the expected semigroup law [?, Section 1.3]

$$635 \quad \partial_{\zeta, \alpha}^{-\mu} \partial_{\zeta, \alpha}^{-\nu} = \partial_{\zeta, \alpha}^{-\mu-\nu} \quad \mu, \nu \in (0, \infty),$$

637 and agrees with ordinary repeated integration when ν is an integer [?, Equation 35].

638 Fractional integration has a smoothing effect: it reduces the sharpness of any power-law
 639 singularity at $\zeta = \alpha$.

640 PROPOSITION 4.1. For each $\nu \in (0, \infty)$, the fractional integral $\partial_{\zeta, \alpha}^{-\nu}$ maps

$$641 \quad \mathcal{H}L_{\sigma, \Lambda}^\infty(\Omega_\alpha) \rightarrow \mathcal{H}L_{\sigma+\nu, \Lambda}^\infty(\Omega_\alpha)$$

642 continuously for all $\sigma > -1$ and $\Lambda \geq 0$.

643 *Proof.* Rewrite the fractional integral as

$$644 \quad \begin{aligned} [\partial_{\zeta, \alpha}^{-\nu} \varphi](a) &= \frac{1}{\Gamma(\nu)} \int_{\zeta=\alpha}^a (\zeta(a) - \zeta)^{\nu-1} \varphi \, d\zeta \\ &= \frac{\zeta(a) - \alpha}{\Gamma(\nu)} \int_{\zeta=\alpha}^a \frac{(\zeta(a) - \zeta)^{\nu-1}}{\zeta(a) - \alpha} \varphi \, d\zeta. \end{aligned}$$

647 The Volterra operator with kernel

$$648 \quad h(a, a') = \frac{1}{\Gamma(\nu)} \frac{(\zeta(a) - \zeta(a'))^{\nu-1}}{\zeta(a) - \alpha}$$

649 satisfies Condition (DIAG \star | γ, λ_Δ) with $\gamma = \nu - 1$ and $\lambda_\Delta = 0$, as long as we loosen the
 650 condition to allow any $\gamma > -1$. Hence, by Proposition 3.4, this operator maps

$$651 \quad \mathcal{H}L_{\sigma, \Lambda}^\infty(\Omega_\alpha) \rightarrow \mathcal{H}L_{\sigma+\nu-1, \Lambda}^\infty(\Omega_\alpha)$$

652 continuously for all $\sigma > -1$ and $\Lambda \geq 0$. Multiplication by $\zeta - \alpha$ then maps

$$653 \quad \mathcal{H}L_{\sigma+\nu-1, \Lambda}^\infty(\Omega_\alpha) \rightarrow \mathcal{H}L_{\sigma+\nu, \Lambda}^\infty(\Omega_\alpha)$$

654 continuously. □

655 From Proposition 4.1, we deduce that when φ belongs to $\mathcal{H}L_{\sigma, \bullet}^\infty(\Omega_\alpha)$ for some $\sigma > -1$,
 656 its fractional integral $\partial_{\zeta, \alpha}^{-\nu} \varphi$ has a well-defined Laplace transform along any ray $\Gamma_{\zeta, \alpha}^\theta \subset \Omega_\alpha$.
 657 Using a 2d integration argument, akin to the one in [?, Theorem 2.39], one can show that

$$658 \quad \mathcal{L}_{\zeta, \alpha}^\theta \partial_{\zeta, \alpha}^{-\nu} \varphi = z^{-\nu} \mathcal{L}_{\zeta, \alpha}^\theta \varphi$$

659 for all $\nu \in (0, \infty)$ and $\varphi \in \mathcal{H}L_{\sigma, \Lambda}^\infty(\Omega_\alpha)$ with $\sigma > -1$. This mirrors the Laplace trans-
 660 form's action on multiplication operators: under the same conditions on φ , you can use
 661 differentiation under the integral to show that [?, Theorem 1.34]

$$662 \quad \mathcal{L}_{\zeta, \alpha}^\theta (\zeta^n \varphi) = \left(-\frac{\partial}{\partial z}\right)^n \mathcal{L}_{\zeta, \alpha}^\theta \varphi$$

663 for all integers $n \geq 0$ and suitable directions θ .

664 *Remark 4.2.* In the relationship between $\partial_{\zeta,\alpha}^{-\nu}$ and multiplication by $z^{-\nu}$ described above,
 665 smoothing out the singularity at $\zeta = \alpha$ in the position domain corresponds to speeding up
 666 the decay as $|z| \rightarrow \infty$ in the frequency domain. This is a general feature of the Laplace
 667 transform.

668 **4.5. Going to the position domain.** Using the properties of the Laplace transform
 669 $\mathcal{L}_{\zeta,\alpha}^\theta$ we can turn differential operators in the frequency domain into Volterra operators in
 670 the position domain.

671 **LEMMA 4.3.** *For each root $-\alpha$ of P , let $\hat{\mathcal{P}}_\alpha$ be the Volterra operator*

$$672 \quad \hat{\mathcal{P}}_\alpha := P(-\zeta) + \partial_{\zeta,\alpha}^{-1} \circ Q(-\zeta) + \partial_{\zeta,\alpha}^{-2} \circ R(\partial_{\zeta,\alpha}^{-1}).$$

673 *If ψ_α satisfies the equation $\hat{\mathcal{P}}_\alpha \psi_\alpha = 0$, then its Laplace transform $\Psi_\alpha := \mathcal{L}_{\zeta,\alpha}^\theta \psi_\alpha$ satisfies*
 674 *the equation $\mathcal{P}\Psi_\alpha = 0$, as long as the Laplace transform is well-defined.*

675 *Remark 4.4.* For the Laplace transform to have any hope of being well-defined, we have
 676 to choose the direction θ so that the ray $\Gamma_{\zeta,\alpha}^\theta$ stays within Ω_α . We'll see later that for our
 677 results to apply, Ω_α can't touch any zero of $P(-\zeta)$ other than $\zeta = \alpha$. As a result, Ω_α might
 678 look like the domain illustrated in Section 1.3.1, reinterpreting the origin as $\zeta = \alpha$.

679 *Proof of Lemma 4.3.* Comparing the definitions of \mathcal{P} and $\hat{\mathcal{P}}_\alpha$, and using the properties
 680 of the Laplace transform discussed in Section 4.4.2, we can work out that $\mathcal{P}\mathcal{L}_{\zeta,\alpha}^\theta = \mathcal{L}_{\zeta,\alpha}^\theta \hat{\mathcal{P}}_\alpha$.
 681 Hence,

$$682 \quad \begin{aligned} \mathcal{P}\Psi_\alpha &= \mathcal{P}\mathcal{L}_{\zeta,\alpha}^\theta \psi_\alpha \\ 683 \quad &= \mathcal{L}_{\zeta,\alpha}^\theta \hat{\mathcal{P}}_\alpha \psi_\alpha. \end{aligned} \quad \square$$

685 We can now state and prove the main result of this section. Choose a simply connected
 686 open set Ω_α that touches $\zeta = \alpha$ and doesn't touch any other roots of $P(-\zeta)$, as illustrated
 687 in Section 1.3.1 under the reinterpretation from Section 4.3. We'll show that the Volterra
 688 equation $\hat{\mathcal{P}}_\alpha \varphi = 0$ has a unique solution of a certain form on Ω_α . By Lemma 4.3, the
 689 Laplace transform of this solution satisfies the equation $\mathcal{P}\Phi = 0$.

690 **THEOREM 4.5.** *Consider a root $-\alpha$ of P where $\tau_\alpha := Q(-\alpha)/P'(-\alpha)$ is real and pos-*
 691 *itive. Choose a simply connected open set Ω_α that touches $\zeta = \alpha$, and doesn't touch any*
 692 *other root of $P(-\zeta)$. Then the equation*

$$693 \quad \hat{\mathcal{P}}_\alpha \psi_\alpha = 0$$

694 *has a unique solution ψ_α in the affine subspace*

$$695 \quad \zeta^{\tau_\alpha-1} + \mathcal{H}L_{\tau_\alpha,\bullet}^\infty(\Omega_\alpha)$$

696 *of the space $\mathcal{H}L_{\tau_\alpha-1,\bullet}^\infty(\Omega_\alpha)$.*

697 *Proof of Theorem 4.5.* Choose a root $-\alpha$ of P where $\tau_\alpha := Q(-\alpha)/P'(-\alpha)$ is real and
 698 positive. For convenience, let $p = P(-\zeta)$ and $q = Q(-\zeta)$, noting that $\zeta = \alpha$ is a root of p .
 699 The conditions laid out in Section 4.2 guarantee that $\frac{\partial}{\partial \zeta} p$ and q are both non-zero $\zeta = \alpha$.

700 To apply Proposition 3.11, we need to rewrite the equation $\hat{\mathcal{P}}_\alpha \varphi = 0$ in the form
 701 $\varphi = \mathcal{V}^\alpha \varphi$, where \mathcal{V}^α is a Volterra operator of the kind described in Section 1.3 under the

reinterpretation from Section 4.3. When we write $\hat{\mathcal{P}}_\alpha$ explicitly as a Volterra operator, the equation $\hat{\mathcal{P}}_\alpha \varphi = 0$ becomes

$$[p \varphi](a) + \int_{\zeta=\alpha}^a q \varphi d\zeta + \int_{\zeta=\alpha}^a k_R(a, \cdot) \varphi d\zeta = 0,$$

where

$$k_R(a, a') = \sum_{j=0}^{\infty} \frac{R_j}{j!} (\zeta(a) - \zeta(a'))^{j+1}.$$

Isolating $p \varphi$ on the left and dividing both sides by p yields the equivalent equation

$$(4.2) \quad \varphi(a) = - \int_{\zeta=\alpha}^a \frac{q}{p(a)} \varphi d\zeta - \int_{\zeta=\alpha}^a \frac{k_R(a, \cdot)}{p(a)} \varphi d\zeta.$$

Let \mathcal{V}_0^α be the Volterra operator with kernel

$$k_0^\alpha(a, a') = -\frac{q(a')}{p(a)},$$

and let \mathcal{V}_*^α be the Volterra operator with kernel

$$k_*^\alpha(a, a') = -\frac{k_R(a, a')}{p(a)}.$$

Now we can write Equation (4.2) as $\varphi = \mathcal{V}^\alpha \varphi$, where $\mathcal{V}^\alpha = \mathcal{V}_0^\alpha + \mathcal{V}_*^\alpha$.

The kernel of \mathcal{V}^α extends meromorphically over all of \mathbb{C}^2 . One part of this observation demands some explanation: we need to confirm that the power series defining k_R converges everywhere. To do this, recall that $R(z^{-1})$ is holomorphic on the disk $|z| > A$. Consequently, for any $\lambda > A$, we have $|R_j| \lesssim \lambda^j$ over all $j \in \{0, 1, 2, \dots\}$. This is enough to show that k_R is well-defined throughout \mathbb{C}^2 . As we'll see later, it also leads to a useful bound on k_R .

Since \mathcal{V}_0^α has a separable kernel of the form described in Section 1.3.2, we're now working in the setting of Proposition 3.11. To apply the theorem, we just need to show that \mathcal{V}_0^α and \mathcal{V}_*^α satisfy the required conditions from Section 1.3.

- Condition (SING | τ) on \mathcal{V}_0^α is satisfied.

Since P is a monic polynomial whose roots are all simple, $1/P(-\zeta)$ has the nice partial fraction decomposition

$$\frac{1}{P(-\zeta)} = \sum_{-\beta \in \mathfrak{B}} \frac{1}{-P'(-\beta)(\zeta - \beta)},$$

where \mathfrak{B} is the zero set of P [?, Section 1.4, Exercise 2]. It follows that

$$\begin{aligned} -\frac{Q(-\zeta(a'))}{P(-\zeta(a))} &= \sum_{-\beta \in \mathfrak{B}} \frac{Q(-\zeta(a'))}{P'(-\beta)(\zeta(a) - \beta)} \\ &= \sum_{\substack{-\beta \in \mathfrak{B} \\ \beta \neq \alpha}} \frac{Q(-\zeta(a'))}{P'(-\beta)(\zeta(a) - \beta)} + \frac{Q(-\zeta(a'))}{P'(-\alpha)(\zeta(a) - \alpha)}. \end{aligned}$$

Now we expand Q around $-\alpha$, using the decomposition $Q(\zeta) = Q(-\alpha) + (\zeta - \alpha)Q_\alpha(\zeta)$ to split up the last term of the expression above:

$$(4.3) \quad -\frac{Q(-\zeta(a))}{P(-\zeta(a))} = \sum_{\substack{-\beta \in \mathfrak{B} \\ \beta \neq \alpha}} \frac{Q(-\zeta(a'))}{P'(-\beta)(\zeta(a) - \beta)} + \frac{(\zeta(a') - \alpha)Q_\alpha(\zeta(a'))}{P'(-\alpha)(\zeta(a) - \alpha)} + \frac{Q(-\alpha)}{P'(-\alpha)(\zeta(a) - \alpha)}.$$

First, we want to show that $-q/p - \tau_\alpha/(\zeta - \alpha)$ is bounded on a neighborhood of $\zeta = \alpha$ in Ω_α . Equation (4.3) tells us that

$$\begin{aligned} -\frac{Q(-\zeta)}{P(-\zeta)} &= \sum_{\substack{-\beta \in \mathfrak{B} \\ \beta \neq \alpha}} \frac{Q(-\zeta)}{P'(-\beta)(\zeta - \beta)} + \frac{(\zeta - \alpha)Q_\alpha(\zeta)}{P'(-\alpha)(\zeta - \alpha)} + \frac{Q(-\alpha)}{P'(-\alpha)(\zeta - \alpha)} \\ &= \sum_{\substack{-\beta \in \mathfrak{B} \\ \beta \neq \alpha}} \frac{Q(-\zeta)}{P'(-\beta)(\zeta - \beta)} + \frac{Q_\alpha(\zeta)}{P'(-\alpha)} + \frac{Q(-\alpha)}{P'(-\alpha)(\zeta - \alpha)} \end{aligned}$$

Recalling that $\tau_\alpha = Q(-\alpha)/P'(-\alpha)$, we conclude that

$$-\frac{Q(-\zeta)}{P(-\zeta)} - \frac{\tau_\alpha}{\zeta - \alpha} = \sum_{\substack{-\beta \in \mathfrak{B} \\ \beta \neq \alpha}} \frac{Q(-\zeta)}{P'(-\beta)(\zeta - \beta)} + \frac{Q_\alpha(\zeta)}{P'(-\alpha)}$$

Since Q and Q_α are polynomials, the right-hand side is bounded on any neighborhood of $\zeta = \alpha$ that avoids the other roots of p and avoids infinity.

Next, we want to show that for any $\sigma > \tau_\alpha$, the bound

$$|\zeta(a) - \alpha| \left| \frac{q(a')}{p(a)} \right| < \sigma$$

holds when a and a' are close enough to $\zeta = \alpha$. Taking the absolute value of both sides of Equation (4.3), and using the assumption that τ_α is real and positive, we see that

$$\begin{aligned} \left| \frac{Q(-\zeta(a'))}{P(-\zeta(a))} \right| &\leq \sum_{\substack{-\beta \in \mathfrak{B} \\ \beta \neq \alpha}} \left| \frac{Q(-\zeta(a'))}{P'(-\beta)(\zeta(a) - \beta)} \right| + \left| \frac{(\zeta(a') - \alpha)Q_\alpha(\zeta(a'))}{P'(-\alpha)(\zeta(a) - \alpha)} \right| + \left| \frac{\tau_\alpha}{\zeta(a) - \alpha} \right| \\ |\zeta(a) - \alpha| \left| \frac{Q(-\zeta(a'))}{P(-\zeta(a))} \right| &\leq \sum_{\substack{-\beta \in \mathfrak{B} \\ \beta \neq \alpha}} \left| \frac{Q(-\zeta(a'))(\zeta(a) - \alpha)}{P'(-\beta)(\zeta(a) - \beta)} \right| + \left| \frac{(\zeta(a') - \alpha)Q_\alpha(\zeta(a'))}{P'(-\alpha)} \right| + \tau_\alpha. \end{aligned}$$

By bringing $\zeta(a')$ closer to α , we can make the middle term

$$\left| \frac{(\zeta(a') - \alpha)Q_\alpha(\zeta(a'))}{P'(-\alpha)} \right|$$

as small as we want. Similarly, by bringing $\zeta(a)$ closer to α , we can make the sum

$$\sum_{\substack{-\beta \in \mathfrak{B} \\ \beta \neq \alpha}} \left| \frac{Q(-\zeta(a'))(\zeta(a) - \alpha)}{P'(-\beta)(\zeta(a) - \beta)} \right|$$

754

as small as we want. Hence, for any $\sigma > \tau_\alpha$, the bound

755

$$|\zeta(a) - \alpha| \left| \frac{Q(-\zeta(a'))}{P(-\zeta(a))} \right| \leq \sigma$$

756

holds over some neighborhood of $(\zeta(a), \zeta(a')) = (\alpha, \alpha)$ in Ω_α^2 .

757

- Condition [\(DIAG₀ | \$\lambda_\Delta\$ \)](#) on \mathcal{V}_0^α is satisfied.

758

Since we're trying to bound $(\zeta(a) - \alpha) k_0^\alpha(a, a')$ with a function of the difference $\omega(a, a') := \zeta(a) - \zeta(a')$, let's rewrite it as a rational function of $\omega(a, a')$ and $\zeta(a)$.

759

First, rewrite

761

$$\begin{aligned} q(a') &= Q(-\zeta(a')) \\ &= Q(\omega(a, a') - \zeta(a)) \end{aligned}$$

762

764

in the form

765

$$q(a') = Q_{d-1}(\zeta(a)) \omega(a, a')^{d-1} + \dots + Q_1(\zeta(a)) \omega(a, a') + Q_0(\zeta(a)),$$

766

where Q_0, \dots, Q_{d-1} are polynomials of degree at most $d-1$. Next, knowing that $-\alpha$ is a root of p , rewrite p in the form $(\zeta - \alpha) P_\alpha(\zeta)$, where P_α is a polynomial of degree $d-1$. We can then write $-(\zeta(a) - \alpha) k_0^\alpha(a, a')$ in the form

767

769

$$(\zeta(a) - \alpha) \frac{q(a')}{p(a)} = \frac{Q_{d-1}(\zeta(a))}{P_\alpha(\zeta(a))} \omega(a, a')^{d-1} + \dots + \frac{Q_1(\zeta(a))}{P_\alpha(\zeta(a))} \omega(a, a') + \frac{Q_0(\zeta(a))}{P_\alpha(\zeta(a))},$$

770

viewing it as a polynomial in $\omega(a, a')$ whose coefficients are rational functions in $\zeta(a)$. If we keep a away from the roots of p other than $\zeta = \alpha$, each coefficient is bounded, so $|\zeta(a) - \alpha| |k_0^\alpha(a, a')|$ is bounded by a polynomial in $|\omega(a, a')|$. It follows that for any $\lambda_\Delta > 0$, and any domain Ω_α that avoids all the roots of p other than $\zeta = \alpha$, we have

773

774

775

$$|\zeta(a) - \alpha| |k_0^\alpha(a, a')| \lesssim e^{\lambda_\Delta |\omega(a, a')|}$$

776

over all $a, a' \in \Omega_\alpha$.

777

For consistency with Condition [\(DIAG_{*} | \$\gamma, \lambda_\Delta\$ \)](#), we choose $\lambda_\Delta > A$.

778

- Condition [\(REG-P | \$B, \epsilon\$ \)](#) on \mathcal{V}_0^α is satisfied.

779

This is true because p is a polynomial.

780

- Condition [\(DIAG_{*} | \$\gamma, \lambda_\Delta\$ \)](#) on \mathcal{V}_*^α is satisfied.

781

First, observe that

782

$$\begin{aligned} |k_*^\alpha(a, a')| &= \left| \frac{k_R(a, a')}{p(a)} \right| \\ &\leq \frac{1}{|p(a)|} \sum_{j=0}^{\infty} \frac{|R_j|}{j!} |\zeta(a) - \zeta(a')|^{j+1} \\ &= \frac{|\zeta(a) - \zeta(a')|}{|p(a)|} \sum_{j=0}^{\infty} \frac{|R_j|}{j!} |\zeta(a) - \zeta(a')|^j. \end{aligned}$$

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785

786 Now, choose $\lambda_\Delta > A$. Since $R(z^{-1})$ is holomorphic on the disk $|z| > A$, we have
 787 $|R_j| \lesssim \lambda_\Delta^j$ over all and $j \in \{0, 1, 2, \dots\}$, as mentioned earlier. It follows that

$$788 \quad \sum_{j=0}^{\infty} \frac{|R_j|}{j!} |\zeta(a) - \zeta(a')|^j \lesssim \sum_{j=0}^{\infty} \frac{\lambda_\Delta^j}{j!} |\zeta(a) - \zeta(a')|^j$$

$$789 \quad \lesssim e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

791 over all $a, a' \in \Omega_\alpha^2$, so

$$792 \quad |k_\star^\alpha(a, a')| \lesssim \frac{|\zeta(a) - \zeta(a')|}{|p(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

793 over all $a, a' \in \Omega_\alpha^2$. Like before, rewrite p in the form $(\zeta - \alpha) P_\alpha(\zeta)$, so we have

$$794 \quad |k_\star^\alpha(a, a')| \lesssim \frac{|\zeta(a) - \zeta(a')|}{|\zeta(a) - \alpha| |P_\alpha(\zeta(a))|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

795 over all $a, a' \in \Omega_\alpha$. Since P_α is a polynomial with a finite number of roots, and Ω_α
 796 doesn't touch any of the roots, $|P_\alpha(\zeta)|^{-1}$ is bounded on Ω_α . We conclude that

$$797 \quad |k_\star^\alpha(a, a')| \lesssim \frac{|\zeta(a) - \zeta(a')|}{|\zeta(a) - \alpha|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

798 over all $a, a' \in \Omega_\alpha$, so Condition (DIAG $\star \mid \gamma, \lambda_\Delta$) is satisfied with $\gamma = 1$.

799 We can now apply Proposition 3.11, yielding the desired result. □