GENERAL THIMBLES INTEGRALS

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1. Proof of Borel regularity

We are going to prove Theorem 5.1 `draft2`. Let X be a $N-$dim manifold, $f: X \to \mathbb{C}$ be a holomorphic Morse function with simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$I(z) := \int_{\mathcal{C}} e^{-zf}\, \nu \tag{1.1}$$

where $\mathcal{C}$ is a suitable contour such that the integral is well defined. Indeed, $I(z)$ represents a pairing between a relative homology class $\mathcal{C} \in H_N^B(X, zf)$ and a cohomology class $\nu \in H^N_{dR}(X, zf)$ (see Section 1.3.1 Thimble integrals in the introduction). Let us restrict to one dimensional X. For any Morse critical points x_α[1] of f, the saddle point approximation allows to compute the asymptotic expansion of $I_\alpha(z)$

$$I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\, \nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)} \sqrt{2\pi} z^{-1/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n} \qquad \text{as } \mathrm{Re}(ze^{i\theta}) \to \infty \tag{1.2}$$

where $\mathcal{C}_\alpha$ is a steepest descent path through the critical point x_α and θ is chosen such that $f(x_\beta) \notin f(x_\alpha) + [0, e^{i\theta}\infty)$ for $\beta \neq \alpha$[2]. Notice that $f \circ \mathcal{C}_\alpha$ lies in the ray $\zeta_\alpha + [0, e^{i\theta}\infty)$, where $\zeta_\alpha := f(x_\alpha)$.

Theorem 1.1. Let $N = 1$. Let $I_\alpha(z)$ defined as in (1.2) for every critical point x_α. Then $\tilde{I}_\alpha$ is Borel regular for $\mathrm{Re}(ze^{i\theta}) > 0$:

(1) The series $\tilde{I}_\alpha(z) = e^{-zf(x_\alpha)} \sqrt{2\pi} z^{-1/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n}$ is Gevrey-1.
(2) The series $\tilde{\iota}_\alpha(\zeta) := \mathcal{B}(\tilde{I}_\alpha)$ converges near $\zeta = \zeta_\alpha$.
(3) If you continue the sum of $\tilde{\iota}_\alpha$ along the ray going rightward from ζ_α in the direction θ, and take its Laplace transform along that ray, you'll recover I_α.

Remark 1.2. (1) We may drop the assumption of non degenerate critical points for f, however the asymptotic expansion of $I_\alpha(z)$ will depend on the order m such that $f^{(m)}(x_\alpha) \neq 0$ and $f^{(j)}(x_\alpha) = 0$ for every $j = 1, \ldots, m-1$ (see [Zorich] Theorem 1 Section 19.2.5).

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[1]By Morse critical points we mean non–degenerate isolated critical points.
[2]Such a θ exists because f has a finite number of critical points.

(2) in [Malgrange74] (see also Chapter 5 of [Mistergard Phd thesis] for a general review), the author computes the asymptotic expansion of exponential integrals for $N > 1$ which get logarithmic terms like

$$\tilde{I}(z) = \sum_{j \in A} \sum_{n \geq 0} \sum_{q=0}^{N-1} a_{n,q,j} z^{-n-j} (\log z)^q,$$

for $A \subset \mathbb{Q}_{\geq 0}$ finite. Due to the presence of logarithmic terms, the definition of Borel transform has to be further extended (see [Mistergard phd] Definition pag 5) and the study of Borel regularity becomes more involved.

Proof. Part (1): Since f is Morse, we can find a holomorphic chart τ around x_α with $\frac{1}{2}\tau^2 = f - \zeta_\alpha$. Let $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$ be the parts of $\mathcal{C}_\alpha$ that go from the past to x_α and from x_α to the future, respectively. We can arrange for τ to be valued in $(-\infty e^{i\theta}, 0]$ and $[0, e^{i\theta}\infty)$ on $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$, respectively. **[We should explicitly spell out and check the conditions that make this possible. I think we're implicitly orienting $\mathcal{C}_\alpha$ so that τ in the upper half-plane.]** Since ν is holomorphic, we can express it as a Taylor series

$$\nu = \sum_{n \geq 0} b_n^\alpha \tau^n \, d\tau$$

that converges in some disk $|\tau| < \varepsilon$.

In coordinates τ the integral $I_\alpha(z)$ can be approximated as

$$I_\alpha(z) \sim e^{-z\zeta_\alpha} \int_{\tau \in [-\varepsilon, \varepsilon]} e^{-z\tau^2/2} \, \nu$$

as $\operatorname{Re}(z e^{i\theta}) \to \infty$ (see Lemma 1 in Section 19.2.2 Zorich). **[I need to learn how this works! Do we get asymptoticity at all orders? —Aaron]** Plugging in the Taylor series above, we get

$$\begin{aligned}
I_\alpha(z) &\sim e^{-z\zeta_\alpha} \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n \geq 0} b_n^\alpha \tau^n \, d\tau \\
&= e^{-z\zeta_\alpha} \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n \geq 0} b_{2n}^\alpha \tau^{2n} \, d\tau \\
&= 2e^{-z\zeta_\alpha} \int_{0}^{\varepsilon} e^{-z\tau^2/2} \sum_{n \geq 0} b_{2n}^\alpha \tau^{2n} \, d\tau.
\end{aligned}$$

By Watson's Lemma (see Lemma 4 Section 19.2.2 Zorich)

$$\begin{aligned}
I_\alpha(z) &\sim e^{-z\zeta_\alpha} \sum_{n \geq 0} b_{2n}^\alpha \Gamma\left(n + \tfrac{1}{2}\right) 2^{n+1/2} z^{-n-1/2} \\
&= e^{-z\zeta_\alpha} \sqrt{2\pi} \sum_{n \geq 0} b_{2n}^\alpha (2n-1)!! z^{-n-1/2}
\end{aligned}$$

FIGURE 1. The contour $\mathcal{C}_\alpha$, its image under f which is the Hankel contour $\mathcal{H}_\alpha = f(\mathcal{C}_\alpha)$ and the ray $[\zeta_\alpha, +\infty]$.

Call the right-hand side $\tilde{I}_\alpha$. We now see that $a_{\alpha,n} = (2n-1)!!\, b^\alpha_{2n}$ in the statement of the theorem. We know from the definition of ε that $\left|b^\alpha_n\right| \varepsilon^n \lesssim 1$. Recalling that $(2n-1)!! \sim (\pi n)^{-1/2}\, 4^n\, n!$ as $n \to \infty$, we deduce that $|a_{\alpha,n}| \lesssim \left(\frac{4}{\varepsilon^2}\right)^n n!$, showing that $\tilde{I}_\alpha$ is Gevrey-1.

Part (2): note that **[explain formally what it means to center at ζ_α]**

$$\tilde{\iota}_\alpha := \mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha = \sqrt{2\pi} \sum_{n \geq 0} (2n-1)!!\, b^\alpha_{2n}\, \frac{(\zeta - \zeta_\alpha)^{n-1/2}}{\Gamma\left(n + \frac{1}{2}\right)}$$

Since $(2n-1)!! = \pi^{-1/2} 2^n \Gamma\left(n + \frac{1}{2}\right)$ and $|b^\alpha_n| \epsilon^n \lesssim 1$, then $\tilde{\iota}_\alpha(\zeta)$ has a finite radius of convergence.

Part (3): Let's recast the integral I_α into the f plane. As ζ goes rightward from ζ_α, the start and end points of $\mathcal{C}_\alpha(\zeta)$ sweep backward along $\mathcal{C}^-_\alpha(\zeta)$ and forward along $\mathcal{C}^+_\alpha(\zeta)$, respectively. Hence, we have

$$\begin{aligned} I_\alpha(z) &= \int_{\mathcal{C}_\alpha} e^{-zf}\, \nu \\ &= \int_{\mathcal{H}_\alpha} e^{-z\zeta} \left(\int_{f^{-1}(\zeta)} \frac{\nu}{df} \right) d\zeta \\ &= \int_{\zeta_\alpha}^{e^{i\theta}\infty} e^{-z\zeta} \left[\frac{\nu}{df} \right]_{\mathrm{start}\, \mathcal{C}_\alpha(\zeta)}^{\mathrm{end}\, \mathcal{C}_\alpha(\zeta)} d\zeta. \end{aligned}$$

where $\mathcal{H}_\alpha$ is the Hanckel contour through the point ζ_α (see Figure [**?**]) with ends in the θ direction. Noticing that the last integral is a Laplace transform for the initial choice of θ, we learn that

$$\hat{\iota}_\alpha(\zeta) = \left[\frac{\nu}{df} \right]_{\mathrm{start}\, \mathcal{C}_\alpha(\zeta)}^{\mathrm{end}\, \mathcal{C}_\alpha(\zeta)}. \tag{1.3}$$

In Ecalle's formalism, $\overset{\triangledown}{\iota}_\alpha := \int_{f^{-1}(\zeta)} \frac{\nu}{df}$ and $\hat{\iota}_\alpha$ are respectively a major and a minor of the singularity and they differ by an holomorphic function (we will see this in the examples Section Airy, Bessel).

We can rewrite our Taylor series for ν as

$$\begin{aligned} \nu &= \sum_{n \geq 0} b^\alpha_n [2(f - \zeta_\alpha)]^{n/2} \frac{df}{[2(f - \zeta_\alpha)]^{1/2}} \\ &= \sum_{n \geq 0} b^\alpha_n [2(f - \zeta_\alpha)]^{(n-1)/2}\, df, \end{aligned}$$

taking the positive branch of the square root on $\mathcal{C}_\alpha^+$ and the negative branch on $\mathcal{C}_\alpha^-$. Plugging this into our expression for $\hat{\iota}_\alpha$, we learn that

$$\begin{aligned}
\hat{\iota}_\alpha(\zeta) &= \left[\sum_{n\geq 0} b_n^\alpha [2(f-\zeta_\alpha)]^{(n-1)/2}\right]_{\mathrm{start}\,\mathcal{C}_\alpha(\zeta)}^{\mathrm{end}\,\mathcal{C}_\alpha(\zeta)} \\
&= \sum_{n\geq 0} b_n^\alpha \Big([2(\zeta-\zeta_\alpha)]^{(n-1)/2} - (-1)^{n-1}[2(\zeta-\zeta_\alpha)]^{(n-1)/2}\Big) \\
&= \sum_{n\geq 0} 2b_{2n}^\alpha [2(\zeta-\zeta_\alpha)]^{n-1/2} \\
&= \sum_{n\geq 0} 2^{n+1/2} b_{2n}^\alpha (\zeta-\zeta_\alpha)^{n-1/2} \\
&= \mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha .
\end{aligned}$$

We have now shown that the sum of $\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha$ is actually equal to $\hat{\iota}_\alpha$ as $\zeta \in \zeta_\alpha + [0, e^{i\theta}\infty)$.

□

Remark 1.3. Different choices of admissible θ correspond to different choices of thimbles $[\mathcal{C}_\alpha] \in H_N^B(X, zf)$, but the Borel transform of $\tilde{I}_\alpha$ does not depend on θ. However, if $\theta_* := \arg(\zeta_\alpha - \zeta_\beta)$ and $\theta_\pm := \theta_* \pm \delta$ for small δ, then $I_\alpha(z)$ jumps on the intersection between $\mathrm{Re}(e^{i\theta_+}z) > 0$ and $\mathrm{Re}(e^{i\theta_-}z) > 0$. This is known as the Stokes phenomenon (see Section resurgence thimbles integrals).

2. 3/2 DERIVATIVE FORMULA

In Theorem 1.1 we have seen that the asymptotic behaviour of $I_\alpha(z)$ has a fractional power contribution namely $\tilde{I}_\alpha(z) = e^{-z\zeta_\alpha} z^{-1/2}\sqrt{2\pi}\sum_{n\geq 0} a_{\alpha,n} z^{-n}$, hence we have used the extended notion of Borel transform to deal with fractional powers. Now we will focus on the formal series $\tilde{\Phi}_\alpha(z) := e^{-z\zeta_\alpha}\sqrt{2\pi}\sum_{n\geq 0} a_{\alpha,n} z^{-n} = z^{1/2}\tilde{I}_\alpha(z)$ which does not contain any fractional power and we prove a fractional derivative formula which relates the Borel transforms $\hat{\varphi}_\alpha(\zeta)$ and $\hat{\iota}_\alpha(\zeta)$. Moreover we show that the $\hat{\varphi}_\alpha(\zeta)$ depends on ν and df as well as $\hat{\iota}_\alpha(\zeta)$ does.

Corollary 2.1. Under the same assumptions of Theorem 1.1, for any ζ on the ray going rightward from ζ_α in the direction of θ, we have

$$\hat{\varphi}_\alpha(\zeta) = \partial^{3/2}_{\zeta \text{ from } \zeta_\alpha}\left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right) = \left(\frac{\partial}{\partial\zeta}\right)^2 \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{\zeta_\alpha}^{\zeta} (\zeta - \zeta')^{-1/2} \left(\int_{\mathcal{C}_\alpha(\zeta')} \nu\right) d\zeta', \tag{2.1}$$

where $\mathcal{C}_\alpha(\zeta)$ is the part of $\mathcal{C}_\alpha$ that goes through $e^{-i\theta} f^{-1}([\zeta_\alpha, \zeta])$. Notice that $\mathcal{C}_\alpha(\zeta)$ starts and ends in $e^{-i\theta} f^{-1}(\zeta)$. **[Be careful about the orientation of $\mathcal{C}_\alpha$.]**

Proof. Theorem **??** tells us that

$$\begin{aligned}\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha &= \mathcal{B}_{\zeta_\alpha} z^{-1/2}\tilde{\varphi}_\alpha \\ &= \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha}\mathcal{B}\tilde{\varphi}_\alpha \\ &= \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha.\end{aligned}$$

It follows, from the proof of part 3 of Theorem 1.1, that

$$\hat{\imath}_\alpha(\zeta) = \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha. \tag{2.2}$$

Since fractional integrals form a semigroup, equation (2.2) implies that

$$\partial^{-1}_{\zeta \text{ from } \zeta_\alpha}\hat{\imath}_\alpha(\zeta) = \partial^{-3/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha.$$

Rewriting equation (1.3) as

$$\hat{\imath}_\alpha(\zeta) = \partial_\zeta\left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right),$$

we can see that

$$\partial^{-1}_{\zeta \text{ from } \zeta_\alpha}\hat{\imath}_\alpha(\zeta) = \int_{\mathcal{C}_\alpha(\zeta)} \nu - \int_{\mathcal{C}_\alpha(0)} \nu.$$

The initial value term vanishes, because the path $\mathcal{C}_\alpha(0)$ is a point. Hence,

$$\int_{\mathcal{C}_\alpha(\zeta)} \nu = \partial^{-3/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha(\zeta).$$

Recalling that the Riemann-Liouville fractional derivative is a left inverse of the fractional integral, we conclude that

$$\partial^{3/2}_{\zeta \text{ from } \zeta_\alpha}\left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right) = \hat{\varphi}_\alpha(\zeta).$$

□

2.1. **Singularities.** From equation (2.2) we see that singularities of $\hat{\imath}_\alpha(\zeta)$ in the Borel plane comes from either poles of ν or zeros of df. Instead, the fractional derivatives formula tells that singularities of $\hat{\varphi}_\alpha$ are given by convolutions of $\zeta^{-1/2}/\Gamma(1/2)$ with $\hat{\imath}_\alpha$. Since $\zeta^{-1/2}/\Gamma(1/2)$ is singular at $\zeta = 0$ the set of singularities of $\hat{\varphi}_\alpha(\zeta)$ is exactly the same as the one of $\hat{\imath}_\alpha(\zeta)$. However, the type of singularities will change and we expect $\hat{\varphi}_\alpha(\zeta)$ to have only simple singularities.

In the examples we noticed that $\hat{\varphi}_\alpha(\zeta)$ is always an hypergeometric function. In particular when there are only two critical values (see Airy, Bessel) the $\hat{\varphi}_\alpha(\zeta)$ is a Gaussian hypergeometric function ${}_2F_1\left(a, b; c; \frac{\zeta}{\zeta_\alpha}\right)$ with $c = 2$ and $a + b = c + 1$. Whereas, in the generalized Airy example (see Section ??) we get generalized hypergeometric functions ${}_3F_2\left(\mathbf{a}; \mathbf{b}; (\frac{\zeta}{\zeta_\alpha} - 1)^2\right)$ and ${}_3F_2\left(\mathbf{a}_0; \mathbf{b}_0; (\frac{\zeta}{\zeta_\alpha})^2\right)$ with $|\mathbf{a}| = |\mathbf{b}| + 1$. This behaviour reflects the resurgence properties of $\hat{\varphi}_\alpha$ (as well as the one of $\hat{\imath}_\alpha$), indeed the analytic continuation of $\hat{\varphi}_\alpha(\zeta)$ at ζ_α is given in terms of $\hat{\varphi}_\beta(\zeta)$, $\zeta_\beta \neq \zeta_\alpha$ when $\hat{\varphi}_\alpha(\zeta), \hat{\varphi}_\beta(\zeta)$ are hypergeometric functions of the previous type.

Lemma 2.2. Let us assume f has only two critical values $\zeta_\alpha = -\zeta_\beta$ and let $\hat{\varphi}_\alpha(\zeta) = {}_2F_1(a,b;2;\frac{\zeta}{\zeta_\alpha})$ with $a+b=c+1$, then across the branch cut

$$\hat{\varphi}_\alpha(\zeta+i0)-\hat{\varphi}_\alpha(\zeta-i0)=C\;{}_2F_1\left(a,b;2;1+\frac{\zeta}{\zeta_\beta}\right) \tag{2.3}$$

$$\hat{\varphi}_\beta(\zeta+i0)-\hat{\varphi}_\beta(\zeta-i0)=-C\;{}_2F_1\left(a,b;2;1+\frac{\zeta}{\zeta_\alpha}\right) \tag{2.4}$$

Proof. It follows from DLMF eq. 15.2.2. □

3. CONTOUR ARGUMENT

As noticed in proof of Theorem 1.1, the integral $I_\alpha(z)$ can be written as

(i) the Laplace transform of $\hat{\imath}_\alpha(\zeta)$

(ii) the Hankel contour integral of the major $\overset{\triangledown}{\imath}_\alpha(\zeta)$

and $\overset{\triangledown}{\imath}_\alpha(\zeta) = \hat{\imath}_\alpha(\zeta) + \text{hol.fct.}$. In the applications we have evidence that $\overset{\triangledown}{\imath}_\alpha(\zeta)$ is an algebraic hypergeometric function and when there are only two critical values, it decomposes as a sum of two germs of holomorphic functions at each critical values respectively (see `airy-resurgence` Section 6.1, 6.3).