Resurgence of modified Bessel functions of second kind

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1 Generalized Airy

In [?] [?] the authors introduce generalized Airy functions $A_k(z), B_0(z), B_k(z)$, $k = 1,2,3$ as approximate solutions of the Orr–Sommerfield fluid equation. They are defined as countour integral ([?] §9.13(ii))

$$
\begin{aligned}
\mathrm{A}_k(z,p) &= \frac{1}{2\pi i}\int_{\Gamma_k} e^{zt-\frac{t^3}{3}}\frac{dt}{t^p} \qquad k=1,2,3\ p\in\mathbb{C}\\
\mathrm{B}_0(z,p) &= \frac{1}{2\pi i}\int_{\Gamma_0} e^{zt-\frac{t^3}{3}}\frac{dt}{t^p} \qquad p\in\mathbb{Z}\\
\mathrm{B}_k(z,p) &= \int_{\gamma_k} e^{zt-\frac{t^3}{3}}\frac{dt}{t^p} \qquad k=1,2,3\ p\in\mathbb{Z}
\end{aligned}
$$

where the countours $\Gamma_k, \Gamma_0, \gamma_k$ are represented in Figure **??**

Each generalize Airy functions is a solution of

$$\left[\partial_z^3 - z\partial_z + (p-1)\right] f(z,p) = 0 \tag{1}$$

and if $p = 0$ they reduced to the classical Airy functions: $\mathrm{Ai}(z) = \mathrm{A}_1(z,0)$ and $\mathrm{Bi}(z) = \mathrm{B}_1(z,0)$.

We define the following exponential integral: let $p \geq 0$, $f(t) = \frac{t^3}{3} - t$ and $\nu_p = \frac{dt}{t^p}$. The critical points of f are $\pm\frac{2}{3}$ (as for the classical Airy case), however the volume form ν_p is meromorphic

$$I_\alpha(z,p) := \frac{1}{2\pi i}\int_{\mathcal{C}_\alpha} e^{-z(\frac{t^3}{3}-t)}\frac{dt}{t^p} \tag{2}$$

where $\mathcal{C}_\alpha$ is the path through the point α, starting and ending at infinity (as in Figure **??**). In particular, $I_+(z,p) = z^{\frac{p-1}{3}}\mathrm{A}_1(z^{2/3},p)$:

$$
\begin{aligned}
I_+(z,p) &= \frac{1}{2\pi i}\int_{\mathcal{C}_+} e^{-z(\frac{t^3}{3}-t)}\frac{dt}{t^p} \\
&= \frac{z^{(p-1)/2}}{2\pi i}\int_{z^{-1/3}\mathcal{C}_+} e^{-z(z^{-1}\frac{s^3}{3}-z^{-1/3}s)}\frac{ds}{s^p} \qquad\qquad t = z^{-\frac{1}{3}}s \\
&= \frac{z^{(p-1)/3}}{2\pi i}\int_{z^{-1/3}\mathcal{C}_+} e^{-\left(\frac{s^3}{3}-z^{2/3}s\right)}\frac{ds}{s^p} \\
&= z^{(p-1)/3}\mathrm{A}_1(z^{2/3},p)
\end{aligned}
$$

It follows that $I_+(z)$ is a solution of

$$
\left[\partial_z^3 - \frac{4}{9}\partial_z + \frac{2-p}{z}\partial_z^2 - \frac{4(1-p)}{9z} - \frac{1+3p-3p^2}{9}\frac{\partial_z}{z^2} + \frac{3+p-3p^2-p^3}{27z^3}\right] I_+(z,p) = 0 \quad (3)
$$

From the general theory of linear ODE, the formal integral solution of (3) is a linear combinations of three generators

$$
\tilde{I}_+(z,p) = U_1 z^{p-1}\tilde{W}_1(z) + U_2 e^{-\frac{2}{3}z}z^{-1/2}\tilde{W}_2(z) + U_3 e^{\frac{2}{3}z}z^{-1/2}\tilde{W}_3(z) \quad (4)
$$

where $\tilde{W}_1, \tilde{W}_2$ and $\tilde{W}_3$ are formal power seirs $\tilde{W}_{\mathbf{k}}(z) = \sum_{j\geq 0} a_{\mathbf{k},j}z^{-j}$, which are the unique (we fix $a_{k,0} = 1$ for $k = 1,2,3$) solution of

$$
\left[\partial_z^3 - \frac{4}{9}\partial_z - \frac{1-2p}{z}\partial_z^2 + \frac{17-30p+12p^2}{9z^2}\partial_z + \frac{8}{27}\frac{p^3-6p^2+11p-6}{z^3}\right]\tilde{W}_1(z) \quad = \quad 0 \quad (5)
$$

$$
\begin{aligned}
\Bigg[\partial_z^3 - 2\partial_z^2 + \frac{8}{9}\partial_z + \frac{1-2p}{2z}\partial_z^2 - \frac{2-4p}{3z}\partial_z + \frac{5+24p+12p^2}{36z^2}\partial_z + \\
-\frac{5+24p+12p^2}{54z^2} - \frac{1}{z^3}\left(\frac{5}{24} + \frac{59}{108}p + \frac{5}{18}p^2 + \frac{1}{27}p^3\right)\Bigg]\tilde{W}_2(z) = 0 \quad (6)
\end{aligned}
$$

$$
\begin{aligned}
\Bigg[\partial_z^3 + 2\partial_z^2 + \frac{8}{9}\partial_z + \frac{1-2p}{2z}\partial_z^2 + \frac{2-4p}{3z}\partial_z + \frac{5+24p+12p^2}{36z^2}\partial_z + \\
+\frac{5+24p+12p^2}{54z^2} - \frac{1}{z^3}\left(\frac{5}{24} + \frac{59}{108}p + \frac{5}{18}p^2 + \frac{1}{27}p^3\right)\Bigg]\tilde{W}_3(z) = 0 \quad (7)
\end{aligned}
$$

Let us first study equation (5): its Borel transform is

$$
\begin{aligned}
-\zeta^3\tilde{w}_1 + \frac{4}{9}\zeta\tilde{w}_1 - (1-2p)\int_0^\zeta \tilde{w}_1(t)t^2dt + \frac{(17-30p+12p^2)}{9}\int_0^\zeta(-t\tilde{w}_1(t))(\zeta-t)dt + \\
+\frac{8}{27}(p^3-6p^2+11p-6)\int_0^\zeta \tilde{w}_1(t)\frac{(\zeta-t)^2}{2}dt = 0
\end{aligned}
$$

We differentiate three times, getting

$$\left(\frac{4}{9}\zeta-\zeta^3\right)\tilde{w}_1^{(3)}(\zeta)+\left(\frac{4}{3}-2(5-p)\zeta^2\right)\tilde{w}_1''(\zeta)-\left(\frac{215}{9}-\frac{34}{3}p+\frac{4}{3}p^2\right)\tilde{w}_1'(\zeta)+$$
$$+\frac{8}{27}\left(p-\frac{9}{2}\right)\left(p-\frac{7}{2}\right)\left(p-\frac{5}{2}\right)\tilde{w}_1(\zeta)=0$$

we set $y=\frac{9}{4}\zeta^2$

$$y^2(1-y)\tilde{w}_1^{(3)}(y)+y\left(\left(p-\frac{13}{2}\right)y+3\right)\tilde{w}_1''(y)-\left(y\left(\frac{305}{36}-\frac{1}{3}(10-p)p\right)-\frac{3}{4}\right)\tilde{w}_1'(y)+$$
$$+\frac{1}{27}\left(p-\frac{5}{2}\right)\tilde{w}_1(y)=0 \quad (8)$$

which is a generalized hypergeomtric equation (16.8.5 [**?**]) with parameters

$$\mathbf{a}_0=\left(\frac{3}{2}-\frac{p}{3};\frac{7}{6}-\frac{p}{3};\frac{5}{6}-\frac{p}{3}\right) \qquad \mathbf{b}_0=\left(\frac{1}{2};\frac{3}{2}\right)$$

therefore, we define $\hat{w}_1(y)=c_1\ {}_3F_2\left(\mathbf{a}_0;\mathbf{b}_0;y\right)$.
Then we look at equation (6): its Borel transform is

$$-\zeta^3\tilde{w}_2(\zeta)-2\zeta^2\tilde{w}_2(\zeta)-\frac{8}{9}\zeta\tilde{w}_2(\zeta)+\frac{1-2p}{2}\int_0^\zeta t^2\tilde{w}_2(t)dt-\frac{2-4p}{3}\int_0^\zeta(-t\tilde{w}_2(t))dt+$$
$$+\frac{5+24p+12p^2}{36}\int_0^\zeta(\zeta-t)(-t\tilde{w}_2(t))dt-\frac{5+24p+12p^2}{54}\int_0^\zeta(\zeta-t)\tilde{w}_2(t)dt+$$
$$-\left(\frac{5}{54}+\frac{59}{108}p+\frac{5}{18}p^2+\frac{1}{27}p^3\right)\int_0^\zeta\frac{(\zeta-t)^2}{2}\tilde{w}_2(t)dt=0 \quad (9)$$

and taking three derivatives it simplifies to

$$\Bigg[\left(\zeta^3+2\zeta^2+\frac{8}{9}\zeta\right)\partial_\zeta^3+\frac{8}{3}\partial_\zeta^2+\left(\frac{17}{2}+p\right)\zeta\left(\zeta+\frac{4}{3}\right)\partial_\zeta^2+$$
$$\left(\frac{p^2}{3}+\frac{14}{3}p+\frac{581}{36}\right)\left(\zeta+\frac{2}{3}\right)\partial_\zeta+\frac{1}{27}\left(p+\frac{7}{2}\right)\left(p+\frac{11}{2}\right)\left(p+\frac{15}{2}\right)\Bigg]\tilde{w}_2(\zeta)=0 \quad (10)$$

define $y=\zeta\left(\zeta+\frac{4}{3}\right)$

$$\Bigg[y\left(y+\frac{4}{9}\right)^2\partial_y^3+2\left(\frac{3}{2}y+\frac{4}{3}+\frac{y}{2}\left(\frac{17}{2}+p\right)\right)\left(y+\frac{4}{9}\right)\partial_y^2+\left(\frac{2}{3}+\frac{y}{4}\left(\frac{17}{2}+p\right)\right)\partial_y$$
$$+\frac{1}{4}\left(\frac{p^3}{3}+\frac{14}{3}p+\frac{581}{36}\right)\left(y+\frac{4}{9}\right)\partial_y+\frac{1}{8\cdot 27}\left(p+\frac{7}{2}\right)\left(p+\frac{11}{2}\right)\left(p+\frac{15}{2}\right)\Bigg]\tilde{w}_2(y)=0$$
$$(11)$$

which now looks like a generalized hypergeometric equation. Indeed setting $t = \frac{9}{4}y + 1$, (11) reads

$$\left[t^2(1-t)\partial_t^3 - t\left((\frac{23}{4}+\frac{p}{2})t - \frac{p}{2} - \frac{11}{4}\right)\partial_t^2 - \left(-\frac{5}{8} - \frac{p}{4} + t\Big(\frac{887}{144} + \frac{17}{12}p + \frac{p^2}{12}\Big)\right)\partial_t + \right.$$
$$\left. - \frac{1}{8\cdot 27}\left(p+\frac{7}{2}\right)\left(p+\frac{11}{2}\right)\left(p+\frac{15}{2}\right)\right]\tilde{w}_2(t) = 0 \quad (12)$$

hence a solution is given by the generalized hypergeometric function ${}_3F_2\left(\mathbf{a};\mathbf{b};y\right)$ with parameters

$$\mathbf{a} = \left(\frac{5}{4}+\frac{p}{6}; \frac{7}{12}+\frac{p}{6}; \frac{11}{12}+\frac{p}{6}\right) \qquad \mathbf{b} = \left(\frac{1}{2}; \frac{5}{4}+\frac{p}{2}\right)$$

and we denote $\hat{w}_2(\zeta) = c_2\ {}_3F_2\left(\mathbf{a};\mathbf{b};\left(\frac{3}{2}\zeta+1\right)^2\right)$. The equation (7) differes from (6) for few signs: we find that its Borel transform is

$$-\zeta^3\tilde{w}_2(\zeta) + 2\zeta^2\tilde{w}_2(\zeta) - \frac{8}{9}\zeta\tilde{w}_2(\zeta) + \frac{1-2p}{2}\int_0^\zeta t^2\tilde{w}_2(t)dt + \frac{2-4p}{3}\int_0^\zeta(-t\tilde{w}_2(t))dt +$$
$$+ \frac{5+24p+12p^2}{36}\int_0^\zeta(\zeta-t)(-t\tilde{w}_2(t))dt + \frac{5+24p+12p^2}{54}\int_0^\zeta(\zeta-t)\tilde{w}_2(t)dt +$$
$$-\left(\frac{5}{54}+\frac{59}{108}p+\frac{5}{18}p^2+\frac{1}{27}p^3\right)\int_0^\zeta\frac{(\zeta-t)^2}{2}\tilde{w}_2(t)dt = 0 \quad (13)$$

and differentiating three times we find a generalized hypergeometric equation with paramters $\mathbf{a},\mathbf{b}$ in the variable $y = \left(\frac{3}{2}\zeta-1\right)^2$, i.e. $\hat{w}_3(\zeta) = c_3\ {}_3F_2\left(\mathbf{a};\mathbf{b};\left(\frac{3}{2}\zeta-1\right)^2\right)$. In particular,

$$\hat{w}_3(\zeta) = C_{23}\hat{w}_2(\zeta - \tfrac{4}{3}) \quad (14)$$

Remark 1 *We notice that if $p = \frac{1}{2}$ then $\hat{w}_1(\zeta), \hat{w}_2(\zeta)$ and $\hat{w}_3(\zeta)$ are generalized hypergeometric functions with the same coefficients:*

$$\mathbf{a}_0 = \left(\frac{3}{2}-\frac{1}{6}, \frac{7}{6}-\frac{1}{6}, \frac{5}{6}-\frac{1}{6}\right) = \left(\frac{4}{3}, 1, \frac{2}{3}\right) \qquad \mathbf{b}_0 = \left(\frac{1}{2}, \frac{3}{2}\right)$$
$$\mathbf{a} = \left(\frac{5}{4}+\frac{1}{12}, \frac{7}{12}+\frac{1}{12}, \frac{11}{12}+\frac{1}{12}\right) = \left(\frac{4}{3}, \frac{2}{3}, 1\right) \qquad \mathbf{b} = \left(\frac{1}{2}, \frac{5}{4}+\frac{1}{4}\right) = \left(\frac{1}{2}, \frac{3}{2}\right)$$

hence for $p = \frac{1}{2}$

$$\hat{w}_1(\zeta) = C_{12}\hat{w}_2(\zeta - \tfrac{2}{3}) = C_{13}\hat{w}_3(\zeta + \tfrac{2}{3}) \quad (15)$$

1.1 Analytic continuation

In [**D.B. Karp and E.G. Prilepkina** formula 3.1 (see also https://arxiv.org/pdf/2110.12219.pdf equation 27)] the authors compute the analytic continuation of generalized hypergeomtric functions across the branch cut

$$ {}_qF_{q-1}\left(\mathbf{a};\mathbf{b};x+i0\right) - {}_qF_{q-1}\left(\mathbf{a};\mathbf{b};x-i0\right) = 2\pi i\frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}G^{q,0}_{q,q}\left(1,\mathbf{b};\mathbf{a};\frac{1}{x}\right). \tag{16} $$

We first consider $\hat{w}_1(\zeta) = c_1\ {}_3F_2\left(\mathbf{a}_0;\mathbf{b}_0;\frac{9}{4}\zeta^2\right)$, it has a branch cut at $\zeta\in[\frac{2}{3},+\infty)$ and another for $\zeta\in(-\infty,-\frac{2}{3}]$. The jumps can be computed as follows

$$ \begin{aligned} {}_3F_2\left(\mathbf{a}_0;\mathbf{b}_0;y+i0\right) - {}_3F_2\left(\mathbf{a}_0;\mathbf{b}_0;y-i0\right) &= 2\pi i\frac{\Gamma(\mathbf{b}_0)}{\Gamma(\mathbf{a}_0)}G^{3,0}_{3,3}\left(1,\mathbf{b}_0;\mathbf{a}_0;\frac{1}{y}\right) \\ &= 2\pi i\frac{\Gamma(\mathbf{b}_0)}{\Gamma(\mathbf{a}_0)}G^{0,3}_{3,3}\left(1-\mathbf{a}_0;0,1-\mathbf{b}_0;y\right) \\ &= 2\pi i\frac{\Gamma(\mathbf{b}_0)}{\Gamma(\mathbf{a}_0)}G^{0,3}_{3,3}\left(\mathbf{a}'_0;0,\frac{1}{2},-\frac{1}{2};y\right) \\ &= 2\pi i\frac{\Gamma(\mathbf{b_0})}{\Gamma(\mathbf{a}_0)}G^{0,3}_{3,3}\left(\mathbf{a}'_0;\mathbf{b}'_0;y\right) \end{aligned} $$

Notice that equation (8) is also the differential equation for Meijer G-funtion $G^{3,0}_{3,3}(\mathbf{a}'_0;\mathbf{b}'_0;y)$ where

$$ \mathbf{a}'_0 = \left(\frac{p}{3}-\frac{1}{2};\frac{p}{3}-\frac{1}{6};\frac{p}{3}+\frac{1}{6}\right) \qquad \mathbf{b}'_0 = \left(0;\frac{1}{2};-\frac{1}{2}\right) $$

1.1.1 $\hat{\mathbf{w}}_2(\zeta)$ and $\hat{\mathbf{w}}_3(\zeta)$

We consider $\hat{w}_2(\zeta)$, it has a branch cut at $\zeta\in[0,+\infty)$ and $\zeta\in(-\infty,-\frac{4}{3}]$,

We first study the equation (5): its unique formal solution $\tilde{w}_1(z)$ has the following coefficients (we compute them with Mathematica)

$$ a_{1,2j} = \frac{\Gamma\left(1+3j-p\right)}{3^j\Gamma(1+j)\Gamma(1-p)} \qquad j\geq 0 \tag{17} $$

Notice that $a_{1,j}$ are well defined only for $p\in\mathbb{C}\setminus\{1,2,3,...\}$. However, for $p=1,2,3$ the solution is well defined and constant $\tilde{w}_1(z)=1$. Therefore the Borel transform of $\tilde{w}_1(z)$ is

$$ \hat{w}_1(\zeta) = \begin{cases} \delta & \text{if } p=1,2,3 \\ \delta+\zeta\frac{\Gamma(4-p)}{3\Gamma(1-p)}{}_3F_2\left(\frac{4-p}{3},\frac{5-p}{3},2-\frac{p}{3};\frac{3}{2},2;\frac{9}{4}\zeta^2\right) & \text{if } p\in\mathbb{C}\setminus\{1,2,3,...\} \end{cases} \tag{18} $$

In particular, the generalized hypergeometric series has a branch cut singularity at $\zeta=\pm\frac{3}{2}$. We expect that $\tilde{w}_1(z)$ is a simple resurgent fuction such that

$$ \begin{aligned} \hat{w}_1(\zeta+\tfrac{2}{3}) &= \delta+\frac{C}{2\pi i\zeta}+\frac{1}{2\pi i}\log(\zeta)\hat{w}_2(\zeta)+\text{hol. fct.} \\ \hat{w}_1(\zeta-\tfrac{2}{3}) &= \delta+\frac{C}{2\pi i\zeta}+\frac{1}{2\pi i}\log(\zeta)\hat{w}_3(\zeta)+\text{hol. fct.} \end{aligned} $$

In [**D.B. Karp and E.G. Prilepkina** formula 3.1 (see also https://arxiv.org/pdf/2110.12219.pdf equation 27)] the authors compute the analytic continuation of generalized hypergeomtric functions across the branch cut

$$ {}_qF_{q-1}\left(\mathbf{a};\mathbf{b};x+i0\right)-{}_qF_{q-1}\left(\mathbf{a};\mathbf{b};x-i0\right)=2\pi i\frac{\Gamma(\mathbf{b}-d+1)}{\Gamma(\mathbf{a}-d+1)}x^{d-1}G^{q,0}_{q,q}\left(d,\mathbf{b};\mathbf{a};\frac{1}{x}\right) \tag{19}$$

which in our case becomes

$$\begin{aligned}
&\frac{3}{2}\zeta\left({}_3F_2\left(\frac{4-p}{3},\frac{5-p}{3},2-\frac{p}{3};\frac{3}{2},2;\frac{9}{4}\zeta^2+i0\right)-{}_3F_2\left(\frac{4-p}{3},\frac{5-p}{3},2-\frac{p}{3};\frac{3}{2},2;\frac{9}{4}\zeta^2-i0\right)\right)=\\
&\quad=2\pi i\frac{\Gamma(\mathbf{b}+\frac{1}{2})}{\Gamma(\mathbf{a}+\frac{1}{2})}G^{3,0}_{3,3}\left(\frac{1}{2},\frac{3}{2},2;\frac{4-p}{3},\frac{5-p}{3},2-\frac{p}{3};\frac{4}{9}\zeta^{-2}\right)\\
&\quad=2\pi i\frac{\Gamma(\mathbf{b}+\frac{1}{2})}{\Gamma(\mathbf{a}+\frac{1}{2})}G^{0,3}_{3,3}\left(\frac{p-1}{3},\frac{p-2}{3},\frac{p}{3}-1;\frac{1}{2},-\frac{1}{2},-1;\frac{9}{4}\zeta^{2}\right)
\end{aligned}$$

When $p=1,2,3$ the solution is a trivial resurgent function, it is constant.

Let us now consider the formal solution $\tilde{w}_2(z)$ and $\tilde{w}_3(z)$. The recursive relation for $a_{2,j},a_{3,j}$ have not a simple expressions as it is for $a_{1,j}$. However, we compute numerically their first coefficients:

	$j=1$	23	45	67
$a_{2,j}$				
$a_{3,j}$				

Then we look at the Borel transform of $\tilde{W}_2,\tilde{W}_3$: their equations have some symmetries in the coefficients, namely they differ by a sing in the even degree coefficients.