Resurgence of the Airy function and other exponential integrals

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June 2, 2022

1 Introduction

1.1 Why does Borel resummation work?

Borel resummation is a way of turning a formal power series

$$\varphi_\bullet = z^\sigma \left(\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \frac{\varphi_3}{z^4} + \dots \right),$$

with $\sigma \in [0, 1)$, into a function which is asymptotic to $\varphi_\bullet$ as $z \to \infty$. Different functions can be asymptotic to the same power series, and Borel resummation picks one of them, performing an implicit regularization [**arXiv:1705.03071, or maybe arXiv:1412.6614**]. When a function matches the Borel sum of its asymptotic series, we'll say it's *Borel regular*. Several familiar kinds of regularity imply Borel regularity, and shed light on why it occurs.

- **Having a good asymptotic approximation**

  Let R_N be the difference between a function and the partial sum

  $$\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \dots + \frac{\varphi_{N-2}}{z^{N-1}}$$

  of its asymptotic series. Watson showed a century ago that the function is Borel regular whenever there's a constant $c \in (0, \infty)$ with

  $$|R_N| \leq \frac{c^{N+1} N!}{|z|^N}$$

  over all orders N and all z in a wide enough wedge around infinity.

- **Satisfying a singular differential equation**

  - Think about conditions where this works.
  - Maybe the correct place is the setting of Ecalle's formal integral. See §5.2.2.1 of Delabaere's *Divergent Series, Summability and Resurgence III.*
  - Say there's a unique solution (up to scaling) that shrinks as you go right; everything else blows up exponentially. Then this is the only solution that can be expressed as a Laplace transform.

- If the Borel-transformed equation has a subexponential solution $\hat{f}$ which is "shifted holomorphic" (we called this having a "fractional power singularity" in `airy-resurgence`), then $\mathcal{L}\hat{f}$ satisfies the original equation, because there are no boundary terms.
- Draw diagram showing formal vs. holomorphic solutions in time vs. frequency domains.

- **Being a thimble integral**

Let X be a translation surface—a Riemann surface carrying a holomorphic 1-form ν. Suppose X is of *meromorphic type*, meaning that we got it by puncturing a compact Riemann surface $\overline{X}$ at finitely many points, and ν has a pole at each puncture. A *translation coordinate* on X is a local coordinate whose derivative is ν.

Take another meromorphic-type translation surface B and a holomorphic Morse[1] map $f\colon \overline{X} \to \overline{B}$ that sends punctures to punctures [actually, don't require this; the Orr–Sommerfeld integrals, for example, don't satisfy it]. Suppose every singularity of B is a critical value of f. [**Typical usage of "Borel plane" seems ambiguous, so maybe we can use "Borel plane" for B and "Borel cover" for the Riemann surface of the Borel-transformed series. How to handle the Orr–Sommerfeld functions (DLMF §9.13)? We know $f = 4u^3 - 3u$ is the pullback of a translation coordinate, but we also need a puncture at $f(0)$...**] For each critical point p, let Γ_p be the ray going rightward from $f(p)$, and let ζ_p be the translation coordinate around Γ_p which vanishes at $f(p)$. These are well-defined as long as Γ_p misses the critical values of f. The preimage $f^{-1}(\Gamma_p)$ is a bunch of disjoint curves, as long as Γ_p misses the other critical values of f. The *Lefschetz thimble* Λ_p is the component of $f^{-1}(\Gamma_p)$ that goes through p, oriented so that shifting it to its left would make its projection run clockwise around Γ_p. The *thimble integral*

$$I_p = \int_{\Lambda_p} e^{-z f^* \zeta_p} \nu$$

is a holomorphic function on the right half-plane parametrized by z, and it turns out [**we hope**] to be Borel regular.

[**Talk about exponential integrals and their decomposition into thimble integrals.**]

In higher-dimensional complex manifolds, integrals over Lefschetz thimbles are still Borel regular [**"Exponential integrals, Lefschetz thimbles and linear resurgence"**][**"Exponential Integral" lectures?**]. This fact plays an important technical role in quantum mechanics, where infinite-dimensional exponential integrals are supposed to give the expectation values of observable quantities. Physicists often use Borel summation and related techniques to assign values to these integrals [**Costin & Kruskal, "On optimal truncation..."**].

Choose a path $\gamma\colon \mathbb{R} \to X$ whose projection $f \circ \gamma$ starts out going leftward out of a puncture, ends up going rightward into a puncture, and never touches a critical value of f. Choose a translation coordinate ζ on B and continue it along $f \circ \gamma$, noting that

[1] This condition means that the critical points of f are isolated (the compactness of $\overline{X}$ guarantees this) and the 2-jet of f is non-zero at every critical point.

it may become multi-valued if $f \circ \gamma$ intersects itself. This data defines the *exponential integral*

$$I = \int_\gamma e^{-zf^*\zeta}\nu,$$

a holomorphic function on the right half-plane parametrized by z. It turns out [**we hope**] that we can get I by summing $e^{-\alpha_p z} I_p$ over various critical points—as long as none of the Γ_p run into each other. [**We get jumps at phases where the Γ_p do hit each other.**] The constants α_p are values of ζ, continued to the critical points along certain paths.

- Each resummation method for asymptotic series makes some implicit assumption that allows us to reconstruct a holomorphic function from its asymptotic behaviour.
- The resummation method works correctly for functions which satisfy that assumption.
- For the modified Bessel function $K_{1/3}$, Borel resummation works because the asymptotic series encodes a second-order differential equation.
  - Different aspects of this example appear in various places (Mariño, Kawai–Takei, Sauzin). We give a detailed, unified treatment.
- We can generalize this argument to all K_ν with $\nu \in \mathbb{Q}$.
- We can also generalize to all third-order exponential integrals.
  - Most of them are equivalent to the $K_{1/3}$ integral, but there's also an interesting degeneration.

1.2 Fractional derivative formula

- Theorem **??** says that for a certain class of exponential integrals

  $$I(z) = \int_\Gamma e^{-zf}\ \nu,$$

  the inverse Laplace [**better to say Borel?**] transform is the $\frac{3}{2}$ derivative of $d\zeta/df$, where $f^* d\zeta = \nu$ [**check**].
- the asymptotic expansion of $I(z)$ is a resurgent function.
- Is it always a *simple* resurgent function?
  - **Maxim belies it is in general, and indeed in our examples we get simple resurgent functions. But how to prove it in general?**

1.3 Stokes phenomenon

- For Bessel functions, we can see explicitly how solutions jump when the Laplace transform angle crosses a critical value.
- The jump comes from the branch cut difference identity for hypergeometric functions.
- Possible interpretation of the Stokes factors as intersections numbers in Morse–Novikov theory [**ask Maxim**]

2 The Laplace and Borel transforms

2.1 The Laplace transform

- Action on differential equations.
  - Can we find a way to prove this when the differential operator spits out a function that's not integrable around zero?
- Global picture?

2.2 The Borel transform

- Action on differential equations.
  - No inhomogeneous terms! How is this consistent with the Laplace transform's action? Is there always an inhomogeneous solution with subexponential asymptotics?

3 Third-order exponential integrals

- Reduce to
$$I(z) = \int \exp\left[-z\big(u^3 + pu + q\big)\right]\, du$$
using change of coordinate.
- When $p \neq 0$, can reduce further to
$$I(z) = p^{1/2} e^{-qz} K_{1/3}(p^{3/2} z).$$
- As p goes to zero, $I(z)$ degenerates to
$$\left(\tfrac{1}{2}\right)^{2/3} e^{-qz}\Gamma(\tfrac{1}{3}) z^{-1/3} = \left(\tfrac{1}{2}\right)^{2/3} e^{-qz}\mathcal{L}_{\zeta,0}(\zeta^{-2/3}) = \left(\tfrac{1}{2}\right)^{2/3} \mathcal{L}_{\zeta_{-q},q}(\zeta^{-2/3}).$$

Veronica's proposal

Title: Borel regularity and Resurgence of Exponential Integrals

1. introduction
   - what are exponential integrals? has to be done
     - motivation
       * In the classical theory of special functions, exponential integrals are often used to express solutions of linear differential and difference equations.
       * In physics ??
       * Geometrically they represent a Poincaré pairing (as explained by Kontsevich in **IHES lectures**).
   - Borel regularity
     - what does it mean being Borel regular?
     - when does it happen?
       * Recall Watson condition (old)
       * State new Borel regularity results
         · Linear, homogeneous ODE with regular singularity at 0 and irregular singularity at infinity [P.Ramis ?, Loday-Richaud ?, big idea in `airy-resurgence`]
         · Thimble integral [`draft2`]
   - State results about resurgence of exponential integrals and Stokes phenomena
     - Thimbles integrals [Kontsevich]: geometric computation of Stokes constants has to be done
     - ODE and fractional derivative formula [`draft2`]
     - if hypergeometric functions appear in a large class of examples: integral formulas for hypergeometric functions has to be done

2. Formalism for Laplace transform [`draft2`, "The geometry of the Laplace transform"]
   (a) Analytic
      i. Introduction
      ii. Brief revew of translation surfaces (we can refer to this from the introduction if we need to)
      iii. The Laplace transform of a holomorphic function
         A. Over an ordinary point
         B. Over a branch point
         C. Differential equation
      iv. Relating differential equations in the frequency domain to integral equations in the position domain
   (b) Formal
      i. Laplace transform of a formal series

ii. Borel transform

iii. Relating differential equations in the frequency variable to integral equations in the position variable

3. Review of integral equations
   - Existence of solutions
   - Fractional integrals and derivatives
   - Going between integral and differential equations (slight functions)

4. General cases

   (a) Borel regularity

   - General ODE of the form

$$\left[P\left(\tfrac{\partial}{\partial z}\right) + z^{-1}Q\left(\tfrac{\partial}{\partial z}\right) + z^{-2}R(z^{-1})\right]\Phi = 0,$$

     where P is a polynomial, Q is a polynomial of one degree lower, and R is an entire function [see `airy-resurgence` and written notes]

     – More generally, for P of degree n, we should be able to handle

$$\left[P\left(\tfrac{\partial}{\partial z}\right) + z^{-1}Q_1\left(\tfrac{\partial}{\partial z}\right) + z^{-2}Q_2\left(\tfrac{\partial}{\partial z}\right) + \ldots + z^{-(n-1)}Q_{n-1}\left(\tfrac{\partial}{\partial z}\right) + z^{-n}R(z^{-1})\right]\Phi = 0,$$

       where Q_k has degree $n-k$. has to be done

       ∗ We want the most general ODE with a regular singularity at $z = 0$ and its only other singularity, typically irregular, at $z = \infty$. has to be done

       ∗ The singularity at ∞ should only be regular for an Euler equation. has to be done

     – Show that we can find a slight solution at each critical value.

     – Show that $\hat{\iota} = \tilde{\iota}$, where:

       ∗ $I = \mathcal{L}\iota$

       ∗ $\hat{\iota}$ is the Taylor expansion of ι

       ∗ $\tilde{I}$ is the asymptotic series of I

       ∗ $\tilde{\iota} = \mathcal{B}\tilde{I}$

       ∗ Idea: Show that $\hat{\iota}$ and $\tilde{\iota}$ have matching asymptotics at $\zeta = 0$. Since they both satisfy the position-domain integral equation, they must coincide.

   - General thimble integral (conditions?)

     – Proof of Borel regularity

     – 3/2-derivative formula

     – Contour argument

   (b) Resurgence

- Explain how Borel regularity relates resurgence of formal series to resurgence of holomorphic functions in the position domain. think more about what we're trying to say here
- Relate to Ecalle's formalism and the alien derivative
- Stokes factors
  - For ODEs
  - For thimble integrals

5. Examples make sure each example contains a computation of the Borel transform, so we can see it matches

   (a) The Airy example

   - $I(z)$ is a solution of a linear ODE. We explicitly find its Borel transform, knowing the nature of singularities and the asymptotic behaviour of a basis of solution for the ODE [`airy-resurgence`]
   - Stokes constants using fractional derivative formula and Borel transform computation [`draft2`]
   - Comparison with the literature has to be done
     - Mariño
     - Sauzin
     - Kontsevich slides
     - Kawai–Takei? [might take too long to understand well enough]

   (b) The Airy–Lucas examples

   - Compute Borel transform [`airy-resurgence`]
   - Compute Stokes constants has to be done

   (c) Bessel 0 (it is different because we have infinite cover)

   - Compute Stokes constants [`draft2`]

   (d) Bessel μ (follows from Bessel 0)

   - Compute Stokes constants [`modified Bessel`]

   (e) The generalized Airy example

   (f) The vibrating beam example

   - In addition to the simple example, maybe we can do an example where the equation on the spatial domain includes fractional integrals, since Andy is interested in that sort of thing