The regularity of ODEs and thimble integrals with respect to Borel summation

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March 4, 2024

Abstract

Through Borel summation, one can often reconstruct an analytic solution of a problem from its asymptotic expansion. We view the effectiveness of Borel summation as a regularity property of the solution, and we show that the solutions of certain differential equation and integration problems are regular in this sense. By taking a geometric perspective on the Laplace and Borel transforms, we also clarify why "Borel regular" solutions are associated with special points on the Borel plane.

The particular classes of problems we look at are level 1 ODEs and exponential period integrals over Lefschetz thimbles. To expand the variety of examples available in the literature, we treat various examples of these problems in detail.

Borel regularity is a new approach to Borel summability, that present Borel summation as a regularization process and it explains the effectiveness of Borel summability, namely that Borel summation can be used to reconstruct analytic functions from their asymptotics. [Highlight analytic perspective more?] Borel regular functions shows up in two classes of examples: as solutions of certain level 1 ODEs and as integrals over Lefschetz thimbles. In addition, we give a geometric description of the Borel plane.

Contents

1 Introduction

1.1 The unreasonable effectiveness of Borel summation

You can often find a formal power series

$$\tilde{\Phi} = \frac{c_0}{z^\tau} + \frac{c_1}{z^{\tau+1}} + \frac{c_2}{z^{\tau+2}} + \frac{c_3}{z^{\tau+3}} + \ldots,$$

with $\tau \in (0, 1]$, that looks or acts like a solution to a problem whose actual solutions are holomorphic functions of z. For example, if you want to understand how the solutions of the holomorphic ordinary differential equation (ODE)

$$\left[\left[\left(\tfrac{\partial}{\partial z}\right)^2 - 1\right] + z^{-1}\tfrac{\partial}{\partial z} - \left(\tfrac{1}{3}\right)^2 z^{-2}\right]\Phi = 0. \tag{1}$$

behave near $z = \infty$, you might start by looking for formal *transmonomial* solutions $e^{-\alpha z}\,\tilde{\Phi}$, where $\tilde{\Phi}$ is a formal power series of the kind above. Setting $\alpha = -1$ and $\tau = \frac{1}{2}$ gives a well-behaved recurrence relation for $\tilde{\Phi}$, which produces the solution

$$e^z\left[\frac{(-1)!!}{z^{1/2}} + \frac{5}{72}\cdot\frac{1!!}{z^{3/2}} + \frac{385}{31104}\cdot\frac{3!!}{z^{5/2}} + \frac{17017}{6718464}\cdot\frac{5!!}{z^{7/2}} + \ldots\right] \tag{2}$$

and its constant multiples (see [45, equation 10.40.1]). As another example, you might rewrite the integral

$$\Phi(z) = \int_\Lambda \exp\left[-z\left(4u^3 - 3u\right)\right]\,du$$

as

$$e^z\,\frac{1}{2\sqrt{3}}\int_{-\infty}^{\infty} e^{-zt^2/2}\left[1 - \frac{\sqrt{3}}{9}t + \frac{5}{72}t^2 - \frac{4\sqrt{3}}{243}t^3 + \frac{385}{31104}t^4 - \frac{7\sqrt{3}}{2187}t^5 + \frac{17017}{6718464}t^6 - \ldots\right]dt$$

using the substitution $\frac{1}{2}t^2 = 4u^3 - 3u + 1$. Naïvely integrating term by term, you again get the transmonomial (2)—up to a constant.see Mathematica document.

Once you have the formal solution $\tilde{\Phi}$, you might try to get an actual solution by applying *Borel summation*, which turns a formal power series into a function asymptotic to it. Borel summation works in three steps.

1. Thinking of z as a "frequency variable," we take the Borel transform (also known as formal inverse Laplace transform) of $\tilde{\Phi}$, producing a formal power series $\tilde{\phi}$ in a new "position variable" ζ.

2. If $\tilde{\phi}$ has a positive radius of convergence we say $\tilde{\Phi}$ is 1-*Gevrey*. We sum $\tilde{\phi}$ to get a holomorphic function $\hat{\phi}$ on a neighborhood of $\zeta = 0$. Then, by analytic continuation, we expand the domain of $\hat{\phi}$ to a Riemann surface B with a distinguished 1-form λ—the continuation of $d\zeta$.

3. Furthermore, if $\hat{\phi}$ grows slowly enough along an infinite ray $b + e^{i\theta}[0, \infty)$ [change, explain, or link to notation] its Laplace transform $\mathcal{L}^\theta_{\zeta,b}\hat{\phi}$ (see Section 3.1.5) turns out to be a holomorphic function of z, well-defined on some sector of the frequency plane. In this case, we say $\tilde{\Phi}$ is *Borel-summable*, and we call $\hat{\Phi} := \mathcal{L}^\theta_{\zeta,b}\hat{\phi}$ its *Borel sum* at b.

The series $\tilde{\Phi}$ and its Borel sum $\hat{\Phi}$ have a special relationship, which is best described in the language of *Gevrey asymptoticity*.

Definition 1. On an open, possibly bounded sector Ω around ∞, a power series

$$a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \frac{a_4}{z^4} + \ldots \tag{3}$$

is *asymptotic* to a holomorphic function F if

$$\left|F - \left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots + \frac{a_n}{z^n}\right)\right| \in o_{\operatorname{Re}(z)\to\infty}\left(|z|^{-n}\right)$$

over all orders n. We'll use the symbol $\sim$ to denote the asymptotic of F, namely we'll write

$$F \sim \sum_{n\geq 1} a_n z^{-n}.$$

Not all funtions are asymptotic to a power series, thus more generally, we'll say a function F is Poincaré asymptotic to a trans-monomial $e^{\lambda z} z^\tau \sum_{n=0}^N a_n z^{-n}$ for some positive $\lambda > 0$ and $\tau \in (0, 1)$ if

$$\left|F - e^{\lambda z} z^\tau\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots + \frac{a_n}{z^n}\right)\right| \in o_{\operatorname{Re}(z)\to\infty}\left(e^{\lambda\operatorname{Re}(z)}|z|^{\tau-n}\right).$$

If two power series of the form (3) are asymptotic to the same function, they must be equal; this can be deduced from the triangle inequality. The power series which is asymptotic to a given function on a given open sector, if it exists, is therefore an intrinsic property of the function: it's the function's *asymptotic expansion* on that sector. The functions that have asymptotic expansions on a given sector form a ring [56, Section A.4], and the map that sends such a function to its asymptotic expansion is a ring homomorphism. We'll denote such map with the symbol æ. More generally, when the asymptotic expansion is taken in the direction θ at infinity, i.e. as $z \to e^{i\theta}\infty$, we'll write æ^θ.

Definition 2. [Define æ^θ homomorphism that sends a function with an asymptotic expansion on a sector that contains the θ-ray to its asymptotic expansion on that sector. . . .]

The *asymptotic expansion* map æ^θ sends a holomorphic function that vanishes at ∞ to the unique power series of the form (3) which is asymptotic to it on the half-plane $\operatorname{Re} e^{i\theta}z > 0$ (see [56, Theorem C.11]).

[State theorem: differentiation commutes with æ^θ under certain circumstances. See Erdelyi's *Asymptotic Expansions*, §1.6. For basic facts about uniform asymptoticity, see §5.7.2 of Mitschi and Sauzin. Theorem 5.20 and its rephrasing say that the Laplace transform of a function is uniformly asymptotic to the formal Laplace transform of the function's Taylor series.]

Definition 3. On an open sector Ω around ∞, a power series of the form (3) is *uniformly* 1-*Gevrey asymptotic* to a holomorphic function F if there's a constant $A \in (0, \infty)$ for which

$$\left| F - \left(\frac{a_1}{z} + \frac{a_2}{z^2} + \ldots + \frac{a_{n-1}}{z^{n-1}} \right) \right| \lesssim \frac{A^n n!}{|z|^n}$$

over all orders n and all $z \in \Omega$. We'll use $\lesssim$ throughout the paper to mean "bounded by a constant multiple of," with the same constant over all the specified variables.

To compare the Borel sum $\hat{\Phi}$ with the original series $\tilde{\Phi}$, let's take it one more step, sending it back into the world of formal power series by taking its asymptotic expansion.

4. By construction, $\text{æ}^\theta \hat{\Phi} = \tilde{\Phi}$. It turns out that $\hat{\Phi}$ is not only asymptotic to $\tilde{\Phi}$, but *uniformly* 1-*Gevrey*-asymptotic [52, Corollary 5.23].

The Borel summation process is summarized in the following diagram:

$$\begin{tikzcd}
& \text{problem} \arrow[dl, dotted] \arrow[dr, "\text{formally solves}"] & \\
\hat{\Phi} \arrow[rr, mapsto, "\text{æ}^\theta", "\text{(1-Gevrey)}"'] & & \tilde{\Phi} \arrow[d, mapsto, "\mathcal{B}"] \\
\hat{\phi} \arrow[u, mapsto, "\mathcal{L}_b^\theta"] & & \tilde{\phi} \arrow[ll, mapsto, "\text{sum}"]
\end{tikzcd}$$

You can't be sure *a priori* that $\hat{\Phi}$ solves your original problem, even if you know that $\tilde{\Phi}$ is asymptotic to an actual solution. After all, $\tilde{\Phi}$ is asymptotic to lots of functions; Borel summation just picks one of them. In many cases, however, Borel summation picks correctly, delivering an actual solution to your problem. The question of how that happens is the starting point for this paper.

1.2 A new perspective: Borel regularity

1.2.1 Introducing Borel regularity

We consider an holomoprhic function Φ, solution of our problem. Taking its asymptotic expansion along the half-plane $\operatorname{Re} e^{i\theta} z > 0$, we get a formal power series $\tilde{\Phi}$.

$$\begin{tikzcd}
 & \text{problem} \arrow[dl, dash, "\text{solves}"'] \arrow[ddr, dash, "\substack{\text{formally}\\ \text{solves}}"] & \\
\Phi \arrow[drr, mapsto, "æ^{\theta}"] & & \\
\hat{\Phi} \arrow[rr, "æ^{\theta}", "\text{(1-Gevrey)}"'] & & \tilde{\Phi} \arrow[d, mapsto, "\mathcal{B}"] \\
\hat{\phi} \arrow[u, mapsto, "\mathcal{L}^{\theta}_{b}"] & & \tilde{\phi} \arrow[ll, mapsto, "\text{sum}"]
\end{tikzcd}$$

If $\tilde{\Phi}$ is Borel-summable, as described in Section 1.1, its Borel sum $\hat{\Phi}$ is a new holomorphic function. Since different functions can be asymptotic to the same power series, taking the Borel sum of the asymptotic series of Φ must smooth away some details. We'll therefore call this process *Borel regularization*. Explicitly, Borel regularization works in four steps.

1. Take the asymptotic expansion $\tilde{\Phi} = æ^{-\theta}\Phi$.
2. Take the Borel transform $\tilde{\phi} = \mathcal{B}\tilde{\Phi}$.
3. Take the sum $\hat{\phi}$ of $\tilde{\phi}$, and expand its domain to a Riemann surface with a distinguished 1-form, as before.
4. Take the Laplace transform $\hat{\Phi} = \mathcal{L}^{\theta}_{\zeta,0}\hat{\phi}$. This is possible, by definition, if $\tilde{\Phi}$ is Borel-summable.[1]

We'll say a function is *Borel-regularizable* if its asymptotic expansion is Borel-summable, ensuring that we can carry out the last two steps.

Defining a regularization process picks out a class of regular functions: the ones that are invariant under regularization. We'll say a function is *Borel regular* if it's Borel-regularizable and Borel regularization leaves it unchanged. In other words, Φ is Borel regular if $\hat{\Phi} = \Phi$.

1.2.2 Borel regularity sometimes explains why Borel summation works

The idea of Borel regularity can help us understand why Borel summation is so effective in some situations. Roughly speaking, Borel summation works well for problems that admit solutions in terms of Laplace transform.

The central goal of this paper is to explain, from this perspective, why Borel summation works well for the two kinds of problems exemplified in Section 1.1.

[1] More generally, we're allowed to take the Laplace transform at another point $b \in B$. This would be natural in our study of ODEs and thimble integrals.

1. Solving a single-level linear ODE, with level 1 [28, Section 2.1][17, Section 5.2.2.1] [awkward?]. Equation (1) is an example. More generally, we'll consider equations of the form $\mathcal{P}\Phi = 0$ given by a differential operator of the form

$$\mathcal{P} = P(\partial_z) + \frac{1}{z}Q(\partial_z) + \frac{1}{z^2}R(z^{-1}),$$

where P is a monic degree-d polynomial, Q is a degree-$(d-1)$ polynomial that's non-zero at every root of P, and $R(z^{-1})$ is holomorphic in some disk $|z| > A$ around $z = \infty$. We'll restrict our attention to the case where the roots of P are simple (see Section 4.1).

The form of $\mathcal{P}$ derives from Equation 2.2.3 of [28, pag. 105], which encompasses both linear and nonlinear ODEs of level 1. Borel summation is more involved for the nonlinear ones, raising the question of whether our analysis generalizes.

2. Evaluating a certain kind of integral: a one-dimensional *thimble integral*

$$I(z) := \int_\Lambda \exp\big[-zf\big]\nu$$

where $f\colon X \to \mathbb{C}$ is a holomorphic functions with isolated and non-degenerate critical points, X is 1-dimensional complex manifold, $\nu \in \Omega^1(X)$ is holomorphic 1-form in X and Λ is a suitable contour such that $I(z)$ is a holomorphic function of z.

These two problems are closely linked. By playing with derivatives of an exponential integral, you can often find a linear ODE that the integral satisfies. Conversely, for many classical ODEs, there are useful bases of exponential integral solutions.

1.3 Goals and Results

First of all, we want to clearly separate the parts of the theory that deal with holomorphic functions and formal series. For ODEs, we proved in [30, Theorem 4] how to build an analytic frame of solutions with an explicit growth behavior, while for thimble integrals we'll prove in part 3 of Theorem 1.6 that there is an explicit analytic function whose Laplace transform is the thimble integral itself. [In other words, we go directly from the frequency–analytic quadrant to the position–analytic quadrant, without passing through the formal side.]

Second of all, we want to explain why it is useful to work in the position domain (the Borel plane): on the one hand integral equations are more regular than differential equations. On the other hand, a thimble integral in the frequency domain can be recast as the Laplace transform of a function in the spatial domain.

1.3.1 Why does Borel summation work for solutions of level 1 ODEs?

Consider a linear level-1 differential operator $\mathcal{P}$ of the form described in Section 1.2.2. This operator will always have an irregular singularity at $z = \infty$.

[These details are right above in Section 1.2.2!]

$$\mathcal{P} = P(\partial_z) + \frac{1}{z}Q(\partial_z) + \frac{1}{z^2}\big[R_0(z^{-1}) + R_1(z^{-1})\,\partial_z + \ldots + R_{d-1}(z^{-1})\,\partial_z^{d-1}\big], \tag{4}$$

where P is a monic degree-d polynomial, Q is a degree-$(d-1)$ polynomial, and $R_j(z^{-1})$ are holomorphic in some disk $|z| > A$ around $z = \infty$, for every $j = 0, \ldots, d-1$.[2] Furthermore we assume P has simple zeros $P(-\alpha_j) = 0$, $j = 1, ..., d$, $P'(-\alpha_j) \neq 0$ for $j = 1, ..., d$ and $Q(-\alpha_j) \neq 0$. Our equation has an irregular singularity at ∞ (see [44, Definition 3.3.2]). Turn the ODE into a system: assume

$$P(\lambda) = P_0 + P_1\lambda + \ldots + P_d\lambda^d$$
$$Q(\lambda) = Q_0 + Q_1\lambda + \ldots + Q_{d-1}\lambda^{d-1}$$

then the ODE $\mathcal{P}\psi = 0$ is equivalent to the system

$$\frac{d}{dz}Y = B(z)Y \tag{5}$$

$$B(z) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & & 1 \\ b_1(z) & & b_j(z) & \ldots & b_{d-1}(z) \end{pmatrix}, \quad Y = \begin{pmatrix} \psi \\ \psi' \\ \vdots \\ \psi^{(d-1)} \end{pmatrix}$$

and $b_j(z) = -P_j - \frac{1}{z}Q_j - \frac{R_j(z)}{z^2}$. A formal system of solutions is given by the matrix

$$Y = \begin{pmatrix} \psi_1 & \psi_2 & \ldots & \psi_d \\ \psi_1' & \psi_2' & \ldots & \psi_d' \\ \vdots & \vdots & \ldots & \vdots \\ \psi_1^{(d-1)} & \psi_2^{(d-1)} & \ldots & \psi_d^{(d-1)} \end{pmatrix}, \qquad \psi_j(z) = e^{-\alpha_j z} z^{-\tau_j} \tilde{f}_j \tag{6}$$

To prove that $z = \infty$ is an irregular singular point we change coordinates and set $w = 1/z$:

$$-w^2 \frac{d}{dw} Y = B(1/w)Y \tag{7}$$

and $-w^{-2}B(1/w)$ has a simple pole at $w = 0$ (i.e. $z = \infty$). Then the formal system of solutions is of the form

$$\psi_j(w) = e^{-\alpha_j/w} w^{\tau_j} \tilde{f}_j(w)$$

hence according to [44, Definition 3.3.2], $w = 0$ is irregular.

[This is now a corollary; moved to historical.] Under these assumptions, the space of formal solutions of $\mathcal{P}\Phi = 0$ has a basis $\tilde{\Psi}_1, ..., \tilde{\Psi}_d$ of the form [61][28, Proposition 2.2.7, p. 111]

$$\tilde{\Psi}_j(z) = e^{-\alpha_j z} z^{-\tau_j} \tilde{F}_j(z) \in e^{-\alpha_j z} z^{-\tau_j}\, \mathbb{C}[\![z^{-1}]\!], \tag{8}$$

where τ_j is the non-zero complex number $Q(-\alpha_j)/P'(-\alpha_j)$. Since the roots $-\alpha_1, \ldots, -\alpha_n$ are distinct, this form determines the basis up to scaling, giving the space of formal solutions a distinguished frame. Add that the formal series is 1-Gevrey. Classically it was proved that the formal solutions of (8) are Borel-Laplace summable and their Borel-Laplace sums

[2] Looking at the existence theorem in [30, Theorem 4], we could apply this reasoning on the analytic side for more general equations, but this particular case makes it easier to talk about the formal side as well.

satisfy the original differential equation (see for example [63, 49, 47, 44][Braaksma,check]). Therefore, the distinguished frame of formal solutions $\tilde{\Psi}_1, ..., \tilde{\Psi}_d$ becomes a distinguished frame of analytic solutions $\hat{\Psi}_1, ..., \hat{\Psi}_d$.

However, can we find a distinguished basis for solutions in a purely analytic way? This should be possible because of our regularity assumptions about the coefficients of $\mathcal{P}$. In fact, the Main Asymptotic Existence Theorem (M.A.E.T.) guarantees the existence of an analytic frame of solutions asymptotic to a formal frame (see [17, Theorem 3.1]), but it is not a constructive method (see [5, Chapter 14]). [Let's try to move most of this detail to the proof section.] Conversely, we can explicitly build a distinguished basis of analytic solutions: first going to the position domain, we can consider d integral operators of the form

$$\hat{P}_j := P(-\zeta) + \partial_{\zeta,\alpha_j}^{-1} \circ Q(-\zeta) + \partial_{\zeta,\alpha_j}^{-2} \circ \sum_{r=0}^{d-1} R_r(\partial_{\zeta,\alpha_j}^{-1}) \circ (-\zeta)^r \tag{9}$$

labeled by the roots of P. This type of operator is the one we studied in [30], hence we know there exists a unique solution ψ_j of $\hat{P}_j \psi_j = 0$, such that ψ_j blows-up as $\zeta_j^{\tau_j - 1}$ at $\zeta_j = 0$ and it has a controlled behavior on an open domain Ω_j which is star-shaped at α_j and avoids the other critical points (see the definition in Section 4.1). In addition, ψ_j can be Laplace transformed, giving a analytic function $\Psi_j := \mathcal{L}^{\theta}_{\zeta_j} \psi_j$. At this point, we may ask what is the relationship between the two frames of analytic solutions, namely the Borel sum $\hat{\Psi}_j$ of the formal solutions $\tilde{\Psi}_j$ and the Laplace transform of ψ_j. Our main result says they are actually proportional, hence proving Borel summation works [add reminder of what it means for Borel summation to "work"].Using the dictionary for the Laplace transform associated with some point $\zeta = \alpha$ in the frequency domain, (link section), we can turn $\mathcal{P}$ into an integral operator $\hat{\mathcal{P}}_\alpha$. If we choose $-\alpha$ to be a root of P, then $\hat{\mathcal{P}}_\alpha$ will have a regular singularity at $\zeta = \alpha$, and we understand a lot of details about its specific form [30]. We show in [30] that $\hat{\mathcal{P}}_\alpha \phi = 0$ has a distinguished solution ψ_α which is characterized by having a power-law singularity with a known exponent $\tau_\alpha - 1$ at $\zeta = \alpha$ and having exponentially bounded growth at infinity.

We know from experience that we can find holomorphic solutions of the equation $\mathcal{P}\Phi = 0$ by taking the Borel sums of formal trans-monomial solutions. However, this only works when there's a coincidence between exponential factor $e^{-\alpha z}$ in the formal solution, the base point $\zeta = \alpha$ we use for Borel summation, and the characteristic equation $P(-\alpha) = 0$. Why?

A first clue is the observation that every solution we get from Borel summation is, by definition, the Laplace transform $\Phi = \mathcal{L}^{\theta}_{\zeta,\alpha} \phi$ of a function on ϕ the position domain. In fact, the differential equation $\mathcal{P}\Phi = 0$ is equivalent to an integral equation $\hat{\mathcal{P}}_\alpha \phi = 0$.

Furthermore, when $-\alpha$ is a root of P, the integral equation $\hat{\mathcal{P}}_\alpha \phi = 0$ has a special solution ψ_α which is distinguished, up to scaling, by its power-law asymptotics at $\zeta = \alpha$. Because this solution is so unique, its Laplace transform Ψ_α turns out to be Borel regular.

Theorem 1.1. *Choose a root $-\alpha$ of P. Choose an open sector Ω_α which has an opening angle of π or less, has $\zeta = \alpha$ at its tip, and doesn't touch any other root of $P(-\zeta)$. The equation $\mathcal{P}\Psi = 0$ has a unique solution Ψ_α in the affine subspace*

$$z^{-\tau_\alpha} + \widehat{\mathcal{H}}L^{\infty}_{-\tau_\alpha - 1, \bullet}(\Omega_\alpha)$$

of the space $\widehat{\mathcal{H}}L^{\infty}_{-\tau_\alpha, \bullet}(\Omega_\alpha)$ from Section 3.2.1.

Theorem 1.2. *The solution Ψ_α from Theorem 1.1 has an asymptotic expansion in the transmonomial space $e^{-\alpha z} z^{-\tau_\alpha}\, \mathbb{C}[\![z^{-1}]\!]$. In fact, up to scaling, it's asymptotic to the Poincaré solution found in that space [not defined yet?].*

Combining our uniqueness and regularity results for ψ_α with uniqueness and regularity results for the Poincaré solutions, we can show that Ψ_α is Borel regular.

Corollary 1.3. *The solution Ψ_α from Theorem 1.1 is Borel regular.*

Theorem 1.4. *Choose a root $-\alpha$ of P. Choose an open sector Ω_α which has an opening angle of π or less, has $\zeta = \alpha$ at its tip, and doesn't touch any other root of $P(-\zeta)$. Equation $\mathcal{P}\Psi = 0$ has a unique solution Ψ_α in the affine subspace*

$$z^{-\tau_\alpha} + \widehat{\mathcal{H}}L^{\infty}_{-\tau_\alpha-1,\bullet}(\Omega_\alpha)$$

of the space $\widehat{\mathcal{H}}L^{\infty}_{-\tau_\alpha,\bullet}(\Omega_\alpha)$ from Section 3.2.1. That solution is Borel regular.

If you prefer to work in the analytic world, you can use this result to prove the existence and Borel summability of special formal transmonomial solutions.

Corollary 1.5. *For each root $-\alpha$ of P, equation $\mathcal{P}\Psi = 0$ has a formal solution in the transmonomial space $e^{-\alpha z} z^{-\tau_\alpha}\, \mathbb{C}[\![z^{-1}]\!]$, which is unique up to scaling. That solution is Borel summable, and its Borel sum is a constant multiple of Ψ_α.*

On the other hand, if you prefer to work in the formal world, you can use Theorem 1.4 to show that the Poincaré solutions are Borel summable. See Section 4.1.2. [In another section, do the "mirror image" argument. Use the Poincaré algorithm to construct a formal transmonomial solution in the classical way. Use the Ramis Index Theorem to show that it's Gevrey-1, and then then use our existence and uniqueness result for regular singular Volterra equations to show that it's not only Gevrey-1, but in fact Borel summable.] [This is sort of a mirror image of the Ramis Index Theorem?] [Change or actually delete]We summarize the main steps of the proof of Theorem 1.4 and Corollary 1.5 (the rigorous proofs are given in Section 4.1). On the right-hand side of the diagram, we have the formal solution $\tilde{\Psi}_\alpha \in e^{-\alpha z} z^{-\tau_\alpha} \mathbb{C}[\![z^{-1}]\!]$ and its Borel transform $\tilde{\psi}_\alpha$.

$$\begin{tikzcd}
 & & \text{ODE} \arrow[dll, tail reversed, no head] \arrow[drr, tail reversed, no head, "\text{formal}"] \arrow[d, "\mathcal{L}^{-1}"'] & & \\
\Psi_\alpha & & & & \tilde{\Psi}_\alpha \arrow[dd, "\mathcal{B}"] \\
 & & \text{IE} \arrow[dll, tail reversed, no head] \arrow[drr, tail reversed, no head] & & \\
\psi_\alpha \arrow[uu, "\mathcal{L}"] \arrow[r, equal] & \hat{\psi}_\alpha & & & \tilde{\psi}_\alpha \arrow[lll, "\text{sum}"]
\end{tikzcd}$$

Conversely, on the left-hand side, by our existence and uniqueness result [30, Theorem 4] we have an analytic function ψ_α with a distinguished behaviour near $\zeta = \alpha$, which solves the integral equation (IE) obtained as the inverse Laplace transform of the (ODE) (see [30, Lemma 2]). The key step is then comparing ψ_α and $\tilde{\psi}_\alpha$. First, by the properties of the Borel transform, it is easy to show that $\tilde{\psi}_\alpha$ is a formal solution of the integral equation (IE). Then, from the explicit formula for $\tilde{\Psi}_\alpha$, we can show that if $\tilde{\psi}_\alpha$ grows as ψ_α near $\zeta = \alpha$. Therefore, by our uniqueness result [30, Theorem 4], we conclude that $\tilde{\psi}_j$ is indeed summable and ψ_α is the analytic continuation of $\tilde{\psi}_\alpha$.

1.3.2 Why does Borel summation work for thimble integrals?

Consider a thimble integral of the form described in Section 1.2.2. We'll define our setting in the language of translation surfaces, to make clear the connection with our geometric picture of the Laplace transform. The codomain of f has a natural structure of translation surface—a Riemann surface carrying a holomorphic 1-form $d\zeta$, and ζ is a translation coordinate. [The important thing about $\mathbb{C}$, where f takes values, is htat it's a translation surface—...] For each critical point a, let Γ_α^θ be the ray going to infinity with an angle θ from the critical value α, and let ζ_α be the translation coordinate around Γ_α^θ which vanishes at α. These are well-defined as long as Γ_α^θ misses the critical values of f. Then, the preimage $f^{-1}(\Gamma_\alpha^\theta)$ is a bunch of disjoint curves (see Figure 1), as long as Γ_α^θ misses the other critical values of f. The Lefschetz thimble Λ_a^θ can be regarded as the component of $f^{-1}(\Gamma_\alpha^\theta)$ that goes through a. We could orient so that shifting Λ_a to its left would make its projection run clockwise around Γ_α. However, this only seems to be possible when X is 1-dimensional.

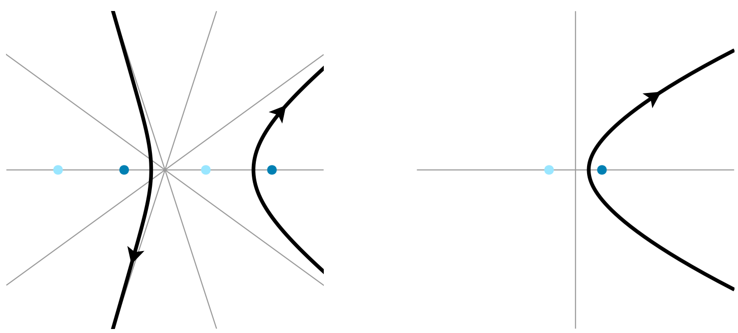


Figure 1: Change the pictures with the one centered at the critical values/points respectivelyThe preimage of a ray departing from the critical value and going to infinity along the positive real axis.

Choose a a ciritcal point of f and $\alpha = f(a)$, let I be the thimble integral through the thimble Λ_a^θ,

$$I(z) := \int_{\Lambda_a^\theta} e^{-zf}\nu \tag{10}$$

our main results show that I is Borel regular as a conseuqence of the fact that thimble integrals can be turned into Laplace transforms. We recall the well-known result

Lemma 1.1 (see Lemma 4.4)**.** *A function ι with $I = \mathcal{L}^\theta_{\zeta,\alpha}\iota$ is given by the* thimble projection formula

$$\iota = \frac{\partial}{\partial\zeta}\left(\int_{\Lambda^\theta(\zeta)} \nu\right), \tag{11}$$

where $\Lambda^\theta(\zeta)$ is the part of Λ_a^θ that goes through $f^{-1}([\alpha, \zeta e^{i\theta}])$. Notice that $\Lambda(\zeta)$ starts and ends in $f^{-1}(\zeta)$. [The thimble Λ needs an orientation, but the orientation is arbitrary.]

Then, Borel regularity for thimble integrals is stated in the following theorem:

Theorem 1.6 (see Theorem 4.6)**.** *If the integral defining I is absolutely convergent, then I is Borel regular. More explicitly:*

1. *As $z \to \infty$ along the ray $z \in e^{-i\theta}\mathbb{R}_{>0}$, the function I is asymptotic to a trans-monomial $\tilde{I} \in e^{-z\alpha} z^{-1/2}\mathbb{C}[\![z^{-1}]\!]$. Recall that θ is the direction of the ray Γ_α^θ.*
2. *The series $\tilde{I}$ is 1-Gevrey. In other words, $\tilde{\iota} := \mathcal{B}_\zeta \tilde{I}$ converges near $\zeta = \alpha$.*
3. *If you continue the sum of $\tilde{\iota}$ along the ray Γ_α^θ, and take its Laplace transform along that ray, you'll recover I.*

We sketch the proofs of the main results in the following diagram (for the rigorous ones see Section 4.2): On the one hand, with a change of coordinates, the thimble integral I can be written as a Laplace type integral, i.e. as the Laplace transform of a function ι, written explicitly in equation (11). On the other hand, we compute the asymptotic expansion of I using the saddle point approximation, which turns out to be a formal 1-Gevrey series in the frequency domain. Consequently, its Borel transform $\tilde{\iota}$ is a germ of holomorphic functions at $\zeta = \alpha$ in the position domain. Finally, we'll show that the Taylor expansion of ι at $\zeta = \alpha$ agrees with $\tilde{\iota}$.

1.3.3 Other results

As part of the treatment, we have made use of some new perspectives on the Laplace transform (see Section 3). Indeed, the Laplace transform is often used to solve ODEs on the frequency domain by relating them to ODEs on the position domain. We find, however, that it is much easier and more natural to relate ODEs on the frequency domain to integral equations on the spatial domain. In particular, working with integral equations in the spatial domain will be our main strategy to prove Borel regularity for ODEs, building on our previous result [30, Theorem 4].

Furthermore, we introduce a geometric picture where the position domain B is a translation surface. If $b \in B$ is non-singular, the frequency domain for $\mathcal{L}^\theta_{\zeta,b}$ is $T^*_{\zeta=b}B$. If b is a conical singularity, the frequency domain is more interesting, as we'll see in our main example (see Figure 2).

We'll illustrate our main results with detailed treatments of several examples: we'll mostly focus on degree two ODEs that admit a frame of solutions expressed as thimble integrals: the Airy–Lucas functions (Section 5.2), the modified Bessel functions, the Airy function (Appendix A), the anharmonic oscillator...). On the one hand, we explicitly solve the integral equation associated with the ODE, building a frame of analytic solutions which are Laplace transforms. On the other hand, our *thimble projection reasoning* shows how to rewrite explicitly thimble integrals as a Laplace transform and it makes Borel regularity evident directly from explicit computations. The thimble projection reasoning is the one we use in Lemma 1.1 and it is crucial for the proof of Theorem 1.6. In the Airy example (see Section A) we'll compute the *thimble projection formula* directly, and by numerical check we show $\hat{\phi} := \partial_{\zeta,\alpha}^{-1/2}\iota$ is a simple resurgent function (see Section A.5.1).

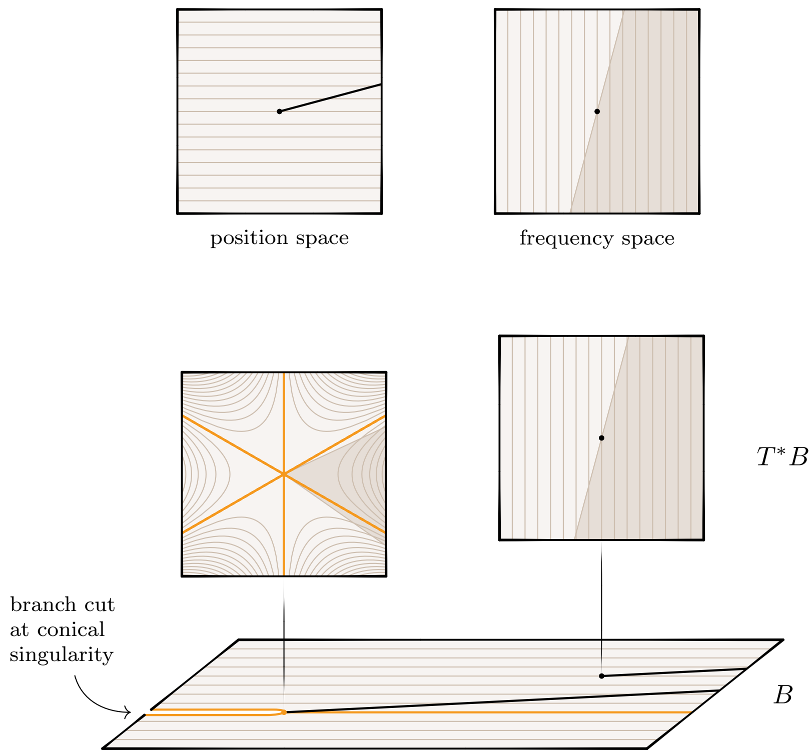


Figure 2: The fiber over an ordinary point of the translation surface B is a copy of the complex plane. However, the fiber over a singular point of B has an interesting structure. Here we represent the example of Airy functions, where the singularity as angle 6π.

Among the examples we study, some of them have been discussed many times, using different approaches and conventions. We try to give an idea of how all these different treatments fit together. For instance, for the Airy function we'll make a comparison with [50, Section 2.2], [52, Section 6.14], and [37, Section 2.2]. The anharmonic oscillator was also discussed in ([7], [2, Appendix B] and [68, Section 2.5.3]). Other examples haven't been discussed much, as for Airy–Lucas functions [11, Equation 3.2].

Recently, resurgence theory (first developed by Écalle in the '80) has attracted interest in mathematics and physics. The resurgence of linear ODEs have been intensively studied [42, 44] and many results are also known for non-linear ODEs [13, 14, 17, 4]. For algebraic thimble integrals of the type we studied in this paper, the resurgence of their asymptotic expansion can be understood geometrically (see [38], [40, Section 6.2]), however for more general exponential integrals (see the last example in [38]) resurgence remains conjectural. Despite their simplicity, our examples of linear ODEs and of 1-dimensional thimble integrals are toy-models that show some features of resurgent functions.

1.4 Plan of the paper

The paper is organized as follows: in Section 2 we review some well-known results concerning Borel regularity which dates back to the classical theory of asymptotics, ODEs and integrals over Lefschetz thimbles. This could help the reader to contextualize our results in the literature. Then, in Section 3 is devoted to define the new geometric picture of the Borel and Laplace transform. The reader who is not familiar with the formalism of Borel and Laplace transforms might start with Sections 3.1.5, the beginning of Section 3.3 and of Section 3.4. Then, Section 4 contains proofs of the main results: Theorem 1.4 and Theorem 1.6. Finally, in Section 5 we give a detailed treatment of different examples both from the ODE perspective and from the thimble integral one. We include Appendix A that contains a detailed treatment of the Airy example, and it is recommended for readers less familiar to the subject. The second one, Appendix B, reviews the main aspects of [30].

1.5 Acknowledgements

This paper is a result of the ERC-SyG project, Recursive and Exact New Quantum Theory (ReNewQuantum) which received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 810573.

We thank Fondation Mathematique Jacques Hadamard for supporting the visit of the second author at IHÉS, under the program *Junior Scientific Visibility*.

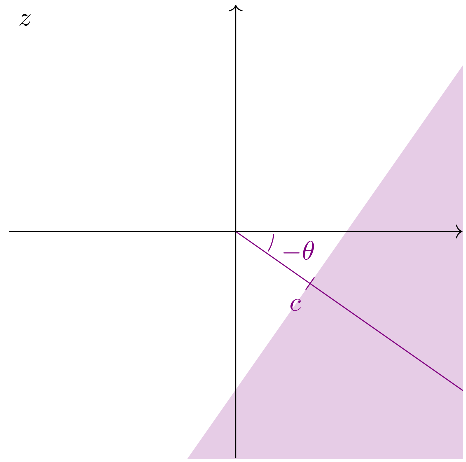
We thank Maxim Kontsevich, David Sauzin, Frédéric Fauvet, Andrew Neitzke, for fruitful discussions and suggestions.

2 Historical context

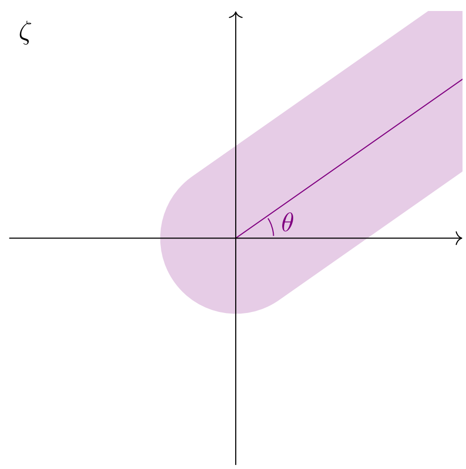
2.1 Borel regularity as a good approximation condition

Borel regular functions can be characterized as functions that are approximated well, asymptotically, by polynomials. Watson showed a century ago [70, Part II, Section 9] that a function F is Borel regular whenever its asymptotic expansion for large $|z|$ is uniformly 1-Gevrey-asymptotic in an obtuse-angled sector at infinity (see Definition 3).

Watson's theorem was soon improved by Nevanlinna [54] (see a modern proof in [55, Theorem B.15] and a generalization to power series with fractional power exponents in [20]), and improved again later by Sokal [67]. These improvements tell us that the obtuse-angled sector around ∞ in the statement above can be replaced with an open disk whose boundary touches infinity—that is, an half-plane which doesn't contain 0, and may be displaced from 0 by some distance c.



When the half-plane extends along the $-\theta$ direction, the sum $\hat{f}$ of the Borel transform of $æ^{-\theta}F$ has an absolutely convergent Laplace transform along the θ direction. In fact, we have $|\hat{f}| \lesssim e^{c|\zeta|}$ uniformly over all ζ in a constant-radius neighborhood of the ray $e^{i\theta}[0, \infty)$.



The Watson–Nevanlinna–Sokal characterization of Borel regular functions is totally general, which means it can't take advantage of any extra structure provided by the problem you're trying to solve. We take the opposite approach, showing that certain functions are Borel regular just because of the extra structure provided by the problems they solve.

2.2 Borel regularity for level 1 ODEs

From a formal perspective, the general theory of linear level 1 ODEs suggests to look for formal solutions $\tilde{\Psi}_\alpha$ of $\mathcal{P}\Phi = 0$ in the transmonomial space $\tilde{\Psi}_\alpha \in e^{-\alpha z} z^{-\tau_\alpha} \mathbb{C}[\![z^{-1}]\!]$, where $\tau_\alpha = Q(-\alpha)/P'(-\alpha)$ [61][28, Proposition 2.2.7, p. 111]. Since the cirtical values are distinct, the transmonomials $\tilde{\Psi}_\alpha$ determine a basis up to scaling, giving the space of formal solutions a distinguished frame. Then, the Main Asymptotic Existence Theorem (M.A.E.T.) guarantees the existence of an analytic frame of solutions asymptotic to the formal one, but it does not give a way to construct that analytic frame (see [5, Chapter 14]). To get a constructive proof, a classical approach was to investigate Borel summability of the formal solutions [43, 44, 46, 47, 49, 63]. For example, the Ramis Index Theorem shows that the $\tilde{\Psi}_\alpha$ are 1-Gevrey series—the first step in proving Borel summability [62]. [Mention connection to index, in Loday-Richaud, and to the way orders get shifted?]

At the same time, a natural question we can ask is whether we can build an analytic frame of solutions, without looking for a formal frame. This is possible, as we proved in [30], and by our construction, these solutions turn out to be uniform 1-Gevrey asymptotic to the formal solutions $\tilde{\Psi}_\alpha$ (see Corollary 1.5). As briefly discussed in Section 1.3.1 in the sketch the proof of Theorem 1.4 and Corollary 1.5, our result is based on going to the position domain and study solutions of an integral equation with a special behaviour at the singular points $\zeta = \alpha$. Although our approach and the classical one of Borel summability both focus on the importance of the position domain, we study different equations: on the one hand, we solve integral equations in the position domain; on the other hand, Malgrange [46] solves differential equations analytically in the position domain, and in [43], Loday-Richaud and Remy solve perturbed integral equations formally in the position domain (following the approach of Écalle [28]).

Remark 2.1. Our approach of solving regular singular Volterra equation—based on the contraction mapping theorem for suitable Banach spaces [30]—is analogous to the one that Braaksma used to solve a different class of non-linear ODEs and difference equations whose coefficients are written as Laplace transforms [10]. What distinguishes our result from the one of Braaksma is to consider the Laplace transform of functions with integrable fractional power singularities, and even if our result has been proved for linear level 1 ODEs, we allow holomorphic coefficents that are not necesserly Laplace transforms.

We leave a possible generalization of our result to other classes of ODEs and difference equations for further pubblications. [Let's do more compare-and-contrast! Our problem space is sort of orthogonal to Braaksma's, but we use the same solution space, so it seems like there could be a cool interaction there.]

Remark 2.2. There are many ways to build a frame of solutions for these kinds of ODEs.

	Analytic	**Formal**
Frequency domain		Poincaré: trans-monomial ansatz [61]
Position domain	Écalle: resurgence $\longleftarrow$ Fixed-point iteration [30]	Écalle: formal perturbation theory [28, 43]

In the late 1800s, Poincaré did it formally in the frequency domain [61]. In the late 1900s, Écalle and later authors did it formally in the position domain [28, 43]. Écalle also showed that each of his formal solutions sums to a resurgent analytic solution, from which a frame of analytic solutions can be extracted. Most recently, we directly constructed a frame of analytic solutions in the position domain [30].

In Section 4.1, we show that we're all finding the same frame. The uniqueness part of [30, Theorem 4] shows that Écalle's formal solutions sum to our analytic ones, and the properties of the Laplace transform guarantee that our solutions, cast into in the frequency domain, are asymptotic to Poincaré's.

2.3 Borel regularity for thimble integrals

Thimble integrals have been studied from different perspectives: in physics, they play an important technical role in quantum mechanics, where infinite-dimensional exponential integrals are supposed to give the expectation values of observable quantities [6, 25, 31, 69]. In this context, physicists often use Borel summation, resurgence and related techniques to assign values to these integrals going beyond their pertubative expansion (see also [8, 9, 14, 36, 35, 59]). For instance, in complex Chern–Simons theory, one could decompose the path integral in thimble integrals and arguing as in the finite dimesional set-up to study the Witten–Reshetikhin–Turaev invariants of 3 manifolds [33, 71]. Take a look at Witten's paper on knots More generally, we could think of a problem that is known to admit a thimble integral solution but that's hard to be defined explicitly. However, we know the perturbative expansion of the expected solution; then Borel regularity tells us that the Borel sum of the pertubative series is the solution we're looking for. In fact, the Laplace transform ray (namely the ray along which the Laplace trasnform is defined) is often a "collapsed" thimble.This is the bit about turning a perturbative series back into a thimble integral, taking advantage of the fact that the Laplace transform ray is often a "collapsed" thimble.

In the algebraic geometric set-up, namely when X is an N-dimensional alegbraic variety over $\mathbb{C}$ and f is a proper map $f\colon X \to \mathbb{C}$, thimble integrals are known as period integrals [21, 39, 58].[3] In fact, the thimbles represent classes in the homology $H_N^{B,\theta}(X, f)$ which are relative to the preimage of $e^{i\theta}\infty$ under the funtion f, for a generic direction θ.[4] Furthermore, if f has non-degenerate critical points, they form a basis for $H_N^{B,\theta}(X, f)$. Then varying θ, $H_N^{B,\theta}(X, f)$ forms a local system over $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ singular at $\theta = \arg(\alpha - \beta)$, and whose monodromy can be computed by Picard–Lefschetz formula (we refer to [3, Section 1] and [58, Section 3.3, Part II]). Equivalently, the monodromy data can be computed as the Stokes phenomena for Laplace transforms; indeed as a consequence of our Borel regularity result Theorem 1.6, thimble integrals can be turned into Laplace transforms.

Classical examples of thimble integrals (with polynomial f) are special functions, thus the Borel summability properties of their asymptotics can be studied either analytically (through the *thimble projection reasoning* see the discussion in Section 5) or geometrically (in terms of a certain Riemann–Hilbert problem [38][40, Section 6.2]).

In the N-dimensional analog of our set-up 1.2.2 the thimbles are actually steepest descent paths from the critical points; thus the asymptotic behaviour as $z \to e^{i\theta}\infty$ can be studied using the steepest descent method [1, 15, 18, 19, 22, 48, 57].[5] However, the asymptotics of thimble integrals is typically a formal power series, and choosing a suitable resummation technique one should be able to recover the original function. This raises the question we address in this paper, namely to understand why Borel summation is effective for 1-dimensional thimbles integrals. In a nutshell, our result follows from the fact that thimble integrals are generalized Laplace transforms, hence they are Borel regular functions. In addition, our approach wants to focus on the role of the position domain and it emphasizes how the geometry of the Laplace transform is the natural framework to describe Borel regularity for thimbles integrals (see Fenyes's lecture [32] and Section 3).

[3] In particular, they find application in mirror symmetry for Fano varieties as they encode the Gromov–Witten invariants.add references—Maxim's mirror symmetry talk, Varchenko.

[4] The relative homology $H_i^{\theta}(X, f)$ is defined as the limit as $c \to e^{i\theta}\infty$, of $H_i^{\theta}(X, f^{-1}(S_c^+))$, where $S_c^+ = \{\zeta \in \mathbb{C} |\, \operatorname{Re}\zeta > c\}$. Generic directions correspond to $\theta \neq \arg(\alpha - \beta)$.

[5] In fact, analougus results hold when f satisfies milder assumptions (see for instance [29, Section 1.2.2]).

3 The Laplace and Borel transforms

3.1 The geometry of the Laplace transform

Classically, the Laplace transform turns functions on the position domain into functions on the frequency domain. In the study of Borel summation and resurgence, it's useful to see the position domain as a *translation surface* B, and the frequency domain as one of its cotangent spaces. Roughly speaking, the Laplace transform lifts holomorphic functions on B to holomorphic functions on T^*B.

3.1.1 Translation surfaces, briefly

A translation surface is a Riemann surface B carrying a holomorphic 1-form λ [72]. A translation chart is a local coordinate ζ with $d\zeta = \lambda$. The standard metric on $\mathbb{C}$ pulls back along translation charts to a flat metric on B, with a conical singularity of angle $2\pi n$ wherever λ has a zero of order $n - 1 > 0$. We'll require B to be finite-type and λ to have a pole at each puncture. This kind of translation surface has a "cylindrical end" at each puncture where λ has order -1, and a "$|2n|$-planar end" at each puncture where λ has order $n - 1 < -1$ [34, Section 2.5] (or cite Aaron's article, which will hopefully present the same background in the translation surface context)(see Figure 3).

Figure 3: On the left an example of translation surface with "cylindrical end". On the right, an example of translation surface with "$|2n|$-planar end".

3.1.2 Direction

The translation structure gives B a notion of direction as well as distance. Away from the zeros of λ, which we'll call *branch points*, we can talk about moving upward, rightward, or at any angle, just as we would on $\mathbb{C}$. At a branch point of cone angle $2\pi n$, we can also talk about moving upward, rightward, or at any angle in $\mathbb{R}/2\pi\mathbb{Z}$, but here there are n directions that fit each description. To make this more concrete, note that around any point $b \in B$, there's a unique holomorphic function ζ_b that vanishes at b and has $d\zeta_b = \lambda$. [If we define "translation parameter" earlier, we can say:] there's a unique translation parameter ζ_b that vanishes at b. This function is a translation chart when b is an ordinary point, and an n-fold branched covering when b is a branch point of cone angle $2\pi n$. In either case, $\zeta_b \in e^{i\theta}[0, \infty)$ is a ray or a set of rays leaving b at angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Near each branch point b, let's fix a coordinate ω_b with $\zeta_b = \frac{1}{n}\omega_b^n$, where $2\pi n$ is the cone angle at b. This lets us label each direction at b with an "extended angle" in $\mathbb{R}/2\pi n\mathbb{Z}$. Of course, there are n different choices for ω_b.

3.1.3 Frequency

The translation structure also gives us an isomorphism $z\colon T_b^*B \to \mathbb{C}$ when $b \in B$ is an ordinary point, and an isomorphism $z\colon T_b^*B^{\otimes n} \to \mathbb{C}$ when b is a branch point of cone angle $2\pi n$ [make it clear that z is an almost-global chart on T^*B—for example, writing

$z\colon T^*B \to \mathbb{C}^2$]. At an ordinary point, we can define z simply as the map

$$\begin{aligned} z\colon T_b^*B &\to \mathbb{C} \\ \lambda\big|_b &\mapsto 1. \end{aligned}$$

To get a definition that generalizes to branch points, though, it's worth taking a fancier point of view. Recall that $T_b^*B = \mathfrak{m}_b/\mathfrak{m}_b^2$, where $\mathfrak{m}_b$ is the ideal of holomorphic functions that vanish at b. Observing that $(f + \mathfrak{m}_b)^n$ lies within $f^n + \mathfrak{m}_b^{n+1}$ for any $f \in \mathfrak{m}_b$, we can identify $T_b^*B^{\otimes n}$ with $\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}$ for $n \geq 1$. When b is an ordinary point, the function ζ_b defined in Section 3.1.2 represents a nonzero element of $\mathfrak{m}_b/\mathfrak{m}_b^2$: the cotangent vector $\lambda\big|_b$. In general, ζ_b represents a nonzero element of $\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}$, where $2\pi n$ is the cone angle at b. We define z as the isomorphism

$$\begin{aligned} z\colon \mathfrak{m}_b^n/\mathfrak{m}_b^{n+1} &\to \mathbb{C} \\ \zeta_b + \mathfrak{m}^{n+1} &\mapsto 1. \end{aligned}$$

When b is a branch point, the coordinate ω_b we chose in Section 3.1.2 gives us an isomorphism

$$\begin{aligned} w_b\colon T_b^*B &\to \mathbb{C} \\ \omega_b + \mathfrak{m}^2 &\mapsto 1 \end{aligned}$$

that makes the diagram

$$\begin{tikzcd} T_b^*B^{\otimes n} \arrow[r, "z"] & \mathbb{C} \\ T_b^*B \arrow[u, "\blacksquare^n"] \arrow[r, "w"'] & \mathbb{C} \arrow[u, "\blacksquare^n"'] \end{tikzcd}$$

commute.

3.1.4 Boundary

Discuss the visual boundary, citing Lemma 3.1 of Dankwart's thesis *On the large-scale geometry of flat surfaces* for the description of geodesics.

3.1.5 The Laplace transform over an ordinary point

Pick a local holomorphic function ζ on B with $d\zeta = \lambda$, and an extended angle $\theta \in \mathbb{R}$. [If we define "translation parameter" earlier, we can say:] Pick a translation parameter ζ. The *Laplace transform* $\mathcal{L}_\zeta^\theta$ turns a local holomorphic function f on B into a local holomorphic function on T^*B. When $b \in B$ is an ordinary point, $\mathcal{L}_\zeta^\theta f$ is defined on T_b^*B by the formula

$$\mathcal{L}_\zeta^\theta f\big|_b = \int_{\Gamma_b^\theta} e^{-z\zeta} f \, d\zeta, \tag{12}$$

where z is the frequency function and Γ_b^θ is the ray that leaves b at angle θ. We'll use the shorthand $\mathcal{L}_{\zeta,b}^\theta f := \mathcal{L}_\zeta^\theta f\big|_{\zeta=b}$ throughout this document.

To make sense of this formula, we ask for the following conditions.

- The base point b is in the domain of ζ. Once we have this, we can continue ζ along the whole ray Γ_b^θ.
- The ray Γ_b^θ avoids the branch points after leaving b.
- The integral converges. We guarantee this by asking for a pair of simpler conditions.
  - With respect to the flat metric, f is uniformly of exponential type Λ along Γ_b^θ, and is locally integrable throughout.[6] Following [30], we require $f \in \mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_b)$ with $\sigma > -1$.[7] The set Ω_b is a tubular neighbourhood of Γ_b^θ.
  - The value of z is in the half-plane H_θ centered around the ray $e^{-i\theta}[0, \infty)$.

3.1.6 The Laplace transform over a branch point

When b is a branch point, we can still use formula (12) to define $\mathcal{L}_\zeta^\theta f$ on T_b^*B, as long as we take care of a few subtleties. Thanks to the labeling choices we made at the end of Section 3.1.2, the extended angle $\theta \in \mathbb{R}$ still picks out a ray Γ_b^θ. The function z is defined on $T_b^*B^{\otimes n}$, where $2\pi n$ is cone angle at b, so we pull it back to T_b^*B along the nth-power map. This amounts to substituting w_b^n for z in formula (12). The half-plane $z \in H_\theta$ in $T_b^*B^{\otimes n}$ pulls back to n sectors of angle π/n in T_b^*B. We only define $\mathcal{L}_\zeta^\theta f$ on one of them: the one centered around the ray $w_b \in e^{-i\theta/n}[0, \infty)$.

3.1.7 Change of translation chart

Let $b \in B$ be an ordinary point, and let ζ_b be the coordinate on B for which $\zeta = \zeta(b) + \zeta_b$. Then the Laplace transform $\mathcal{L}_{\zeta_b,0}$, which turns functions on B into functions on T_b^*B, is compatible with $\mathcal{L}_{\zeta,b}$.

Lemma 3.1. *Let $\varphi \in \mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_b)$ with $\sigma > -1$ and for some $\Lambda > 0$, then*

$$e^{-bz}\mathcal{L}_{\zeta_b,0}\varphi = \mathcal{L}_{\zeta,b}\varphi. \tag{13}$$

In other words, the diagram

$$\begin{tikzcd} \mathcal{O}_{T^*B}(H) \arrow[rr, "e^{bz}"] & & e^{bz}\,\mathcal{O}_{T^*B}(H) \\ & \mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_b) \arrow[ul, "\mathcal{L}_{\zeta_b,0}"] \arrow[ur, "\mathcal{L}_{\zeta,b}"'] & \end{tikzcd}$$

commutes, where H is the half-plane $\mathrm{Re}(z) > 0$ (the coordinate z is a global coordinate on T^*B).

[6] Recall that a function is of exponential type Λ if for every $\varepsilon > 0$ there is a constant A_ε (which depends on ε) such that $|f(x)| \leq A_\varepsilon e^{\Lambda+\varepsilon}$. We instead require that there exists a uniform constant A such that for every $\varepsilon > 0$ the bound holds.

[7] The condition $\sigma > -1$ guarantees the integrability at $\zeta = b$.

Proof. With a change of variable in the integral that defines the Laplace transform, we see that

$$\begin{aligned}\mathcal{L}_{\zeta,b}\varphi &= \int_{\Gamma_{\zeta,b}} e^{-z\zeta}\,\varphi\,d\zeta \\ &= \int_{\Gamma_{\zeta_b,0}} e^{-z(b+\zeta_b)}\,\varphi\,d\zeta_b \\ &= e^{-bz}\int_{\Gamma_{\zeta_b,0}} e^{-z\zeta_b}\,\varphi\,d\zeta_b \\ &= e^{-bz}\mathcal{L}_{\zeta_b,0}\varphi.\end{aligned}$$

□

We now consider a rescale of the translation structure of B, expanding displacements by a factor of $\mu \in (0,\infty)$. The coordinate $\xi = \mu\zeta$ is a chart for the new translation structure. The corresponding frequency coordinate $x\colon T^*B \to B$ is given by $d\xi \mapsto 1$, so $x = \mu^{-1}z$.

Lemma 3.2. *Let $\varphi \in \mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega)$ with $\sigma > -1$ and for some $\Lambda > 0$, then*

$$\mathcal{L}_{\xi,0}\varphi = \mu\,\mathcal{L}_{\zeta,0}\varphi.$$

Proof. From the computation

$$\begin{aligned}\mathcal{L}_{\xi,0}\varphi &= \int_{\Gamma_{\xi,0}} e^{-x\xi}\,\varphi\,d\xi \\ &= \int_{\Gamma_{\zeta,0}} e^{-z\zeta}\,\varphi\,\mu\,d\zeta \\ &= \mu\,\mathcal{L}_{\zeta,0}\varphi\end{aligned}$$

we get the desired result. □

3.2 Analysis of the Laplace transform

3.2.1 Regularity and decay properties

Let $\Omega_b \subset B$ be an open, simply connected set that touches but does not contain $\zeta = b$ (see Figure 4). Suppose Ω_b contains the ray $\Gamma^\theta_{\zeta,b}$. In [30] we introduce the function space $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_b)$ of holomorphic functions on Ω_b which are uniformly of exponential type Λ and blow up like $|\zeta|^\sigma$ as ζ approaches b. When $\sigma > -1$, the singularity at $\zeta = b$ is integrable. Hence, the Laplace transform $\mathcal{L}^\theta_{\zeta,b}$ turns elements of $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_b)$ into well-defined holomorphic functions on the half-plane $\operatorname{Re}(e^{i\theta}z) > 0$ in the fiber $T^*_{\zeta=b}B$ [52, Section 5.6].

$\mathcal{H}L^\infty_{\sigma,\bullet}(\Omega^\Lambda_\alpha)$ Suppose Ω_α is an open sector with $\zeta = \alpha$ at its tip, and an opening angle of π or less. In this case, for any $\Lambda \in \mathbb{R}$, let $\widehat{\Omega}^\Lambda_\alpha$ be the union of the half-planes $\operatorname{Re}(e^{i\theta}z) > \Lambda$ over all angles θ in the opening of Ω_α. Then, let $\widehat{\mathcal{H}L}^\infty_{\sigma,\Lambda}(\widehat{\Omega}^\Lambda_\alpha)$ be the space of holomorphic functions F on $\widehat{\Omega}^\Lambda_\alpha$ with $|F| \lesssim \Delta^\sigma$, where Δ is the distance to the boundary of $\widehat{\Omega}^\Lambda_\alpha$. The norm $\|F\|_{\sigma,\Lambda} = \sup_{\widehat{\Omega}^\Lambda_\alpha} \Delta^{-\sigma}|F|$ turns $\widehat{\mathcal{H}L}^\infty_{\sigma,\Lambda}(\Omega_\alpha)$ into a Banach space. Cite [68] better for the theorem.

Proposition 3.3 (following [68]). *Let Ω_α be an open sector with $\zeta = \alpha$ at its tip, and an opening angle of π or less. For any $\sigma > 0$ and $\Lambda \geq 0$, and any angle θ in the opening of Ω_α, the Laplace transform $\mathcal{L}^\theta_{\zeta_\alpha,0}$ is a continuous map $\mathcal{H}L^\infty_{\sigma-1,\Lambda}(\Omega_\alpha) \to \widehat{\mathcal{H}}L^\infty_{-\sigma,\Lambda}(\Omega_\alpha)$, with a norm of at most $\Gamma(\sigma)$.*

Proof. Given some $f \in \mathcal{H}L^\infty_{\sigma-1,\Lambda}(\Omega_\alpha)$, we compute

$$\begin{aligned}
|\mathcal{L}^\theta_{\zeta_\alpha,0} f| &= \left| \int_{\Gamma^\theta_{\zeta,\alpha}} e^{-z\zeta_\alpha} f \, d\zeta_\alpha \right| \\
&\leq \int_{\Gamma^\theta_{\zeta,\alpha}} e^{-\operatorname{Re}(z\zeta_\alpha)} |\zeta_\alpha|^{\sigma-1} e^{\Lambda|\zeta_\alpha|} \|f\|_{\sigma-1,\Lambda} \, |d\zeta_\alpha| \\
&\leq \int_{\Gamma^\theta_{\zeta,\alpha}} e^{(\Lambda - c_{z,\theta}|z|)|\zeta_\alpha|} |\zeta_\alpha|^{\sigma-1} \|f\|_{\sigma-1,\Lambda} \, |d\zeta_\alpha|,
\end{aligned}$$

where $c_{z,\theta}$ is the cosine of $\arg(z) + \theta$. When $|z|$ is large and $c_{z,\theta}$ is positive, the integrand shrinks exponentially as $|\zeta|$ grows. This shows that for each angle θ in the opening of Ω_α, the integral defining $\mathcal{L}^\theta_{\zeta_\alpha,0} f$ converges in some region of $\widehat{\Omega}^\Lambda_\alpha$. It also shows that for different angles θ, the functions $\mathcal{L}^\theta_{\zeta_\alpha,0} f$ match where their domains overlap. We can therefore glue these functions together into one big Laplace transform of f, defined at large values of $|z|$ across the whole opening angle of $\widehat{\Omega}^\Lambda_\alpha$.

We can now simplify the calculation of $\mathcal{L}^\theta_{\zeta_\alpha,0} f$ by looking at $\arg(z)$ and using the closest angle θ in the opening of Ω_α. This keeps $\Lambda - c_{z,\theta}|z|$ equal to Δ, the distance to the boundary of $\widehat{\Omega}^\Lambda_\alpha$. It follows that

$$\begin{aligned}
|\mathcal{L}^\theta_{\zeta_\alpha,0} f| &\leq \int_{\Gamma^{\arg(z)}_{\zeta,\alpha}} e^{-\Delta|\zeta_\alpha|} |\zeta_\alpha|^{\sigma-1} \|f\|_{\sigma-1,\Lambda} \, |d\zeta_\alpha| \\
&= \int_0^\infty e^{-\Delta t} t^{\sigma-1} \|f\|_{\sigma-1,\Lambda} \, dt.
\end{aligned}$$

The integral on the last line converges throughout $\widehat{\Omega}^\Lambda_\alpha$. We can evaluate it by observing that this bound on the Laplace transform of f is itself a Laplace transform:

$$|\mathcal{L}^\theta_{\zeta_\alpha,0} f| \leq \Gamma(\sigma)\Delta^{-\sigma} \|f\|_{\sigma-1,\Lambda}.$$

In terms of the metric on $\widehat{\mathcal{H}}L^\infty_{-\sigma,\Lambda}(\Omega_\alpha)$ defined above, this bound says that

$$\|\mathcal{L}^\theta_{\zeta_\alpha,0} f\|_{\sigma,\Lambda} \leq \Gamma(\sigma) \|f\|_{\sigma-1,\Lambda},$$

which is what we wanted to show. □

Proposition 3.4. *Let Ω_α be an open sector of the kind described in Proposition 3.3, and let $\Omega^\varepsilon_\alpha \subset \Omega_\alpha$ be the open sector created by cutting a sector of angle $\varepsilon > 0$ off each edge of Ω_α. Choose any $\lambda' > \Lambda$. Under the conditions of Proposition 3.3, the Laplace transform*

$$\mathcal{L}^\theta_{\zeta_\alpha,0} \colon \mathcal{H}L^\infty_{\sigma-1,\Lambda}(\Omega_\alpha) \to \widehat{\mathcal{H}}L^\infty_{-\sigma,\Lambda}(\widehat{\Omega}^\Lambda_\alpha)$$

has a continuous left inverse

$$\left(\mathcal{L}^{\theta}_{\zeta_\alpha,0}\right)^{-1} : \widehat{\mathcal{H}}L^{\infty}_{-\sigma,\Lambda}(\widehat{\Omega}^{\Lambda}_{\alpha}) \to \mathcal{H}L^{\infty}_{\sigma-1,\lambda'}(\Omega^{\varepsilon}_{\alpha}),$$

with a norm of at most

$$\frac{\Gamma(1-\sigma)}{\pi\,\sin(\varepsilon/2)}.$$

Proof. Define the sector $\Omega^{\varepsilon/2}_{\alpha} \subset \Omega_\alpha$ similarly to $\Omega^{\varepsilon}_{\alpha}$. Choosing some $\lambda' > \Lambda$, let $\widehat{\Omega}^{\varepsilon/2,\lambda'}_{\alpha} \subset \widehat{\Omega}^{\Lambda}_{\alpha}$ be the union of the half-planes $\operatorname{Re}(e^{i\theta}z) > \lambda'$ over all angles θ in the opening of $\Omega^{\varepsilon/2}_{\alpha}$. The boundary of $\widehat{\Omega}^{\varepsilon/2,\lambda'}_{\alpha}$ forms a path Λ, which we orient so that the boundary of $\widehat{\Omega}^{\Lambda}_{\alpha}$ is on its left. Parameterize Λ using the arc length parameter t which is zero at the midpoint of the circular arc part of Λ. Along Λ, we have

$$\Delta \geq \mu + \sin(\varepsilon/2)\,|t|,$$

for some $\mu \in (\lambda, \lambda')$, where Δ is still the distance to the boundary of $\widehat{\Omega}^{\Lambda}_{\alpha}$. On $\Omega^{\varepsilon}_{\alpha} \times \Lambda$, we have

$$\operatorname{Re}(z\zeta_\alpha) \leq |\zeta_\alpha|\big(\lambda' - \sin(\varepsilon/2)\,|t|\big).$$

The inverse Laplace transform is given by the formula

$$\left(\mathcal{L}^{\theta}_{\zeta_\alpha,0}\right)^{-1} F = \frac{1}{2\pi i}\int_{\Lambda} e^{z\zeta_\alpha} F\,dz.$$

When F is in $\widehat{\mathcal{H}}L^{\infty}_{-\sigma,\Lambda}(\Omega_\alpha)$, we have the bound

$$\begin{aligned}
\left|\left(\mathcal{L}^{\theta}_{\zeta_\alpha,0}\right)^{-1} F\right| &\leq \frac{1}{2\pi}\int_{\Lambda} e^{\operatorname{Re}(z\zeta_\alpha)}\Delta^{-\sigma}\|F\|_{\sigma,\Lambda}\,dz \\
&\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{|\zeta_\alpha|\left(\lambda' - \sin(\varepsilon/2)\,|t|\right)}\big(\mu + \sin(\varepsilon/2)\,|t|\big)^{-\sigma}\|F\|_{\sigma,\Lambda}\,dt \\
&= e^{|\zeta_\alpha|\lambda'}\|F\|_{\sigma,\Lambda}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\zeta_\alpha|\,\sin(\varepsilon/2)\,|t|}\big(\mu + \sin(\varepsilon/2)\,|t|\big)^{-\sigma}\,dt,
\end{aligned}$$

which we can rewrite as

$$\begin{aligned}
\left|\left(\mathcal{L}^{\theta}_{\zeta_\alpha,0}\right)^{-1} F\right| &\leq e^{|\zeta_\alpha|\lambda'}\|F\|_{\sigma,\Lambda}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\zeta_\alpha|\,|s|}\big(\mu + |s|\big)^{-\sigma}\,\frac{ds}{\sin(\varepsilon/2)} \\
&= e^{|\zeta_\alpha|\lambda'}\|F\|_{\sigma,\Lambda}\,\frac{1}{\pi\,\sin(\varepsilon/2)}\int_{0}^{\infty} e^{-|\zeta_\alpha|s}\big(\mu + s\big)^{-\sigma}\,ds \\
&\leq e^{|\zeta_\alpha|\lambda'}\|F\|_{\sigma,\Lambda}\,\frac{1}{\pi\,\sin(\varepsilon/2)}\int_{0}^{\infty} e^{-|\zeta_\alpha|s}s^{-\sigma}\,ds
\end{aligned}$$

using the new parameter $s = \sin(\varepsilon/2)\,t$. Recognizing the integral in the last line as itself a Laplace transform, we have

$$\left|\left(\mathcal{L}^{\theta}_{\zeta_\alpha,0}\right)^{-1} F\right| \leq e^{|\zeta_\alpha|\lambda'}\|F\|_{\sigma,\Lambda}\,\frac{\Gamma(1-\sigma)}{\pi\,\sin(\varepsilon/2)}|\zeta_\alpha|^{\sigma-1},$$

which is what we wanted to show.

[Oops, looks like we can't keep Δ constant along the integration path if we want to recover something with a $|\zeta_\alpha|^{\sigma-1}$ singularity.] ∎

[We might be able to remove the decay properties completely!] The asymptotics of f at the starting point of $\Gamma^{\theta}_{\zeta,\alpha}$ control the asymptotics of $\mathcal{L}^{\theta}_{\zeta,\alpha} f$ at the infinite end of $\Gamma_{z,0}$. Once we see how this works for $\alpha = 0$, Section 3.1 will do the rest. In addition, we'll set $\theta = 0$ to keep the notation simpler, but the results hold for every angle θ. Let $F = \mathcal{L}_{\zeta,0} f$. Equation 1.8 of [66] shows[8] that

$$f \in O_{\zeta,0\leftarrow}(1) \quad \Longrightarrow \quad F \in O_{z,0\rightarrow}\left(\frac{1}{z}\right).$$

More generally, for $\sigma > -1$ [prove or cite],

$$f \in O_{\zeta,0\leftarrow}(\zeta^{\sigma}) \quad \Longrightarrow \quad F \in O_{z,0\rightarrow}\left(\frac{1}{z^{1+\sigma}}\right).$$

Exact power law asymptotics relate similarly [prove or cite]:

$$f \sim \zeta^{\tau} \text{ at the start of } \Gamma_{\zeta,0} \quad \Longrightarrow \quad F \sim \frac{\Gamma(1+\tau)}{z^{1+\tau}} \text{ at the end of } \Gamma_{z,0}.$$

[The big-O asymptotics dictionary is interesting, but we might not need it. Consider dropping.]

3.3 The geometry of the Borel transform

The Laplace transform $\mathcal{L}_{\zeta,0}$ acts in an especially simple way on powers of the coordinate ζ:

$$\mathcal{L}_{\zeta,0}\left[\frac{\zeta^n}{n!}\right] = z^{-n-1}.$$

Here, we're thinking of z as the standard coordinate on $T^*_{\zeta=0}B$, as described in Section 3.1.3. We can get a function on $T^*_{\zeta=b}B$ instead by taking the Laplace transform with respect to the coordinate ζ_b defined by $\zeta = \zeta_b + b$:

$$\mathcal{L}_{\zeta_b,0}\left[\frac{\zeta_b^n}{n!}\right] = z^{-n-1}.$$

On each cotangent space, we can define a formal inverse of the Laplace transform by turning negative powers of z back into powers of the appropriate translation coordinate. This formal inverse is called the *Borel transform*. To be more precise, the Borel transform $\mathcal{B}_{\zeta}$ on $T^*_{\zeta=0}B$ is the inverse of $\mathcal{L}_{\zeta,0}$ on monomials

$$\begin{array}{c} \{z^{-1}, z^{-2}, z^{-3}, z^{-4}, \ldots\} \\ {\scriptstyle \mathcal{L}_{\zeta,0}}\big\uparrow\big\downarrow{\scriptstyle \mathcal{B}_{\zeta}} \\ \{1, \zeta, \tfrac{1}{2!}\zeta^2, \tfrac{1}{3!}\zeta^3, \ldots\} \end{array}$$

[8]The argument cited still works in our generality. For holomorphic f, one can also use [68, Equation 1.5].

and it extends to formal power series by linearity

$$\begin{aligned} \mathcal{B}_\zeta \colon z^{-1}\mathbb{C}[\![z^{-1}]\!] &\to \mathbb{C}[\![\zeta]\!] \\ \sum_{n=0} a_n z^{-n-1} &\mapsto \sum_{n=0}^{\infty} a_n \frac{\zeta^n}{n!}. \end{aligned}$$

Remark 3.5. The definition of the Borel transform can be easily extended to fractional power of z: indeed notice that

$$\mathcal{L}_{\zeta,0}\big[\zeta^\sigma\big] = \Gamma(\sigma+1) z^{-\sigma-1}$$

for every $\sigma \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$. Thus

$$\mathcal{B}_\zeta(z^{-\sigma-1}) := \frac{\zeta^\sigma}{\Gamma(\sigma+1)}$$

and then extended by linearity to $z^{-\sigma}\mathbb{C}[\![z^{-1}]\!]$.

If a function φ belongs to $\mathcal{H}L^\infty_{0,\bullet}(\Omega)$, and its Taylor series

$$\tilde{\varphi} = a_1 + a_2\zeta + a_3\frac{\zeta^2}{2!} + a_4\frac{\zeta^3}{3!} + \dots$$

has an infinite radius of convergence, then the Taylor coefficients decay fast enough for us to take the Laplace transform term by term [52, Theorem 5.20]:

$$\begin{aligned} \mathcal{L}_{\zeta,0}\varphi &= \mathcal{L}_{\zeta,0}\left[\sum_{n=0}^{\infty} a_{n+1}\frac{\zeta^n}{n!}\right] \\ &= \sum_{n=0}^{\infty} a_{n+1}\,\mathcal{L}_{\zeta,0}\left[\frac{\zeta^n}{n!}\right] \\ &= \sum_{n=0}^{\infty} a_{n+1} z^{-n-1} \end{aligned}$$

As a result, we can recover the Taylor series $\tilde{\varphi}$ from the series

$$\tilde{\Phi} = a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4} + \dots$$

using the Borel transform:

$$\begin{aligned} \mathcal{B}_\zeta\tilde{\Phi} &= \mathcal{B}_\zeta\left[\sum_{n=0}^{\infty} a_{n+1} z^{-n-1}\right] \\ &= \sum_{n=0}^{\infty} a_{n+1}\frac{\zeta^n}{n!}. \end{aligned}$$

We took the Borel transform with respect to the coordinate ζ because $\tilde{\Phi}$ represents a holomorphic function on the cotangent space $T^*_{\zeta=0}B$.

If we'd expanded φ as a Taylor series φ_b in the translated coordinate ζ_b from above, we could have taken the Laplace transform $\mathcal{L}_{\zeta_b,0}$ instead, giving a power series in z^{-1} that represents a function on $T^*_{\zeta=b}B$. Using the Borel transform with respect to ζ_b, defined like the formal inverse of $\mathcal{L}_{\zeta_b,0}$

$$\begin{array}{c} \{z^{-1}, z^{-2}, z^{-3}, z^{-4}, \dots\} \\ \mathcal{L}_{\zeta_b,0}\Big\uparrow\Big\downarrow\mathcal{B}_{\zeta_b} \\ \{1, \zeta_b, \frac{1}{2!}\zeta_b^2, \frac{1}{3!}\zeta_b^3, \dots\} \end{array}$$

and extended to formal power series by linearity $\mathcal{B}_{\zeta_b} \colon z^{-1}\mathbb{C}[\![z^{-1}]\!] \to \mathbb{C}[\![\zeta_b]\!]$, we could again recover φ_b.

3.3.1 Action on transseries

A transseries is a generalization of usual power series, whose building blocks (called trans-monomials) are of the form $e^{\alpha_n z}\varphi_n$ where α_n is a sequence of complex numbers such that $0 = \alpha_0 e^{-i\theta} < \alpha_1 e^{-i\theta} < ... < \alpha_n e^{-i\theta} < ...$ and $\varphi_n \in \mathbb{C}[\![z^{-1}]\!]$ for every n:

$$T_{\alpha,\varphi} := \sum_{n=0}^{\infty} e^{\alpha_n z}\varphi_n(z)$$

The main feature of $T_{\alpha,\varphi}$ is that its asymptotics as $\operatorname{Re} e^{i\theta}z \to +\infty$ is φ_0. In fact, they play a crucial role in the theory of ODEs with irregular singularities.

It's natural to ask whether we can extend the definition of $\mathcal{B}_\zeta$ to transseries, and we'll achieve it using the relationship between $\mathcal{L}_{\zeta,\alpha}$ and $\mathcal{L}_{\zeta_\alpha,0}$ in identity (13).

Definition 3.6. The action of $\mathcal{B}_\zeta$ on trans-monomials is the formal inverse of $\mathcal{L}_{\zeta,\alpha}$

$$\mathcal{L}_{\zeta,\alpha}\mathcal{B}_\zeta\left[e^{-z\alpha}z^{-n-1}\right] = e^{-z\alpha}z^{-n-1}$$

and it extends by linearity to transseries.

In particular, from identity (13) we deduce

$$\begin{aligned} e^{-z\alpha}z^{-n-1} &= e^{-z\alpha}\mathcal{L}_{\zeta_\alpha,0}\mathcal{B}_\zeta\left[e^{-\zeta\alpha}z^{-n-1}\right] \\ z^{-n-1} &= \mathcal{L}_{\zeta_\alpha,0}\mathcal{B}_\zeta\left[e^{-\zeta\alpha}z^{-n-1}\right] \end{aligned}$$

and taking the inverse of $\mathcal{L}_{\zeta_\alpha,0}$ we find

$$\mathcal{B}_\zeta\left[e^{-z\alpha}z^{-n-1}\right] = \frac{\zeta_\alpha^n}{n!}.$$

In other words, the following diagram

$$\begin{tikzcd} \{z^{-1}, z^{-2}, z^{-3}, z^{-4}, \dots\} \arrow[r, "e^{-z\alpha}"] \arrow[d, "\mathcal{B}_\zeta"'] & e^{-\alpha z}\{z^{-1}, z^{-2}, z^{-3}, z^{-4}, \dots\} \arrow[d, shift left, "\mathcal{B}_\zeta"] \\ \{1, \zeta, \zeta^2, \zeta^3, \dots\} \arrow[r, "\mathsf{T}^*_{-\alpha}"'] & \{1, \zeta_\alpha, \zeta_\alpha^2, \zeta_\alpha^3, \dots\} \arrow[u, shift left, "\mathcal{L}_{\zeta,\alpha}"] \arrow[ul, "\mathcal{L}_{\zeta_\alpha,0}"] \end{tikzcd}$$

commutes, where $\mathsf{T}_{-\alpha}$ denotes traslation by $-\alpha$. Notice that the functions of the varible z are meant to be on the fiber $T^*_{\zeta=0}B$.

3.3.2 Change of translation chart

We'll show that the Borel transform is compatible with the change of translation chart for the Laplace transform in Section 3.1.7. To be more precise, we'll show that the diagram

$$\begin{tikzcd}
e^{-zb}\{z^{-1}, z^{-2}, z^{-3}, z^{-4}, \dots\} \arrow[rr, "e^{zb}"] & & \{z^{-1}, z^{-2}, z^{-3}, z^{-4}, \dots\} \arrow[d, "\mathcal{B}_{\zeta_b}"] \\
\{1, \zeta, \zeta^2, \zeta^3, \dots\} \arrow[u, "\mathcal{L}_{\zeta,b}"] \arrow[rr, "\text{change chart}"'] & & \{1, \zeta_b, \zeta_b^2, \zeta_b^3, \dots\}
\end{tikzcd}$$

commutes, where the functions of z are functions on the fiber $T^*_{\zeta=b}B$.

Proof. We want to show that

$$\mathcal{B}_{\zeta_b}\left[e^{zb}\mathcal{L}_{\zeta,b}\big[\zeta^n\big]\right] = \zeta^n.$$

Recall that $\mathcal{B}_{\zeta_b}$ is the formal inverse of $\mathcal{L}_{\zeta_b,0}$ on the cotangent fibre over b. Thus, taking the Borel transform on both side of the identity (13) we find

$$\begin{aligned}
\mathcal{B}_{\zeta_b}\left[e^{zb}\mathcal{L}_{\zeta,b}\big[\zeta^n\big]\right] &= \mathcal{B}_{\zeta_b}\mathcal{L}_{\zeta_b,0}\big[\zeta^n\big] \\
&= \mathcal{B}_{\zeta_b}\mathcal{L}_{\zeta_b,0}\left[\sum_{k=0}^{n}\binom{n}{k}\zeta_b^k b^{n-k}\right] \\
&= \sum_{k=0}^{n}\binom{n}{k} b^{n-k}\mathcal{B}_{\zeta_b}\mathcal{L}_{\zeta_b,0}\big[\zeta_b^k\big] \\
&= \sum_{k=0}^{n}\binom{n}{k} b^{n-k}\zeta_b^k \\
&= \zeta^n.
\end{aligned}$$

□

3.3.3 Action on translations in the frequency domain

Since $\mathcal{B}_\zeta$ is the formal inverse of $\mathcal{L}_{\zeta,0}$, we can deduce how $\mathcal{B}_\zeta$ acts on translations in the frequency domain, namely we can show that for $\varphi = \sum_{n=0}^{\infty} a_n z^{-n-1}$

$$\mathcal{B}_\zeta \mathsf{T}^*_{-c}\varphi = e^{-c\zeta}\mathcal{B}_\zeta\varphi.$$

Indeed, the following diagram

$$\begin{tikzcd}
\{z^{-1}, z^{-2}, z^{-3}, z^{-4}, \dots\} \arrow[r, "\mathsf{T}^*_{-c}"] \arrow[d, shift left, "\mathcal{B}_\zeta"] & \{(z+c)^{-1}, (z+c)^{-2}, (z+c)^{-3}, (z+c)^{-4}, \dots\} \arrow[d, shift left, "\mathcal{B}_\zeta"] \\
\{1, \zeta, \zeta^2, \zeta^3, \dots\} \arrow[u, shift left, "\mathcal{L}_{\zeta,0}"] \arrow[r, "e^{-c\zeta}"'] & e^{-c\zeta}\{1, \zeta, \zeta^2, \zeta^3, \dots\} \arrow[u, shift left, "\mathcal{L}_{\zeta,0}"]
\end{tikzcd}$$

commutes, where the functions of the variable z belong to the fiber $T^*_{\zeta=0}B$. If z is a function on the fiber over $\zeta = b$, then it's enough to replace $\mathcal{B}_\zeta$ with $\mathcal{B}_{\zeta_b}$ and to use $\mathcal{L}_{\zeta_b,0}$ as the inverse.

3.4 Borel and Laplace transforms as algebra homomorphisms

The space of formal power series $\mathbb{C}[\![z^{-1}]\!]$ is a $\mathbb{C}$-algebra with unit 1, thus by definition the Borel transform is a $\mathbb{C}$-algebra homomorphisms between $\mathbb{C}[\![z^{-1}]\!]$ and $\mathbb{C}\delta + \mathbb{C}[\![\zeta]\!]$, where $\delta := \mathcal{B}_\zeta(1)$ denotes the formal unit on $\mathbb{C}[\![\zeta]\!]$.

In fact, since $\mathcal{B}_\zeta$ is deifned as the formal inverse of $\mathcal{L}_{\zeta,0}$ we can introduce the *formal Laplace transform* as the map from $\mathbb{C}[\![\zeta]\!]$ to $z^{-1}\mathbb{C}[\![z^{-1}]\!]$ which acts as $\mathcal{L}_{\zeta,0}$ on monomials ζ^k and then it extends by linearity. As a result, we find that the

$$\mathcal{B}_\zeta \colon \mathbb{C}[\![z^{-1}]\!] \to \mathbb{C}\delta + \mathbb{C}[\![\zeta]\!]$$

is a $\mathbb{C}$-algebra isomorphism whose inverse is the formal Laplace transform.

However, as we discussed in Section 3.2.1, the Laplace transform $\mathcal{L}_{\zeta,0}$ is naturally defined on the space of integrable, holomorphic functions, namely $\mathcal{H}L^\infty_{\sigma,\bullet}(\Omega) \subset \zeta^\sigma\mathbb{C}\{\zeta\}$ for $\sigma > -1$, hence we might expect it extends to a $\mathbb{C}$-algebra homomorphism from $\mathcal{H}L^\infty_{\sigma,\bullet}(\Omega)$ to $\mathcal{O}(H)$. Indeed, that's the case, but it requires to introduce the convolution product on the space of analytic functions $\zeta^{\sigma-1}\mathbb{C}\{\zeta\}$. We'll discuss it in Section 3.4.1.

Consequently, also at the formal level we can restric to a subalgebra of $\mathbb{C}[\![z^{-1}]\!]$ whose image under $\mathcal{B}_\zeta$ lies in $\mathbb{C}\{\zeta\}$. This subalgebra is the so called space of 1-Gevrey series $\mathbb{C}[\![z^{-1}]\!]_1 \subset \mathbb{C}[\![z^{-1}]\!]$ and $\mathcal{B}_\zeta \colon \mathbb{C}[\![z^{-1}]\!]_1 \to \mathbb{C}\delta + \mathbb{C}\{\zeta\}$ is a $\mathbb{C}$-algebra homomorphism (see Section 3.4.2).

Finally in Section 3.4.3, we'll consider the action of $\mathcal{B}_\zeta$ and $\mathcal{L}_{\zeta,0}$ with respect to the action of derivatives both in the frequency and the position domain, and fractional integrals/derivatives $\partial^\lambda_{\zeta,0}$ in the position domain.

We end this paragraph introducing the notation we'll be using in the following.

Notation. *We want to distinguish between formal series and holomorphic functions, as well as between the spatial domain (Borel plane) and the frequency domain (z-plane). Therefore we adopt the following notation:*

- Φ*: upper-case letters are holomorphic functions on the frequency domain;*
- $\tilde{\Phi}$*: tilde stands for formal series, so an upper-case letter with tilde is a formal series in the frequency domain;*
- ϕ*: lower-case letters are holomorphic functions on the frequency domain. Since the Laplace transform* $\mathcal{L}_{\zeta,0}$ *turns functions on the spatial domain into functions on the frequency domain, we might write* $\mathcal{L}_{\zeta,0}\phi = \Phi$*.*
- $\tilde{\phi}$*: lower-case letter with tilde are formal series in the Borel plane. As we'll see, the Borel transform of* $\tilde{\Phi}$ *is* $\mathcal{B}_\zeta\tilde{\Phi} =: \tilde{\phi}$*;*
- $\hat{\phi}$*: lower-case letters with hat are the sum of the formal series* $\tilde{\phi}$*, when it exists.*

3.4.1 The convolution product

Let $\tilde{\phi}, \tilde{\psi} \in \mathbb{C}\{\zeta\}$ be two convergent formal series in a neighbourhood of $\zeta = 0$, and let $\hat{\phi}$ and $\hat{\psi}$ be respectively their sum in a common region of convergence, say $|\zeta| < R$. Then the convolution product $\hat{\phi} * \hat{\phi}$ is defined as

$$\hat{\phi} * \hat{\psi} := \int_0^{\zeta} \hat{\phi}(\zeta - \zeta')\hat{\psi}(\zeta')\, d\zeta' \tag{14}$$

and it admits a convergent power series expansion for $|\zeta| < R$ (see [52, Lemma 5.14]).

In addition, we can relax the assumption on $\tilde{\phi}, \tilde{\psi}$ by allowing integrable singularties at $\zeta = 0$, namely $\tilde{\phi} \in \zeta^{\sigma}\mathbb{C}\{\zeta\}$ and $\tilde{\psi} \in \zeta^{\tau}\mathbb{C}\{\zeta\}$ with $\sigma, \tau > -1$. In this case, $\hat{\phi}$ and $\hat{\psi}$ will be respectively the sum of the series $\tilde{\phi}$ and $\tilde{\psi}$ in an open domain $\Omega \cap \{|\zeta| < R\}$ (that may not contain $\zeta = 0$). In fact, this leads to the definition of the convolution product for functions in $\mathcal{H}L^{\infty}_{\sigma,\Lambda}(\Omega)$:

Lemma 3.7. *Let $\phi, \psi \in \mathcal{H}L^{\infty}_{\sigma,\Lambda}(\Omega)$ for $\sigma > -1$, then $\phi * \psi \in \mathcal{H}L^{\infty}_{\sigma,\Lambda}(\Omega)$.*

Proof.

$$\begin{aligned}
|\tilde{\phi} * \tilde{\psi}| &= \left| \int_0^{\zeta} \tilde{\phi}(\zeta - \zeta')\tilde{\psi}(\zeta')\, d\zeta' \right| \\
&\leq \int_0^{\zeta} |\tilde{\phi}(\zeta - \zeta')||\tilde{\psi}(\zeta')|\, d\zeta' \\
&\leq ||\tilde{\psi}||_{\sigma,\Lambda} \int_0^{\zeta} \tilde{\phi}(\zeta - \zeta')|\zeta'|^{\sigma} e^{\Lambda|\zeta'|}\, d\zeta' \\
&\leq ||\tilde{\psi}||_{\sigma,\Lambda}|\zeta|^{\sigma} e^{\Lambda|\zeta|} \int_0^{\zeta} \tilde{\phi}(\zeta - \zeta')|\zeta - \zeta'|^{\sigma} e^{\Lambda|\zeta - \zeta'|}\, d\zeta' \\
&\leq |\zeta|^{\sigma} e^{\Lambda|\zeta|} ||\tilde{\psi}||_{\sigma,\Lambda}||\tilde{\phi}||_{\sigma,\Lambda}.
\end{aligned}$$

□

In addition, we can show that $\mathcal{L}_{\zeta,0}$ is a $\mathbb{C}$-algebra homomorphism from $\mathcal{H}L^{\infty}_{\sigma,\Lambda}(\Omega)$ to the space of holomorphic functions on the right-half-plane $\operatorname{Re} z > 0$.

Proposition 3.8. *Let $\sigma > -1$ and $\Lambda > 0$. The Laplace transform $\mathcal{L}_{\zeta,0}$ is $\mathbb{C}$-algebra homomorphism from $(\mathcal{H}L^{\infty}_{\sigma,\Lambda}(\Omega), *)$ to to the space of holomorphic functions on the right-half-plane $\operatorname{Re} z > 0$ with the standard multiplication of functions.*

Proof. It is enough to show that

$$\mathcal{L}_{\zeta,0}\big[\phi * \psi\big] = \mathcal{L}_{\zeta,0}\big[\phi\big]\mathcal{L}_{\zeta,0}\big[\psi\big],$$

which indeed follows from simple integral computations (see for instance [66, Theorem 2.39]). □

Remark 3.9. it is possible to define an inverse Laplace transform, so that $\mathcal{L}_{\zeta,0}$ becomes an algebra isomorphisms. However we'll refer to [standard reference] for details. [see also section on Regularity for Laplace]

3.4.2 Borel transforms on 1-Gevrey series

So far, our discussion on the Borel transform was at the level of formal series, however there is a special class of formal series that behaves well under Borel transform, meaning that their Borel transform gives a germ of holomorphic functions at $\zeta = 0$. These formal series are the so-called 1-Gevrey series:

$$\tilde{\Phi} = \sum_{n \geq 0} a_n z^{-n} \in \mathbb{C}[\![z^{-1}]\!]_1$$

is 1-Gevrey if there exists $A > 0$ such that $|a_n| \lesssim A^n n!$ for all $n \geq 0$.[9]

Lemma 3.10. *Let $\tilde{\Phi} \in \mathbb{C}[\![z^{-1}]\!]$. Then, $\tilde{\Phi} \in \mathbb{C}[\![z^{-1}]\!]_1$ is 1-Gevrey if and only if $\mathcal{B}_\zeta \tilde{\Phi} \in \mathbb{C}\{\zeta\}$ is a germ of holomorphic functions.*

Let $\tilde{\Phi}, \tilde{\Psi} \in \mathbb{C}[\![z^{-1}]\!]_1$ (assuming they are series of z on the fiber over $\zeta = 0$), then following [52, Definition 5.12], the convolution product $*$ of $\tilde{\phi} := \mathcal{B}_\zeta \tilde{\Phi}$ and $\tilde{\psi} := \mathcal{B}_\zeta \tilde{\Psi}$ is defined as

$$\mathcal{B}_\zeta(\tilde{\Psi}\tilde{\Phi}) =: \tilde{\psi} * \tilde{\phi}.$$

Furthermore, since $\tilde{\phi}$ and $\tilde{\psi}$ are convergent formal series, it follows that the sum of $\tilde{\psi} * \tilde{\phi}$ is $\hat{\psi} * \hat{\phi}$ (see the discussion in Section 3.4.1).

3.4.3 Action on fractional derivatives and fractional integrals

As we'll see in Section 5, it is useful to study the action of the Laplace and Borel transform on both fractional integral and fractional derivative operators.

Definition 4. Let $\nu \in (-\infty, 1)$, the *fractional integral* $\partial_{\zeta,b}^{\nu-1}$ is defined by

$$[\partial_{\zeta,b}^{\nu-1} f](p) := \frac{1}{\Gamma(1-\nu)} \int_{\zeta=b}^{p} \big(\zeta(p) - \zeta\big)^{-\nu} f \, d\zeta,$$

where p is any point on the position domain B.[10]

Notice that the fractional integral obeys the expected semigroup law [53, Section 1.3]

$$\partial_{\zeta,b}^{\lambda} \, \partial_{\zeta,b}^{\mu} = \partial_{\zeta,b}^{\lambda+\mu} \qquad \lambda, \mu \in (-\infty, 0),$$

and agrees with ordinary repeated integration when ν is an integer (see [53, Equation 35]).

For $\alpha \in (0,1)$ and integers $n \geq 0$, fractional derivatives $\partial_{\zeta,0}^{n+\alpha}$ are defined by composing $\partial_{\zeta,0}^{\alpha-1}$ with powers of $\frac{\partial}{\partial \zeta}$. However, $\partial_{\zeta,0}^{\alpha-1}$ and $\frac{\partial}{\partial \zeta}$ don't commute [53, equation 54]. Various ordering conventions give various definitions of $\partial_{\zeta,0}^{n+\alpha} f$, which differ by operators that act on the germ of f at zero (see [53, Section 1.3]—original source [60]). We'll use the *Riemann-Liouville* convention.

[9]In asymptotic analysis k-Gevrey series $\sum_{n \geq 0} a_n z^{-n}$ have coefficients that grow as $(n!)^k$, i.e. there exists $A > 0$ such that $|a_n| \leq A^n (n!)^k$ for every $n \geq 0$. The Borel transform can be generalized in order to obtain germs of holomorphic function for higher Gevrey series.

[10]For completeness we also give the definition of the fractional integral $\partial_{\zeta_b,0}^{-\lambda}$:

$$\partial_{\zeta_b,0}^{-\lambda} \varphi(p) := \frac{1}{\Gamma(\lambda)} \int_{\zeta_b=0}^{p} (\zeta_b(p) - \zeta_b)^{\lambda-1} \varphi(\zeta_b + b) \, d\zeta_b.$$

Definition 3.11. For $\alpha \in (0,1)$ and integers $n \geq 0$, the *Riemann-Liouville fractional derivative* $\partial_{\zeta,0}^{n+\alpha}$ is defined by

$$\partial_{\zeta,0}^{n+\alpha} := \left(\tfrac{\partial}{\partial\zeta}\right)^{n+1} \partial_{\zeta,0}^{\alpha-1}.$$

The symbol $\partial_{\zeta,0}^{\mu}$ is now defined for any $\mu \in \mathbb{R} \smallsetminus \{0,1,2,3,\ldots\}$. It denotes a fractional integral when μ is negative, and a fractional derivative when μ is a positive non-integer.

The Riemann-Liouville fractional derivative is a left inverse of the fractional integral, in the sense that $\partial_{\zeta,0}^{\lambda}\,\partial_{\zeta,0}^{-\lambda} = \mathrm{Id}$ for all $\lambda \in (0,\infty)$. This extends the semigroup law:

$$\partial_{\zeta,0}^{\lambda}\,\partial_{\zeta,0}^{\mu} = \partial_{\zeta,0}^{\lambda+\mu} \qquad\qquad \lambda \in \mathbb{R} \smallsetminus \{0,1,2,3,\ldots\}, \quad \mu \in (-\infty,0).$$

Proposition 3.12. *Let $\phi \in \mathcal{H}L^{\infty}_{\sigma,\Lambda}(\Omega)$ with $\sigma > -1$ and $\Lambda > 0$, then*

(i) [fractional derivative] *for every $\mu \in (0,1)$, $\mathcal{L}_{\zeta,0}\big[\partial_{\zeta,0}^{\mu}\phi\big] = z^{\mu}\mathcal{L}_{\zeta,0}\phi$;*

(ii) [fractional integral] *$\mathcal{L}_{\zeta,0}\big[\partial_{\zeta,0}^{\lambda}\phi\big] = z^{\lambda}\mathcal{L}_{\zeta,0}\phi$, for every $\lambda \in \mathbb{R}_{<0}$;*

(iii) [derivative] *let $n \in \mathbb{Z}_{>0}$ and assume that $\phi^{(k)}(0) = 0$ for every $k = 0,\ldots,n$. Then $\mathcal{L}_{\zeta,0}\big[\big(\tfrac{\partial}{\partial\zeta}\big)^{n}\phi\big] = z^{n}\mathcal{L}_{\zeta,0}\phi$;*

(iv) *$\mathcal{L}_{\zeta,0}\big[\zeta^{n}\phi\big] = \big(-\tfrac{\partial}{\partial z}\big)^{n}\mathcal{L}_{\zeta,0}\phi$, for every $n \in \mathbb{Z}_{\geq 0}$.*

Proof. The result of *(i)* and *(ii)* follow from the properties of the Laplace transform with respect to the convolution product: for every $\tau \in \mathbb{R}_{<0} \cup (0,1)$

$$\begin{aligned} z^{\tau}\mathcal{L}_{\zeta,0}\phi &= \mathcal{L}_{\zeta,0}\Big[\frac{\zeta^{-\tau-1}}{\Gamma(-\tau)}\Big]\mathcal{L}_{\zeta,0}\phi \\ &= \mathcal{L}_{\zeta,0}\Big[\frac{\zeta^{-\tau-1}}{\Gamma(-\tau)} * \phi\Big] \\ &= \mathcal{L}_{\zeta,0}\int_0^{\zeta} \frac{(\zeta-\zeta')^{-\tau-1}}{\Gamma(-\tau)}\phi(\zeta')d\zeta' \\ &= \mathcal{L}_{\zeta,0}\partial_{\zeta,0}^{\tau}\phi \end{aligned}$$

The result of (iii) follows by repetedly using integration by part and the fact that boundary terms vanish:

$$\begin{aligned} z^{n}\mathcal{L}_{\zeta,0}\phi &= \int_0^{\infty} e^{-z\zeta} z^{n}\phi\, d\zeta \\ &= (-1)^{n}\int_0^{\infty} \partial_{\zeta}^{n}\big[e^{-z\zeta}\big]\phi\, d\zeta \\ &= \mathcal{L}_{\zeta,0}\big[\partial_{\zeta}^{n}\phi\big]. \end{aligned}$$

Finally, the proof of (iv) follows by integration by parts and the fact that $\zeta^{n}\phi \in \mathcal{H}L^{\infty}_{\sigma,\Lambda}(\Omega)$. ☐

Lemma 3.13. *For any non-integer $\mu \in (0,\infty)$ and any integer $k \geq 0$,*

$$\partial_{\zeta}^{\mu}\left[\mathcal{B}_{\zeta}\left(z^{-(k+1)}\right)\right] = \mathcal{B}_{\zeta}\left(z^{\mu}z^{-(k+1)}\right).$$

Proof. We'll show that for any $\alpha \in (0,1)$ and any integer $n \geq 0$, the claim holds with $\mu = n + \alpha$. First, evaluate

$$\begin{aligned}
\partial_\zeta^{\alpha-1}\left[\mathcal{B}_\zeta\left(z^{-(k+1)}\right)\right] &= \frac{1}{\Gamma(1-\alpha)}\int_0^\zeta (\zeta-\zeta')^{-\alpha}\frac{{\zeta'}^k}{\Gamma(k+1)}\,d\zeta' \\
&= \frac{1}{\Gamma(1-\alpha)\,\Gamma(k+1)}\int_0^1 (\zeta-\zeta t)^{-\alpha}(\zeta t)^k\,\zeta\,dt \\
&= \frac{\zeta^{k-(\alpha-1)}}{\Gamma(1-\alpha)\,\Gamma(k+1)}\int_0^1 (1-t)^{-\alpha}t^k\,dt \\
&= \frac{\zeta^{k-(\alpha-1)}}{\Gamma\big(k-(\alpha-1)+1\big)}
\end{aligned}$$

by reducing the integral to Euler's beta function (see [45, Identity 5.12.1]). This establishes that

$$\left(\tfrac{\partial}{\partial\zeta}\right)^{n+1}\partial_\zeta^{\alpha-1}\left[\mathcal{B}_\zeta\left(z^{-(k+1)}\right)(\zeta)\right] = \frac{\zeta^{k-(n+\alpha)}}{\Gamma\big(k-(n+\alpha)+1\big)} \tag{15}$$

for $n = -1$. If (15) holds for $n = m$, it also holds for $n = m+1$, because

$$\begin{aligned}
\tfrac{\partial}{\partial\zeta}\left(\tfrac{\partial}{\partial\zeta}\right)^{m+1}\partial_\zeta^{\alpha-1}\left[\mathcal{B}_\zeta\left(z^{-(k+1)}\right)(\zeta)\right] &= \tfrac{\partial}{\partial\zeta}\left(\frac{\zeta^{k-(m+\alpha)}}{\big(k-(m+\alpha)\big)\,\Gamma\big(k-(m+\alpha)\big)}\right) \\
&= \frac{\zeta^{k-(m+1+\alpha)}}{\Gamma\big(k-(m+\alpha)\big)}
\end{aligned}$$

Hence, (15) holds for all $n \geq -1$, and the desired result quickly follows. The condition $\alpha \in (0,1)$ saves us from the trouble we'd run into if $k-(m+\alpha)$ were in $\mathbb{Z}_{\leq 0}$. This is how we avoid the initial value corrections that appear in ordinary derivatives of Borel transforms. ☐

We can now prove the properties of the Borel transform analogous to the one of the Laplace transform from Proposition 3.12.

Proposition 3.14. *Let* $\tilde{\Phi} \in \mathbb{C}[\![z^{-1}]\!]_1$

(i) [fractional derivative] $\mathcal{B}_\zeta\left[z^\mu\,\tilde{\Phi}\right] = \partial_{\zeta,0}^\mu\,\mathcal{B}_\zeta\tilde{\Phi}$, *for every* $\mu \in \mathbb{R}_{>0}$*;*

(ii) [fractional integral] $\mathcal{B}_\zeta\left[z^\lambda\,\tilde{\Phi}\right] = \partial_{\zeta,0}^\lambda\,\mathcal{B}_\zeta\tilde{\Phi}$, *for every* $\lambda \in \mathbb{R}_{<0}$*;*

(iii) let $k \in \mathbb{Z}_{\geq 0}$ *and* $\tilde{\Phi} \in z^{-k-1}[\![z^{-1}]\!]_1$*, then* $\mathcal{B}_\zeta\left[\partial_z^k\tilde{\Phi}\right] = (-\zeta)^k\mathcal{B}_\zeta\tilde{\Phi}$.[11]

Proof. The proof of (i) follows from Lemma 3.13.

Then, to prove (ii), first notice that for $\lambda \in \mathbb{R}_{<0}$

$$\partial_{\zeta,0}^\lambda\zeta^k = \zeta^{k-\lambda}\frac{k!}{\Gamma(k-\lambda+1)}.$$

[11]This property holds more generally for $\tilde{\Phi} \in \mathbb{C}[\![z^{-1}]\!]$, without restricting to 1-Gevrey series.

Hence, if $\tilde{\Phi} = \sum_{n\geq 0} a_n z^{-n}$

$$\mathcal{B}_\zeta\left[z^\lambda \tilde{\Phi}\right] = \sum_{n\geq 0} a_n \frac{\zeta^{n-\lambda}}{\Gamma(n-\lambda+1)} = \sum_{n\geq 0} a_n \frac{\zeta^{n-\lambda}}{n!}\frac{n!}{\Gamma(n-\lambda+1)} = \sum_{n\geq 0} \frac{a_n}{n!}\,\partial^\lambda_{\zeta,0}\zeta^n = \partial^\lambda_\zeta \mathcal{B}_\zeta \tilde{\Phi}$$

where in the last step, we use that $\mathcal{B}_\zeta\tilde{\Phi}$ is convergent.

Finally, (iii) follows from a simple computation:

$$\begin{aligned}
\mathcal{B}_\zeta\left[\partial_z^k \tilde{\Phi}\right] &= \mathcal{B}_\zeta\left[\sum_{n=k+1}^{\infty} a_n \frac{(n+k-1)!}{(n-1)!}(-1)^k z^{-n-k}\right] \\
&= (-1)^k \sum_{n=k+1}^{\infty} a_n \frac{(n+k-1)!}{(n-1)!}\frac{\zeta^{n+k-1}}{(n+k-1)!} \\
&= (-\zeta)^k \sum_{n=k+1}^{\infty} a_n \frac{\zeta^{n-1}}{(n-1)!} \\
&= (-\zeta)^k \mathcal{B}_\zeta \tilde{\Phi}
\end{aligned}$$

□

Remark 3.15. We notice that properties (i) and (ii) of Proposition 3.14 are special cases of the convolution product: for every $\tau \in \mathbb{R}\setminus\mathbb{Z}_{>0}$

$$\partial^\tau_{\zeta,0}\left[\mathcal{B}_\zeta\tilde{\Phi}\right] = \mathcal{B}_\zeta\left[z^\tau\tilde{\Phi}\right] = \mathcal{B}_\zeta(z^\tau) * \mathcal{B}_\zeta\tilde{\Phi}.$$

$$\begin{aligned}
\partial^\tau_{\zeta,0}\mathcal{B}_\zeta\tilde{\Phi} = \mathcal{B}_\zeta(z^\tau\tilde{\Phi}) &= \mathcal{B}_\zeta(z^\tau) * \mathcal{B}_\zeta\tilde{\Phi} \\
&= \frac{\zeta^{-\tau-1}}{\Gamma(-\tau)} * \sum_{n=0}^{\infty} a_n \frac{\zeta^n}{n!} \\
&= \int_0^\zeta \frac{(\zeta')^{-\tau-1}}{\Gamma(-\tau)} \sum_{n=0}^{\infty} a_n \frac{(\zeta-\zeta')^n}{n!} d\zeta' \\
&= \sum_{n=0}^{\infty} \frac{a_n}{n!}\frac{1}{\Gamma(-\tau)}\zeta^{n-\tau}\frac{\Gamma(n+1)\Gamma(-\tau)}{\Gamma(n-\tau+1)}
\end{aligned}$$

Recall from Lemma 3.13 that for every $\tau > 0$ and $\tau \notin \mathbb{Z}_{>0}$,

$$\frac{\zeta^{n-\tau}}{\Gamma(n-\tau+1)} = \partial^\tau_{\zeta,0}\left[\frac{\zeta^n}{n!}\right]$$

and if $\tau < 0$, by definition

$$\partial^\tau_{\zeta,0}\left[\frac{\zeta^n}{n!}\right] = \frac{1}{\Gamma(-\tau)}\int_0^\zeta (\zeta')\frac{(\zeta-\zeta')^n}{n!}d\zeta'$$

therefore, we deduce $\partial^\tau_{\zeta,0}\mathcal{B}_\zeta\tilde{\Phi} = \frac{\zeta^{\tau-1}}{\Gamma(-\tau)} * \mathcal{B}_\zeta\tilde{\Phi}$.

3.4.4 Compatibility of the convolution product with the geometry

The convolution product is compatible with the change of translation chart: if $\tilde{\Phi}, \tilde{\Psi} \in \mathbb{C}[\![z^{-1}]\!]_1$ be series on $T^*_{\zeta=b}B$ and let $\tilde{\psi} := \mathcal{B}_{\zeta_b}\tilde{\Psi}$ and $\tilde{\phi} := \mathcal{B}_{\zeta_b}\tilde{\Phi}$

$$\mathcal{B}_{\zeta_b}(\tilde{\Psi}\tilde{\Phi}) = \hat{\psi} * \hat{\phi} = \int_0^{\zeta_b} \hat{\psi}(\zeta_b - \zeta')\hat{\phi}(\zeta')d\zeta'$$

Furthermore, we can extend the previous definition to trans-series: let $\tilde{\Phi}, \tilde{\Psi} \in e^{-z\alpha}\mathbb{C}[\![z^{-1}]\!]_1$ be series on $T^*_{\zeta=0}B$, we define

$$\mathcal{B}_\zeta(\tilde{\Psi}\tilde{\Phi}) =: \tilde{\psi} *_\alpha \tilde{\phi}.$$

Then the sum of $\tilde{\psi} *_\alpha \tilde{\phi}$ can be explicitly written as

$$\hat{\psi} *_\alpha \hat{\phi} = \int_\alpha^\zeta d\zeta' \hat{\psi}(\zeta - \zeta')\hat{\phi}(\zeta') \tag{16}$$

where $\hat{\phi}$ and $\hat{\psi}$ denote respectively the sum of $\tilde{\phi}$ and $\tilde{\psi}$ in a neighbourhood of $\zeta = \alpha$.

We also introduce δ_α as the unit for the product $*_\alpha$

$$\delta_\alpha *_\alpha \mathcal{B}_\zeta(e^{-\alpha z} z^{-n-1}) := \mathcal{B}_\zeta(1 \cdot e^{-\alpha z} z^{-n-1}) = \mathcal{B}_\zeta \mathcal{L}_{\zeta,\alpha} \frac{\zeta_\alpha^n}{n!} = \frac{\zeta_\alpha^n}{n!},$$

where in the last step we used the properties of $\mathcal{B}_\zeta$ acting on transseries (see Section 3.3.1). More generally, if $\mathcal{B}_\zeta\tilde{\Phi}$ extends to an analytic function $\hat{\phi} \in \mathcal{H}L^\infty_{\sigma,\bullet}(\Omega_\alpha)$ for $\sigma > -1$, we deduce that $\delta_\alpha *_\alpha \hat{\phi} = \hat{\phi} *_\alpha \delta_\alpha = \hat{\phi}$.

Remark 3.16. We can express the fractional integrals/derivatives $\partial^\lambda_{\zeta,\alpha}$ and $\partial^\lambda_{\zeta_\alpha,0}$ as convolution products, and we'll see how the definitions vary depending on the fibers of T^*B in consideration.

We first assume $\Phi = \mathcal{L}_{\zeta,\alpha}\phi$ for $\phi \in \mathcal{H}L^\infty_{\sigma,\bullet}(\Omega_\alpha)$, and assume z is a function on $T^*_{\zeta=0}B$. Then, for every $\lambda \in (-1, \infty)$

$$\mathcal{B}_\zeta(z^{-\lambda}\Phi) = \mathcal{B}_\zeta(z^{-\lambda}\mathcal{L}_{\zeta,\alpha}\phi) = \mathcal{B}_\zeta(\mathcal{L}_{\zeta,\alpha}\partial^{-\lambda}_{\zeta,\alpha}\phi) = \partial^{-\lambda}_{\zeta,\alpha}\phi = \frac{1}{\Gamma(\lambda)}\int_\alpha^\zeta (\zeta - t)^{\lambda-1}\phi(t)dt = \mathcal{B}_\zeta z^{-\lambda} *_\alpha \phi$$

where we use the identity $\mathcal{L}_{\zeta,\alpha}\,\partial^{-\lambda}_{\zeta,\alpha}\varphi = z^{-\lambda}\mathcal{L}_{\zeta,\alpha}\varphi$, that generalizes properties (i) and (ii) in Proposition 3.12. Hence

$$\mathcal{B}_\zeta(z^{-\lambda}) *_\alpha \mathcal{B}_\zeta\Phi = \partial^{-\lambda}_{\zeta,\alpha}\left[\mathcal{B}_\zeta\Phi\right].$$

Then, we study the case when Φ and z are functions on the fiber over $\zeta = b$, and we assume $\Phi = \mathcal{L}_{\zeta_b,0}\phi$ for $\phi \in \mathcal{H}L^\infty_{\sigma,\bullet}(\Omega_b)$. For every $\lambda \in (-1, \infty)$

$$\begin{aligned}\mathcal{B}_{\zeta_b}\left[z^{-\lambda}\Phi\right] = \mathcal{B}_{\zeta_b}\left[z^{-\lambda}\mathcal{L}_{\zeta_b,0}\phi\right] = \mathcal{B}_{\zeta_b}\left[\mathcal{L}_{\zeta_b,0}\partial^{-\lambda}_{\zeta_b,0}\phi\right] = \partial^{-\lambda}_{\zeta_b,0}\phi = \frac{1}{\Gamma(\lambda)}\int_0^{\zeta_b} (\zeta_b - t)^{\lambda-1}\phi(t)dt \\ = \mathcal{B}_{\zeta_b}(z^{-\lambda}) * \phi\end{aligned}$$

hence,

$$\partial^{-\lambda}_{\zeta_b,0}\left[\mathcal{B}_{\zeta_b}\Phi\right] = \mathcal{B}_{\zeta_b}(z^{-\lambda}) * \mathcal{B}_{\zeta_b}\Phi.$$

4 Proof of main results

4.1 Borel regularity for ODEs

[Explain the contents of the section]

4.1.1 Regularity results

Let's recall the setting from Section 1.2.2. Let $\mathcal{P}$ be a linear differential operator of the form

$$\mathcal{P} = P(\partial_z) + \frac{1}{z}Q(\partial_z) + \frac{1}{z^2}R(z^{-1}), \tag{17}$$

where

1. P is a monic degree-d polynomial whose roots are all simple;
2. Q is a degree-$(d-1)$ polynomial that's non-zero at every root of P;
3. $R(z^{-1})$ is holomorphic in some disk $|z| > A$ around $z = \infty$. In particular, the power series

   $$R(z^{-1}) = \sum_{j=0}^{\infty} R_j z^{-j}$$

   converges in the region $|z| > A$.

[Discuss terminology, like in `reg-sing-volterra`?]

Theorem 4.1 (restatement of 1.1)**.** *Choose a root $-\alpha$ of P. Choose an open sector Ω_α which has an opening angle of π or less, has $\zeta = \alpha$ at its tip, and doesn't touch any other root of $P(-\zeta)$. The equation $\mathcal{P}\Psi = 0$ has a unique solution Ψ_α in the affine subspace*

$$z^{-\tau_\alpha} + \widehat{\mathcal{H}L}^{\infty}_{-\tau_\alpha - 1, \bullet}(\Omega_\alpha)$$

of the space $\widehat{\mathcal{H}L}^{\infty}_{-\tau_\alpha, \bullet}(\Omega_\alpha)$ from Section 3.2.1.

Proof. After rescaling by a constant, Theorem 4 of [30] tells us that the equation $\hat{\mathcal{P}}_\alpha \psi_\alpha = 0$ has a unique solution ψ_α in the affine subspace

$$\frac{\zeta^{\tau_\alpha - 1}}{\Gamma(\tau_\alpha)} + \mathcal{H}L^{\infty}_{\tau_\alpha, \bullet}(\Omega_\alpha)$$

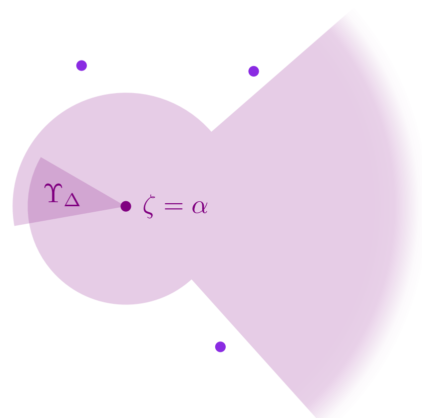
of the space $\mathcal{H}L^{\infty}_{\tau_\alpha - 1, \bullet}(\Omega_\alpha)$. The same is true on any smaller sector created by shaving a sector off each edge of Ω_α. It follows, by the results of Section 3.2.1 [the domain Ω_α needs to be a sector for this], that the equation $\mathcal{P}\Psi_\alpha = 0$ has a unique solution Ψ_α in the affine subspace

$$z^{-\tau_\alpha} + \widehat{\mathcal{H}L}^{\infty}_{-\tau_\alpha - 1, \bullet}(\Omega_\alpha)$$

of the space $\widehat{\mathcal{H}L}^{\infty}_{-\tau_\alpha, \bullet}(\Omega_\alpha)$. $\square$

[...] Our arguments in [30] took place on $\mathbb{C}$, but they work work just as well on a covering of $\mathbb{C}$ which is branched at the roots of $P(-\zeta)$. Let's use the covering $\pi \colon \widetilde{B}_{P(-\zeta)} \to \mathbb{C}$ built by puncturing $\mathbb{C}$ at the roots of $P(-\zeta)$, taking the universal cover, and then gluing the roots back in as infinite-order branch points.

Lemma 4.1. *Choose a root $-\alpha$ of P. Consider a simply connected open set $\Upsilon \subset \widetilde{B}_{P(-\zeta)}$ which touches but doesn't contain $\zeta = \alpha$, is star-shaped around $\zeta = \alpha$, and doesn't touch any other root of $P(-\zeta)$. Suppose that $\pi \colon \Upsilon \to \mathbb{C}$ overlaps itself at the edges: it's two-to-one over a simply connected open set $\Upsilon_\Delta \subset \mathbb{C}$ which is star-shaped around $\zeta = \alpha$, and it's one-to-one everywhere else.*



For any real numbers σ, Λ and any $\delta \in [0, 1)$, the intersection of $\zeta^{\sigma+\delta}\,\mathcal{O}(\pi\Upsilon)$ with $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Upsilon)$ is a closed subspace of the latter.

Proof. Each point $b \in \Upsilon_\Delta$ has two preimages $b_+, b_- \in \Upsilon$, where b_+ is 2π counterclockwise of b_- around $\zeta = \alpha$. The operator $\operatorname{hol} \colon \mathcal{H}L^\infty_{\sigma,\Lambda}(\Upsilon) \to \mathcal{H}L^\infty_{\sigma,\Lambda}(\Upsilon_\Delta)$ defined by

$$\operatorname{hol} \varphi\big|_b = \varphi(b_+) - \varphi(b_-)$$

is bounded. The intersection of $\zeta^{\sigma+\delta}\,\mathcal{O}(\Upsilon_\Delta)$ with $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Upsilon)$ is the kernel of the bounded operator $\operatorname{hol} - e^{2\pi i\delta}$, so it's closed. □

Theorem 4.2 (restatement of 1.2)**.** *The solution Ψ_α from Theorem 4.1 has an asymptotic expansion in the transmonomial space $e^{-\alpha z} z^{-\tau_\alpha}\,\mathbb{C}[\![z^{-1}]\!]$. In fact, up to scaling, it's asymptotic to the Poincaré solution found in that space [not defined yet?].*

Proof. Take an open sector Ω_α of the kind used in Theorem 4.1, and expand it to an open set $\Upsilon \subset \widetilde{B}_{P(-\zeta)}$ of the kind used in Lemma 4.1. By Theorem 4 of [30], the equation $\hat{\mathcal{P}}_\alpha \psi_\alpha = 0$

has a unique solution ψ_α in the affine subspace

$$\frac{\zeta^{\tau_\alpha - 1}}{\Gamma(\tau_\alpha)} + \mathcal{H}L^\infty_{\tau_\alpha,\bullet}(\Upsilon)$$

of the space $\mathcal{H}L^\infty_{\tau_\alpha-1,\bullet}(\Upsilon)$. Restricting this solution to Ω_α gives the analogous solution from the proof of Theorem 4.1, as we can see from the uniqueness part of Theorem 4. To prove that Ψ_α has a the desired asymptotic expansion, we'll show that ψ_α lies in $\zeta^{\tau_\alpha-1}\,\mathcal{O}(\pi\Upsilon)$.

For convenience, let $p = P(-\zeta)$ and $q = Q(-\zeta)$. In [30], to prove Theorem 4, we rewrite the equation $\hat{\mathcal{P}}_\alpha f = 0$ as a regular singular Volterra equation $f = \mathcal{V}^\alpha f$ that satisfies the conditions of Theorem 3. The operator $\mathcal{V}^\alpha$ is the sum of the "prototypical" part

$$\mathcal{V}^\alpha_0 = -\tfrac{1}{p} \circ \partial^{-1}_{\zeta,\alpha} \circ q$$

and the perturbation

$$\mathcal{V}^\alpha_\star = \tfrac{1}{p} \circ \partial^{-2}_{\zeta,\alpha} \circ R\big(\partial^{-1}_{\zeta,\alpha}\big).$$

- Prove Theorem 3 using Picard iteration, using the *prototype solution*

  $$f_0(a) = \frac{1}{p(a)} \left(\frac{\zeta(a)}{\zeta(b)}\right)^\tau \exp\left[-\int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta}\right)\, d\zeta\right],$$

  given by an arbitary point $b \in \Omega_\alpha$ as our initial guess.
- This expression for the prototype solution, taken from sec:construction Section 3.2.1 of [30], makes it clear that the prototype solution belongs to $\zeta^{\tau_\alpha-1}\,\mathcal{O}(\pi\Upsilon)$.
- The operator $\mathcal{V}^\alpha$ preserves the intersection of $\zeta^{\sigma+\delta}\,\mathcal{O}(\pi\Upsilon)$ with $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Upsilon)$. Hence, the Picard iteration sequence must lie in this subspace.
- We know from Lemma 4.1 that the intersection of $\zeta^{\tau_\alpha-1}\,\mathcal{O}(\pi\Upsilon)$ with $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Upsilon)$ is a closed subspace of the latter. Hence, the solution ψ_α that Picard iteration converges to must also lie within this subspace.
- Use Laplace transforms of truncated power series for ψ_α to show that Ψ_α has an asymptotic series?

□

Theorem 4.3 (restatement of 1.4)**.** *Choose a root $-\alpha$ of P. Choose an open sector Ω_α which has an opening angle of π or less, has $\zeta = \alpha$ at its tip, and doesn't touch any other root of $P(-\zeta)$. Equation*

$$\mathcal{P}\Psi = 0 \tag{18}$$

has a unique solution Ψ_α in the affine subspace

$$z^{-\tau_\alpha} + \widehat{\mathcal{H}}L^\infty_{-\tau_\alpha-1,\bullet}(\Omega_\alpha)$$

of the space $\widehat{\mathcal{H}}L^\infty_{-\tau_\alpha,\bullet}(\Omega_\alpha)$ from Section 3.2.1.

Proof. As a consequence of Theorem 4 [30], after rescaling by a constant, the equation $\hat{\mathcal{P}}_\alpha \psi_\alpha = 0$ has a unique solution ψ_α in the affine subspace

$$\frac{\zeta^{\tau_\alpha - 1}}{\Gamma(\tau_\alpha)} + \mathcal{H}L^\infty_{\tau_\alpha,\bullet}(\Omega_\alpha)$$

of the space $\mathcal{H}L^\infty_{\tau_\alpha-1,\bullet}(\Omega_\alpha)$. The same is true on any smaller sector created by shaving a sector off each edge of Ω_α. It follows, by the results of Section 3.2.1 [the domain Ω_α needs to be a sector for this], that the equation $\mathcal{P}\Psi_\alpha = 0$ has a unique solution Ψ_α in the affine subspace

$$z^{-\tau_\alpha} + \widehat{\mathcal{H}}L^\infty_{-\tau_\alpha-1,\bullet}(\Omega_\alpha)$$

of the space $\widehat{\mathcal{H}}L^\infty_{-\tau_\alpha,\bullet}(\Omega_\alpha)$. □

Theorem 4.4. *Assuming Ψ_α from Theorem 4.3 admits an asymptotic expansion as $\operatorname{Re} z \to e^{i\theta}\infty$ along some rays contained in the sector Ω_α, then Ψ_α is Borel regular. That solution Ψ_α from Theorem 4.3 is Borel regular.*

Proof. If Ψ_α admits an asymptotic expansion $\tilde{\Psi}_\alpha$, then Ψ_α is a formal solution of Equation (17), thus by Ramis Index Theorem is 1-Gevrey. The Borel transform of $\tilde{\Psi}_\alpha$ is a germ of convergent power series, and by the properties of the Borel transform, it's a formal solution of $\hat{\mathcal{P}}_\alpha \phi = 0$. [check the exponential term $e^{-z\alpha}$, otherwise get different points]

- the asymptotitics of Ψ_α [exist and] are 1-Gevrey
- The solution Ψ_α has an asymptotic expansion indeed it's the Laplace transform of ψ_α, thus if we consider the Taylor expansion of ψ_α at $\zeta = \alpha$ and then we integrate term by term we can compute the asymptotic. Notice that Poincaré asymptotic doesn't have to be to a power series, but it can be mor generally to a trans-monomial. I think we should state the definition for trans-monomials.

$$\Psi_\alpha = \int_\alpha^\infty e^{-z\zeta} \sum_{n=0}^\infty a_n (\zeta-\eta)^n d\zeta \tag{19}$$

$$= \sum_{n=0}^N a_n \int_\alpha^\infty e^{-z\zeta}(\zeta-\eta)^n d\zeta + \int_\alpha^\infty e^{-z\zeta} \sum_{n=N+1}^\infty a_n(\zeta-\eta)^n d\zeta \tag{20}$$

$$= e^{-z\eta} \sum_{n=0}^N a_n \int_\alpha^\infty e^{-z(\zeta-\eta)}(\zeta-\eta)^n d\zeta + e^{-z\eta} \int_\alpha^\infty e^{-z(\zeta-\eta)} \sum_{n=N+1}^\infty a_n(\zeta-\eta)^n d\zeta \tag{21}$$

$$\left| \Psi_\alpha - e^{-z\eta} \sum_{n=0}^N a_n \int_\alpha^\infty e^{-z(\zeta-\eta)}(\zeta-\eta)^n d\zeta \right| \le e^{-\operatorname{Re}(z)|\eta|} \int_\alpha^R |e^{-z(\zeta-\eta)}| \sum_{n=N+1}^\infty R^{-n} |(\zeta-\eta)|^n d\zeta +$$
$$e^{-\operatorname{Re}(z)|\eta|} \int_R^\infty |e^{-z(\zeta-\eta)}||\psi_\alpha| d\zeta \qquad \text{[use that } \psi_\alpha \text{ is exponentially bounded]}$$

$$\Psi_\alpha = \lim_{b \to (\zeta=\alpha)} \int_b^\infty e^{-z\zeta} f \, d\zeta$$

- Say $\left|\Psi_\alpha - \int_b^\infty\right| \le \epsilon$ over all z. Then $|z|^n\ \left|\Psi_\alpha - \int_b^\infty\right| \le |z|^n\epsilon$
- $$\int_\alpha^\infty e^{-z\zeta}|z|^n|\psi_\alpha - \text{poly}(\zeta)|d\zeta$$
$$\int_\alpha^\infty \left[\left(\frac{\partial}{\partial\zeta}\right)^n e^{-z\zeta}\right]|\psi_\alpha - \text{poly}(\zeta)|d\zeta$$
- Need $\lim_{z\to\infty} z^n|\Psi_\alpha - \text{poly}|$ for all $n \in \mathbb{Z}_{\ge 0}$
- $\tilde{\psi}_\alpha \in \zeta_\alpha^{\tau_\alpha}\mathbb{C}\{\zeta_\alpha\}$ and solves $\hat{\mathcal{P}}_\alpha\phi = 0$
- Integral looks like
$$\begin{aligned}\int_0^\infty e^{-z\zeta} f\, d\zeta &= \int_0^r e^{-z\zeta} f\, d\zeta + \int_r^\infty e^{-z\zeta} f\, d\zeta \\ &\text{“}\approx\text{”} \int_0^r e^{-z\zeta} f\, d\zeta + e^{-z\,\text{Re}(\zeta)}\|f\|_1 \\ &\text{“}\approx\text{”}\|f\|_1 + e^{-zr}\|f\|_1\end{aligned}$$
- $\hat{\mathcal{P}}_\alpha\phi = 0$ admits a unique formal solution
- $\tilde{\psi}_\alpha$ is proportional to the Taylor expansion of ψ_α [which exists]

□

The key to our proof is recasting Equation (18) as an integral equation in the position domain. This integral equation belongs to a well-behaved class of regular singular Volterra equations, which we study in [30]. In particular, it can be solved using Picard iteration, which refines an initial guess toward the unique solution in its basin of attraction. We'll use the guess $\frac{1}{\Gamma(\tau_\alpha)}\zeta^{\tau_\alpha-1}$ [actually, use prototype solution instead] to find a solution ψ_α with the same singularity at $\zeta_\alpha = 0$. of the space $\mathcal{H}L^\infty_{\tau_\alpha-1,\bullet}(\Omega_\alpha)$.

For the Borel regularity part of the proof, we need to make a connection between the analytic and formal worlds. To do this in the position domain, we again use Picard iteration. [This does seem to generalize Delabaere's argument! We should confirm and mention that.]

Lemma 4.2. *In Theorem 4 of [30], the solution ψ_α can be expressed as the sum of a power series in $\zeta_\alpha^{\tau_\alpha-1}\mathbb{C}\{\zeta_\alpha\}$.*

Proof. The ring of formal power series $\mathbb{C}\{\zeta_\alpha\}$ is isomorphic to the ring of germs of holomorphic functions at $\zeta = \alpha$. This isomorphism identifies the analytic and formal versions of the integration operator $\partial_{\zeta,\alpha}^{-1}$. In the argument that follows, we'll only distinguish the analytic and formal interpretations of $\mathbb{C}\{\zeta_\alpha\}$ when we need to.

For convenience, let $p = P(-\zeta)$ and $q = Q(-\zeta)$. In [30], to prove Theorem 4, we rewrite the equation $\hat{\mathcal{P}}_\alpha f = 0$ as a regular singular Volterra equation $f = \mathcal{V}^\alpha f$ that satisfies the conditions of Theorem 3. The operator $\mathcal{V}^\alpha$ is the sum of the "prototypical" part

$$\mathcal{V}_0^\alpha = -\tfrac{1}{p} \circ \partial_{\zeta,\alpha}^{-1} \circ q$$

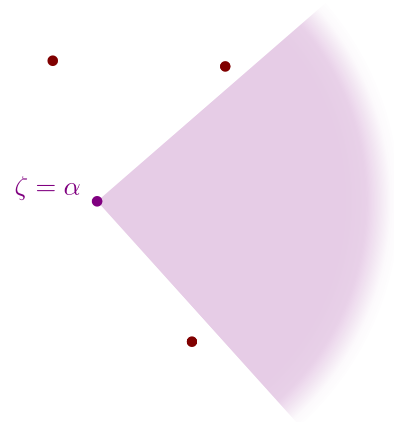


Figure 4: The domain Ω_α

and the perturbation

$$\mathcal{V}^\alpha_\star = \tfrac{1}{p} \circ \partial_{\zeta,\alpha}^{-2} \circ R\big(\partial_{\zeta,\alpha}^{-1}\big).$$

Let's run through the proof of Theorem 3, specialized to the case we consider in Theorem 4. First, picking an arbitrary point $b \in \Omega_\alpha$, we show that the *prototype solution*

$$f_0(a) = \frac{1}{p(a)} \exp\left(-\int_b^a \frac{q}{p}\, d\zeta\right)$$

lies in the space $\mathcal{H}L^\infty_{\tau_\alpha-1,\bullet}(\Omega_\alpha)$ and satisfies the equation $f_0 = \mathcal{V}^\alpha_0 f_0$. We then look for a perturbation $f_\star$ that makes $f = f_0 + f_\star$ a solution of the Volterra equation we're trying to solve. This is equivalent to solving the inhomogeneous equation

$$f_\star = \mathcal{V}^\alpha_\star f_0 + \mathcal{V}^\alpha f_\star \tag{22}$$

The central idea of the proof is to show that $\mathcal{V}^\alpha_\star$ maps f_0, and in fact all of $\mathcal{H}L^\infty_{\tau_\alpha-1,\bullet}(\Omega_\alpha)$, into $\mathcal{H}L^\infty_{\tau_\alpha,\bullet}(\Omega_\alpha)$ [check], and that $\mathcal{V}^\alpha$ is a contraction of $\mathcal{H}L^\infty_{\tau_\alpha,\Lambda}(\Omega_\alpha)$ when Λ is large enough. It follows, by the contraction mapping theorem, that equation (22) has a unique solution $f_\star$ in $\mathcal{H}L^\infty_{\tau_\alpha,\Lambda}(\Omega_\alpha)$. More explicitly, we can solve equation (22) by Picard iteration. Starting with any initial guess $f_\star^{(0)} \in \mathcal{H}L^\infty_{\tau_\alpha,\Lambda}(\Omega_\alpha)$ and defining

$$
\begin{aligned}
f_\star^{(1)} &= \mathcal{V}_\star^\alpha f_0 + \mathcal{V}^\alpha f_\star^{(0)} \\
f_\star^{(2)} &= \mathcal{V}_\star^\alpha f_0 + \mathcal{V}^\alpha f_\star^{(1)} \\
f_\star^{(3)} &= \mathcal{V}_\star^\alpha f_0 + \mathcal{V}^\alpha f_\star^{(2)} \\
&\vdots
\end{aligned}
$$

we get a sequence of functions that converges in $\mathcal{H}L^\infty_{\tau_\alpha,\Lambda}(\Omega_\alpha)$ to a solution.

[...]

Since p has a simple root at $\zeta = \alpha$, the prototypical part $\mathcal{V}_0^\alpha$ sends $\mathbb{C}\{\zeta_\alpha\}$ to itself. In fact, for any real number σ that's not a negative integer, $\mathcal{V}_0^\alpha$ maps $\zeta^\sigma\mathbb{C}\{\zeta_\alpha\}$ to itself. Under the same condition on σ, the operator $R(\partial_{\zeta,\alpha}^{-1})$ is well-defined as a map from $\zeta_\alpha^\sigma\mathbb{C}\{\zeta_\alpha\}$ to itself,[12] so the perturbation can be seen as a map $\mathcal{V}_\star^\alpha \colon \zeta_\alpha^\sigma\mathbb{C}\{\zeta_\alpha\} \to \zeta_\alpha^{\sigma+1}\mathbb{C}\{\zeta_\alpha\}$.

[...]

$\square$

[We'll prove a formal analogue of Theorem 4 from [30]. We use essentially the same technique], replacing the function spaces $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_\alpha)$ used in [30] with the formal power series spaces $\zeta_\alpha^\sigma\mathbb{C}[\![\zeta_\alpha]\!]$ under the non-archimedean norm

$$\left|c_n\zeta_\alpha^{\sigma+n} + c_{n+1}\zeta_\alpha^{\sigma+n+1} + c_{n+2}\zeta_\alpha^{\sigma+n+2} + \ldots\right|_\lambda = \lambda^{-(\sigma+n)}$$

the weighted L^1 norm [whoops, not defined for all formal power series]

$$\left|c_n\zeta_\alpha^{\sigma+n}+c_{n+1}\zeta_\alpha^{\sigma+n+1}+c_{n+2}\zeta_\alpha^{\sigma+n+2}+\ldots\right|_\lambda = |c_n|\lambda^{-(\sigma+n)}+|c_{n+1}|\lambda^{-(\sigma+n+1)}+|c_{n+2}|\lambda^{\sigma+n+2}+\ldots$$

for some $\lambda > 1$. [My hope was to use a non-archimedean norm to give a contraction mapping theorem formulation of an argument that more and more terms match, but it's hard to split up the operator in a way that makes more and more terms match...]

Lemma 4.3. *The equation $\hat{\mathcal{P}}_\alpha\tilde{\psi}_\alpha = 0$ has a unique formal solution $\tilde{\psi}_\alpha$ in the affine subspace*

$$\frac{\zeta_\alpha^{\tau_\alpha-1}}{\Gamma(\tau_\alpha)} + \zeta_\alpha^{\tau_\alpha}\mathbb{C}[\![\zeta_\alpha]\!]$$

of the space $\zeta_\alpha^{\tau_\alpha-1}\mathbb{C}[\![\zeta_\alpha]\!]$.

Proof. In [30], to prove Theorem 4, we rewrite the equation $\hat{\mathcal{P}}_\alpha f = 0$ as a regular singular Volterra equation $f = \mathcal{V}^\alpha f$ that satisfies the conditions of Theorem 3. Instead of looking for an analytic solution, like we do in [30], we'll seek a formal solution $\tilde{f}$. First, we have to make sure the equation makes sense in the space of formal power series. Let $\tilde{p} = P(-\zeta)$

[12] This is easiest to see from the formal point of view. Multiplication by $R(\zeta)$ sends convergent series to convergent series, and $\partial_{\zeta,\alpha}^{-1}$ acts like multiplication by ζ followed by the map $\zeta^\rho \mapsto \frac{1}{\Gamma(\rho+1)}\zeta^\rho$. A convergent series stays convergent under any operation that leaves its coefficients the same or smaller. [Discuss convergence of the operator series from the analytic point of view?].

and $\tilde{q} = Q(-\zeta)$, thinking of the polynomials as formal power series, and let $\tilde{R}$ be the Taylor series of R. The operator $\mathcal{V}^\alpha$ is the sum of the "prototypical" part

$$\mathcal{V}_0^\alpha = -\tfrac{1}{\tilde{p}} \circ \partial_{\zeta,\alpha}^{-1} \circ \tilde{q}$$

and the perturbation

$$\mathcal{V}_\star^\alpha = \tfrac{1}{\tilde{p}} \circ \partial_{\zeta,\alpha}^{-2} \circ \tilde{R}(\partial_{\zeta,\alpha}^{-1}),$$

[have to get a contraction somehow for this to work like I imagined] which are both written here as operators on $\mathbb{C}[\![\zeta_\alpha]\!]$.

Now, let's run through the proof of Theorem 3, replacing the function spaces $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_\alpha)$ used in [30] with the formal power series spaces $\zeta_\alpha^\sigma \mathbb{C}[\![\zeta_\alpha]\!]$ in the "λ-adic" norms described above. For simplicity, we'll specialize the proof to the case we consider in Theorem 4. First, picking an arbitrary point $\zeta = \eta$ in Ω_α, we show that the *prototype solution*

$$\tilde{f}_0 = \frac{1}{\tilde{p}} \left(\frac{\zeta}{\eta}\right)^{\tau_\alpha} \exp\left[-\partial_{\zeta,\eta}^{-1}\left(\frac{\tilde{q}}{\tilde{p}} + \frac{\tau_\alpha}{\zeta}\right)\right]$$

satisfies the equation $\tilde{f}_0 = \mathcal{V}_0^\alpha \tilde{f}_0$. Our expression for the prototype solution, taken from `sec:construction` Section 3.2.1 of [30], gives a convergent power series expansion of the analytic prototype solution defined in [30].

This definition of the prototype solution is taken from an intermediate step of the calculate in Section 3.2.1 `sec:construction` of [30], in which

to express the prototype solution as a formal power series that satisfies the same equation. The definition of τ_α ensures that $\tilde{q}/\tilde{p} + \tau_\alpha/\zeta$ is in $\mathbb{C}[\![\zeta_\alpha]\!]$, so $\tilde{f}_0$ is in $\zeta_\alpha^{\tau_\alpha - 1}\mathbb{C}[\![\zeta_\alpha]\!]$.

Next, we look for a perturbation $\tilde{f}_\star$ that makes $\tilde{f} = \tilde{f}_0 + \tilde{f}_\star$ a solution of the Volterra equation we're trying to solve. This is equivalent to solving the inhomogeneous equation

$$\tilde{f}_\star = \mathcal{V}_\star^\alpha \tilde{f}_0 + \mathcal{V}^\alpha \tilde{f}_\star \tag{23}$$

The central idea of the proof is to show that $\mathcal{V}_\star^\alpha$ maps $\zeta_\alpha^{\tau_\alpha - 1}\mathbb{C}[\![\zeta_\alpha]\!]$ into $\zeta_\alpha^{\tau_\alpha}\mathbb{C}[\![\zeta_\alpha]\!]$ and that $\mathcal{V}^\alpha$ is a contraction of $\zeta_\alpha^{\tau_\alpha - 1}\mathbb{C}[\![\zeta_\alpha]\!]$.

f_0 lies in $\mathcal{H}L^\infty_{\tau_\alpha - 1,\bullet}(\Omega_\alpha)$, that $\mathcal{V}_\star^\alpha$ maps $\mathcal{H}L^\infty_{\tau_\alpha - 1,\bullet}(\Omega_\alpha)$ into $\mathcal{H}L^\infty_{\tau_\alpha,\bullet}(\Omega_\alpha)$ [check], and that $\mathcal{V}^\alpha$ is a contraction of $\mathcal{H}L^\infty_{\tau_\alpha,\Lambda}(\Omega_\alpha)$ when Λ is large enough. It follows, by the contraction mapping theorem, equation (22) has a unique solution $f_\star$ in $\mathcal{H}L^\infty_{\tau_\alpha,\Lambda}(\Omega_\alpha)$. More explicitly, we can solve equation (22) by Picard iteration. Starting with any initial guess $\tilde{f}_\star^{(0)} \in \mathcal{H}L^\infty_{\tau_\alpha,\Lambda}(\Omega_\alpha)$ and defining

$$\begin{aligned}
\tilde{f}_\star^{(1)} &= \mathcal{V}^\alpha \tilde{f}_\star^{(0)} \\
\tilde{f}_\star^{(2)} &= \mathcal{V}^\alpha \tilde{f}_\star^{(1)} \\
\tilde{f}_\star^{(3)} &= \mathcal{V}^\alpha \tilde{f}_\star^{(2)} \\
&\vdots
\end{aligned}$$

we get a sequence of functions that converges in $\zeta_\alpha^{\tau_\alpha}\mathbb{C}[\![\zeta_\alpha]\!]$ to a solution.

[Argument of Loday-Richaud. . .] □

Proof. Let $\hat{\mathcal{P}}_\alpha$ be the Volterra operator defined in Section 4.5 of [30]. Take any holomorphic function ϕ on Ω_α with a well-defined Laplace transform $\Phi = \mathcal{L}^{\theta}_{\zeta,\alpha}\phi$ [choose θ]. Lemma 2 of [30] shows that $\hat{\mathcal{P}}_\alpha \phi = 0$ if and only if $\mathcal{P}\Phi = 0$ [the lemma only states one direction; change it before publication to state both directions].

As a consequence of Theorem **??**, the equation $\hat{\mathcal{P}}_\alpha \psi_\alpha = 0$ has a unique solution ψ_α in the affine subspace

$$\frac{\zeta^{\tau_\alpha - 1}}{\Gamma(\tau_\alpha)} + \mathcal{H}L^\infty_{\tau_\alpha,\bullet}(\Omega_\alpha)$$

of the space $\mathcal{H}L^\infty_{\tau_\alpha - 1,\bullet}(\Omega_\alpha)$. The same is true on any smaller sector created by shaving a sector off each edge of Ω_α. It follows, by the results of Section 3.2.1 [the domain Ω_α needs to be a sector for this], that the equation $\mathcal{P}\Psi_\alpha = 0$ has a unique solution Ψ_α in the affine subspace

$$z^{-\tau_\alpha} + \widehat{\mathcal{H}}L^\infty_{-\tau_\alpha - 1,\bullet}(\Omega_\alpha)$$

of the space $\widehat{\mathcal{H}}L^\infty_{-\tau_\alpha,\bullet}(\Omega_\alpha)$.

To show that Ψ_α is Borel regular, first observe that by [Erdelyi §1.6—cite in appdendix; to get uniformity, we might need a constant-width strip, rather than a sector], the asymptotic expansion $\tilde{\Psi}_\alpha = æ^\theta \Psi_\alpha$ satisfies the same differential equation that Ψ_α does. In addition, the leadeing term of $\tilde{\Psi}_\alpha$ is $z^{-\tau_\alpha}$. Thus, $\tilde{\psi}_\alpha = \mathcal{B}_\zeta \tilde{\Psi}_\alpha$ satisfies the same intergral equation that ψ_α does, and its leading term is $\frac{1}{\Gamma(\tau_\alpha)}\zeta^{\tau_\alpha - 1}$. □

4.1.2 A new proof of the Borel summability of the Poincaré solutions

Given a differential operator $\mathcal{P}$ as in Equation (17), Poincaré showed how to build a frame of formal solutions of the equation $\mathcal{P}\Psi = 0$ with a simple algorithm: for every root α of P, let $\tau_\alpha := Q(-\alpha)/P'(-\alpha)$ and consider the following ansatz

$$\tilde{\Psi}_\alpha(z) := e^{-z\alpha} z^{-\tau_\alpha} \sum_{n=0}^{\infty} a_{\alpha,n}\, z^{-n} \in e^{-z\alpha} z^{-\tau_\alpha} \mathbb{C}[\![z^{-1}]\!]. \tag{24}$$

Then solving order by order in n it's possible to compute the coefficients of the formal series $\sum_{n=0}^{\infty} a_{\alpha,n}\, z^{-n}$, which will turn out to be a 1-Gevrey series because of Ramis Index Theorem [62]. When applicable, Borel summability is an effective method to produce an analytic frame of solutions from the formal Poincaré frame $\tilde{\Psi}_\alpha$. In particular, as a consequence of our result in Theorem 4.3 we deduce that the Poincaré frame of solutions is Borel summable (proving Corollary 1.5).

Theorem 4.5. *For each root α of P, let $\tilde{\Psi}_\alpha(z) \in e^{-z\alpha} z^{-\tau_\alpha}\mathbb{C}[\![z^{-1}]\!]_1$ be the formal solution of $\mathcal{P}\Psi = 0$ with $a_{\alpha,0} = 1$. Then $\tilde{\Psi}_\alpha(z)$ is Borel summable and its Borel sum $\hat{\Psi}_\alpha(z)$ is proportional to the analytic solution Ψ_α of Theorem 4.3.*

Proof. We divide the proof in the following steps:

(a) let $\tilde{\psi}_\alpha := \mathcal{B}_\zeta \tilde{\Psi}_\alpha$, then $\hat{\mathcal{P}}_\alpha \tilde{\psi}_\alpha = 0$.

(b) $\tilde{\psi}_\alpha \in \mathbb{C}\{\zeta_\alpha\}$ can be analytically continued and its sum $\hat{\psi}_\alpha$ solves $\hat{\mathcal{P}}_\alpha \hat{\psi}_\alpha = 0$.

(c) $\hat{\psi}_\alpha$ admits a Laplace transform $\mathcal{L}^\theta_{\zeta,\alpha}\hat{\psi}_\alpha =: \hat{\Psi}_\alpha$ and that $\hat{\Psi}_\alpha \propto \Psi_\alpha$.

Step (a) follows from the properties of the Borel transform: recall from Section 3.4.4 that $\tilde{\Psi}_\alpha$ is a transmonomial, thus the Borel transform of $z^{-\lambda}\tilde{\Psi}_\alpha$ is written in terms of the convolution product $*_\alpha$:

$$\begin{aligned}
\mathcal{B}_\zeta\left(\mathcal{P}\tilde{\Psi}_\alpha\right) &= \left[P(-\zeta) + 1 *_\alpha Q(-\zeta) + \zeta *_\alpha \left(\sum_{r=0}^{d-1} \mathcal{B}_\zeta(R_r(z^{-1})) *_\alpha (-\zeta)^r\right)\right] \tilde{\psi}_\alpha(\zeta_\alpha) \\
&= \left[P(-\zeta) + \partial^{-1}_{\zeta,\alpha} \circ Q(-\zeta) + \partial^{-2}_{\zeta,\alpha}\left(\sum_{r=0}^{d-1} R_r(\partial^{-1}_{\zeta,\alpha})(-\zeta)^r\right)\right] \tilde{\psi}_\alpha(\zeta_\alpha) \\
&= \hat{\mathcal{P}}_\alpha \tilde{\psi}_\alpha(\zeta_\alpha)
\end{aligned}$$

Hence, the Borel transform of $\tilde{\Psi}_\alpha$ is a solution of $\hat{\mathcal{P}}_\alpha\tilde{\psi}_\alpha$=0.

To prove Step (b) we'll use the properties of the integral equation $\hat{\mathcal{P}}_\alpha\psi = 0$. First, recall that ψ_α is the uniques solution of $\hat{\mathcal{P}}_\alpha\psi = 0$ such that $\psi_\alpha \in \mathcal{H}L^\infty_{\tau_\alpha-1,\bullet}(\Omega_\alpha)$ [30, Theorem 4]. Then, as shown in Lemma 4.2, the Taylor expansion of ψ_α is the formal solution of $\hat{\mathcal{P}}_\alpha\psi = 0$ in $\zeta_\alpha^{\tau_\alpha-1}\mathbb{C}\{\zeta_\alpha\}$. Since by direct computations $\tilde{\psi}_\alpha \in \zeta_\alpha^{\tau_\alpha-1}\mathbb{C}\{\zeta_\alpha\}$

$$\tilde{\psi}_\alpha = \sum_{n=0}^{\infty} a_{\alpha,n}\frac{\zeta_\alpha^{n+\tau_\alpha-1}}{\Gamma(n+\tau_\alpha)},$$

our uniquess result ensures that $\tilde{\psi}_\alpha$ is proportional to the Taylor expansion of ψ_α in their common region of convergence. Thus if $\hat{\psi}_\alpha$ denotes the sum of $\tilde{\psi}_\alpha$, we deduce that $\tilde{\psi}_\alpha \propto \psi_\alpha$.

Finally, we can Laplace transform $\hat{\psi}_\alpha$ along a ray Γ^θ_α in Ω_α, proving the $\tilde{\Psi}_\alpha$ is Borel summable. In addition, since $\tilde{\psi}_\alpha \propto \psi_\alpha$, their Laplace transform will be proportional too $\mathcal{L}^\theta_{\zeta,\alpha}\hat{\psi}_\alpha \propto \Psi_\alpha$. This ends the proof of Step (c).

□

4.2 Borel regularity for thimble integrals

Recall the setting from Section 1.2.2: let X be a 1-dimensional complex manifold equipped with a volume form $\nu \in \Gamma(X,\Omega^1)$ and a holomorphic function $f\colon X \to \mathbb{C}$ with non–degenerate critical points. Let I be the thimble integral

$$I := \int_{\Lambda^\theta_a} e^{-zf}\nu \tag{25}$$

where Λ^θ_a is the thimble through the critical point a. In this section, we'll prove that thimble integrals of the form (25) are Borel regular. The first step will be to rewrite I as the Laplace transform of a function ι which is explicitly given by the *thimble projection formula.*

Lemma 4.4. *[Well-known result—see, e.g., §3.3 of [58], which uses the variation ν/df instead of derivative of integral.] A function ι with $I = \mathcal{L}^\theta_{\zeta,\alpha}\iota$ is given by the* thimble projection formula

$$\iota = \frac{\partial}{\partial\zeta}\left(\int_{\Lambda^\theta_\alpha(\zeta)}\nu\right), \tag{26}$$

where $\Lambda_\alpha^\theta(\zeta)$ is the part of Λ_a^θ that goes through $f^{-1}([\alpha, \zeta e^{i\theta}])$. Notice that $\Lambda_\alpha^\theta(\zeta)$ starts and ends in $f^{-1}(\zeta)$. [The thimble Λ_a^θ needs an orientation, but the orientation is arbitrary.]

Proof. Let us recast the integral I into the position domain. As ζ goes rightward from α with angle θ, the start and end points of $\Lambda_\alpha^\theta(\zeta)$ sweep backward along $\Lambda_\alpha^-(\zeta)$ and forward along $\Lambda_\alpha^+(\zeta)$, respectively. Hence, we have

$$\begin{aligned} I(z) &= \int_{\Lambda_a^\theta} e^{-zf}\nu \\ &= \int_\alpha^{e^{i\theta}\infty} e^{-z\zeta}\left[\frac{\nu}{df}\right]_{\text{start }\Lambda_\alpha^\theta(\zeta)}^{\text{end }\Lambda_\alpha^\theta(\zeta)} d\zeta. \end{aligned}$$

Noticing that the right-hand side is a Laplace transform, we learn that

$$\iota(\zeta) = \left[\frac{\nu}{df}\right]_{\text{start }\Lambda_\alpha^\theta(\zeta)}^{\text{end }\Lambda_\alpha^\theta(\zeta)}. \tag{27}$$

□

We'll now prove that I is Borel regular:

Theorem 4.6. *If the integral defining I is absolutely convergent, then I is Borel regular. More explicitly:*

1. *As $z \to \infty$ along the ray $z \in e^{-i\theta}\mathbb{R}_{>0}$, the function I is asymptotic to a trans-monomial $\tilde{I} \in e^{-z\alpha}z^{-1/2}\mathbb{C}[\![z^{-1}]\!]$.*
2. *The series $\tilde{I}$ is 1-Gevrey. In other words, $\tilde{\iota} := \mathcal{B}_\zeta\tilde{I}$ converges near $\zeta = \alpha$.*
3. *If you continue the sum of $\tilde{\iota}$ along the ray Γ_α^θ, and take its Laplace transform along that ray, you'll recover I.*

Proof. Since f has non–degenerate critical points, we can find a holomorphic chart τ around a with $\frac{1}{2}\tau^2 = f - f(a)$. Let Λ_α^- and Λ_α^+ be the parts of Λ_a^θ that go from the past $(-e^{-i\theta}\infty)$ to a and from a to the future $(+e^{-i\theta}\infty)$, respectively. We can arrange for τ to be valued in $(-\infty, 0]$ and $[0, \infty)$ on Λ_α^- and Λ_α^+, respectively. [Point out that we can always accomplish this by changing the sign of τ.] Since ν is holomorphic, we can express it as a Taylor series

$$\nu = \sum_{m\geq 0} b_m^a \tau^m \, d\tau$$

that converges in some disk $|\tau| < \varepsilon$.

Let's write $\approx$ when two functions are asymptotic at all orders around the base point; by the steepest descent method,[13]

$$e^{z\alpha}I \approx \int_{\tau\in[-\varepsilon,\varepsilon]} e^{-z\tau^2/2}\nu$$

[13] The details can be found in [51]: follow the proof of Proposition 2.1 in through equation (2.9).

as $z \to \infty$. Plugging in the Taylor series for ν, we get

$$\begin{aligned} e^{z\alpha} I &\approx \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{m \geq 0} b^a_m \tau^m \, d\tau \\ &= \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n \geq 0} b^a_{2n} \tau^{2n} \, d\tau. \end{aligned}$$

By the dominated convergence theorem,[14]

$$\begin{aligned} e^{z\alpha} I &\approx \sum_{n \geq 0} b^a_{2n} \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \tau^{2n} \, d\tau \\ &= \sum_{n \geq 0} (2n-1)!! \, b^a_{2n} \left[\sqrt{2\pi} \, z^{-(n+1/2)} \operatorname{erf}\left(\varepsilon\sqrt{z/2}\right) - 2e^{-z\varepsilon^2/2} \sum_{\substack{k \in \mathbb{N}_+ \\ k \leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{-n+k-1} \right]. \end{aligned}$$

The annoying $e^{-z\varepsilon^2/2}$ correction terms are crucial for the convergence of the sum, but we can get rid of them and still have a formal series asymptotic to $e^{-z\alpha} I(z)$. In other words,

$$e^{z\alpha} I(z) \sim \sqrt{2\pi} \sum_{n \geq 0} (2n-1)!! \, b^a_{2n} \, z^{-(n+1/2)} \operatorname{erf}\left(\varepsilon\sqrt{z/2}\right).$$

To see why, cut the sum off after N terms, and observe that

$$\left| \sum_{n=0}^{N} (2n-1)!! b^a_{2n} 2e^{-z\varepsilon^2/2} \sum_{\substack{k \in \mathbb{N}_+ \\ k \leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{-n+k-1} \right| \leq 2e^{-\operatorname{Re}(z)\varepsilon^2/2} \sum_{n=0}^{N} (2n-1)!! b^a_{2n} n |z|^{-n}.$$

The right-hand side goes to 0 as $\operatorname{Re}(z)$ grows. The differences $1 - \operatorname{erf}\left(\varepsilon\sqrt{z/2}\right)$ shrink exponentially as z grows [12, inequality (5)], allowing the simpler estimate

$$e^{z\alpha} I(z) \sim \sqrt{2\pi} \sum_{n \geq 0} (2n-1)!! \, b^a_{2n} \, z^{-(n+1/2)}.$$

Hence, defining the formal series $\tilde{I}$

$$\tilde{I}(z) := e^{-z\alpha} z^{-1/2} \sqrt{2\pi} \sum_{n \geq 0} (2n-1)!! \, b^a_{2n} \, z^{-n}$$

we get $\tilde{I} \in e^{-z\alpha} z^{-1/2} \mathbb{C}[\![z^{-1}]\!]$. This ends the proof of part 1.

Using the porperties of the Borel tranform acting on trans-monomials 3.3.1 we get

$$\begin{aligned} \tilde{\imath} &= \sqrt{2\pi} \sum_{n \geq 0} (2n-1)!! \, b^a_{2n} \frac{\zeta_\alpha^{n+\frac{1}{2}}}{\Gamma(n+\frac{1}{2})} \\ &= \sqrt{2\pi} \sum_{n \geq 0} \frac{2^n}{\sqrt{\pi}} \, b^a_{2n} \, \zeta_\alpha^{n+\frac{1}{2}} \end{aligned}$$

[14] Notice that the sum over k is empty when $n = 0$. Following convention, we extend the double factorial to all odd integers by its recurrence relation, giving $(-1)!! = 1$.

We know from the definition of ε that $|b_n^a|\,\varepsilon^n \lesssim 1$, thus we deduce that $\tilde{\iota}$ has a finite radius of convergence. This ends the proof of part 2.

[Analytic/position version starts here.] The proof of part 3 relies on the thimble projection formula (26): we'll show that the Taylor expansion of ι at $\zeta = \alpha$ agrees with $\tilde{\iota}$. We can rewrite the Taylor series for ν as

$$\nu = \sum_{n \geq 0} b_n^a [2(f-\alpha)]^{(n-1)/2}\, df,$$

taking the positive branch of the square root on Λ_a^+ and the negative branch on Λ_a^-. Plugging this into our expression for ι (in Theorem 4.4), we learn that

$$\begin{aligned}
\iota &= \left[\sum_{m \geq 0} b_m^a [2(f-\alpha)]^{(m-1)/2} \right]_{\text{start}\,\Lambda_a(\zeta)}^{\text{end}\,\Lambda_a(\zeta)} \\
&= \sum_{m \geq 0} b_m^a \left([2\zeta_\alpha]^{(m-1)/2} - (-1)^{m-1} [2\zeta_\alpha]^{(m-1)/2} \right) \\
&= \sum_{n \geq 0} 2 b_{2n}^a [2\zeta_\alpha]^{n-1/2} \\
&= \sum_{n \geq 0} 2^{n+1/2} b_{2n}^a \zeta_\alpha^{n-1/2}
\end{aligned}$$

In particular, $\tilde{\iota}$ can be analytically continued along Γ_α^θ and its sum is given by ι.

Lemma 4.4 tells us that

$$\begin{aligned}
I_j &= \mathcal{L}^\theta_{\zeta,\alpha_j} \iota_j \\
e^{z\alpha_j} I_j &= \mathcal{L}^\theta_{\zeta_j,0} \iota_j \\
&= \mathcal{L}^\theta_{\zeta_j,0} \left(\sum_{n=0}^{N} 2^{n+1/2} b_{2n}^j \zeta_j^{n-1/2} + \zeta^{N+1/2} f \right) \\
&= \mathcal{L}^\theta_{\zeta_j,0} \left(\sum_{n=0}^{\infty} 2^{n+1/2} b_{2n}^j \zeta_j^{n-1/2} \right).
\end{aligned}$$

Over any finite number of terms, we can exchange the sum with the Laplace transform, giving

$$\begin{aligned}
e^{z\alpha_j} I_j &= \Delta_n + \sum_{n=0}^{N} 2^{n+1/2} b_{2n}^j\, \mathcal{L}^\theta_{\zeta_j,0} \zeta_j^{n-1/2} \\
&= \Delta_n + \sum_{n=0}^{N} 2^{n+1/2} b_{2n}^j \Gamma(n + \tfrac{1}{2})\, z^{-n-1/2} \\
&= \Delta_n + \sum_{n=0}^{N} \sqrt{2\pi}\, (2n-1)!!\, b_{2n}^j\, z^{-n-1/2},
\end{aligned}$$

where

$$\Delta_n = \mathcal{L}^{\theta}_{\zeta_j,0}\left(\sum_{n=N+1}^{\infty} 2^{n+1/2} b^j_{2n} \zeta_j^{n-1/2}\right).$$

$$\begin{aligned} 2^n\Gamma(n+\tfrac{1}{2}) &= \sqrt{\pi}\,(2n-1)!! \\ 2^{n+1/2}\Gamma(n+\tfrac{1}{2}) &= \sqrt{2\pi}\,(2n-1)!! \end{aligned}$$

Proposition 4.5. *Consider a function*

$$\varphi = \sum_{\kappa \in K} a_\kappa \zeta^\kappa$$

defined by a power series with real exponents $K \subset (-1, \infty)$, with a positive lower bound on the distances between the exponents. If the series converges throughout the disk $|\zeta| < \varepsilon$, then

$$\partial^{-\nu}_{\zeta,0}\varphi = \sum_{\kappa \in K} \tfrac{\Gamma(\kappa+1)}{\Gamma(\kappa+\nu+1)} \zeta^{\kappa+\nu}$$

on that disk.

Proof. The convergence of the series for φ in the disk $|\zeta| < \varepsilon$ implies that for each $\eta \in (0, \varepsilon)$, we have $|a_\kappa| \lesssim \eta^{-\kappa}$ over all $\kappa \in K$. The ☐

Reference for monomials: "Evaluation of Fractional Integrals and Derivatives of Elementary Functions"

For convenience, think of K as sequence, in increasing order. At any point $\zeta = \rho$ with $|\rho| < \varepsilon$, the series defining φ converges, implying that

$$\begin{aligned} &\lim_{\kappa \in K} |a_\kappa \rho^\kappa| = 0 \\ &\lim_{\kappa \in K} |a_\kappa||\rho|^\kappa = 0 \\ &|a_\kappa||\rho|^\kappa \lesssim 1 \text{ over all } \kappa \in K \\ &|a_\kappa| \lesssim |\rho|^{-\kappa} \text{ over all } \kappa \in K. \end{aligned}$$

In other words, for every $r \in (0, \varepsilon)$, we have $|a_\kappa| \lesssim r^{-\kappa}$ over all $\kappa in K$.

Hence, on the region $|\zeta| < |\rho|$, we have

$$\begin{aligned} |\varphi| &\le \sum_{\kappa \in K} |a_n||\zeta|^\kappa \\ &\lesssim \sum_{\kappa \in K} r^{-\kappa}|\zeta|^\kappa \\ &= \sum_{\kappa \in K} |\zeta/r|^\kappa. \end{aligned}$$

In the disk $|\zeta| < r$, the ratio $|\zeta/r|$ is smaller than 1, so increasing its exponent always decreases the value.

Let $c > 0$ be the lower bound on the distances between the exponents. Choose some $\lambda \in (-1, \infty)$ so that $K \subset (\lambda, \infty)$. For each $n \in \mathbb{N}_{\geq 0}$, the nth element of K must be greater than $\lambda + nc$, so

$$\begin{aligned} |\varphi| &\lesssim \sum_{n=0}^{\infty} |\zeta/r|^{\lambda+nc} \\ &= |\zeta/r|^{\lambda} \sum_{n=0}^{\infty} (|\zeta/r|^{c})^{n} \\ &= \frac{|\zeta/r|^{\lambda}}{1 - |\zeta/r|^{c}} \\ &= \frac{1}{|\zeta/\rho| - |\zeta/r|^{1+c}}. \end{aligned}$$

We now see that the series defining φ converges absolutely on $|\zeta| < |\rho|$.

The kernel of $\partial_{\zeta,0}^{\nu-1}$ is $\big(\zeta(p) - \zeta\big)^{\nu-1}$. It has an integrable singularity at p, and it's non-singular at $\zeta = 0$.

Proposition 4.6. *Suppose $f_0, f_1, f_2, \ldots$ is a sequence of holomorphic functions on some connected domain Ω, and the series $\sum_{k=0}^{\infty} |f_k|$ converges at each point in Ω. Then, for any $\nu > 0$,*

$$\partial_{\zeta,\alpha}^{-\nu}\left(\sum_{k=0}^{\infty} f_k\right) = \sum_{k=0}^{\infty} \partial_{\zeta,\alpha}^{-\nu} f_k$$

throughout Ω.

Proof. Given any point $p \in \Omega$, choose a path γ that runs through Ω from $\zeta = \alpha$ to p. Since γ is the continous image of a closed interval, it's compact, so ζ is bounded

Recalling that

$$[\partial_{\zeta,\alpha}^{\nu-1} f](p) := \frac{1}{\Gamma(1-\nu)} \int_{\zeta=\alpha}^{p} \big(\zeta(p) - \zeta\big)^{-\nu} f \, d\zeta,$$

[...] □

By the steepest descent method,[15] [define ≈]

$$e^{z\alpha_j} I_j \approx \int_{\tau \in [-\varepsilon, \varepsilon]} e^{-z\tau^2/2} \nu$$

as $z \to \infty$ along the ray $e^{-i\theta}\,\mathbb{R}_{>0}$. □

[15] The details can be found in [51]: follow the proof of Proposition 2.1 in through equation (2.9).

Remark 4.7. [Spirit of the remark: if you want to run the argument above for Φ, you can, but there's one subtlety you should be aware of...] For some applications, it might be more convenient to study Borel regularity of $\Phi = z^{-1/2} I$, as its asymptotics won't have fractional power singularities. The proof follows the same argument we chose for I, but there's one subtulty to be aware of when rewriting Φ as the Laplace transform of a function ϕ. Indeed, first notice that

$$\phi = \partial_{\zeta,\alpha}^{-1/2} \iota$$

Then, we can take the 1/2 fractional integral on both sides:

$$\begin{aligned} \partial_{\zeta,\alpha}^{-1/2} \phi &= \partial_{\zeta,\alpha}^{-1} \iota \\ &= \partial_{\zeta,\alpha}^{-1} \partial_\zeta \left(\int_{\Lambda_\alpha^\theta(\zeta)} \nu \right) \\ &= \int_{\Lambda_\alpha^\theta(\zeta)} \nu - \int_{\Lambda_\alpha^\theta(\alpha)} \nu. \end{aligned}$$

The second term vanishes because $\Lambda_\alpha^\theta(\alpha)$ is a single point, leaving

$$\partial_{\zeta,\alpha}^{-1/2} \phi = \int_{\Lambda_\alpha^\theta(\zeta)} \nu.$$

Finally, we apply the 1/2 derivative of both sides, recalling that it's a left inverse of the 1/2 integral, and we find

$$\phi = \partial_{\zeta,\alpha}^{1/2} \left(\int_{\Lambda_\alpha^\theta(\zeta)} \nu \right). \tag{28}$$

[This argument works for any function that vanishes at α, so maybe we should instead prove it generally in the section on fractional calculus.]

Remark 4.8. We can use the 1/2 derivative formula (28) to see that the functions ϕ are all resurgent, in the sense of Écalle [26, Section 1]. Let's say $\Gamma_\alpha^\theta(\beta)$ is the straight path from $\zeta = \alpha$ to $\zeta = \beta$ in $\mathbb{C}$. As long as this path avoids the critical values of f, it lifts uniquely along f to a path $\Lambda_\alpha^\theta(\beta)$ in X. This lets us analytically continue $\int_{\Lambda_\alpha^\theta(\zeta)} \nu$ to a star-shaped domain $\Omega_\alpha \subset \mathbb{C}$. Intuitively, Ω_α is constructed by drawing rays from $\zeta = \alpha$ through all of the other critical values, and then cutting $\mathbb{C}$ along the parts of those rays that go from the other critical values to ∞ (see Figure 5). Let's calculate the jump of $\int_{\Lambda_\alpha^\theta(\zeta)} \nu$ across the cut starting at $\zeta = \beta$. Write β as $\alpha + re^{i\theta_\beta}$, and let β^+ and β^- be the points we get by infinitesimally increasing or decreasing θ_β. The jump is then given by

$$\int_{\Lambda_\alpha^\theta(\beta+)} \nu - \int_{\Lambda_\alpha^\theta(\beta-)} \nu$$

and it is which turns out to be proportional to $\tilde{\phi}_\beta$, namely to the germ of analytic functions at $\zeta_\alpha = \beta$. We'll see an example of this behaviour in Appendix A.5.1.

Initially, the function $\int_{\Lambda_j(\zeta)} \nu$ is only defined for ζ on the ray $[\alpha_j, \infty)$, but it can be analytically continued to points off the ray by homotopy of the path $\Lambda_j(\zeta)$. Since ν is

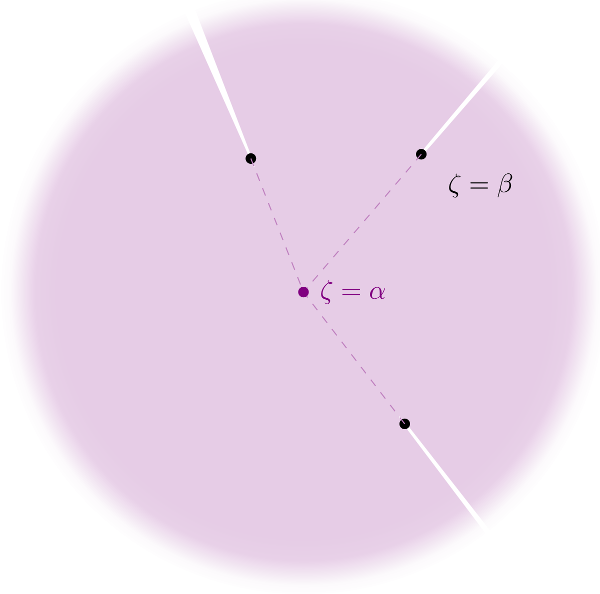


Figure 5: The domain Ω_α

holomorphic, the only way for $\int_{\Lambda_j(\zeta)} \nu$ to be singular is for something to go wrong with the hotmotopy of $\Lambda_j(\zeta)$. At a critical point of f, the ends of the homotoped path $\Lambda_j(\zeta)$ come together, telling us that.

Corollary 4.7. *If f and ν are polynomials, the function $\phi = \partial_{\zeta,\alpha}^{-1/2} \iota$ defined in Remark 4.7 has simple resurgent singularities at $\zeta = \beta$, for all β roots of P such that $\beta \neq \alpha$.*

Proof. Recall ϕ_α is the sum of the Borel transform of $z^{1/2}\tilde{I}_j$, namely

$$\hat{\phi}_j(\zeta) = \partial_{\zeta,\alpha_j}^{1/2} \hat{\iota}_j = \int_{\alpha_j}^{\zeta} (\zeta - s)^{-3/2} \hat{\iota}_j(s)\, ds$$

Then by Pham's result (see (4.9)) we know

$$\hat{\iota}_j(\zeta_j) = \frac{(-\zeta_j)^{-1/2}}{2\pi i \Gamma(-\frac{1}{2})} \sum_{k=0}^{\infty} c_k (-\zeta_j)^{-k} + \text{ hol. funct.}$$

hence

$$\hat{\iota}_j(\zeta) = \int_{\alpha_j}^{\zeta} (\zeta - s)^{-3/2} \frac{(-s+\alpha_j)^{-1/2}}{2\pi i \Gamma(-\frac{1}{2})} \sum_{k=0}^{\infty} c_k (-s+\alpha_j)^{-k}\, ds + \text{ hol. funct.}$$

□

Remark 4.9. When f and ν are polynomials and X is N-dimensional, a result of Pham [58, Equation 2.4, II partie] allows to write ι explicitly as

$$(-1)^{\frac{N(N-1)}{2}} \sum_{k=0}^{\infty} c_k \, \delta^{-\frac{N}{2}-k} + \text{ hol. funct.} \tag{29}$$

for some coefficients c_k which depends on f and ν, and with $\delta^{-\ell}$ be defined as

$$\delta^{-\ell} := \begin{cases} \frac{\zeta^{\ell-1}}{-2\pi i(\ell-1)!} \log(\zeta) + \text{ hol. funct.} & \text{if } \ell \in \mathbb{N}^* \\ \\ \frac{(-\zeta)^{\ell-1}}{2\pi i\,\Gamma(-\ell)} + \text{ hol. funct.} & \text{if } \ell \notin \mathbb{N}^* \end{cases}$$

In particular, if $N = 1$, the inverse Laplace transform of I has a fractional power singularity of order $1/2$ at the origin, as we compute explicitly in the proof of Theorem 4.6.

5 Examples

5.1 Airy

The Airy equation is

$$\left[\left(\frac{\partial}{\partial y}\right)^2 - y\right] \psi = 0. \tag{30}$$

One solution is given by the Airy function,

$$\mathrm{Ai}(y) = \frac{1}{2\pi i} \int_\Gamma \exp\left(\tfrac{1}{3}t^3 - yt\right) \, dt,$$

where Γ is a path that comes from ∞ at $-60°$ and goes to ∞ at $60°$.

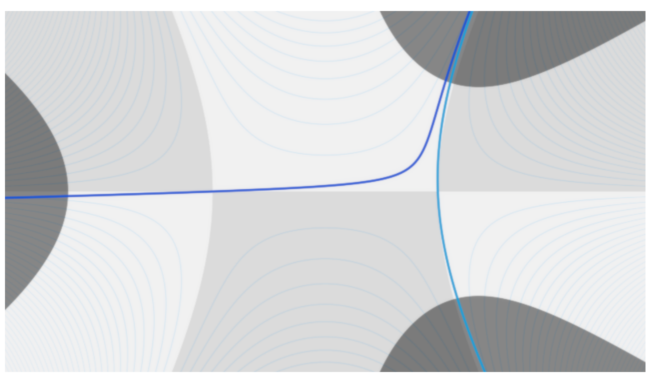


Figure 6: The contour Λ_1 in the u plane is represented by the light blue path. More generally, the dark blue contour corresponds to the preimage of the ray $+1 + e^{i\theta}\infty$ and $\theta =$. While the light blue contour corresponds to the preimage of the ray $-1 + e^{i\theta}\infty$. The gray regions correspond to the preimage of $\operatorname{Re}\zeta >$ *const*. [for Aaron: could you please add the correct constant in the description of the figure, and explain what are the different colored regions? Thanks]

With the substitution $t = 2uy^{1/2}$, we can rewrite the Airy integral as

$$\mathrm{Ai}(y) = y^{1/2}\,\frac{1}{\pi i}\int_{y^{-1/2}\Lambda_1} \exp\left[\tfrac{2}{3}y^{3/2}\left(4u^3 - 3u\right)\right] du.$$

We have rescaled the contour by a factor of two, but it still approaches ∞ in the desired way. Note that $4u^3 - 3u$ is the third Chebyshev polynomial.

By considering other Chebyshev polynomials, we can situate the Airy function within the family of *Airy-Lucas functions*, so we'll go straight to the general case. However, since the Airy function is a classic example in the study of Borel summation and resurgence, it may be useful to see it on its own. For this reason, in Appendix A we give a detailed treatment of the Airy function, specializing the general argument of all Airy-Lucas functions.

5.2 Airy–Lucas

The Airy-Lucas equation is

$$\left[\left(\tfrac{\partial}{\partial y}\right)^2 - (m-1)y^{-1}\tfrac{\partial}{\partial y} - y^{n-2}\right]\psi = 0 \tag{31}$$

with $n \in \{3, 4, 5, \ldots\}$ and $m \in \{1, 2, \ldots, n-1\}$. A few solutions are given by the Airy-Lucas functions [11, equation 3.6]

$$\widehat{\mathrm{Ai}}^{(k)}_{n,m-1}(y) = \left\{\begin{array}{ll} 1 & j \text{ even} \\ i & j \text{ odd}\end{array}\right\}\frac{y^{m/2}}{\pi}\int_{\Lambda^{(j)}} \exp\left[\tfrac{2}{n}y^{n/2}\,T_n(u)\right] U_{m-1}(u)\,du,$$

where Λ_j is the Lefschetz thimble through $u = \cos\left(\frac{j}{n}\pi\right)$ (see Figure 7).

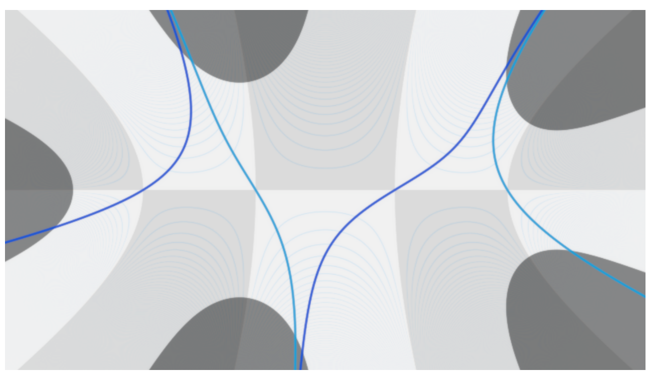


Figure 7: Integration contour Λ_j for the Airy–Lucas function with $n = 5$. The dark blue contours correspond to the preimage of the ray $1 + e^{i\theta}\infty$ and $\theta =$???. While the light blue contours correspond to the preimage of the ray $-1 + e^{i\theta}\infty$.

5.2.1 Rewriting as a modified Bessel $\frac{m}{n}$ equation

We can distill the most interesting parts of the Airy-Lucas function by writing

$$\widehat{\mathrm{Ai}}^{(k)}_{n,m-1}(y) = \text{const.}\; y^{m/2}\,K\left(\tfrac{2}{n}y^{n/2}\right),$$

where

$$K_{m/n}(z) = \text{const.} \int_{z^{-1/n}\Lambda^{(k)}} \exp\left[zT_n(u)\right] U_{m-1}(u)\, du. \tag{32}$$

Saying that $\widehat{\mathrm{Ai}}^{(k)}_{n,m-1}$ satisfies the Airy-Lucas equation is equivalent to saying that K satisfies the modified Bessel $\frac{m}{n}$ equation [Why put the order m/n in the name? Is that standard?]

$$\left[z^2\left(\tfrac{\partial}{\partial z}\right)^2 + z\tfrac{\partial}{\partial z} - \left[\left(\tfrac{m}{n}\right)^2 + z^2\right]\right] K = 0. \tag{33}$$

In fact, as we'll see in Section 5.2.8, K is the modified Bessel function $K_{m/n}$.

Let's put equation (33) in the form (18):

$$\left[\left[\left(\tfrac{\partial}{\partial z}\right)^2 - 1\right] + z^{-1}\tfrac{\partial}{\partial z} - \left(\tfrac{m}{n}\right)^2 z^{-2}\right] K = 0. \tag{34}$$

5.2.2 Asymptotic analysis

From the general theory of ODE of Poincaré rank 1, we know that the space of trans-series solutions of (34) has a basis of trans-monomials

$$\{e^{-\alpha z} z^{-\tau_\alpha}\, \tilde{W}_\alpha \mid \alpha^2 - 1 = 0\}$$

where the $\tilde{W}_\alpha \in \mathbb{C}[\![z^{-1}]\!]$ are formal power series in z^{-1} and $\tau_\alpha = 1/2$. From [45, Equations 10.40.2 and 10.17.1], we learn that $K_{m/n}(z) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-z} z^{-1/2}\, \tilde{W}_1$, with

$$\tilde{W}_1 = 1 - \frac{(\frac{1}{2} - \frac{m}{n})_1(\frac{1}{2} + \frac{m}{n})_1}{2^1 \cdot 1!}\, z^{-1} + \frac{(\frac{1}{2} - \frac{m}{n})_2(\frac{1}{2} + \frac{m}{n})_2}{2^2 \cdot 2!}\, z^{-2} - \frac{(\frac{1}{2} + \frac{m}{n})_3(\frac{1}{2} + \frac{m}{n})_3}{2^3 \cdot 3!}\, z^{-3} + \ldots \tag{35}$$

The holomorphic analysis in Section 5.2.3 will give us holomorphic solutions

$$\{e^{-\alpha z} z^{-\tau_\alpha}\, W_\alpha \mid \alpha^2 - 1 = 0\},$$

which seem analogous to the trans-monomials above. Borel summation makes the analogy precise. We'll see in Section 5.2.6 that each $z^{-\tau_\alpha}\, W_\alpha$ is proportional to the Borel sum of $z^{-\tau_\alpha}\, \tilde{W}_\alpha$. This is evidence of Theorem 4 from [30].

5.2.3 The big idea

We're going to look for functions v_α whose Laplace transforms $\mathcal{L}_{\zeta,\alpha} v_\alpha$ satisfy equation (34). We'll succeed when $\alpha^2 - 1 = 0$, and we'll see that $K_{m/n}$ is a scalar multiple of $\mathcal{L}_{\zeta,1} v_1$.

We can see from Section **??** that $\mathcal{L}_{\zeta,\alpha} v$ satisfies the differential equation (34) if and only if v satisfies the integral equation

$$\left[\left[\zeta^2 - 1\right] - \partial^{-1}_{\zeta,\alpha} \circ \zeta - \left(\tfrac{m}{n}\right)^2 \partial^{-2}_{\zeta,\alpha}\right] v = 0. \tag{36}$$

It's tempting to differentiate both sides of this equation until we get

$$\left[\left(\tfrac{\partial}{\partial \zeta}\right)^2 \circ \left[\zeta^2 - 1\right] - \tfrac{\partial}{\partial \zeta} \circ \zeta - \left(\tfrac{m}{n}\right)^2\right] v = 0, \tag{37}$$

which is easier to solve. Unfortunately, a solution of equation (37) won't satisfy equation (36) in general. However, as we show in Section B.2, a solution of equation (37) *will* satisfy equation (36) if it belongs to $\mathcal{H}L^\infty_{\sigma,\bullet}(\Omega_j)$, for some $\sigma \in (0,1)$.

This is great news, because equation (37) has a regular singularity at each root of $\zeta^2 - 1$, and the Frobenius method often gives a solution of order $\sigma = 1/2$ at each regular singular point. We can see the regular singularities by moving the derivatives to the right:

$$\left[(\zeta^2 - 1)\left(\tfrac{\partial}{\partial \zeta}\right)^2 + 3\zeta \tfrac{\partial}{\partial \zeta} + \left[1 - \left(\tfrac{m}{n}\right)^2\right]\right] v = 0.$$

In Sections 5.2.4 – 5.2.5, we'll see this approach succeed. For each root α_j, we'll find a solution v_j of equation (37) which is $\mathcal{H}L^\infty_{\sigma,\bullet}(\Omega_j)$, for some σ, and Ω_j as in Figure 4. We know the function $\mathcal{L}_{\zeta,\alpha} v_\alpha$ will satisfy equation (34), and we can even find its asymptotics from the order τ_α of v_α. We learned in Section 3.1 that

$$\mathcal{L}_{\zeta,\alpha_j} v_j = e^{-\alpha_j z} V_j$$

where $V_j = \mathcal{L}_{\zeta_j,0} v_\alpha$ and $\zeta = \alpha_j + \zeta_j$. We can see from Section 3.2.1 that V_j is asymptotic to a scalar multiple of $z^{-1-\tau_j}$ at $z = \infty$, so the further decomposition

$$\mathcal{L}_{\zeta,\alpha_j} v_j = e^{-\alpha_j z} z^{-\tau_j} W_j,$$

makes W_j is asymptotic to a scalar multiple of z^{-1} at $z = \infty$.

5.2.4 Focus on $\zeta = 1$

Let's find a solution of equation (37) which belongs to $\mathcal{H}L^\infty_{\sigma,\bullet}(\Omega_1)$. Define a new coordinate ζ_1 on $\mathbb{C}$ so that $\zeta = 1 + \zeta_1$. In this coordinate, equation (37) looks like

$$\left[\zeta_1(2 + \zeta_1)\left(\tfrac{\partial}{\partial \zeta_1}\right)^2 + 3(1 + \zeta_1)\tfrac{\partial}{\partial \zeta_1} + \left[1 - \left(\tfrac{m}{n}\right)^2\right]\right] v = 0. \tag{38}$$

With another change of coordinate, given by $\zeta_1 = -2\xi_1$, we can rewrite equation (37) as the hypergeometric equation

$$\left[\xi_1(1 - \xi_1)\left(\tfrac{\partial}{\partial \xi_1}\right)^2 + 3(\tfrac{1}{2} - \xi_1)\tfrac{\partial}{\partial \xi_1} - \left[1 - \left(\tfrac{m}{n}\right)^2\right]\right] v = 0. \tag{39}$$

Looking through the twenty-four expressions for Kummer's six solutions, we find one [45, formula 15.10.12] which belongs to $\mathcal{H}L^\infty_{1/2,\bullet}$ at $\xi_1 = 0$:

$$\begin{aligned} v_1 &= \qquad \xi_1^{-1/2}\, {}_2F_1\left(\tfrac{1}{2} - \tfrac{m}{n}, \tfrac{1}{2} + \tfrac{m}{n}; \tfrac{1}{2}; \xi_1\right) \\ &= -i\sqrt{2}\,\zeta_1^{-1/2}\, {}_2F_1\left(\tfrac{1}{2} - \tfrac{m}{n}, \tfrac{1}{2} + \tfrac{m}{n}; \tfrac{1}{2}; -\tfrac{1}{2}\zeta_1\right) \end{aligned}$$

From the argument in Section 5.2.3, we know that $\mathcal{L}_{\zeta,1} v_1$ satisfies equation (33), and can be written as $e^{-z} V_1$, where $V_1 = \mathcal{L}_{\zeta_1,0} v_1$. Since v_1 has order $-1/2$, the decomposition $V_1 = z^{-1/2} W_1$ makes W_1 asymptotic to a scalar multiple of $1 + O(z^{-1})$ at $z = \infty$.

5.2.5 Focus on $\zeta = -1$

Let's find a solution of equation (37) which belongs to $\mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega_{-1})$. In the rescaled coordinate from Section 5.2.4, this is the point $\xi_1 = 1$. Looking again through Kummer's table of solutions, we find another expression [45, formula 15.10.14] which belongs to $\mathcal{H}L^{\infty}_{1/2,\bullet}$ at $\xi_1 = 1$:

$$\begin{aligned} v_{-1} &= (1-\xi_1)^{-1/2}\,{}_2F_1\big(\tfrac{1}{2}-\tfrac{m}{n},\tfrac{1}{2}+\tfrac{m}{n};\tfrac{1}{2};1-\xi_1\big) \\ &= \quad \sqrt{2}\,\zeta_{-1}^{-1/2}\,{}_2F_1\big(\tfrac{1}{2}-\tfrac{m}{n},\tfrac{1}{2}+\tfrac{m}{n};\tfrac{1}{2};\tfrac{1}{2}\zeta_{-1}\big) \end{aligned}$$

where ζ_{-1} is the coordinate with $\zeta = -1+\zeta_{-1}$. From the argument in Section 5.2.3, we know that $\mathcal{L}_{\zeta,-1}v_{-1}$ satisfies equation (33), and can be written as $e^{z}V_{-1}$, where $V_{-1} = \mathcal{L}_{\zeta_{-1},0}v_{-1}$. Since v_{-1}, like our other solution, has order $-1/2$, the same decomposition $V_{-1} = z^{-1/2}W_{-1}$ makes W_{-1} asymptotic to a scalar multiple of $1+O(z^{-1})$ at $z=\infty$.

In this example, v_1 and v_{-1} happen to be related by a symmetry: the Möbius transformation that pulls ζ back to $-\zeta$. Kummer's solutions typically come from six different hypergeometric equations, which are related by the Möbius transformations that permute their singularities. In our case, though, exchanging 1 with -1 keeps equation (37) the same.

5.2.6 Abstract argument for Borel regularity

The analysis in Sections 5.2.3–5.2.5 picks out a frame in the space of analytic solutions of (34). The frame is generated by solutions of the form $\mathcal{L}_{\zeta,1}v_1$ and $\mathcal{L}_{\zeta,-1}v_{-1}$, with $v_j \in \mathcal{H}L^{\infty}_{1/2,\bullet}(\Omega_j)$. We get v_j from the existence result in Section 4.1, which gives $v_j = \zeta^{\tau-1} + v'_j$ with $\tau \in (0,\infty)$ and $v'_j \in \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega)$ [What is this space? Doesn't match notation from `reg-sing-volterra`] for some $\epsilon \in (0,1]$. Looking back at the definitions of our weighted L^{∞} spaces, we can see that

$$\begin{aligned} v'_j \in \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega) &\Longrightarrow \zeta_j^{1-\tau-\epsilon}\,v'_j \text{ is bounded near } \zeta_j = 0 \\ &\Longrightarrow \zeta_j^{\epsilon}\,\zeta_j^{1-\tau-\epsilon}\,v'_j \text{ goes to 0 as } \zeta_j \text{ goes to 0} \\ &\Longrightarrow \zeta_j^{1-\tau}\,v'_j \text{ goes to 0 as } \zeta_j \text{ goes to 0} \\ &\Longrightarrow v'_j/\zeta_j^{\tau-1} \text{ goes to 0 as } \zeta_j \text{ goes to 0} \\ &\Longleftrightarrow v'_j \in o\big(\zeta_j^{\tau-1}\big). \end{aligned}$$

In this case, $\tau = \frac{1}{2}$.

The Poincaré algorithm, as we saw in Section 5.2.2, picks out a frame in the space of formal trans-series solutions of equation (34). The frame is generated by trans-monomial solutions of the form $e^{-z}z^{-1/2}\,\tilde{W}_1$ and $e^{z}z^{-1/2}\,\tilde{W}_{-1}$, with $\tilde{W}_j \in \mathbb{C}[\![z^{-1}]\!]$. We'll now show that if these solutions are Borel-summable, their Borel sums generate the same frame as $\mathcal{L}_{\zeta,1}v_1$ and $\mathcal{L}_{\zeta,-1}v_{-1}$.

The Borel transform maps $e^{-\alpha_j z}z^{-1/2}\,\mathbb{C}[\![z^{-1}]\!]$ into $\zeta_j^{-1/2}\,\mathbb{C}[\![\zeta_j]\!]$ (see Section 3.3.1), and it turns formal trans-monomial solutions of equation (34) into formal power series solutions of equation (36). Summation sends convergent power series in $\zeta_j^{-1/2}\,\mathbb{C}[\![\zeta_j]\!]$ into $\zeta_j^{-1/2} + o\big(\zeta_j^{-1/2}\big)$ [since the argument above is one-way, isn't this weaker than being in $\mathcal{H}L^{\infty}_{1/2,\bullet}$?], and it turns convergent power series that satisfy a given fractional integral equation into holomorphic functions that satisfy the same equation [30, Lemma 2]. Thus, if $e^{-z} z^{-1/2}\, \tilde{W}_1$ and $e^{z} z^{-1/2}\, \tilde{W}_{-1}$ are 1-Gevrey, their Borel transforms sum to solutions of equation (36), which lie in $\mathcal{H}L^{\infty}_{1/2,\bullet}$ [needs domain] for $\alpha = 1$ and $\alpha = -1$, respectively.

When α is 1 or -1, equation (36) becomes the kind of singular integral equation discussed in Section 4.1. It therefore has exactly one solution in $\mathcal{H}L^{\infty}_{1/2,\bullet}$ [needs domain], up to scaling. That means $\mathcal{B}\big[e^{-\alpha_j z} z^{-1/2}\, \tilde{W}_j\big]$ must sum to a scalar multiple of v_j. Thus, if $e^{-\alpha_j z} z^{-1/2}\, \tilde{W}_j$ is Borel-summable, its Borel sum is a scalar multiple of $\mathcal{L}_{\zeta,\alpha_j} v_j$.

5.2.7 Confirmation of Borel regularity

We can confirm the conclusion of Section 5.2.6 using our explicit expressions for the formal power series $\tilde{W}_\alpha$ and the functions v_α. We found in Section 5.2.2 that

$$\begin{aligned} \tilde{W}_1 &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{m}{n}\right)_k \left(\frac{1}{2} + \frac{m}{n}\right)_k}{k!} \left(-\frac{1}{2}\right)^k z^{-k} \\ \tilde{W}_{-1} &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{m}{n}\right)_k \left(\frac{1}{2} + \frac{m}{n}\right)_k}{k!} \left(\frac{1}{2}\right)^k z^{-k}. \end{aligned}$$

Computing

$$\begin{aligned} \mathcal{B}_\zeta\left[e^{-z} z^{-1/2}\, \tilde{W}_1\right] &= \mathcal{B}_\zeta\left[e^{-z} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{m}{n}\right)_k \left(\frac{1}{2} + \frac{m}{n}\right)_k}{k!} \left(-\frac{1}{2}\right)^k z^{-k-\frac{1}{2}}\right] \\ &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{m}{n}\right)_k \left(\frac{1}{2} + \frac{m}{n}\right)_k}{k!} \left(-\frac{1}{2}\right)^k \frac{\zeta_1^{k-\frac{1}{2}}}{\Gamma\left(k+\frac{1}{2}\right)} \\ &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{m}{n}\right)_k \left(\frac{1}{2} + \frac{m}{n}\right)_k}{k!} \left(-\frac{1}{2}\right)^k \frac{\zeta_1^{k-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)_k} \\ &= \frac{\zeta_1^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{m}{n}\right)_k \left(\frac{1}{2} + \frac{m}{n}\right)_k}{\left(\frac{1}{2}\right)_k} \left(-\frac{1}{2}\right)^k \frac{\zeta_1^k}{k!}, \end{aligned}$$

we see that $\mathcal{B}\big[e^{-z} z^{-1/2}\, \tilde{W}_1\big]$ sums to

$$\tfrac{1}{\Gamma(1/2)}\, \zeta_1^{-1/2}\, {}_2F_1\big(\tfrac{1}{2} - \tfrac{m}{n}, \tfrac{1}{2} + \tfrac{m}{n}; \tfrac{1}{2}; -\tfrac{1}{2}\zeta_1\big).$$

Looking back at Section 5.2.4, we recognize this as a scalar multiple of v_1. In fact, with the thimble projection reasoning, there is no ambiguity in the choice of the multiplicative constant.

Through a similar calculation, we see that $\mathcal{B}\big[e^{z} z^{-1/2}\, \tilde{W}_{-1}\big]$ sums to

$$\tfrac{1}{\Gamma(1/2)}\, \zeta_{-1}^{-1/2}\, {}_2F_1\big(\tfrac{1}{2} - \tfrac{m}{n}, \tfrac{1}{2} + \tfrac{m}{n}; \tfrac{1}{2}; \tfrac{1}{2}\zeta_{-1}\big).$$

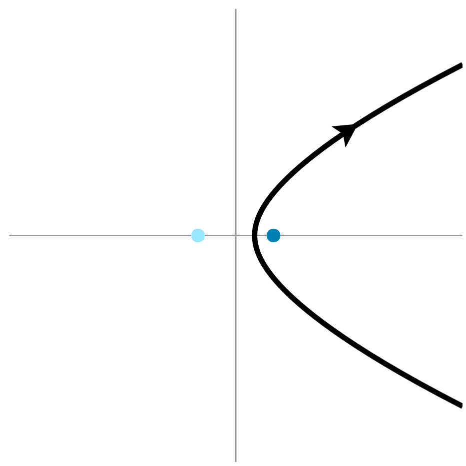
Looking back at Section 5.2.5, we recognize this as a scalar multiple of v_{-1}.

5.2.8 Thimble projection reasoning for Airy-Lucas functions

[**New phrasing for rewrite:** The critical point splits the thimble Λ into two pieces: the *incoming* branch, where the orientation of the thimble runs toward the critical point, and the *outgoing* branch, where the orientation points away.] We follow the reasoning of the proof of Theorem 1.6. We can recast integral (32) into the ζ plane by setting $-\zeta = T_n(u)$, which implies that $-d\zeta = nU_{n-1}(u)\,du$. Projecting $z^{-1/n}\Lambda^{(k)}$ to a contour γ_z in the ζ plane and choosing the branch of u that lifts γ_z back to $z^{-1/n}\Lambda^{(k)}$, we get [sign flipped]

$$\begin{aligned} K_{m/n}(z) &= \frac{n}{2\sinh\left(\frac{m}{n}\,i\pi\right)} \int_{z^{-1/n}\Lambda^{(k)}} \exp\left[zT_n(u)\right]\, U_{m-1}(u)\,du \\ &= -\frac{1}{2\sinh\left(\frac{m}{n}\,i\pi\right)} \int_{\gamma_z} e^{-z\zeta}\, \frac{U_{m-1}(u)}{U_{n-1}(u)}\,d\zeta \end{aligned}$$

For $z \in (0,\infty)$, the contour γ_z runs counterclockwise around $[1,\infty)$, as shown below, so we have to choose the negative sign above (?). Let's assume $z \in (0,\infty)$ for the rest of the section. [Our conclusions should probably hold whenever $\mathrm{Re}(z) > 0$.]



The contour γ_1 [reversed] in the ζ plane, slightly off from the critical value $\zeta = 1$.

Now we observe, that by properties of Chebyshev polynomials the integrand can be written as an explicit function of ζ:

$$\begin{aligned} \frac{U_{m-1}(\cos(\phi))}{U_{n-1}(\cos(\phi))} &= \frac{\sin(m\phi)}{\sin(\phi)}\,\frac{\sin(\phi)}{\sin(n\phi)} \\ &= \frac{\sin(m\phi)}{\sin(n\phi)} \\ &= \frac{m}{n}\;{}_2F_1\left(\frac{1}{2} - \frac{m}{2n}, \frac{1}{2} + \frac{m}{2n}; \frac{3}{2}; \sin^2(n\phi)\right) \\ &= \frac{m}{n}\;{}_2F_1\left(\frac{1}{2} - \frac{m}{2n}, \frac{1}{2} + \frac{m}{2n}; \frac{3}{2}; 1 - \zeta^2\right) \end{aligned}$$

where in the last step we use $-\zeta = T_n(u) = T_n(\cos\phi) = \cos(n\phi)$. Applying identity 15.8.4, 15.8.27 and 15.8.28 from [45],

$$
\begin{aligned}
&{}_2F_1\left(\frac{1}{2}-\frac{m}{2n},\frac{1}{2}+\frac{m}{2n};\frac{3}{2};1-\zeta^2\right) = \\
&\quad = \frac{\pi}{\Gamma\left(1-\frac{m}{2n}\right)\Gamma\left(1+\frac{m}{2n}\right)}\ {}_2F_1\left(\frac{1}{2}-\frac{m}{2n},\frac{1}{2}+\frac{m}{2n};\frac{1}{2};\zeta^2\right)+ \\
&\qquad\qquad - \frac{\pi\zeta}{\Gamma\left(\frac{1}{2}-\frac{m}{2n}\right)\Gamma\left(\frac{1}{2}+\frac{m}{2n}\right)}\ {}_2F_1\left(1-\frac{m}{2n},1+\frac{m}{2n};\frac{3}{2};\zeta^2\right) \\
&\quad = {}_2F_1\left(1-\frac{m}{n},1+\frac{m}{n};\frac{3}{2};\frac{1}{2}-\frac{\zeta}{2}\right)+{}_2F_1\left(1-\frac{m}{n},1+\frac{m}{n};\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)+ \\
&\qquad + \frac{1}{2}{}_2F_1\left(1-\frac{m}{n},1+\frac{m}{n};\frac{3}{2};\frac{1}{2}-\frac{\zeta}{2}\right)-\frac{1}{2}{}_2F_1\left(1-\frac{m}{n},1+\frac{m}{n};\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right) \\
&\quad = \frac{3}{2}{}_2F_1\left(1-\frac{m}{n},1+\frac{m}{n};\frac{3}{2};\frac{1}{2}-\frac{\zeta}{2}\right)+\frac{1}{2}{}_2F_1\left(1-\frac{m}{n},1+\frac{m}{n};\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)
\end{aligned}
$$

In addition, recall that ${}_2F_1\left(1-\frac{m}{n},1+\frac{m}{n};\frac{3}{2};\frac{1}{2}-\frac{\zeta}{2}\right)$ has a branch cut singularity at $\zeta=-1$ while ${}_2F_1\left(1-\frac{m}{n},1+\frac{m}{n};\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)$ has a branch cut at $\zeta = 1$; hence

$$
\begin{aligned}
K_{m/n}(z) &= -\frac{m}{4n\sinh(\frac{m}{n}i\pi)}\int_{\gamma_z} e^{-z\zeta}{}_2F_1\left(1-\frac{m}{n},1+\frac{m}{n};\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)d\zeta \\
&= -\frac{1}{2}\int_1^{+\infty} e^{-z\zeta}\left(-\frac{1}{2}+\frac{\zeta}{2}\right)^{-1/2}{}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};\frac{1}{2}-\frac{\zeta}{2}\right)d\zeta
\end{aligned}
$$

where in the second step we use the analytic continuation of the hypergeometric function [45, Equation 15.2.3].

Finally, comparing this expression for $K_{m/n}$ with the expression for v_1 computed in Section A.2.2 we notice that

$$K_{m/n}(z) = \frac{i}{2}\mathcal{L}_{\zeta,1}v_1.$$

5.2.9 A flavor of resurgence: the Stokes phenomena in the position domain

In the previous sections, we have seen how to compute the Borel transform of a formal frame of solutions of the modified Bessel equation with parameter m/n (Section 5.2.7) or equivalently of the asymptotic expansion of thimble integrals $K_{m/n}$ (Section 5.2.8). In both cases we find

$$v_1(\zeta) = i\left(-\frac{1}{2}+\frac{\zeta}{2}\right)^{-1/2}{}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};\frac{1}{2}-\frac{\zeta}{2}\right)$$

which has a fractional power singularity at $\zeta = 1$. In addition, the hypergeometric function ${}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};\frac{1}{2}-\frac{\zeta}{2}\right)$ is convergent at $\zeta=1$ therefore v_1 has the typical form of a *singularity* in the formalism of Écalle's resurgent functions. More precisely, we can consider ${}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};\frac{1}{2}-\frac{\zeta}{2}\right)$ as a new holomorphic function (analytic continuation of a new germ at the origin), which has a branch cut singularity at $\zeta=-1$. Using equation [45, 15.2.3], we can compute the analytic continuation of this germ at $\zeta=-1$:

$$\begin{aligned}
&{}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};-\frac{\zeta_1}{2}+i\varepsilon\right)-{}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};-\frac{\zeta_1}{2}-i\varepsilon\right)=\\
&=\frac{2\pi i}{\Gamma\left(\frac{1}{2}-\frac{m}{n}\right)\Gamma\left(\frac{1}{2}+\frac{m}{n}\right)}\left(-\frac{\zeta_{-1}}{2}\right)^{-1/2}{}_2F_1\left(\frac{m}{n},-\frac{m}{n};\frac{1}{2};\frac{\zeta_{-1}}{2}\right)\\
&=2i\cos(\pi m/n)\left(-\frac{\zeta_{-1}}{2}\right)^{-1/2}{}_2F_1\left(\frac{m}{n},-\frac{m}{n};\frac{1}{2};\frac{\zeta_{-1}}{2}\right)\\
&=2\cos(\pi m/n)\left(\frac{\zeta_{-1}}{2}\right)^{-1/2}{}_2F_1\left(\frac{m}{n},-\frac{m}{n};\frac{1}{2};\frac{\zeta_{-1}}{2}\right)\\
&=2\cos(\pi m/n)\left(\frac{\zeta_{-1}}{2}\right)^{-1/2}\left(-\frac{\zeta_1}{2}\right)^{1/2}{}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};\frac{\zeta_{-1}}{2}\right)
\end{aligned}$$

Hence, we deduce

$$v_1(\zeta+1+i\varepsilon)-v_1(\zeta+1+i\varepsilon)=2i\cos(\pi m/n)v_{-1}(\zeta) \tag{40}$$

namely, that the germ of the singularity at $\zeta=1$ knows about the germ of the singularity at $\zeta=-1$. This is an instance of *resurgent functions*, introduced by Écalle in [26, 27, 28][16]. Although we'll not discuss more about resurgence, we would like to point out that among the great advantages of resurgence (for instance compared to Borel summability) there are the *formalism of singularities* and the *Alien calculus* which allow to understand the Stokes phenomena working in the position domain. The same phenomena were described by Berry and Howls [8] who noticed that divergent asymptotic expansions of thimble integrals encode the contribution of other critical values.

We can repeat the same reasoning with v_{-1}:

$$v_{-1}(\zeta)=\left(\frac{1}{2}+\frac{\zeta}{2}\right)^{-1/2}{}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};\frac{1}{2}+\frac{\zeta}{2}\right)$$

and ${}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};\frac{1}{2}+\frac{\zeta}{2}\right)$ is singular at $\zeta=1$. Hence

$$\begin{aligned}
&{}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};\frac{\zeta_{-1}}{2}+i\varepsilon\right)-{}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};\frac{\zeta_{-1}}{2}-i\varepsilon\right)=\\
&=\frac{2\pi i}{\Gamma\left(\frac{1}{2}-\frac{m}{n}\right)\Gamma\left(\frac{1}{2}+\frac{m}{n}\right)}\left(\frac{\zeta_1}{2}\right)^{-1/2}{}_2F_1\left(\frac{m}{n},-\frac{m}{n};\frac{1}{2};-\frac{\zeta_1}{2}\right)\\
&=2i\cos(\pi m/n)\left(\frac{\zeta_1}{2}\right)^{-1/2}{}_2F_1\left(\frac{m}{n},-\frac{m}{n};\frac{1}{2};-\frac{\zeta_1}{2}\right)\\
&=2i\cos(\pi m/n)\left(\frac{\zeta_1}{2}\right)^{-1/2}\left(\frac{\zeta_{-1}}{2}\right)^{1/2}{}_2F_1\left(\frac{1}{2}-\frac{m}{n},\frac{1}{2}+\frac{m}{n};\frac{1}{2};-\frac{\zeta_1}{2}\right)
\end{aligned}$$

[16] If the reader is interested in resurgence, it may be useful to look at the following review [52, 23, 2].

and the analogue of (40) is

$$v_{-1}(\zeta-1+i\varepsilon)-v_{-1}(\zeta-1-i\varepsilon)=-2i\cos(\pi m/n)v_1(\zeta). \tag{41}$$

Equations (40) and (41) are a manifestation of the Stokes phenomena, already in the position domain. Indeed, when we start varying the contour of integration for the Laplace transform, we know that $\mathcal{L}^{\pi}_{\zeta,1}$ and $\mathcal{L}^{0}_{\zeta,-1}$ are not well defined because the contour hits the other singularity (respectively at $\zeta=-1$ and $\zeta=1$). The Stokes phenomenon consists of comparing the Laplace transform below and above the critical direction and it quantifies the jump as a constant (the so-called Stokes constant) times another function. In our example:

$$\begin{aligned}\mathcal{L}^{\pi+\varepsilon}_{\zeta,1}v_1-\mathcal{L}^{\pi-\varepsilon}_{\zeta,1}v_1&=2i\cos(\pi m/n)\,e^{-z}\,\mathcal{L}^{\pi}_{\zeta,-1}v_{-1}\\ \mathcal{L}^{\varepsilon}_{\zeta,-1}v_{-1}-\mathcal{L}^{-\varepsilon}_{\zeta,-1}v_{-1}&=-2i\cos(\pi m/n)\,e^{z}\,\mathcal{L}_{\zeta,1}v_1\end{aligned}$$

thus the Stokes constants are respectively $\pm 2i\cos(\pi m/n)$[17].

It may be surprising that the Stokes constants depend on $\cos(\pi m/n)$, as the Stokes constants are the intersection number of *dual pairs of thimbles*, according to Picard–Lefschetz formula (in particular, they are integers in a suitable normalization), see [58, Section 5] and [3, Chapter ??]. However, in the Airy–Lucas example, f is not Morse (unless $n=3$), hence we have to interpret the $\cos(\pi m/n)$ as a consequence of the ambiguity on the lift of a path in the position domain. I think we should argue we have the action of the permutation which leaves the base unchanged but gives a

5.3 Modified Bessel

The modified Bessel equation is a generalization of equation (33) where $\frac{m}{n}$ is replaced by a complex parameter μ

$$\left[z^2\left(\tfrac{\partial}{\partial z}\right)^2+z\tfrac{\partial}{\partial z}-\left[\mu^2+z^2\right]\right]\varphi=0 \tag{42}$$

On the one hand, this new condition does not really affect the argument we present in Section 5.2.6, so we briefly state the main results:

- **Asymptotic analysis**: equation (42) admits a basis of formal solutions

$$\tilde{I}_{\mu,1}(z)=e^{-z}z^{-1/2}\tilde{W}_{\mu,1}\qquad\text{and}\qquad\tilde{I}_{\mu,-1}(z)=e^{z}z^{-1/2}\tilde{W}_{\mu,-1}$$

  with

$$\begin{aligned}\tilde{W}_{\mu,1}&=1-\frac{\left(\frac12-\mu\right)\left(\frac12+\mu\right)}{2\cdot 1!}z^{-1}+\frac{\left(\frac12-\mu\right)_2\left(\frac12+\mu\right)_2}{2^2\cdot 2!}z^{-2}-\frac{\left(\frac12-\mu\right)_3\left(\frac12+\mu\right)_3}{2^3\cdot 3!}z^{-3}+\dots\\ \tilde{W}_{\mu,-1}&=1+\frac{\left(\frac12-\mu\right)\left(\frac12+\mu\right)}{2\cdot 1!}z^{-1}+\frac{\left(\frac12-\mu\right)_2\left(\frac12+\mu\right)_2}{2^2\cdot 2!}z^{-2}+\frac{\left(\frac12-\mu\right)_3\left(\frac12+\mu\right)_3}{2^3\cdot 3!}z^{-3}+\dots\end{aligned}$$

- **Frame of analytic solutions**: there exist two functions $v_{\mu,1},v_{\mu,-1}$ such that $\mathcal{L}_\alpha v_{\mu,\alpha}$ satisfies equation (42) and they are explicitly

$$\begin{aligned}v_{\mu,1}&=-i\sqrt{2}\,\zeta_1^{-1/2}\,{}_2F_1\left(\tfrac12-\mu,\tfrac12+\mu;\tfrac12;-\tfrac12\zeta_1\right)\\ v_{\mu,-1}&=\sqrt{2}\,\zeta_{-1}^{-1/2}\,{}_2F_1\left(\tfrac12-\mu,\tfrac12+\mu;\tfrac12;\tfrac12\zeta_{-1}\right)\end{aligned}$$

[17] If we think of the Laplace transforms of v_1 and v_{-1} as a frame of solutions for the Airy–Lucas differential equation, the equations above define the so-called Stokes matrices for the ODE.

- **Borel regularity** We can compare the Borel transform of the formal solutions $\tilde{W}_{\mu,\pm1}$ with the analytic solutions $v_{\mu,\pm1}$. They agree (up to the choice of a constant), hence we have another example where Borel regularity can be explicitly verified.

On the other hand, the thimble projection reasoning we describe in Section 5.2.8 has to be generalized, as we'll discuss in the following Section 5.3.1.

5.3.1 Lifting to a countable cover

Formula (32) expresses the modified Bessel function $K_{m/n}$ as an exponential integral on a finite cover of $\mathbb{C}$. Lifting to a countable cover reveals this formula as a special case of a general integral formula for modified Bessel functions.

Setting $u = \cosh(t/n)$ and recalling that

$$\begin{aligned}\cosh(n\tau) &= T_n(\cosh(\tau))\\ \sinh(m\tau) &= U_{m-1}(\cosh(\tau))\sinh(\tau),\end{aligned}$$

we can rewrite formula (32) as [switching to the conventional sign for the projection map, so $\Lambda^{(3)}$ now comes from ∞ at $-60°$ and goes to ∞ at $60°$]

$$\begin{aligned}K_{m/n}(z) &= \frac{n}{2\sinh\left(\frac{m}{n}i\pi\right)}\int_{z^{-1/n}\Lambda^{(k)}}\exp\left[zT_n(u)\right]U_{m-1}(u)\,du\\ &= \frac{n}{2\sinh\left(\frac{m}{n}i\pi\right)}\int_{\Lambda_j}\exp\left[z\cosh(t)\right]U_{m-1}(\cosh(t/n))\sinh(t/n)\,d(t/n)\\ &= \frac{1}{2\sinh\left(\frac{m}{n}i\pi\right)}\int_{\Lambda_j}\exp\left[z\cosh(t)\right]\sinh\left(\frac{m}{n}t\right)dt.\end{aligned}\tag{43}$$

where Λ_j is the path coming from infinity along $-i\pi+(2\pi i)j+(-\infty,0]$ and going to infinity along $i\pi+(2\pi i)j+[0,+\infty)$, as represented in Figure 8. In fact, since $\cosh(t)$ is periodic, the integral has the same value for every choice of Λ_j with $j\in\mathbb{Z}$.

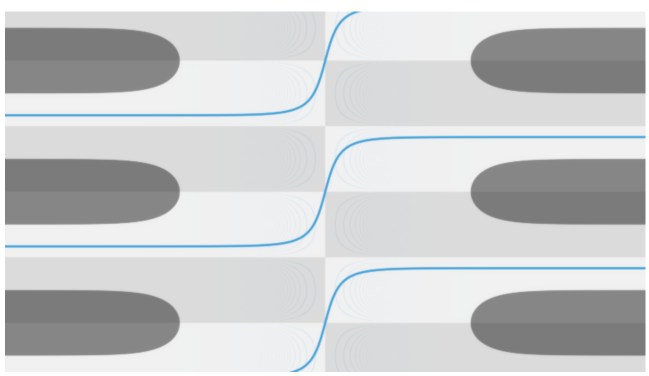


Figure 8: The paths Λ_j.

For any $\mu \in \mathbb{C} \setminus \mathbb{Z}$, we have a thimble integral representation of K_μ given by formulas 10.32(ii) and 10.27.4 of [45][Cite Watson too?]

$$K_\mu(z) = \frac{1}{2\sinh(\mu\, i\pi)} \int_{\Lambda_0} \exp\left[z\cosh(t)\right] \sinh(\mu t)\, dt, \tag{44}$$

Following Watson *A treatment of Bessel function*, 1944, pp. 181 (it's the reference in [45])

$$\begin{aligned}
2\pi i\, I_\mu(z) &= \int_{+\infty}^{0} e^{z\cosh(-\pi i+a)-\mu(-i\pi+a)} da + \int_{0}^{+\infty} e^{z\cosh(\pi i+a)-\mu(i\pi+a)} da + \int_{-\pi}^{\pi} e^{z\cosh(ib)-\mu ib} db \\
&= -\int_{-\infty}^{0} e^{z\cosh(-\pi i-a)+\mu(i\pi+a)} da + \int_{0}^{+\infty} e^{z\cosh(\pi i+a)-\mu(i\pi+a)} da + \int_{0}^{\pi} e^{z\cosh(ib)} \cosh(i\mu b) db \\
&= -\int_{-\infty}^{0} e^{z\cosh(\pi i+a)+\mu(i\pi+a)} da + \int_{0}^{+\infty} e^{z\cosh(\pi i+a)-\mu(i\pi+a)} da + \int_{0}^{\pi} e^{z\cosh(ib)} \cosh(i\mu b) db
\end{aligned}$$

hence

$$\begin{aligned}
4i\sin(\mu\pi)K_\mu(z) &= 2\pi i\big[I_{-\mu}(z) - I_\mu(z)\big] \\
&= \int_{-\infty}^{0} e^{z\cosh(\pi i+a)} \left[-e^{-\mu(i\pi+a)} + e^{\mu(i\pi+a)}\right] da + \int_{0}^{+\infty} e^{z\cosh(\pi i+a)} \left[e^{\mu(i\pi+a)} - e^{-\mu(i\pi+a)}\right] da \\
&= 2\int_{-\infty}^{0} e^{z\cosh(\pi i+a)} \sinh(a+i\pi) da + 2\int_{0}^{+\infty} e^{z\cosh(\pi i+a)} \sinh(i\pi+a) da \\
&= 2\int_{-\infty}^{0} e^{z\cosh(-\pi i+a)} \sinh(a-i\pi) da + 2\int_{0}^{+\infty} e^{z\cosh(\pi i+a)} \sinh(i\pi+a) da
\end{aligned}$$

The integral converges when z is in the right half-plane. We get formula (43) when we choose a rational parameter $\mu = m/n$.

We can now follow Section 5.2.8 to compute the Borel transform of the asymptotic of K_μ as an inverse Laplace transform: let $\zeta = -\cosh(t)$

$$\begin{aligned}
K_\mu(z) &= -\frac{1}{2\sinh(\mu\, i\pi)} \int_{\gamma_z} e^{-z\zeta} \frac{\sinh(\mu t)}{\sinh(t)}\, d\zeta \\
&= -\frac{\mu}{2\sinh(\mu\, i\pi)} \int_{\gamma_z} e^{-z\zeta}\, {}_2F_1\left(\frac{1+\mu}{2}, \frac{1-\mu}{2}; \frac{3}{2}; 1-\zeta^2\right) d\zeta \qquad [45, 15.4.16]
\end{aligned}$$

where γ_z is Hankel contour coming from ∞ to 1 and then going back to ∞. Using formula [45, 15.8.4], followed by [45, 15.8.27] and [45, 15.8.28]

$$\begin{aligned}
&{}_2F_1\left(\frac{1+\mu}{2}, \frac{1-\mu}{2}; \frac{3}{2}; 1-\zeta^2\right) = \\
&\qquad = \frac{3}{2}\, {}_2F_1\left(1-\mu, 1+\mu, \frac{3}{2}; \frac{1-\zeta}{2}\right) + \frac{1}{2}\, {}_2F_1\left(1-\mu, 1+\mu, \frac{3}{2}; \frac{1+\zeta}{2}\right)
\end{aligned}$$

hence

$$K_\mu(z) = \frac{\mu\, i}{4\sin(\mu\pi)} \int_{\gamma_z} e^{-z\zeta}\, {}_2F_1\left(1-\mu, 1+\mu; \frac{3}{2}; \frac{1+\zeta}{2}\right) d\zeta$$

Equation [45, 15.2.3] gives the analytic continuation of hypergeometric functions across the branch cut:

$$
\begin{aligned}
K_\mu(z) &= -\frac{1}{2}\int_1^\infty e^{-z\zeta}\left(\frac{\zeta-1}{2}\right)^{-1/2} {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{1}{2}; \frac{1-\zeta}{2}\right) d\zeta \\
&= -\frac{i}{2}\mathcal{L}_{\zeta,1} v_{\mu,1}.
\end{aligned} \tag{45}
$$

5.3.2 The case $\mu = 0$

When μ goes to 0, formula (44) becomes

$$K_0(z) = \frac{1}{2\pi i}\int_\Omega \exp\left[z\cosh(t)\right]\, t\, dt.$$

Choosing Ω to be the unit-speed path that runs from ∞ leftward to $-i\pi$, upward to $i\pi$, and rightward back to ∞, we can rewrite this formula as

$$
\begin{aligned}
K_0(z) &= \frac{1}{2\pi i}\int_0^\infty \exp\left[-z\cosh(t)\right]\, 2\pi i\, dt \\
&= \int_0^\infty \exp\left[-z\cosh(t)\right]\, dt \\
&= \int_1^\infty \exp\left[-z\,\tfrac{1}{2}\left(s+\tfrac{1}{s}\right)\right]\,\frac{ds}{s},
\end{aligned}
$$

with $s = e^t$. This is a special case of formula 10.32.9 from [45]. Then equation (45) gives

$$K_0(z) = -\frac{1}{2}\int_1^\infty e^{-z\zeta}\left(\frac{\zeta-1}{2}\right)^{-1/2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{1-\zeta}{2}\right) d\zeta = \int_1^\infty \frac{e^{-z\zeta}}{\sqrt{\zeta^2-1}}\, d\zeta. \tag{46}$$

We can also verify equation (46) using the thimble projection reasoning: from the expression of K_0

$$K_0(z) = \int_0^\infty \exp(-z\cosh(t))dt \tag{47}$$

we set $\zeta = \cosh(t)$, and choosing a branch for the square root

$$
\begin{aligned}
d\zeta &= \sinh(t)dt \\
&= \sqrt{\zeta^2-1}\, dt
\end{aligned}
$$

we finally get

$$\int_1^\infty e^{-z\zeta}\frac{d\zeta}{\sqrt{\zeta^2-1}}$$

5.4 Generalized Airy

In [64] and [24, Appendix] the authors introduce generalized Airy functions A_k, B_0, B_k, $k = 1, 2, 3$ as approximate solutions of the Orr–Sommerfield fluid equation. They are defined as contour integral [45, Section 9.13(ii)]

$$\begin{aligned}
\mathrm{A}_k(y,p) &= \frac{1}{2\pi i}\int_{\Gamma_k} e^{yt-\frac{t^3}{3}}\frac{dt}{t^p} \qquad k=1,2,3\ p\in\mathbb{C}\\
\mathrm{B}_0(y,p) &= \frac{1}{2\pi i}\int_{\Gamma_0} e^{yt-\frac{t^3}{3}}\frac{dt}{t^p} \qquad p\in\mathbb{Z}\\
\mathrm{B}_k(y,p) &= \int_{\gamma_k} e^{yt-\frac{t^3}{3}}\frac{dt}{t^p} \qquad k=1,2,3\ p\in\mathbb{Z}
\end{aligned}$$

where the contours $\Gamma_k, \Gamma_0, \gamma_k$ are represented in Figure 9.

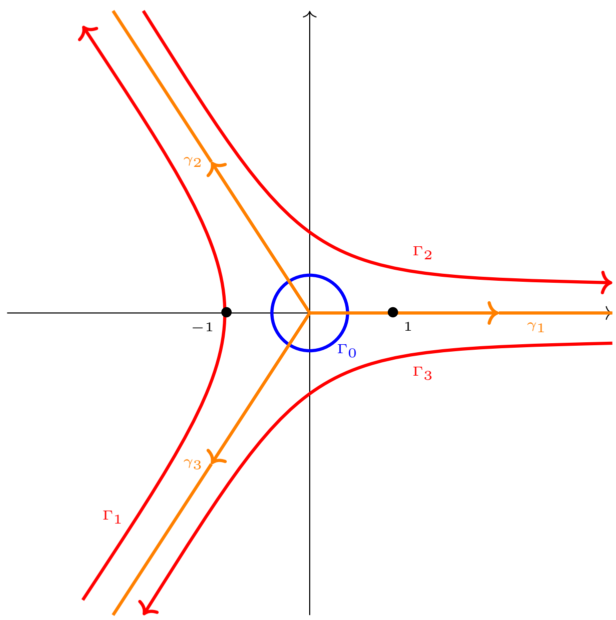


Figure 9: Representation of the path of integration for $\mathrm{A}_k, \mathrm{B}_0, \mathrm{B}_k$ respectively in red, blue and orange.

Each generalized Airy function is a solution of

$$\left[\partial_y^3 - y\partial_y + (p-1)\right] f = 0 \tag{48}$$

and if $p = 0$ they reduce to the classical Airy functions.

Following the treatment of the Airy–Lucas functions, we'll rewrite the generalized Airy function $\mathrm{A}_1(y,p)$ as thimble integrals with the function $f = 4u^3 - 3u$. This follows from the change of coordinates $z = \frac{2}{3}y^{3/2}$, and gives $\mathrm{A}_1(y,p) = (12z)^{\frac{1-p}{3}} I_+(z,p)$ with

$$I_+(z,p) := \frac{1}{2\pi i}\int_{z^{-1/3}\Gamma_1} e^{-z(4u^3-3u)}\frac{du}{u^p}$$

$$\begin{aligned}
I_+(z,p) &= \frac{1}{2\pi i}\int_{\Lambda_+} e^{-z(4u^3-3u)}\,\frac{du}{u^p} \\
&= \frac{1}{2\pi i}(12z)^{(p-1)/3}\int_{z^{1/3}\Lambda_+} e^{-z(\frac{4}{12}\frac{t^3}{z}-3(12z)^{-1/3}t)}\,\frac{dt}{t^p} \qquad\qquad u = (12z)^{-\frac{1}{3}}t \\
&= \frac{1}{2\pi i}(12z)^{(p-1)/3}\int_{z^{1/3}\Lambda_+} e^{-\left(\frac{t^3}{3}-(\frac{3}{2}z)^{2/3}t\right)}\,\frac{dt}{t^p} \\
&= (12z)^{(p-1)/3}\mathrm{A}_1((\tfrac{3}{2}z)^{2/3},p)
\end{aligned}$$

Notice that, the critical points of f are $\alpha_\pm = \pm\frac{1}{2}$, as in the Airy case. However, the volume form ν_p is meromorphic, hence f should be regarded as a function from $\mathbb{C}^*$ to $\mathbb{C}$. In particular, in this example we'll see that there is a new singularity at the origin due to the singularity of ν_p. For simplicity, we'll restrict to the case $p=1$. Then, $I_+(z,1)$ solves

$$\left[\partial_z^3 - \partial_z + \frac{1}{z}\partial_z^2 - \frac{1}{9}\frac{\partial_z}{z^2}\right]\varphi = 0 \tag{49}$$

or equivalently, $\partial_z\varphi$ solves the modified Bessel equation (33) with parameter $1/3$. Therefore, following the reasoning of Section 5.2.3 and the results of Sections 5.2.4 and 5.2.5, we get two analytic solutions of (49): $\mathcal{L}_{\zeta,1}v_{1,1}$ and $\mathcal{L}_{\zeta,-1}v_{-1,1}$

$$v_{1,1} = -i\sqrt{2}\,(\zeta_1+1)^{-1}\,\zeta_1^{-1/2}\,{}_2F_1\!\left(\tfrac{1}{6},\tfrac{5}{6};\tfrac{1}{2};-\tfrac{1}{2}\zeta_1\right)$$

$$v_{-1,1} = -\sqrt{2}\,(1-\zeta_{-1})^{-1}\,\zeta_{-1}^{-1/2}\,{}_2F_1\!\left(\tfrac{1}{6},\tfrac{5}{6};\tfrac{1}{2};\tfrac{1}{2}\zeta_{-1}\right)$$

where $\zeta_1 = \zeta - 1$ and $\zeta_{-1} = \zeta + 1$. From the result about the Bessel equation, we know there exists two analytic functions v_1 and v_{-1} such that φ_1 and φ_2 defined by $\partial_z\varphi_1 = \mathcal{L}_{\zeta,1}v_1$ and $\partial_z\varphi_2 = \mathcal{L}_{\zeta,-1}v_{-1}$, give a frame of analytic solutions of (49). Then we argue as follows:

$$\begin{aligned}
\partial_z\varphi_1 &= \mathcal{L}_{\zeta,1}v_1 \\
\varphi_1 - \varphi_1(a) &= \int_a^z\int_1^\infty e^{-y\zeta}v_1 d\zeta\,dy \\
&= \int_1^\infty d\zeta\,v_1\int_a^z e^{-y\zeta}\,dy \\
&= \int_1^\infty d\zeta\,v_1\left[\frac{e^{-\zeta a}}{\zeta} - \frac{e^{-z\zeta}}{\zeta}\right] \\
&= \mathcal{L}_{\zeta,1}\tfrac{v_1}{\zeta} - \left[\mathcal{L}_{\zeta,1}\tfrac{v_1}{\zeta}\right](a)
\end{aligned}$$

Notice that $v_{1,1} \in \mathcal{H}L^\infty_{1/2,\bullet}(\Omega_1)$, it has a simple pole at $\zeta_1 = -1$ and it is a log–singularity at $\zeta_1 = -2$. Similarly, $v_{-1,1} \in \mathcal{H}L^\infty_{1/2,\bullet}(\Omega_{-1})$, it has a simple pole at $\zeta_{-1} = 1$ and it is a log–singularity at $\zeta_{-1} = 2$.

Notice that $\mathcal{L}_{\zeta,1}v_{1,1}$ and $\mathcal{L}_{\zeta,-1}v_{-1,1}$ are not a frame of solutions; the equation (49) is of degree three, but it is enough to consider the constant function $1 = \mathcal{L}_\zeta\delta$ to build a frame of analytic solutions, where δ is the formal unit (see the definition in Section 3.4.1). In particular, $1 = B_0(z, 1)$ and confirms Borel regularity for the constant solution.

5.4.1 Thimble projection formula

In Theorem **??**, we proved a 3/2-derivative formula to compute the Borel transform of the asymptotics of a thimble integral.

5.5 Third-order thimble integrals

Let $f: \mathbb{C} \to \mathbb{C}$ be a degree three polynomial[18], we are going to consider thimble integrals of the form

$$I_j(z) = \int_{\Lambda_j} \exp\left[z\big(u^3 + pu + q\big)\right] \, du$$

where Λ_j is a thimble in $H_1^{B,z}$ through a critical point of f. We can distinguish between the case when f has two distinct critical points x_+, x_- and only one critical point x, which correspond respectively to the case $p \neq 0$ and $p = 0$. [19] Recall that translations and scaling-rotations act naturally on the image of $f: X \to \mathbb{C}$ (f not necessarily a degree three polynomial), and their action transforms the thimble integral I_j in the following ways:

- **Action of translations:** let $q \in \mathbb{C}$, post-composing with translation of q maps $f \to f + q$, thus

$$\int_{\Lambda_j} e^{z(f+q)}\nu = e^{qz} I_j(z)$$

- **Action of scaling-rotations:** let $\gamma \in \mathbb{C}$, post-composing with scaling-rotations by γ maps $f \to \gamma \cdot f$, thus

$$\int_{\Lambda_j} e^{z(\gamma \cdot f)}\nu = I_j(z\gamma)$$

In particular, we can always reduce to study the thimble integral in simpler situations. As an example, every degree three polynomial with distinct critical points can be transformed in the Chebyshev polynomial $T_3(t) = 4t^3 - 3t$, implying that the thimble integral I_+ can

[18]Every degree three polynomial $a_3u^3 + a_2u^2 + a_1u + a_0$ can be written in the form $a_3u^3 + pu + q$, after a change of coordinates $u \to u - \frac{a_2}{3a_3}$. Furthermore, we can restrict to monic polynomials by setting $a_3 = 1$.

[19]Equivalently, the difference between the generic case ($p \neq 0$) and the degenerate one ($p = 0$) is visible from the ODE perspective as in the former case I is a solution of degree two ODE

$$\left[\partial_z^2 + 2q\partial_z + \left(\frac{4p^3}{27} + q^2\right) + \frac{1}{z}\left(\partial_z + q\right) - \frac{1}{9z^2}\right]\Phi = 0,$$

while in the latter, I solves a degree one ODE (see (50)).

be transformed into the Airy–Lucas integral with parameter $m = 1, n = 3$. This follows by post-composing f with a translation of $-q$ followed by a scaling-rotation of $\frac{-i3^{3/2}}{2p^{3/2}}$

$$\begin{aligned} f - q &= u^3 + pu \\ (f-q) \cdot \frac{-i3^{3/2}}{2p^{3/2}} &= 4\Big(\sqrt{-\tfrac{3}{p}}\tfrac{u}{2}\Big)^3 - 3\Big(\sqrt{-\tfrac{3}{p}}\tfrac{u}{2}\Big) \\ &= 4t^3 - 3t \end{aligned}$$

with $t = \sqrt{-\frac{3}{p}}\frac{u}{2}$. Then the thimble integral I_+ gets transformed into

$$\begin{aligned} e^{-zq} I_+\left(-\tfrac{i3^{3/2}}{2p^{3/2}} z\right) &= \int_{\Lambda_+} e^{z(4t^3-3t)} du \\ &= 2\sqrt{-\tfrac{p}{3}} \int_{\Lambda_+} e^{z(4t^3-3t)} dt \\ &= 2\sqrt{-\tfrac{p}{3}} K_{1/3}(z). \end{aligned}$$

In particular, when $p \neq 0$, the thimble integral I_+ can be studied following the analysis of Appendix A.

We then consider the case when $p = 0$, and we study it both as a solution of an ODE and as a thimble integral. First, notice that $I(z)$ solves

$$\left[\partial_z + q + \frac{1}{3z}\right] I = 0 \tag{50}$$

and we can find a Borel regular solution following the argument of Section 5.2.3: $\mathcal{L}^{\theta}_{\zeta,\alpha} v$ is a solution of (50) if and only if v is a solution of

$$\left[(\zeta - q)\partial_\zeta + \frac{2}{3}\right] v = 0$$

namely $v = (\zeta - q)^{-2/3} = \zeta_q^{-2/3} = {}_2F_1\left(a, \frac{2}{3}; a; 1 - \zeta_q\right)$, where $\zeta_q = \zeta - q$ and $a \in \mathbb{C}$ is a free parameter.

The result is confirmed by the thimble projection reasoning which in this simple case reads

$$\begin{aligned} \int_\Lambda e^{-z(u^3+q)} du &= e^{-zq} e^{i\pi/3} \int_0^{+e^{i\theta}\infty} e^{-z\zeta} \zeta^{-2/3} d\zeta \\ &= e^{i\pi/3} \int_q^{+e^{i\theta}\infty} e^{-z\zeta}\, \zeta_q^{-2/3}\, d\zeta \end{aligned}$$

where $e^{\pi i/3}$ accounts for the monodromy of $\zeta^{-2/3}$ at the origin.

Remark 5.1. The degenerate case $p = 0$ can be recovered from the generic case, taking the limit of $I_+(z)$ as $p \to 0$. Indeed,

$$e^{-zq} I_+(z) = 2\sqrt{-\frac{p}{3}} K_{1/3}\Big(\frac{2p^{3/2}}{i3^{3/2}} z\Big)$$

and its limit as $p \to 0$ can be computed from equations 10.25.2 and 10.27.4 in [45] giving

$$e^{\pi i/3}\Gamma\big(\tfrac{1}{3}\big)e^{-zq}z^{-1/3}.$$

$$\begin{aligned} 2\sqrt{\frac{-p}{3}}K_{1/3}\Big(\frac{2ip^{3/2}}{3^{3/2}}z\Big) &= 2\sqrt{\frac{-p}{3}}\left[\frac{\pi}{2\sin(\pi/3)}\Big(I_{-1/3}\big(\tfrac{2ip^{3/2}}{3^{3/2}}z\big) - I_{1/3}\big(\tfrac{2ip^{3/2}}{3^{3/2}}z\big)\Big)\right] \\ &= \frac{2}{3}\pi i p^{1/2}\left[\Big(\frac{ip^{3/2}}{3^{3/2}}z\Big)^{-1/3}\sum_{k\geq 0}\frac{(-\frac{p^3}{3}z^2)^k}{k!\Gamma(k+\frac{2}{3})} - \Big(\frac{ip^{3/2}}{3^{3/2}}z\Big)^{1/3}\sum_{k\geq 0}\frac{(-\frac{p^3}{3}z^2)^k}{k!\Gamma(k+\frac{4}{3})}\right] \\ &= \frac{2\pi}{\sqrt{3}}e^{\pi i/3}z^{-1/3}\sum_{k\geq 0}\frac{(-\frac{p^3}{3}z^2)^k}{k!\Gamma(k+\frac{2}{3})} - \frac{2\pi}{3\sqrt{3}}e^{2\pi i/3}z^{1/3}p\sum_{k\geq 0}\frac{(-\frac{p^3}{3}z^2)^k}{k!\Gamma(k+\frac{4}{3})} \end{aligned}$$

Now we take the limit as $p \to 0$ and we find

$$\frac{2\pi}{\sqrt{3}}e^{\pi i/3}z^{-1/3}\frac{1}{\Gamma\big(\frac{2}{3}\big)} = e^{\pi i/3}\Gamma\big(\tfrac{1}{3}\big)z^{-1/3}$$

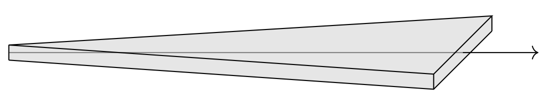
From the properties of the Laplace transform, we then deduce

$$e^{\pi i/3}\Gamma\big(\tfrac{1}{3}\big)e^{-zq}z^{-1/3} = e^{\pi i/3}e^{-qz}\mathcal{L}_{\zeta,0}(\zeta^{-2/3}) = e^{\pi i/3}\mathcal{L}_{\zeta_{-q},q}(\zeta^{-2/3})$$

which agrees with the thimble integral representation of I computed above.

5.6 The triangular cantilever

A *triangular cantilever* is a flexible strip with constant thickness and linearly tapered width, clamped at the broad end so it sticks out horizontally like a diving board.



If you strike the strip from above, it vibrates up and down. Let's suppose the vibrations are small, and uniform across the width of the strip, so we can describe them using the Euler–Bernoulli beam model [**?**, §12.4]. The vibration modes of frequency ω are described by the vertical displacement profiles Φ that satisfy the equation

$$\Big[\Big[\big(\tfrac{\partial}{\partial z}\big)^4 - \omega^2\Big] + \tfrac{2}{z}\big(\tfrac{\partial}{\partial z}\big)^3\Big]\,\Phi = 0, \tag{51}$$

where z is the distance from the tip along the strip's axis.[20]

The vertical displacement of a vibrating beam is annihilated by the operator

$$\big(\tfrac{\partial}{\partial z}\big)^2 \circ EI_{\text{roll}} \circ \big(\tfrac{\partial}{\partial z}\big)^2 + m\big(\tfrac{\partial}{\partial t}\big)^2,$$

[20]To keep the equation simple, we've adjusted the units of time so that the strip's elasticity, density, and taper are absorbed into the frequency parameter.

as described in equation 12.58 of [**?**]. Here, m is the mass per unit length, E is the elastic modulus, and I_{roll} is the second moment of area about the axis the beam "rolls" around as it vibrates. We can write the displacement as $\cos(\omega t)$ times a time-independent envelope, which is annihilated by the operator

$$\mathcal{P} = \left(\tfrac{\partial}{\partial z}\right)^2 \circ EI_{\text{roll}} \circ \left(\tfrac{\partial}{\partial z}\right)^2 - m\omega^2.$$

We're assuming the cantilever has a rectangular cross-section with half-width s and half-thickness t, so its second moment of area about the side-to-side axis is $\frac{4}{3}st^3$, mass per unit length isproportional to st, giving

$$\mathcal{P} = \left(\tfrac{\partial}{\partial z}\right)^2 \circ \tfrac{4}{3}Et^3 s \circ \left(\tfrac{\partial}{\partial z}\right)^2 - t\omega^2 s.$$

For the triangular cantilever, t is constant, and $s = \sigma z$ for some constant σ. Letting $c_1 = \frac{4}{3}\sigma E t^3$ and $c_2 = t\sigma\omega^2$, we have

$$\begin{aligned}
\mathcal{P} &= \left(\tfrac{\partial}{\partial z}\right)^2 \circ c_1 z \circ \left(\tfrac{\partial}{\partial z}\right)^2 - c_2 z \\
&= c_1 \tfrac{\partial}{\partial z} \circ \left[1 + z\tfrac{\partial}{\partial z}\right] \circ \left(\tfrac{\partial}{\partial z}\right)^2 - c_2 z \\
&= c_1 \circ \left[\tfrac{\partial}{\partial z} + \tfrac{\partial}{\partial z} + z\left(\tfrac{\partial}{\partial z}\right)^2\right] \circ \left(\tfrac{\partial}{\partial z}\right)^2 - c_2 z \\
&= c_1 2\left(\tfrac{\partial}{\partial z}\right)^3 + c_1 z\left(\tfrac{\partial}{\partial z}\right)^4 - c_2 z \\
\tfrac{1}{c_1 z}\mathcal{P} &= \tfrac{2}{z}\left(\tfrac{\partial}{\partial z}\right)^3 + \left(\tfrac{\partial}{\partial z}\right)^4 - \tfrac{c_2}{c_1}.
\end{aligned}$$

Now, set $\mu = \frac{c_2}{c_1} = \frac{3}{4}\frac{1}{E}\left(\frac{\omega}{t}\right)^2$. This example works because the linear density of the beam and the second moment of area about the side-to-side axis are proportional, so they both end up in the $P(\partial/\partial z)$ term.

We're going to look for functions v and points $\zeta = \alpha$ for which the Laplace transform $\mathcal{L}_{\zeta,\alpha} v$ satisfies the differential equation (51). We can see from Section **??** that $\mathcal{L}_{\zeta,\alpha} v$ satisfies equation (51) if and only if v satisfies the integral equation

$$\left[\left[\zeta^4 - \omega^2\right] - 2\partial_{\zeta,\alpha}^{-1} \circ \zeta^3\right] v = 0. \tag{52}$$

Observing that

$$\tfrac{\partial}{\partial\zeta}\sqrt{\zeta^4 - \omega^2} = \frac{2\zeta^3}{\sqrt{\zeta^4 - \omega^2}},$$

we learn that

$$v_{\text{uni}} = \frac{1}{\sqrt{\zeta^4 - \omega^2}}$$

satisfies equation (52) whenever $\alpha^4 - \omega = 0$. Thus, a single "universal solution" in the position domain leads to four linearly independent solutions $V_\alpha = \mathcal{L}_{\zeta,\alpha} v_{\text{uni}}$ of equation (51), indexed by the fourth roots of ω^2.

As we'll show in Section B.2, for $j = 1, \ldots, 4$, v_j solves (**??**) if and only if v_j solves the differential equation

$$\left[\left[\zeta^4 + \mu\right]\tfrac{\partial}{\partial\zeta} - 2\zeta^3\right] v_j = 0 \tag{53}$$

In particular, at $\zeta_j = \zeta - \alpha_j$ we find

$$v_j \propto \frac{1}{\zeta_j^{1/2}} \prod_{k \neq j} (\zeta_j - \alpha_k + \alpha_j)^{-1/2}.$$

[Stokes? Thimbles?]

A The Airy equation

A.0.1 Rewriting as a modified Bessel equation

We can distill the most interesting part of the Airy function by writing

$$\operatorname{Ai}(y) = \tfrac{1}{\pi\sqrt{3}}\, y^{1/2}\, K\big(\tfrac{2}{3} y^{3/2}\big),$$

where

$$K(z) = i\sqrt{3} \int_{z^{-1/3}\Lambda_1} \exp\left[z\left(4u^3 - 3u\right)\right]\, du. \tag{54}$$

and the contour Λ_1 is represented in Figure 6. Saying that Ai satisfies the Airy equation is equivalent to saying that K satisfies the modified Bessel equation

$$\left[z^2 \big(\tfrac{\partial}{\partial z}\big)^2 + z \tfrac{\partial}{\partial z} - \left[\big(\tfrac{1}{3}\big)^2 + z^2\right]\right] K = 0. \tag{55}$$

In fact, K is the modified Bessel function $K_{1/3}$ [45, equation 9.6.1]. Like we did in equation (34), we can rewrite the modified Bessel equation above as

$$\left[\left[\big(\tfrac{\partial}{\partial z}\big)^2 - 1\right] + z^{-1} \tfrac{\partial}{\partial z} - \big(\tfrac{1}{3}\big)^2 z^{-2}\right] K = 0. \tag{56}$$

A.1 Asymptotic analysis

From the general theory of ODE of Poincaré rank 1, we know that the space of trans-series solutions of (56) has a basis of trans-monomials

$$\{e^{-\alpha z} z^{-\tau_\alpha}\, \tilde{W}_\alpha \mid \alpha^2 - 1 = 0\}$$

where the $\tilde{W}_\alpha \in \mathbb{C}[\![z^{-1}]\!]$ are formal power series in z^{-1}. From equations 10.40.2 and 10.17.1 of [45], we learn that $K \sim \big(\tfrac{\pi}{2}\big)^{1/2} e^{-z} z^{-1/2}\, \tilde{W}_1$, with

$$\tilde{W}_1 = 1 - \frac{(\frac{1}{6})_1 (\frac{5}{6})_1}{2^1 \cdot 1!}\, z^{-1} + \frac{(\frac{1}{6})_2 (\frac{5}{6})_2}{2^2 \cdot 2!}\, z^{-2} - \frac{(\frac{1}{6})_3 (\frac{5}{6})_3}{2^3 \cdot 3!}\, z^{-3} + \dots \tag{57}$$

The holomorphic analysis in Section A.2.1 will give us holomorphic solutions

$$\{e^{-\alpha z} z^{-\tau_\alpha}\, W_\alpha \mid \alpha^2 - 1 = 0\},$$

which seem analogous to the trans-monomials above. Borel summation makes the analogy precise. We'll see in Section A.3 that each $z^{-\tau_\alpha}\, W_\alpha$ is proportional to the Borel sum of $z^{-\tau_\alpha}\, \tilde{W}_\alpha$. This is an evidence of [30, Theorem 4].

A.2 Building a distinguished frame of analytic solutions

A.2.1 Going to the spatial domain

We are going to look for functions v_j whose Laplace transforms $\mathcal{L}_{\zeta,\alpha_j} v_j$ satisfy equation (56). We'll succeed when $\alpha^2 - 1 = 0$, and we'll see that K is a scalar multiple of $\mathcal{L}_{\zeta,1} v_1$.

We can see from Section **??** that $\mathcal{L}_{\zeta,\alpha} v$ satisfies the differential equation (56) if and only if v satisfies the integral equation

$$\left[\left[\zeta^2 - 1\right] - \partial_{\zeta,\alpha}^{-1} \circ \zeta - \left(\tfrac{1}{3}\right)^2 \partial_{\zeta,\alpha}^{-2}\right] v = 0. \tag{58}$$

It's tempting to differentiate both sides of this equation until we get

$$\left[\left(\tfrac{\partial}{\partial\zeta}\right)^2 \circ \left[\zeta^2 - 1\right] - \tfrac{\partial}{\partial\zeta} \circ \zeta - \left(\tfrac{1}{3}\right)^2\right] v = 0, \tag{59}$$

which is easier to solve. However, as we learned in Section B.2, a solution of equation (59) satisfies equation (58) if it's slight and locally integrable at $\zeta = \alpha$.

This is great news, because equation (59) has a regular singularity at each root of $\zeta^2 - 1$, and the Frobenius method often gives a slight solution at each regular singular point. We can see the regular singularities by moving the derivatives to the right:

$$\left[(\zeta^2 - 1)\left(\tfrac{\partial}{\partial\zeta}\right)^2 + 3\zeta \tfrac{\partial}{\partial\zeta} + \left[1 - \left(\tfrac{1}{3}\right)^2\right]\right] v = 0.$$

In Sections A.2.2–A.2.3, we'll see this approach succeed. For each root α, we'll find a solution v_α of equation (59) which is slight and locally integrable at $\zeta = \alpha$. We know the function $\mathcal{L}_{\zeta,\alpha} v_\alpha$ will satisfy equation (56), and we can even find its asymptotics from the order τ_α of v_α. We learned in Section 3.1 that

$$\mathcal{L}_{\zeta,\alpha} v_\alpha = e^{-\alpha z} V_\alpha$$

where $V_\alpha = \mathcal{L}_{\zeta_\alpha,0} v_\alpha$ and $\zeta = \alpha + \zeta_\alpha$. We can see from Section 3.2.1 that V_α is asymptotic to a scalar multiple of $z^{-\tau_\alpha}$ at $z = \infty$, so the further decomposition

$$\mathcal{L}_{\zeta,\alpha} v_\alpha = e^{-\alpha z} z^{-\tau_\alpha} W_\alpha,$$

makes W_α is asymptotic to a scalar multiple of $1 + O(z^{-1})$ at $z = \infty$.

A.2.2 Focus on $\zeta = 1$

Let's find a solution of equation (59) which is slight and locally integrable at $\zeta = 1$. Define a new coordinate ζ_1 on $\mathbb{C}$ so that $\zeta = 1 + \zeta_1$. In this coordinate, equation (59) looks like

$$\left[\zeta_1(2 + \zeta_1)\left(\tfrac{\partial}{\partial\zeta_1}\right)^2 + 3(1 + \zeta_1)\tfrac{\partial}{\partial\zeta_1} + \left[1 - \left(\tfrac{1}{3}\right)^2\right]\right] v = 0. \tag{60}$$

With another change of coordinate, given by $\zeta_1 = -2\xi_1$, we can rewrite equation (59) as the hypergeometric equation

$$\left[\xi_1(1 - \xi_1)\left(\tfrac{\partial}{\partial\xi_1}\right)^2 + 3(\tfrac{1}{2} - \xi_1)\tfrac{\partial}{\partial\xi_1} - \left[1 - \left(\tfrac{1}{3}\right)^2\right]\right] v = 0. \tag{61}$$

Looking through the twenty-four expressions for Kummer's six solutions, we find one [45, formula 15.10.12] which is manifestly slight and locally integrable at $\xi_1 = 0$:

$$\begin{aligned} v_1 &= \xi_1^{-1/2}\, {}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; \xi_1\right) \\ &= -i\left(\frac{\zeta_1}{2}\right)^{-1/2} {}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; -\tfrac{1}{2}\zeta_1\right) \end{aligned}$$

From the argument in Section A.2.1, we know that $\mathcal{L}_{\zeta,1} v_1$ satisfies equation (56), and can be written as $e^{-z} V_1$, where $V_1 = \mathcal{L}_{\zeta_1,0} v_1$. Since v_1 has order $-1/2$, the decomposition $V_1 = z^{1/2} W_1$ makes W_1 asymptotic to a scalar multiple of z^{-1} at $z = \infty$.

A.2.3 Focus on $\zeta = -1$

Let's find a solution of equation (59) which is slight and locally integrable at $\zeta = -1$. In the rescaled coordinate from Section A.2.2, this is the point $\xi_1 = 1$. Looking again through Kummer's table of solutions, we find another expression [45, formula 15.10.14] which is manifestly slight and locally integrable at $\xi_1 = 1$:

$$\begin{aligned} v_{-1} &= (1-\xi_1)^{-1/2} F\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; 1-\xi_1\right) \\ &= \left(\frac{\zeta_{-1}}{2}\right)^{-1/2} F\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; \tfrac{1}{2}\zeta_{-1}\right) \end{aligned}$$

where ζ_{-1} is the coordinate with $\zeta = -1 + \zeta_{-1}$. From the argument in Section 5.2.3, we know that $\mathcal{L}_{\zeta,-1} v_{-1}$ satisfies equation (42), and can be written as $e^{z} V_{-1}$, where $V_{-1} = \mathcal{L}_{\zeta_{-1},0} v_{-1}$. Since v_{-1}, like our other solution, has order $-1/2$, the same decomposition $V_{-1} = z^{1/2} W_{-1}$ makes W_{-1} asymptotic to a scalar multiple of z^{-1} at $z = \infty$.

In this example, v_1 and v_{-1} happen to be related by a symmetry: the Möbius transformation that pulls ζ back to $-\zeta$. Kummer's solutions typically come from six different hypergeometric equations, which are related by the Möbius transformations that permute their singularities. In our case, though, exchanging 1 with -1 keeps equation (59) the same.

A.3 Borel regularity

Recall that $\tilde{W}_1$ is a formal solution of (56)

$$\tilde{W}_1(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!} \frac{(-1)^n}{2^n} z^{-n}. \tag{62}$$

Our goal is to prove that the Borel sum of $z^{-1/2}\tilde{W}_1$ is proportional to $V_1 = \mathcal{L}_{\zeta_1,0} v_1$. Let us compute the Borel transform of $z^{-1/2}\tilde{W}_1$: notice that the appropriate coordinate is ζ_1

$$\begin{aligned} \mathcal{B}\left[z^{-1/2}\tilde{W}_1(z)\right](\zeta_1) &= \mathcal{B}\left[\sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!} \frac{(-1)^n}{2^n} z^{-n-\frac{1}{2}}\right](\zeta_1) \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!} \frac{(-1)^n}{2^n} \frac{\zeta_1^{n-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})} \\ &= \zeta_1^{-1/2}\, {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; -\frac{\zeta_1}{2}\right) \end{aligned}$$

Comparing with the solution v_1 computed in Section A.2.2 we notice that

$$v_1(\zeta_1) \propto \mathcal{B}\left[z^{-1/2}\tilde{W}_1\right](\zeta_1)$$

therefore the Borel Laplace sum of $z^{-1/2}\tilde{W}_1$ is proportional to V_1. In particular, both K and $e^{-z}V_1$ are analytic solutions of the Airy equation (56) and they have the same asymptotics at ∞ up to a multiplicative constant.

Similarly, if we consider the formal power series

$$\tilde{W}_{-1}(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!} \frac{1}{2^n} z^{-n} \tag{63}$$

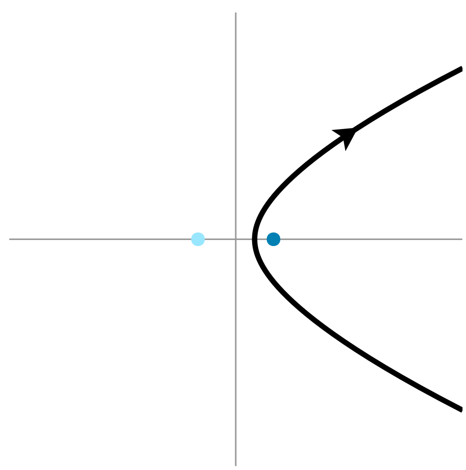
analogous computations give that $v_{-1}(\zeta_{-1}) \propto \mathcal{B}\left[z^{-1/2}\tilde{W}_{-1}\right](\zeta_{-1})$.

A.4 Thimble projection reasoning for the Airy function

Generalizing from Section 5.2.8, we can recast integral (54) into the ζ plane by setting $-\zeta = 4u^3 - 3u$. Projecting $z^{-1/3}\Lambda_1$ to a contour $\mathcal{H}_z^1$ in the ζ plane and choosing the branch of u that lifts $\mathcal{H}_z^1$ back to $z^{-1/3}\Lambda_1$, we have

$$K = \frac{i}{\sqrt{3}} \int_{\mathcal{H}_z^1} e^{-z\zeta} \frac{d\zeta}{4u^2 - 1}. \tag{64}$$

For $z \in (0, \infty)$, the contour $\mathcal{H}_z^1$ runs clockwise around $[1, \infty)$, as shown below. Let's assume $z \in (0, \infty)$ for the rest of the section. [Our conclusions should probably hold whenever $\mathrm{Re}(z) > 0$.]



The contour $\mathcal{H}_z^1$ in the ζ plane.

From formula 15.4.14 in [45], we learn that for our desired branch of u,

$$\frac{1}{4u^2 - 1} = -F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\right),$$

so we can rewrite integral (64) as

$$K = \frac{1}{i\sqrt{3}} \int_{\mathcal{H}_z^1} e^{-z\zeta} F\big(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\big)\, d\zeta.$$

This gives us an alternate route to the conclusion of Section A.2.1, which we'll follow below.

In addition to the solutions v_1 and v_{-1} from Section A.2.2– A.2.3, equation (59) has the solutions

$$\begin{aligned} g_1 &= {}_2F_1\big(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; \xi_1\big) \\ g_{-1} &= {}_2F_1\big(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; 1-\xi_1\big), \end{aligned}$$

given by formulas 15.10.11 and 15.10.13 from [45].

The quadratic transformation identity 15.8.27 from [45] shows that[21]

$$F\big(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\big) = \tfrac{1}{3}(g_1 + g_{-1}),$$

so we have

$$K = \frac{1}{i\,3\sqrt{3}} \int_{\mathcal{H}_z^1} e^{-z\zeta} (g_1 + g_{-1})\, d\zeta.$$

The solution g_{-1} is holomorphic on $\zeta \in [1, \infty)$, so it integrates to zero. The solution g_1, in contrast, is non-meromorphic at $\zeta = 1$. Along the branch cut $\zeta \in [1, \infty)$, its above-minus-below difference is $-\frac{3\sqrt{3}}{2}\, v_1$, as given[22] by equation 15.2.3 from [45]. Hence,

$$\begin{aligned} K &= \frac{i}{2} \int_1^\infty e^{-z\zeta} v_1\, d\zeta \\ e^z K &= \frac{i}{2} \int_1^\infty e^{-z(\zeta-1)} v_1\, d\zeta \\ e^z K &= \tfrac{i}{2} \mathcal{L}_{\zeta_1} v_1, \end{aligned}$$

just as we found in Section A.3. However, in this case, we get an exact equality (compared to the proportionality of Section A.3).

A.4.1 Another solution

Section A.4 associates the solution K of equation (42) with the solution g_1 of equation (61), which contributes the pole at $\zeta = 1$ of

$$\frac{du}{d\zeta} = \frac{1}{4u^2 - 1} = \tfrac{1}{3}(g_1 + g_{-1}).$$

The solution g_{-1}, which contributes the pole at $\zeta = -1$, is associated with another solution of equation (56). To express this other solution as a Laplace transform, following the method of Section A.4, we would use the solution

$$v_{-1} = (1-\xi)^{-1/2}\ {}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; 1-\xi\big)$$

[21] Note that $2\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}) = 2\Gamma(\frac{1}{2})\,\frac{1}{2}\Gamma(\frac{1}{2}) = \pi$ and $\big[\Gamma(\frac{5}{6})\Gamma(\frac{7}{6})\big]^{-1} = \big[\Gamma(\frac{5}{6})\,\frac{1}{6}\Gamma(\frac{1}{6})\big]^{-1} = \frac{6\sin(\frac{1}{6}\pi)}{\pi} = \frac{3}{\pi}$.

[22] Note that $\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})^{-1} = \frac{1}{2}$ and $\big[\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})\big]^{-1} = \big[\Gamma(\frac{2}{3})\,\frac{1}{3}\Gamma(\frac{1}{3})\big]^{-1} = \frac{3\sin(\frac{1}{3}\pi)}{\pi} = \frac{3\sqrt{3}}{2\pi}$.

of equation (59), given by formula 15.10.14 from [45]. This is the only solution, up to scale, which has a fractional power singularity at $\zeta = -1$.

In summary, the thimble projection technique of solving equation (56) is associated with the basis

$$\begin{aligned} g_1 &= {}_2F_1\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; \xi\right) \\ g_{-1} &= {}_2F_1\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; 1-\xi\right) \end{aligned}$$

of solutions for equation (59), given by formulas 15.10.11 and 15.10.13 from [45]. These solutions contribute the poles at $\xi = 1$ and $\xi = 0$, respectively, of a generic solution.

The Laplace transformation method of solving equation (56), on the other hand, is associated with the basis

$$\begin{aligned} v_{-1} &= (1-\xi)^{-1/2}\, {}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; 1-\xi\right) \\ v_1 &= \xi^{-1/2}\, {}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; \xi\right) \end{aligned}$$

given by formulas 15.10.14 and 15.10.12 from [45]. These solutions, up to scale, are the only ones with fractional power singularities.

Identities 15.10.18, and 15.10.22 from [45] give the change of basis

$$\begin{aligned} v_{-1} &= \tfrac{1}{\sqrt{3}}\, g_{-1} + \tfrac{1}{2}\, v_1 \\ v_1 &= \tfrac{1}{\sqrt{3}}\, g_1 + \tfrac{1}{2}\, v_{-1}. \end{aligned}$$

Summing these identities, we see that

$$g_1 + g_{-1} = \tfrac{\sqrt{3}}{2}\, (f_1 + f_{-1}),$$

giving the alternate decomposition

$$\frac{du}{d\zeta} = \tfrac{1}{2\sqrt{3}}\, (f_1 + f_{-1}).$$

A.5 Thimble projection formula

In the Airy-Lucas integral, f is a Chebyshev polynomial, so we can do a thimble projection technique (see Section 5.2.8) or apply the thimble projection formula (**??**) using trigonometric substitution. As an example of how one might reason in the latter case, we look at the Airy function, given by $f = 4u^3 - 3u$.

[...] this time applying the thimble projection formula to the generalized Cardano's formula.

Recall that the roots of $f(u) - \zeta$ can be written explicitly as

$$\begin{aligned} u(\zeta) &= \cos\left(\frac{1}{3}\arccos\zeta - \frac{2\pi k}{3}\right) \quad k = 0, 1, 2 \\ &= {}_2F_1\left(-\frac{1}{6}, \frac{1}{6}; \frac{1}{2}; 1 - \zeta^2\right) \end{aligned}$$

[Add the numerical check to show simple resurgence. For example, maybe graph exponential of the magnitude of function on position domain] The thimble Λ_1 can be parametrized by the path $\theta \mapsto \cosh(\theta - \frac{2}{3}\pi i)$. Since $4u^3 - 3u$ is the third Chebyshev polynomial and cosh is 2π-periodic in the imaginary direction, the start and end points of $\Lambda_1(\zeta)$ are then characterized by

$$\begin{aligned} u &= \cosh(\mp\theta - \tfrac{2}{3}\pi i) \\ \zeta &= \cosh(3\theta). \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\Lambda_1(\zeta)} \nu &= \int_{\Lambda_1(\zeta)} du \\ &= u\Big|_{\text{start}\,\Lambda_1(\zeta)}^{\text{end}\,\Lambda_1(\zeta)} \\ &= \cosh(\theta - \tfrac{2}{3}\pi i) - \cosh(-\theta - \tfrac{2}{3}\pi i) \\ &= \big[\cosh(\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(\theta)\sinh(-\tfrac{2}{3}\pi i)\big] \\ &\quad - \big[\cosh(-\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(-\theta)\sinh(-\tfrac{2}{3}\pi i)\big] \\ &= 2\sinh(\theta)\sinh(-\tfrac{2}{3}\pi i) \\ &= -i\sqrt{3}\,\sinh(\theta). \end{aligned}$$

Let $\xi = \frac{1}{2}(1 - \zeta)$, and notice that $\xi = -\sinh\big(\frac{3}{2}\theta\big)^2$ at the start and end points. The identity 15.4.16 [45]

$$\sinh(\theta) = \tfrac{1}{3}\sinh\big(\tfrac{3}{2}\theta\big)\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{3}{2}; -\sinh\big(\tfrac{3}{2}\theta\big)^2\big)$$

then shows us that

$$\frac{i}{\sqrt{3}} \int_{\Lambda_1(\zeta)} \nu = (-\xi)^{1/2}\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{3}{2}; \xi\big).$$

Now we can evaluate the half-integral of $\int_{\Lambda_1} \nu$ using Bateman's fractional integral formula for hypergeometric functions see [41, Section 4.1].

$$\begin{aligned} \partial_{\zeta,1}^{-1/2}\left(\int_{\Lambda_1(\zeta)} \nu\right) &= \frac{1}{\Gamma\big(\frac{1}{2}\big)} \int_1^{\zeta} (\zeta - \zeta')^{-1/2} \left(\int_{\Lambda_1(\zeta')} \nu\right) d\zeta' \\ &= \frac{1}{\Gamma\big(\frac{1}{2}\big)} \int_0^{\xi} (\xi' - \xi)^{-1/2} \Big[-i\sqrt{3}\,(-\xi)^{1/2}\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{3}{2}; \xi\big)\Big]\,(-\,d\xi') \\ &= -i\frac{\Gamma\big(\frac{3}{2}\big)}{\Gamma(2)}(-\xi)\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; 2; \xi\big) \\ &= \frac{i}{2}\sqrt{\pi}\,\xi\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; 2; \xi\big). \end{aligned}$$

Finally, we differentiate twice using 15.5.4 and 15.5.1 from [45].

$$\begin{aligned}
\partial^{3/2}_{\zeta,1}\left(\int_{\Lambda_1(\zeta)}\nu\right) &= \left(-\tfrac{\partial}{\partial\xi}\right)^2\left[\frac{i\sqrt{\pi}}{2}\,\xi\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big)\right]\\
&= \tfrac{i\sqrt{\pi}}{2}\left(\tfrac{\partial}{\partial\xi}\right)^2\left[\xi\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big)\right]\\
&= \tfrac{i\sqrt{\pi}}{2}\,\tfrac{\partial}{\partial\xi}\left[{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};1;\xi\big)\right]\\
&= \tfrac{i\sqrt{\pi}}{2}\,\tfrac{5}{12}\,{}_2F_1\big(\tfrac{7}{6},\tfrac{11}{6};2;\xi\big).
\end{aligned}$$

For completeness, we repeat the computations for $\hat{\phi}_{-1}$: the thimble Λ_{-1} can be parametrized by the path $\theta \mapsto -\cosh(\theta - \frac{2}{3}\pi i)$. Hence,

$$\begin{aligned}
\int_{\Lambda_{-1}(\zeta)}\nu &= \int_{\Lambda_{-1}(\zeta)} du\\
&= u\Big|_{\text{start}\,\Lambda_{-1}(\zeta)}^{\text{end}\,\Lambda_{-1}(\zeta)}.
\end{aligned}$$

The start and end points of Λ_{-1} are characterized by

$$\begin{aligned}
u &= -\cosh(\mp\theta - \tfrac{2}{3}\pi i)\\
\zeta &= -\tfrac{2}{3}\cosh(3\theta),
\end{aligned}$$

so

$$\begin{aligned}
\int_{\Lambda_{-1}(\zeta)}\nu &= -\cosh(\theta - \tfrac{2}{3}\pi i) + \cosh(-\theta - \tfrac{2}{3}\pi i)\\
&= -\big[\cosh(\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(\theta)\sinh(-\tfrac{2}{3}\pi i)\big]\\
&\quad + \big[\cosh(-\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(-\theta)\sinh(-\tfrac{2}{3}\pi i)\big]\\
&= 2\sinh(\theta)\sinh(\tfrac{2}{3}\pi i)\\
&= i\sqrt{3}\,\sinh(\theta)
\end{aligned}$$

Let $\xi = \frac{1}{2}(1+\zeta)$, and notice that $\xi = -\sinh(\theta)^2$ at the start and end points. The identity 15.4.16 in [45]

$$\sinh(\theta) = \sinh(\theta)\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};-\sinh(\theta)^2\big)$$

then shows us that

$$-\frac{i}{\sqrt{3}}\int_{\mathcal{C}_-(\zeta)}\nu = (-\xi)^{1/2}\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};\xi\big).$$

Now we can evaluate the half-integral of $\int_{\Lambda_{-1}}\nu$ using Bateman's fractional integral formula for hypergeometric functions:

$$\begin{aligned}
\partial_{\zeta,-1}^{-1/2}\left(\int_{\Lambda_{-1}(\zeta)}\nu\right) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{\zeta}(\zeta-\zeta')^{-1/2}\left(\int_{\Lambda_{-1}(\zeta')}\nu\right)d\zeta' \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{\xi}(\xi-\xi')^{-1/2}\left[i\sqrt{3}\,(-\xi')^{1/2}\,{}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};\xi'\right)\right]\left(d\xi'\right) \\
&= \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}(-\xi)\,{}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6};2;\xi\right) \\
&= -\tfrac{\sqrt{\pi}}{2}\,\xi\,{}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6};2;\xi\right).
\end{aligned}$$

Finally, we differentiate twice using 15.5.4 and 15.5.1 from [45]

$$\begin{aligned}
\partial_{\zeta,-1}^{3/2}\left(\int_{\Lambda_{-1}(\zeta)}\nu\right) &= \left(\tfrac{\partial}{\partial\xi}\right)^2\left[-\tfrac{\sqrt{\pi}}{2}\,\xi\,{}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6};2;\xi\right)\right] \\
&= -\tfrac{\sqrt{\pi}}{2}\left(\tfrac{\partial}{\partial\xi}\right)^2\left[\xi\,{}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6};2;\xi\right)\right] \\
&= -\tfrac{\sqrt{\pi}}{2}\,\tfrac{\partial}{\partial\xi}\left[{}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6};1;\xi\right)\right] \\
&= -\tfrac{\sqrt{\pi}}{2}\,\tfrac{5}{12}\,{}_2F_1\left(\tfrac{7}{6},\tfrac{11}{6};2;\xi\right).
\end{aligned}$$

A.5.1 An eye on resurgence

[Add link to general references in intro]. Resurgent functions in the position domain are characterized by being "endlessly analytically continuable" away from their singularities [16, Section 1.2.2]{this paper only defines resurgent functions in the frequency domain}{For definition in the position domain, cite §6.1 of Mitschi and Sauzin [52]}. [Focus on endlessly continuable functions instead? See [Section 2, "Variations on the resurgence of the gamma function".] They have the special property that by studying their analytic continuation around one singularity, we can typically discover the germ of another singularity. They have the very special property that by studying one germ, we can typically reconstruct the others by looking at the behavior near the singular points. [They are "self-reproducing" in a certain remarkable way involving holonomy around singularities, as we'll see in the example below…] A prototypical class of resurgent functions is the so-called simple resurgent functions [16, Section 1.2.3]{oops, this source defines simple resurgent functions in the frequency domain}. namely an [resurgent] analytic function $\hat{\phi}$ in the position domain with singularities ω such that

[…] resurgent function with the property that at each singularity ω, we have an expansion of the form

$$\hat{\phi} = \frac{C_\omega}{\zeta_\omega} + \frac{S_\omega}{2\pi i}\log(\zeta_\omega)\,\tilde{\phi}_\omega + \text{ hol. funct.}[\text{convergent series in } \zeta_\omega]$$

where $\zeta = \omega + \zeta_\omega$, $\tilde{\phi}_\omega$ is a series in ζ_ω C_ω, S_ω are constants and S_ω is typically assumed to be integral (in a suitable normalization) and called the Stokes constant of the singularity ω. The goal of resurgence is to investigate the location of the singularities together with their Stokes constants and the corresponding germs $\tilde{\phi}_\omega$.

Thimbles integrals are good candidates to understand the structure of resurgent functions; indeed the functions $\hat{\phi}_j$ in equation (**??**) are resurgent. Their resurgent structure can be easily reconstructed: if ν is holomorphic, then $\hat{\phi}_j$ is singular at $\zeta_j = \alpha_k$ for $k \neq j$. Hence the set of singularities is given by the critical values of f. Then, the computation of the Stokes constants and of the other germs $\tilde{\phi}_k$ is done by studying $\hat{\phi}_j$ near its singular points. For example, for the Airy function, we can expand $\hat{\phi}_1$ near $\zeta = -1$:

$$\hat{\phi}_1(\zeta - 1) = \frac{36}{5\pi}\frac{1}{\zeta} + \frac{1}{2\pi}\log(\zeta)\left(1 + \frac{77}{144}\zeta + \frac{17017}{62208}\zeta^2 + \frac{7436429}{53747712}\zeta^3 + \dots\right) + \text{ hol. funct.}$$

Notice that $(1+\frac{77}{144}\zeta+\frac{17017}{62208}\zeta^2+\frac{7436429}{53747712}\zeta^3+\dots)$ is the Taylor series of $\hat{\phi}_{-1}$ at the origin. So we see that expanding $\hat{\phi}_1$ at the singularity we see the germ of another singularity $\tilde{\phi}_{-1}$. Analogously, if we expand $\hat{\phi}_{-1}$ near $\zeta = 1$ we find

$$\hat{\phi}_{-1}(\zeta + 1) = -\frac{36}{5\pi}\frac{1}{\zeta} + \frac{1}{2\pi}\log(\zeta)\left(1 - \frac{77}{144}\zeta + \frac{17017}{62208}\zeta^2 - \frac{7436429}{53747712}\zeta^3 + \dots\right) + \text{ hol. funct.}$$

where the germ $(1 - \frac{77}{144}\zeta + \frac{17017}{62208}\zeta^2 - \frac{7436429}{53747712}\zeta^3 + \dots)$ is the Taylor expansion of $\hat{\phi}_1$ at the origin. From these expressions, we also noticed that $\hat{\phi}_i$ are simple resurgent functions.

A.6 Comparison with other treatments of the Airy equation

A.6.1 Different Borel transform convention

Physicists often use a different version of the Borel transform:

$$\begin{aligned}\mathcal{B}_{phys}\colon \mathbb{C}[\![z^{-1}]\!] &\to \mathbb{C}[\![\zeta]\!]\\ z^{-n} &\mapsto \frac{\zeta^n}{n!}.\end{aligned}$$

This version avoids sending 1 to the convolution unit δ, at the cost of no longer mapping multiplication to convolution or inverting the formal Laplace transform. It's sometimes convenient to $\mathcal{B}_{phys}(f) = \mathcal{B}(z^{-1}f)$.

For problems involving a small parameter $\hbar$ rather than a large parameter z, physicists also define

$$\begin{aligned}\mathcal{B}_{phys}\colon \mathbb{C}[\![\hbar]\!] &\to \mathbb{C}[\![\zeta]\!]\\ \hbar^n &\mapsto \frac{\zeta^n}{n!}.\end{aligned}$$

From a combinatorial perspective, this is just the transformation that sends an ordinary generating function to the corresponding exponential generating function.

In [50], the author studies the Airy functions as an example of resurgent functions. He starts with the formal trans-monomial solutions of the Airy equation:

$$\begin{aligned}\tilde{\Phi}_{\mathrm{Ai}}(x) &= \frac{1}{2\sqrt{\pi}}x^{-1/4}e^{-\frac{2}{3}x^{3/2}}\tilde{W}_1(x^{-3/2})\\ \tilde{\Phi}_{\mathrm{Bi}}(x) &= \frac{1}{2\sqrt{\pi}}x^{-1/4}e^{\frac{2}{3}x^{3/2}}\tilde{W}_2(x^{-3/2})\end{aligned}$$

where

$$\tilde{W}_{1,2}(\hbar) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(\mp\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} \hbar^n.$$

Then, by applying the two definitions of the Borel transform $\mathcal{B}_{phys}$ and $\mathcal{B}$, on the one hand we have

$$\begin{aligned}
w_{1,2}(\zeta) &:= \mathcal{B}_{phys}(\tilde{W}_{1,2})(\zeta) \\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(\mp\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} \frac{\zeta^n}{n!} \\
&= {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \mp\frac{3}{4}\zeta\right)
\end{aligned}$$

on the other hand, we find

$$\begin{aligned}
\mathcal{B}(\tilde{W}_{1,2})(\zeta) &= \frac{1}{2\pi}\delta + \sum_{n=1}^{\infty} \frac{1}{2\pi} \left(\mp\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} \frac{\zeta^{n-1}}{(n-1)!} \\
&= \frac{1}{2\pi}\delta + \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(\mp\frac{3}{4}\right)^{n+1} \frac{\Gamma(n+1+\frac{5}{6})\Gamma(n+1+\frac{1}{6})}{(n+1)!} \frac{\zeta^n}{n!} \\
&= \frac{1}{2\pi}\delta \mp \frac{3}{4} \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(\mp\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{11}{6})\Gamma(n+\frac{7}{6})}{\Gamma(n+2)} \frac{\zeta^n}{n!} \\
&= \frac{1}{2\pi}\delta \mp \frac{5}{48} {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; \mp\frac{3}{4}\zeta\right)
\end{aligned}$$

and comparing the two solutions we notice that up to the factor of δ

$$\mathcal{B}(\tilde{W}_{1,2})(\zeta) - \frac{1}{2\pi}\delta = \mp\frac{5}{48} {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; \mp\frac{3}{4}\zeta\right) = \frac{d}{d\zeta}\mathcal{B}_{phys}(\tilde{W}_{1,2})(\zeta) \tag{65}$$

More generally, if $\tilde{\Phi}(z) \in z^{-1}\mathbb{C}[\![z^{-1}]\!]$, i.e. it has no constant term, then

$$\frac{d}{d\zeta} \circ \mathcal{B}_{phys}\tilde{\Phi} = \mathcal{B}\tilde{\Phi}. \tag{66}$$

In particular, $\frac{d}{d\zeta} \circ \mathcal{B}_{phys}\left[z^{-1/2}\tilde{W}_1\right]\left(\frac{2}{3}\zeta\right) = v_1(\zeta)$.

A.6.2 Integral formula for hypergeometric functions

In [52] the author studies summability and resurgent properties of solutions of the Airy equation.

He considers the formal power series

$$\tilde{\Phi}_{\pm}(z) := \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(\mp\frac{1}{2}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} z^{-n}$$

such that

$$\tilde{\Phi}_{\mathrm{Ai}}(y) = \frac{1}{2\sqrt{\pi}} y^{-1/4} e^{-\frac{2}{3}y^{3/2}} \tilde{\Phi}_{+}\left(\tfrac{2}{3}y^{3/2}\right)$$
$$\tilde{\Phi}_{\mathrm{Bi}}(y) = \frac{1}{2\sqrt{\pi}} y^{-1/4} e^{\frac{2}{3}y^{3/2}} \tilde{\Phi}_{-}\left(\tfrac{2}{3}y^{3/2}\right)$$

are formal solutions of the Airy equation. Notice that compared to Marino's formal solutions, Sauzin adopts a different change of coordinates $z = \frac{2}{3}y^{3/2}$.

By looking for solutions of the Borel transformed equation, he wrote the Borel transform of $\tilde{\Phi}_{\pm}$ as a convolution product:

$$\hat{\phi}_{+}(\zeta) := \mathcal{B}\tilde{\Phi}_{+} = \delta + \frac{d}{d\zeta}\hat{\chi}(\zeta) \qquad \hat{\phi}_{-}(\zeta) := \mathcal{B}\tilde{\Phi}_{-} = \delta - \frac{d}{d\zeta}\hat{\chi}(-\zeta)$$

where $\hat{\chi}(\zeta) = \frac{2^{1/6}}{\Gamma(1/6)\Gamma(5/6)}(2\zeta+\zeta^2)^{-1/6} * \zeta^{-5/6}$.

Notice that the function $\hat{\chi}(\zeta)$ is an hypergeometric function:

$$\begin{aligned}
\hat{\chi}(\zeta) &= \frac{2^{1/6}}{\Gamma(1/6)\Gamma(5/6)}(2\zeta+\zeta^2)^{-1/6} * \zeta^{-5/6} \\
&= \frac{2^{1/6}}{\Gamma(1/6)\Gamma(5/6)}\int_0^{\zeta}(2\zeta'+\zeta'^2)^{-1/6}(\zeta-\zeta')^{-5/6}d\zeta' \\
&= \frac{2^{1/6}}{\Gamma(1/6)\Gamma(5/6)}\int_0^{1}(\zeta t)^{-1/6}(2+\zeta t)^{-1/6}(\zeta-\zeta t)^{-5/6}\zeta dt \\
&= \frac{2^{1/6}}{\Gamma(1/6)\Gamma(5/6)}\int_0^{1}t^{-1/6}2^{-1/6}(1+\frac{\zeta}{2}t)^{-1/6}(1-t)^{-5/6}d\zeta' \\
&= \frac{1}{\Gamma(1/6)\Gamma(5/6)}\int_0^{1}t^{-1/6}(1+\frac{\zeta}{2}t)^{-1/6}(1-t)^{-5/6}d\zeta' \\
&= {}_2F_1\left(\frac{1}{6},\frac{5}{6};1;-\frac{\zeta}{2}\right)
\end{aligned}$$

where in the last step we use the Euler formula for hypergeometric functions [23]. Then, by taking derivatives we recover $\hat{\phi}_{\pm}(\zeta)$:

$$\hat{\phi}_{+}(\zeta) = \delta - \frac{1}{2}\frac{5}{36}\ {}_2F_1\left(\frac{7}{6},\frac{11}{6};2;-\frac{\zeta}{2}\right) = \delta - \frac{2}{3}\frac{5}{48}{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;-\frac{\zeta}{2}\right)$$
$$\hat{\phi}_{-}(\zeta) = \delta + \frac{1}{2}\frac{5}{36}\ {}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\frac{\zeta}{2}\right) = \delta + \frac{2}{3}\frac{5}{48}{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\frac{\zeta}{2}\right)$$

[23] The Euler formula is

$$ {}_2F_1\left(a,b;c;x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt. \tag{67}$$

and up to a multiplicative constant they match our computations for $\mathcal{B}(\tilde{W}_{1,2})$ (see equation (65)). The main advantage of writing Gauss hypergeometric functions as a convolution product relies on Écalle's singularity theory. Indeed $(2\zeta+\zeta^2)^{-1/6}$ extends analytically to the universal cover of $\mathbb{C}\setminus\{0,-2\}$ and the convolution with $\zeta^{-5/6}$ does not change the set of singularities (see part c of [52, Section 6.14.5]). Furthermore, the author proves that $\hat{\phi}_{\pm}(\zeta)$ are simple resurgent functions (see [52, Lemma 6.106]).

A.6.3 Comparison with exact WKB

Kawai and Takei study the WKB analysis of the Airy–type Schrodinger equation

$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) = 0 \tag{68}$$

as $\eta\to\infty$. They define $\psi_B(x,y)$ as the inverse Laplace transform of $\psi(x,\eta)$ with respect to η. In the coordinate $t=\frac{3}{2}yx^{-3/2}$ they find an explicit formula for $\psi_B(x,y)$ in terms of Gauss hypergeometric functions:

$$\psi_{+,B}(x,y) = \frac{1}{x}\phi_+(t) = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{1}{x}s^{-1/2}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};s\right)$$

$$\psi_{-,B}(x,y) = \frac{1}{x}\phi_-(t) = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{1}{x}(1-s)^{-1/2}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};1-s\right)$$

where $s=t/2+1/2$. The same hypergeometric functions have been computed in Section 69 as the Borel transform of the formal solutions of the Airy equation

$$\left[\left(\frac{d}{dw}\right)^2 - w\right]f(w) = 0. \tag{69}$$

Although the two equations look closely related (they are equivalent by the change of coordinates $w=x\eta^{2/3}$), the Borel transform of ψ is computed with respect to $\frac{2}{3}\eta x^{3/2}$ (which is the conjugate variable of t) while the Borel transform of $f(w)$ is computed with respect to w. So we need to find a different change of coordinates to explain why the Borel transforms of $\psi(x,\eta)$ and $f(w)$ are given by the same hypergeometric function.

First of all notice that if η and y are conjugate variables under Borel transform, meaning

$$\sum_{n\geq 0} a_n\eta^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n\geq 0}\frac{a_n}{n!}y^n$$

then $t=\frac{3}{2}yx^{-3/2}$ is the conjugate variable of $q=\frac{2}{3}\eta x^{3/2}$ up to correction by a factor of $\frac{3}{2}x^{-3/2}$

$$\sum_{n\geq 0} a_n q^{-n-1} = \sum_{n\geq 0} a_n x^{-3/2(n+1)}\left(\frac{2}{3}\eta\right)^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n\geq 0}\frac{a_n x^{-3/2(n+1)}}{n!}\left(\frac{3}{2}\right)^{n+1}y^n = \frac{3}{2}x^{-3/2}\sum_{n\geq 0}\frac{a_n}{n!}t^n.$$

In addition, $\psi_{B,\pm}(x,y) = \frac{1}{x}\phi_\pm(t)$, therefore we expect that $\psi(x,\eta) = x^{1/2}\Phi(q)$. Assume that $\psi(x,y)$ is a solution of (68), then $\Phi(q)$ solves

$$\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi(q) = 0 \tag{70}$$

Proof.

$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) = 0$$

$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]x^{1/2}\Phi(q) = 0$$

$$\frac{d}{dx}\left[\frac{1}{2}x^{-1/2}\Phi + x^{1/2}\frac{d}{dx}\Phi\right] - \eta^2 x^{3/2}\Phi = 0$$

$$-\frac{1}{4}x^{-3/2}\Phi + \frac{1}{2}x^{-1/2}\frac{d}{dx}\Phi + \frac{1}{2}x^{-1/2}\frac{d}{dx}\Phi + x^{1/2}\left(\frac{d}{dx}\right)^2\Phi - \eta^2 x^{3/2}\Phi = 0$$

$$\left[x^{1/2}\left(\frac{d}{dx}\right)^2 + x^{-1/2}\frac{d}{dx} - \frac{1}{4}x^{-3/2} - \eta^2 x^{3/2}\right]\Phi = 0$$

$$\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0$$

$\square$

Now rewrite (70) in the coordinates $q = \frac{2}{3}\eta x^{3/2}$:

$$\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0$$

$$\left[\eta^2 x\left(\frac{d}{dq}\right)^2 + \frac{1}{2}\eta x^{-1/2}\frac{d}{dq} + x^{-1}\cdot\eta\, x^{1/2}\frac{d}{dq} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0$$

$$\left[\eta^2 x\left(\frac{d}{dq}\right)^2 + \frac{3}{2}\eta x^{-1/2}\frac{d}{dq} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0$$

$$\left[\left(\frac{d}{dq}\right)^2 + \frac{3}{2}\eta^{-1}x^{-3/2}\frac{d}{dq} - \frac{1}{4}\eta^{-2}x^{-3} - 1\right]\Phi = 0$$

$$\left[\left(\frac{d}{dq}\right)^2 + q^{-1}\frac{d}{dq} - \frac{1}{9}q^{-2} - 1\right]\Phi = 0$$

therefore $\Phi(q)$ is a solution of the transformed Airy equation (56).

Remark A.1. The change of coordinate $q = \frac{2}{3}\eta x^{3/2}$ is not casual: recall that the WKB ansatz for a Schrodinger type equation is

$$\psi(x,\eta) = \exp\left(\int_{x_0}^{x} S(\eta, x')dx'\right) \tag{71}$$

and $S(\eta, x) = \sum_{k\geq -1} S_k(x)\eta^{-k}$. In addition, for the Airy–type Schrodinger equation

$$S_{-1}^2 = x$$

hence, up to a choice of sign for the square root

$$q = \frac{2}{3}\eta x^{3/2} = \eta \int_0^x \sqrt{x'}dx' = \eta \int_0^x S_{-1}(x')dx'.$$

We expect that the change of coordinates $q = \eta \int_0^x S_{-1}(x')dx'$ would explain the analogies between the Borel transform of the WKB solution of a Schrodinger equation and the Borel transform of the associated ODE.

B Facts about differential and integral equations

Shall we review the content of [30]?

B.1 Passing between the analytic world and the formal world

Show that the Taylor expansion of an analytic solution of an integral equation satisfies the same equation (or just cite the statement in Erdelyi). Cite the theorem of Erdelyi which shows that a solution of a differential equation (under certain conditions) has an asymptotic expansion, which satisfies the corresponding differential equation.

B.2 Order shifting

Consider holomorphic functions on a simply connected open domain Ω that touches but doesn't contain $\zeta = 0$. As we've seen in Section 3.2.1, functions in $\mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$ with $\sigma > -1$ play a special role in Laplace transform methods for linear differential equations. This is because differential equations in the frequency domain arise most naturally from integral equations in the spatial domain, but we'd like to work with differential equations in the spatial domain too. In $\mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$, differential and integral equations enjoy their simplest equivalence.

Proposition B.1. *Let $\psi \in \mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$, and $\sigma > -1$*

$$\left[\sum_{k=0}^{n} \left(\tfrac{\partial}{\partial\zeta}\right)^k \circ h_k + \sum_{k=1}^{m} \partial_{\zeta,0}^{-k} \circ h_{-k}\right]\psi = 0,$$

if and only if

$$\left[\sum_{k=1}^{n} \left(\tfrac{\partial}{\partial\zeta}\right)^{k-1} \circ h_k + \sum_{k=0}^{m} \partial_{\zeta,0}^{-k-1} \circ h_{-k}\right]\psi = 0,$$

where $h_n, \ldots, h_{-m}$ are holomorphic functions.

Proof. The reverse implication holds without any special condition on ψ, because $\frac{\partial}{\partial\zeta}\,\partial_{\zeta,0}^{-1}$ acts as the identity on all differentiable functions.

To prove the forward implication, rewrite the first equation in the statement as

$$\frac{\partial}{\partial\zeta}\left[\sum_{k=1}^{n}\left(\frac{\partial}{\partial\zeta}\right)^{k-1}\circ h_k\right]\psi = -\left[h_0+\sum_{k=1}^{m}\partial_{\zeta,0}^{-k}\circ h_{-k}\right]\psi. \tag{72}$$

The function

$$\phi = \left[\sum_{k=1}^{n}\left(\frac{\partial}{\partial\zeta}\right)^{k-1}\circ h_k\right]\psi$$

belongs to the function space $\mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$. Looking at the right-hand side of equation (72), we can see that ϕ' belongs to $\mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$, and by assumption its order is greater than -1. Hence, ϕ belongs to $\mathcal{H}L^{\infty}_{\sigma+1,\bullet}(\Omega)$, which means it vanishes at $\zeta = 0$.

Integrating both sides of equation (72), we get

$$\partial_{\zeta,0}^{-1}\,\frac{\partial}{\partial\zeta}\,\phi = -\left[\partial_{\zeta,0}^{-1}\circ h_0+\sum_{k=1}^{m}\partial_{\zeta,0}^{-k-1}\circ h_{-k}\right]\psi.$$

Since $\partial_{\zeta,0}^{-1}\,\frac{\partial}{\partial\zeta}$ acts as the identity on functions that vanish at $\zeta = 0$, this simplifies to

$$\phi = -\left[\sum_{k=0}^{m}\partial_{\zeta,0}^{-k-1}\circ h_{-k}\right]\psi,$$

which rearranges to the second equation in the statement. $\square$

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