RESURGENCE AND BOREL REGULARITY FOR ODES
SEMINAR AT MATAMZEE

VERONICA FANTINI

1. What is resurgence

Theory of resurgence was introduced by Ecalle in the '80 and it deals with divergent power series

$$\tilde{\Phi}(z) = \sum_{n\geq 0} a_n z^{-n-1} \in \mathbb{C}[\![z^{-1}]\!], \quad \text{with } a_n \sim n! \tag{1.1}$$

they have zero radius of convergence.

1.1. Paradigma of Borel–Laplace sum.

$$\mathbb{C}[\![z^{-1}]\!] \ni \tilde{\Phi} := \textstyle\sum_{n\geq 0} a_n z^{-n-1} \xrightarrow{\ \mathcal{B}\ } \tilde{\phi}(\zeta) = \textstyle\sum_{n\geq 0} \frac{a_n}{n!}\zeta^n \in \mathbb{C}\{\zeta\}$$

[make drawings of the Borel plane]

Study the analytic continuation of $\tilde{\phi}(\zeta)$ and if $\hat{\phi}(\zeta)$ **behaves well** you can go back to the z-plane via Laplace transform

$$\begin{tikzcd}
\mathbb{C}[\![z^{-1}]\!] \ni \tilde{\Phi} := \sum_{n\geq 0} a_n z^{-n-1} \arrow[rr, "\mathcal{B}"] & & \tilde{\phi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!}\zeta^n \in \mathbb{C}\{\zeta\} \arrow[dl, "\mathcal{L}^\theta"] \\
& \Phi \in \mathcal{O}(H_\theta) &
\end{tikzcd}$$

[make drawings of the z-plane]

The relation with $\tilde{F}$ is via asymptotic expansion as $\Re z e^{i\theta} \to \infty$

$$\begin{tikzcd}
\mathbb{C}[\![z^{-1}]\!] \ni \tilde{\Phi} := \sum_{n\geq 0} a_n z^{-n-1} \arrow[rr, "\mathcal{B}"] & & \tilde{\phi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!}\zeta^n \in \mathbb{C}\{\zeta\} \arrow[dl, "\mathcal{L}^\theta"] \\
& \Phi \in \mathcal{O}(H_\theta) \arrow[ul, dashed, "\text{asymptotics}"] &
\end{tikzcd}$$

The Laplace transform is defined

$$F(z) \cong \mathcal{L}^\theta f(z) = \int_0^{+\infty e^{i\theta}} e^{-z\zeta} f(\zeta) d\zeta \tag{1.2}$$

for $\theta \in [0, 2\pi)$ and $F \in \mathcal{O}(H_\theta)$ if and only if $|f| \leq Ae^{-c|z|}$ for every z in a tubular neighbourhood. In general one has to check

- $\hat{f}$ can be defined through infinity;
- $\hat{f}$ has the right decaying at ∞. HARDER

For the first one, we have to study the singularities of $\hat{f}$.

1.1.1. *Ecalle's resurgent function.* $f \in \mathbb{C}\{\zeta\}$ is resurgent if it endlessly analytically continuable, i.e.

[draw picture]

He defined the **theory of singularities** and **relaxed the definition of Laplace transform** to deal with log- singularities, square root, poles, etc.

Examples of (*minor* of) singularities

$$\overset{\vee}{I}_c(\zeta) := \frac{\zeta^{c-1}}{(1-e^{-2\pi i c})\Gamma(c)} \qquad c \in \mathbb{C} \setminus \mathbb{Z}_{>0}$$

$$\overset{\vee}{I}_c(\zeta) := \frac{\zeta^{c-1}}{2\pi i \Gamma(c)} \log \zeta \qquad c \in \mathbb{Z}_{>0}$$

draw singularity in a row (like Painlevé)

1.1.2. *The role of θ.* Varying θ, as long as $|\theta - \theta'| < \pi$, $\mathcal{L}^\theta f = \mathcal{L}^{\theta'}$ on $H_\theta \cap H_{\theta'}$. More generally, they disagree

$$\mathcal{L}^\theta - \mathcal{L}^{\theta'} = S_{\theta\theta'} \mathcal{L}^{\theta'} \qquad S_{\theta\theta'} \in \mathbb{C}$$

These phenomena is called the Stokes phenomena. Computing the Stokes constant is crucial to understand the structure of F, and Ecalle developed the *Alien calculus* to study the Stokes phenomena.

Summarizing: the core of resurgence theory is

(1) study the Borel plane;
(2) compute the Stokes constants.

stress that resurgence is about Borel plane, but knows about the z-plane.

1.2. **Motivation: why divergent series?**

- **thimbles integrals**:

  COOL: if f is algebraic, $I(z)$ is a period, i.e. it is a geometric object studied (Deligne, Malgrange, Pham, Kontsevich–Soibelman $\sim$ generalization when f is local coordinate for 1-form $f = \int \alpha$) and the resurgence of $\tilde{I}$ has a geometric nature

  - critical values of $f \sim$ singularity in the Borel plane

  - Picard–Lefschetz theory ~ Stokes phenomena
  - gradient lines ~ Stokes indexes
- **ODEs** with irregular singularity at ∞: they admits formal solutions, and if the ODE is regular enough M.A.E.T. assures the existence of a holomorphic solution asymptotic to the formal one.
  - non linear ODEs have interesting behaviours (maybe be resonant)
- **q–difference equation**: like ODEs,

$$n! \rightsquigarrow q^{n(n+1)/2} \; |q|>1 \tag{1.3}$$

there is a dictionary between ODEs and q-difference equation.

2. ODEs

2.1. Which class of ODEs we consider.

$$[P(\partial/\partial_z)+\frac{1}{z}Q(\partial/\partial_z)+\sum_{j=2}^{d} z^{-j}R_j(\partial/\partial_z)]\Phi(z)=0 \tag{2.1}$$

with

- $P(\lambda)$ a degree d polynomial
- $Q(\lambda)$ a degree $d-1$ polynomial
- $R_j(\lambda)$ are degree $d-j$ polynomials

they are defined by Poincaré as *series normal de Ier ordre.* As a general fact, if $P(-\lambda)$ has simple roots $\alpha_1,\dots,\alpha_d$ then (2.1) admits d formal solution of the form

$$\tilde{\Phi}_j(z)=e^{-\alpha_j z}z^{-\tau_j}\tilde{\phi}_j(z)\in e^{-\alpha_j z}z^{-\tau_j}\mathbb{C}[[z^{-1}]][\log(z)] \tag{2.2}$$

where $\tau_j=-Q(\alpha_j)/P'(\alpha_j)$.

The M.A.E.T theorem, guarantees under suitable assumptions on the ODE, the existence of an holomorphic solution $\Phi_j(z)$ asymptotic to $\tilde{\Phi}_j$ in a suitable sector.

Our goal is to prove that $\Phi_j(z)\propto\mathcal{L}^{\theta}_{\zeta_j}\mathcal{B}\tilde{\Phi}_j$ for some angle θ.

Equivalently said, Borel–Laplace summability picks an actual solution of the ODE.

2.2. **Borel regularity for ODEs.**

$$\begin{tikzcd}
& P\left(\frac{\partial}{\partial_z}\right)+\frac{1}{z}Q\left(\frac{\partial}{\partial_z}\right)+\sum_{j=2}^{d} z^{-j}R_j\left(\frac{\partial}{\partial_z}\right) & \\
\Phi_j \arrow[ur, red] & & \tilde{\Phi}_j \arrow[ul] \arrow[d, "\mathcal{B}"] \\
\hat{\phi}_j \arrow[u, "\mathcal{L}^{\theta}_{\zeta,\alpha_j}"] & & \tilde{\phi}_j(\zeta) \arrow[ll, "\text{sum}"']
\end{tikzcd}$$

a priori it is not guarantee that $\mathsf{L}^{\theta}_{\zeta,\alpha_j}\mathcal{B}\tilde{\Phi}_j$ is a solution of (2.1), because different functions may have the same asymptotic.

Idea of the proof is based on the following diagram:

$$\begin{tikzcd}
& P\left(\frac{\partial}{\partial_z}\right)+\frac{1}{z}Q\left(\frac{\partial}{\partial_z}\right)+\sum_{j=2}^{d} z^{-j}R_j\left(\frac{\partial}{\partial_z}\right) \arrow[dd, "\mathcal{B}"] & \\
\Phi_j \arrow[ur, red] & & \tilde{\Phi}_j \arrow[ul] \arrow[dd, "\mathcal{B}"] \\
& P(-\zeta)+\partial_\zeta^{-1}\circ Q(-\zeta)+\sum_{j=2}^{d}\partial_\zeta^{-j}\circ R_j(-\zeta) & \\
\hat{\phi}_j \arrow[uu, "\mathcal{L}^{\theta}_{\zeta,\alpha_j}"] & & \tilde{\phi}_j(\zeta) \arrow[ul] \arrow[ll, "\text{sum}"']
\end{tikzcd}$$

there exists a solution $\tilde{\phi}_j(\zeta)$ which is also slight, and in suitable coordinates it is a convergent series in $\overset{\vee}{I}_{n+\tau_j}(\zeta_j)$.

$$\begin{tikzcd}
& P\left(\frac{\partial}{\partial_z}\right)+\frac{1}{z}Q\left(\frac{\partial}{\partial_z}\right)+\sum_{j=2}^{d} z^{-j}R_j\left(\frac{\partial}{\partial_z}\right) \arrow[dd, leftrightarrow, "\mathcal{L}"', "\mathcal{B}"] & \\
\Phi_j \arrow[ur, red] & & \tilde{\Phi}_j \arrow[ul] \arrow[dd, "\mathcal{B}"] \\
& P(-\zeta)+\partial_\zeta^{-1}\circ Q(-\zeta)+\sum_{j=2}^{d}\partial_\zeta^{-j}\circ R_j(-\zeta) & \\
\hat{\phi}_j \arrow[uu, "\mathcal{L}^{\theta}_{\zeta,\alpha_j}"] & & \tilde{\phi}_j(\zeta) \arrow[ul] \arrow[ll, "\text{sum}"']
\end{tikzcd}$$

the Laplace transform along a Hankel contour gives an inverse for $\mathcal{B}$.

[repeat again the idea: a solution of (2.1) is Borel regular, because its Borel–Laplace sum gives an actual solution.]

Remark 2.1. [Extend the definition of $\mathcal{B}$ and $\mathcal{L}$]

- As far as $\Gamma(\tau)$ is well defined, we can allow $\tau \in \mathbb{R}$ and extend the definition of $\mathcal{B}$.
- Ecalle's theory of singularity introduces a generalized Laplace transform which take as contour a Hankel contour rather than a straight line. This is not the only generalization of Laplace transform Ecalle introduced.

2.3. **Proof of Borel regularity.**

- Borel transform the ODE_z and we get IE_ζ.
  - IE are not easy to be solved so usually in the application we differentiate them to get an ODE_ζ. If $\hat{\phi}$ is slight we don't loose informations by differentiating.
- by Prop 1, there exists a solution $\hat{\phi}(\zeta_j)$

$$\begin{aligned}
\hat{\phi}(\zeta_j) &= \zeta_j^{\tau_j-1} + \tilde{g}_j & \tilde{g}_j \in \mathcal{HL}^{\infty,1-\tau_j-\epsilon} \\
&= \sum_{k\geq 0} a_k \zeta_j^{\tau_j-1+k} + \text{h.f.} \\
&= (1-e^{-2\pi i\tau_j}) \sum_{n\geq 0} \tilde{a}_n \overset{\vee}{I}_{\tau_j+n}(\zeta_j) & \tilde{a}_n = a_n \Gamma(\tau_j+n), \\
& & \overset{\vee}{I}_c(\xi) = \frac{\xi^{c-1}}{(1-e^{-2\pi i c})\Gamma(c)}
\end{aligned}$$

- $\hat{\phi}(\zeta_j)$ is a germ of meromorphic function

$$\begin{aligned}
\limsup_{n\to\infty} \sqrt[n]{\frac{|\tilde{a}_n|}{\Gamma(\tau_j+n)(1-e^{-2\pi i(\tau_j+n)})}} &= \limsup_{n\to\infty} \sqrt[n]{\frac{a_n\Gamma(\tau_j+n)}{\Gamma(\tau_j+n)(1-e^{-2\pi i(\tau_j+n)})}} = \\
&= \limsup_{n\to\infty} \sqrt[n]{\frac{a_n}{(1-e^{-2\pi i(\tau_j+n)})}} < +\infty
\end{aligned}$$

where in the last step we use

$$\infty + \quad > \quad \|\tilde{g}_j \zeta_j^{\tau_j+\epsilon-1}\|_\infty \quad = \quad \|\sum_{n\geq 1} \zeta_j^{\tau_j-1+n} \zeta_j^{-(\tau_j+\epsilon)+1}\|_\infty \quad = \quad \|\sum_{n\geq 1} a_n \zeta_j^{n-\epsilon}\|_\infty \tag{2.3}$$

- by Ecalle definition of Laplace transform [see Sauzin], $\hat{\phi}(\zeta_j)$ has a well defined Laplace transform which is asymptotic to $\sum_{n\geq 0} \tilde{a}_n z^{-\tau_j-n}(1-e^{-2\pi i\tau_j}) = (1-e^{-2\pi i\tau_j})\sum_{n\geq 0} a_n \Gamma(\tau_j+n) z^{-\tau_j-n}$

$$(1-e^{-2\pi i\tau_j}) \sum_{n\geq 0} a_n \Gamma(\tau_j+n) z^{-\tau_j-n} \sim \mathcal{L}^\theta_{\zeta_j} \hat{\phi}_j = e^{\alpha_j z} \mathcal{L}^\theta_{\zeta,\alpha_j} \hat{\phi}$$

hence

$$\mathcal{L}^{\theta}_{\zeta,\alpha_j}\hat{\phi} \sim e^{-\alpha_j z} z^{-\tau_j} \sum_{n\geq 0} a_n \Gamma(\tau_j + n) z^{-n} \propto \tilde{\Phi}_j(z) \tag{2.4}$$

2.4. Corollary: construct an explicit holomorphic solution predicted by M.A.E.T..

Corollary 2.2. The holomorphic solution which exists by M.A.E.T. can be characterized as the Borel–Laplace sum of $\tilde{\Phi}_j$.

3. Q-DIFFERENCE EQUATIONS

3.1. Formal solutions vs actual solutions: paradigma of q-Borel Laplace summability.

3.1.1. *From ODEs to q - difference equations.*

ODE	q-difference, $\lvert q\rvert > 1$
$\sum_{n\geq 0}^{d} a_n(z)\frac{\partial^n}{\partial z^n}\Phi = a \in \mathbb{C}(z)$ and $a_n \in \mathbb{C}(z)$	$\sum_{n\geq 0} a_{q,n}\sigma_q^n f = a_q \in \mathbb{C}(x)$ and $a_{q,n} \in \mathbb{C}(x)$
$z^2\Phi'(z) + \Phi(z) = z$ $\tilde{\Phi}(z) = \sum_{n\geq 0}(-1)^{n+1} n! z^{-n-1}$	$x\sigma_q f(x) + f(x) = x$ $\hat{f}(x) = \sum_{n\geq 0}(-1)^n q^{n(n+1)/2} x^{n+1}$
General notation	
$z\frac{d}{dz}$	$\sigma_q f(x) = f(qx)$
$\frac{d}{dz}$	$\delta_q f(x) = \frac{f(qx)-f(x)}{(q-1)x}$
$n!$	$q^{n(n+1)/2}$
e^z	$e_q(x) = \log(1/q)\Theta_q(x)$

Remark 3.1. Why $n!$ corresponds to $q^{-n(n+1)/2}$

$$\Gamma(n) = \int_0^{+\infty} e^{-t} t^n \frac{dt}{t} \tag{3.1}$$

$$q^{-n(n-1)/2} = \int_0^{+\infty} \frac{t^n}{e_q(t)} \frac{dt}{t} \tag{3.2}$$

There is also another interesting relation between q- factorial and $n!$

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n} \to_{q\to 1} n! \tag{3.3}$$

$$\begin{tikzcd}
\hat f(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{C}[[x]]_{(q;1)} \arrow[rr, "\mathcal{B}_{(q;1)}"] & & \varphi(\xi) = \sum_{n\geq 0} a_n q^{-n(n-1)/2}\xi^n \arrow[dl, "\mathsf{L}^{[\lambda]}_{q;1}"] \\
& f(x) \arrow[ul, dashed, "\text{asymptotics}"] &
\end{tikzcd}$$

3.2. **Borel regularity: Dreyfus's theorem.** Let $\mathcal{B}_\mu$ be the q- Borel transform for q-Gevrey μ series

$$\mathcal{B}_\mu\colon \sum_{n\geq 0} a_n x^n \to \sum_{n\geq 0} a_n q^{-n(n-1)/(2\mu)}\xi^n \tag{3.4}$$

and let $\mathsf{L}^{[\lambda]}_{\mu,\kappa}$ be the q- Laplace transform with parameters $\mu \in \mathbb{Q}_{>0}$, $\kappa \in \mathbb{N}^*$ ($\mathsf{L}^{[\lambda]}_{q;1} := \mathsf{L}^{[\lambda]}_{1,1}$)

$$\mathsf{L}^{[\lambda]}_{\mu,\kappa}\varphi(x) = \frac{\mu}{\kappa}\sum_{l\in\kappa^{-1}\mathbb{Z}} \frac{\varphi(q^l\lambda)}{\Theta_{q^{1/\mu}}(\frac{q^{1/\mu+l}\lambda}{x})} \quad \lambda \in \mathbb{C}^*/q^{\kappa^{-1}\mathbb{Z}} \tag{3.5}$$

Let $\mathbb{H}^{[\lambda]}_{\mu,\kappa}$ be the space of functions $\varphi \in \mathcal{M}(\mathbb{C}^*)$, such that there exists $\varepsilon > 0$, $\Omega \subset \mathbb{C}$ connected

- $\bigcup_{l\in\kappa^{-1}\mathbb{Z}}\{x \in \mathbb{C}^* \| x - \lambda q^l| < \varepsilon|\lambda q^l|\} \subset \Omega$
- φ can be continued analytically in Ω with $q^{1/\mu}$ exponential growth

$$|\varphi(\xi)| < C|\Theta_{|q|^{1/\mu}}(A|\xi|)|$$

For every $\varphi(\xi) \in \mathbb{H}^{[\lambda]}_{\mu,\kappa}$, $\mathsf{L}^{[}_{\mu,\kappa}\lambda]\varphi(x) \in$.

Under suitable assumptions on the q-difference equation, the following result holds true

Theorem 3.2. Let $\hat h$ be a formal power series solution of a linear q-difference equation with coefficients in $\mathbb{C}(x)$. There exist $\kappa_1,\ldots,\kappa_r \in \mathbb{Q}_{>0}$, $n, K \in \mathbb{N}^*$ and a finite set $\Sigma \subset \mathbb{C}^*/q^{n^{-1}\mathbb{Z}}$, we may compute from the q-difference equation, such that for all $\lambda \in (\mathbb{C}^*/q^{n^{-1}\mathbb{Z}}) \setminus \mathbb{Z}$,

$$\mathsf{S}^{[\lambda]}(\hat h) := \mathsf{L}^{[\lambda]}_{\kappa_r,n} \circ \mathsf{L}^{[\lambda]}_{\kappa_{r-1},K} \circ \ldots \circ \mathsf{L}^{[\lambda]}_{\kappa_1,K} \circ \mathcal{B}_{\kappa_1} \circ \ldots \circ \mathcal{B}_{\kappa_r}\hat h$$

is meromorphic on $\mathbb{C}^*$, and is solution of the same equation as $\hat h$. Moreover, $\mathsf{S}^{[\lambda]}(\hat h)$ is asymptotic to $\hat h$ and, for $|x|$ close to 0 it has poles of order at most 1 that are contained in $\lambda q^{n-1}, n \in \mathbb{Z}$.

Idea of the proof: let $P(\hat f) = \left[\sum_{n=0}^m a_n \sigma_q^n\right]\hat f$

$$\begin{tikzcd}
 & P\hat{f} = a \arrow[dd] & \\
f(x) \arrow[ur, red] & & \hat{f}(x) \arrow[ul] \arrow[dd, "\mathcal{B}_{\kappa_1}\circ\ldots\circ\mathcal{B}_{\kappa_r}"] \\
 & q-\text{difference equation} & \\
\varphi(\xi)\in\mathbb{H}^{[\lambda]}_{\kappa_1,K},\ \text{for some } \lambda \arrow[uu, "\mathsf{L}^{[\lambda]}_{\kappa_r,n}\circ\ldots\circ\mathsf{L}^{[\lambda]}_{\kappa_1,K}"] \arrow[rr, Leftrightarrow] & & \varphi(\xi)\in\mathbb{C}\{\xi\},\ \text{of slope } 0 \arrow[ul]
\end{tikzcd}$$

3.3. **Summary.**

Formal series	
$\tilde{\Phi}(z) = \sum_{n\geq 0} a_n z^{-n-1} \in \mathbb{C}[\![z^{-1}]\!]_1$	$\hat{f}(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{C}[\![x]\!]_{(q;1)}$
Borel transform	
$\tilde{\phi}(\zeta) = \sum_{n\geq 0} a_n \frac{\zeta^n}{n!} \in \mathbb{C}\{\zeta\}$	$\varphi(\xi) = \sum_{n\geq 0} a_n q^{n(n-1)/2} \xi^n \in \mathbb{C}(\!(\xi)\!)$
Laplace transform	
$\mathcal{L}^\theta_\zeta \hat{\phi}(z) = \int_0^{+e^{i\theta}\infty} e^{-z\zeta} \hat{\phi}(\zeta) d\zeta \in \mathcal{O}(H_\theta)$	$\mathcal{L}^\theta_{q;1}\varphi(x) = \int_0^{e^{i\theta}\infty} \frac{\varphi(\xi)}{e_q(\frac{\xi}{x})} d\xi$ Θ function: $\Theta_q(x) = \sum_{n\in\mathbb{Z}} q^{-n(n+1)/2} x^n$ $\mathsf{L}^{[\lambda]}_{q;1}\varphi(x) = \sum_{n\geq 0} \frac{\varphi(\lambda q^n)}{\Theta_q(\frac{\lambda q^n}{x})} \in \mathcal{O}(H^\lambda)$, $\lambda \in \mathbb{C}^*$ $H^\lambda = D^*_r \setminus \{-\lambda q^n, n \in \mathbb{Z}\}$
Domain of definition	
$\hat{\phi}(\zeta)$ s.t. $\exists C, a > 0$ $\lvert\hat{\phi}(\zeta)\rvert < C e^{a\lvert\zeta\rvert}$, $\zeta \in S_\delta$ S_δ is an half-strip	$\varphi(\xi)$ s.t. $\exists C, a > 0$ $\lvert\varphi(\xi)\rvert < C\lvert\xi\rvert^a q^{\frac{1}{2}\left(\frac{\log\lvert\xi\rvert}{\log q}\right)^2}$, $\xi \in S_\delta \cap \mathbb{C}^*$ $\varphi(\xi) \in \mathbb{H}^{[\lambda]}_{\mu,\kappa}$
Gevrey asymptotics	
$\mathcal{L}^\theta_\zeta \hat{\phi}(z) \sim_1 \tilde{\Phi}(z)$	$\mathsf{L}^{[\lambda]}_{q;1}\varphi(x) \sim_{q;1} \hat{f}(x)$
Stokes phenomena	
varying θ, $\mathcal{L}^\theta$ jumps	varying $\mu \not\equiv_q \lambda$, $\mathsf{L}^{[\mu]}_{q;1}$ jumps
Borel regularity	
M.A.E.T: existence of hol. solutions F,Feynes : BL sum gives an actual solution *slight functions*	[Praagman, 86]: existence of merom. solutions [Dreyfus, 14]: q–BL gives an actual solution *slope* 0 operators