# Airy function: Kawai+Takei vs. Mariño

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Kawai and Takei want to solve

$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) = 0.$$

They define $\psi_B(x,y)$ as the inverse Laplace transform of $\psi(x,\eta)$ with respect to $\eta$.

With $w = x\eta^{2/3}$, the equation above is equivalent to

$$\left[\left(\frac{d}{dw}\right)^2 - w\right]\psi(w\eta^{-2/3},\eta) = 0.$$

Proof: substitute back to get

$$\begin{aligned}
\left[\eta^{-4/3}\left(\frac{d}{dx}\right)^2 - \eta^{2/3}x\right]\psi(x,\eta) &= 0\\
\left[\eta^{-4/3}\left(\frac{d}{dx}\right)^2 - \eta^{-4/3}\eta^2 x\right]\psi(x,\eta) &= 0\\
\eta^{-4/3}\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) &= 0.
\end{aligned}$$

Hence, $\psi(w\eta^{-2/3},\eta) = k(\eta)\mathsf{Ai}(w)$ is a solution for any holomorphic function $k$.

## 1 Veronica's change of coordinates

Kawai and Takei study the WKB analysis of the equation

$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) = 0 \tag{1}$$

as $\eta \to \infty$. They define $\psi_B(x,y)$ as the inverse Laplace transform of $\psi(x,\eta)$ with respect to $\eta$. In the coordinates $t = yx^{-3/2}$ they find an explicit formula for $\psi_B(x,y)$ in terms of

Gauss hypergeometric functions:

$$\psi_{+,B}(x,y) = \frac{1}{x}\phi_+(t) = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{1}{x}s^{-1/2}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};s\right)$$
$$\psi_{-,B}(x,y) = \frac{1}{x}\phi_-(t) = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{1}{x}(1-s)^{-1/2}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};1-s\right)$$

where $s = 3t/4 + 1/2$. The same hypergeometric functions have been computed in Section ?? as the Borel transform of the formal solutions of the Airy equation

$$\left[\left(\frac{d}{dw}\right)^2 - w\right] f(w) = 0. \tag{2}$$

Although the two equations look closely related (they are equivalent by the change of coordinates $w = x\eta^{2/3}$), the Borel transform of $\psi$ is computed with respect to $\eta x^{3/2}$ (which is the conjugate variable of $t$) while the Borel transform of $f(w)$ is computed with respect to $w$. So we need to find a different change of coordinates to explain why the Borel transforms of $\psi(x,\eta)$ and $f(w)$ are given by the same hypergeometric function.

First of all notice that if $\eta$ and $y$ are conjugate variables under Borel transform, meaning

$$\sum_{n\geq 0} a_n \eta^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n\geq 0} \frac{a_n}{n!} y^n$$

then $t = yx^{-3/2}$ is the conjugate variable of $q = \eta x^{3/2}$ up to correction by a factor of $x^{-3/2}$

$$\sum_{n\geq 0} a_n q^{-n-1} = \sum_{n\geq 0} a_n x^{-3/2(n+1)} \eta^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n\geq 0} \frac{a_n x^{-3/2(n+1)}}{n!} y^n = x^{-3/2} \sum_{n\geq 0} \frac{a_n}{n!} t^n.$$

In addition, $\psi_{B,\pm}(x,y) = \frac{1}{x}\phi_\pm(t)$, therefore we expect that $\psi(x,\eta) = x^{1/2}\Phi(q)$. Assume that $\psi(x,y)$ is a solution of (1), then $\Phi(q)$ solves

$$\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi(q) = 0 \tag{3}$$

*Proof.*

$$
\begin{aligned}
&\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) = 0\\
&\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]x^{1/2}\Phi(q) = 0\\
&\frac{d}{dx}\left[\frac{1}{2}x^{-1/2}\Phi + x^{1/2}\frac{d}{dx}\Phi\right] - \eta^2 x^{3/2}\Phi = 0\\
&-\frac{1}{4}x^{-3/2}\Phi + \frac{1}{2}x^{-1/2}\frac{d}{dx}\Phi + \frac{1}{2}x^{-1/2}\frac{d}{dx}\Phi + x^{1/2}\left(\frac{d}{dx}\right)^2\Phi - \eta^2 x^{3/2}\Phi = 0\\
&\left[x^{1/2}\left(\frac{d}{dx}\right)^2 + x^{-1/2}\frac{d}{dx} - \frac{1}{4}x^{-3/2} - \eta^2 x^{3/2}\right]\Phi = 0\\
&\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0
\end{aligned}
$$

□

Now rewrite (3) in the coordinates $q = \eta x^{3/2}$:

$$
\begin{aligned}
&\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0\\
&\left[\frac{9}{4}\eta^2 x\left(\frac{d}{dq}\right)^2 + \frac{3}{4}\eta x^{-1/2}\frac{d}{dq} + x^{-1}\cdot\frac{3}{2}\,\eta\,x^{1/2}\frac{d}{dq} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0\\
&\left[\eta^2 x\left(\frac{d}{dq}\right)^2 + \frac{1}{3}\eta x^{-1/2}\frac{d}{dq} + \frac{2}{3}\,\eta\,x^{-1/2}\frac{d}{dq} - \frac{1}{9}x^{-2} - \frac{4}{9}\eta^2 x\right]\Phi = 0\\
&\left[\eta^2\left(\frac{d}{dq}\right)^2 + \eta x^{-3/2}\frac{d}{dq} - \frac{1}{9}x^{-3} - \frac{4}{9}\eta^2\right]\Phi = 0\\
&\left[\left(\frac{d}{dq}\right)^2 + \eta^{-1}x^{-3/2}\frac{d}{dq} - \frac{1}{9}\eta^{-2}x^{-3} - \frac{4}{9}\right]\Phi = 0\\
&\left[\left(\frac{d}{dq}\right)^2 + q^{-1}\frac{d}{dq} - \frac{1}{9}q^{-2} - \frac{4}{9}\right]\Phi = 0
\end{aligned}
$$

therefore $\Phi(q)$ is a solution of the transform Airy equation (see draft2).

## 2 Weber equation: WKB vs modifield Bessel ODE

In [Takei] the author studied the Borel summation of WKB solutions of the harmonic oscillator

$$\left[\frac{d^2}{dx^2} - \eta^2 x^2 - \lambda\eta\right]\psi = 0 \tag{4}$$

with a parameter $\lambda$. He proved that the WKB solutions

$$\psi_\pm(x,\eta) = e^{\pm\eta x^2/2} \sum_{n=0}^{\infty} \frac{\psi_{\pm,n}}{x^{2n+(1\pm\lambda)/2}} \eta^{-\left(\frac{1}{2}\pm\frac{\lambda}{4}+n\right)} \tag{5}$$

where $\psi_{\pm,n}$ are constants independent of $x$ and $\eta$ have Borel transform (in the variable $y$ conjugate to $\eta$)

$$\psi_{+,B}(x,y) = \frac{x^{-3/2}}{\Gamma\left(\frac{1}{2}+\frac{\lambda}{4}\right)} s^{-1/2+\lambda/4} F\left(\frac{1}{4}+\frac{\lambda}{4}, \frac{3}{4}+\frac{\lambda}{4}; \frac{1}{2}+\frac{\lambda}{4}; s\right) \tag{6}$$

$$\psi_{-,B}(x,y) = \frac{x^{-3/2}}{\Gamma\left(\frac{1}{2}+\frac{\lambda}{4}\right)} (s-1)^{-1/2+\lambda/4} F\left(\frac{1}{4}+\frac{\lambda}{4}, \frac{3}{4}+\frac{\lambda}{4}; \frac{1}{2}+\frac{\lambda}{4}; 1-s\right) \tag{7}$$

where $s = y/x^2 + 1/2$.

**Lemma 1.** *Set $\lambda = 0$. Let $\eta = 2zx^{-2}$ then $\psi_-(x,\eta) = \frac{1}{2\sqrt{\pi}} x^{1/2} \tilde{K}_{1/4}(z)$ is a solution of (4) if and only if $\tilde{K}_{1/4}(z)$ is a formal solution of the modified Bessel equation*

$$\left[\frac{d^2}{dz^2} - 1 + \frac{1}{z}\frac{d}{dz} - \frac{1}{16z^2}\right]\tilde{K}_{1/4}(z) = 0 \tag{8}$$

*Proof.* We start with the Weber equation (4)

$$\begin{aligned}
&\left[\frac{d^2}{dx^2} - \eta^2 x^2\right]\psi_- = 0 \\
&\left[\frac{d^2}{dx^2} - \eta^2 x^2\right] x^{1/2}\tilde{K}_{1/4}(z) = 0 \\
&\frac{d}{dx}\left(\frac{1}{2}x^{-1/2}\tilde{K}_{1/4}(z) + x^{1/2}\frac{d}{dx}\tilde{K}_{1/4}(z)\right) - \eta^2 x^{5/2}\tilde{K}_{1/4}(z) = 0 \\
&-\frac{1}{4}x^{-3/2}\tilde{K}_{1/4}(z) + x^{-1/2}\frac{d}{dx}\tilde{K}_{1/4}(z) + x^{1/2}\frac{d^2}{dx^2}\tilde{K}_{1/4}(z) - \eta^2 x^{5/2}\tilde{K}_{1/4}(z) = 0 \\
&-\frac{1}{4}x^{-2}\tilde{K}_{1/4}(z) + x^{-1}\frac{d}{dx}\tilde{K}_{1/4}(z) + \frac{d^2}{dx^2}\tilde{K}_{1/4}(z) - \eta^2 x^2\tilde{K}_{1/4}(z) = 0
\end{aligned}$$

Under the change of coordinates $\eta = 2zx^{-2}$ we have

$$\frac{d}{dx} = \eta x \frac{d}{dz} \qquad\qquad \frac{d^2}{dx^2} = \eta\frac{d}{dz} + \eta^2 x^2 \frac{d^2}{dz^2}$$

therefore

$$
\begin{aligned}
&-\frac{1}{4}x^{-2}\tilde{K}_{1/4}(z)+x^{-1}\frac{d}{dx}\tilde{K}_{1/4}(z)+\frac{d^2}{dx^2}\tilde{K}_{1/4}(z)-\eta^2x^2\tilde{K}_{1/4}(z)=0\\
&-\frac{1}{4}x^{-2}\tilde{K}_{1/4}(z)+\eta\frac{d}{dz}K_{1/4}(z)+\left(\eta\frac{d}{dz}+\eta^2x^2\frac{d^2}{dz^2}\right)\tilde{K}_{1/4}(z)-\eta^2x^2\tilde{K}_{1/4}(z)=0\\
&-\frac{1}{4}\frac{\eta}{2z}\tilde{K}_{1/4}(z)+\eta\frac{d}{dz}\tilde{K}_{1/4}(z)+\eta\frac{d}{dz}\tilde{K}_{1/4}(z)+2z\eta\frac{d^2}{dz^2}\tilde{K}_{1/4}(z)-2z\eta\tilde{K}_{1/4}(z)=0\\
&-\frac{1}{4}\frac{1}{4z^2}K_{1/4}(z)+\frac{1}{z}\frac{d}{dz}\tilde{K}_{1/4}(z)+\frac{d^2}{dz^2}K_{1/4}(z)-\tilde{K}_{1/4}(z)=0\\
&\qquad\qquad\qquad\left[\frac{d^2}{dz^2}-1+\frac{1}{z}\frac{d}{dz}-\frac{1}{16z^2}\right]\tilde{K}_{1/4}(z)=0
\end{aligned}
$$

$\square$

**Lemma 2.** *Set $\lambda = 0$. Let $\eta = 2zx^{-2}$ then*

$$\psi_{-,B}(x,y) = x^{-3/2}\hat{\kappa}_{1/4}(\zeta)$$

*where $y = \frac{\zeta}{2}x^2$ and $\hat{\kappa}_{1/4}(\zeta) = \mathcal{B}(\tilde{K}_{1/4})(\zeta)$.*

*Proof.* We move to the Borel plane: let $\zeta$ be the variable conjugate to $z$ and let $y$ the conjugate to $\eta$, i.e.

$$
\begin{aligned}
\mathcal{B}\colon \sum_{n\geq 0} a_n z^{-n-1} &\to \sum_{n\geq 0} a_n \frac{\zeta^n}{n!}\\
\mathcal{B}\colon \sum_{n\geq 0} a_n \eta^{-n-1} &\to \sum_{n\geq 0} a_n \frac{y^n}{n!}
\end{aligned}
$$

if $\eta = 2zx^{-2}$ then $y = \frac{\zeta}{2}x^2$, indeed

$$
\begin{tikzcd}
\sum_{n\geq 0} a_n \eta^{-n-1} \arrow[rr, "\mathcal{B}"] \arrow[d, equal] & & \sum_{n\geq 0} a_n \frac{y^n}{n!} \\
\sum_{n\geq 0} a_n \left(\frac{2}{x^2}\right)^{-n-1} z^{-n-1} \arrow[r, equal] & \frac{x^2}{2}\sum_{n\geq 0} a_n \left(\frac{2}{x^2}\right)^{-n} z^{-n-1} \arrow[r, "\mathcal{B}"] & \frac{x^2}{2}\sum_{n\geq 0} a_n \left(\frac{x^2}{2}\right)^{n} \frac{\zeta^n}{n!}
\end{tikzcd}
$$

In particular, $\tilde{K}_{1/4}(z)$ is known as the asymptotics of the modified Bessel function $K_{1/4}(z)$ which is equal to

$$\tilde{K}_{1/4}(z) = \sqrt{\pi}e^{-z}\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{n!}(2z)^{-n-1/2}$$

Then we can compute its Borel transform:

$$
\begin{aligned}
\hat{\kappa}_{1/4} &= \sqrt{\pi} \sum_{n \geq 0} \frac{(-1)^n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{\Gamma(n+1/2) n!} \left(\frac{\zeta - 1}{2}\right)^{n-1/2} \\
&= \sqrt{\pi} \left(\frac{\zeta - 1}{2}\right)^{-1/2} \sum_{n \geq 0} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{\Gamma(n+1/2) n!} \left(\frac{1 - \zeta}{2}\right)^{n} \\
&= \sqrt{2} (\zeta - 1)^{-1/2} F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \frac{1-\zeta}{2}\right)
\end{aligned}
$$

hence

$$
\begin{tikzcd}
\psi_-(x, \eta) \arrow[r, "\mathcal{B}"] \arrow[d, equal] & \psi_{-,B}(x, y) = \frac{x^{-3/2}}{\sqrt{\pi}} (s-1)^{-1/2+\lambda/4} F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2} +; 1 - s\right) \arrow[d, equal, "\cdot \frac{x^2}{2}"] \\
\frac{1}{2\sqrt{\pi}} x^{1/2} \tilde{K}_{1/4}(z) \arrow[r, "\mathcal{B}"] & \frac{1}{2\sqrt{\pi}} x^{1/2} \sqrt{2} (\zeta - 1)^{-1/2} F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \frac{1-\zeta}{2}\right)
\end{tikzcd}
$$

□

Furthermore, the modified Bessel equation (8) has another holomorphic solution $I_{1/4}(z)$ (which together with $K_{1/4}$ forms a basis) whose asymptotic behaviour is

$$
\tilde{I}_{1/4}(z) = \frac{1}{\sqrt{\pi}} e^z \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!} (2z)^{-n-1/2} \tag{9}
$$

# 3 Physical picture

Here are some vague physical ideas about why the change of coordinates described above works. They're pretty incoherent, and they should all be taken with a grain of salt.

Consider an equation of the form

$$
\left[\left(\frac{\partial}{\partial x}\right)^2 - \eta^2 x^{n-2} (1 + f) - \lambda \eta\right] \psi = 0, \tag{10}
$$

where $n \in \{3, 4, 5, 6, \dots\}$ and the function $f$ goes to zero as $x \to \infty$. It's reasonable to look for trans-series solutions with the exponential factor $e^{\pm z}$, where $z = \frac{2}{n} \eta x^{n/2}$, because (I don't think the following calculation works)

$$
\begin{aligned}
\left[\left(\frac{\partial}{\partial x}\right)^2 - \eta^2 x^{n+2}\right] e^{\pm z} &= -\left(\tfrac{n}{2} - 1\right) \eta x^{n/2 - 2} e^{\pm z} \\
&= -\left(\tfrac{n}{2} - 1\right) \eta \left(\tfrac{n}{2} \eta^{-1} z\right)^{1-4/n} e^{\pm z} \\
&= -\left(\tfrac{n}{2} - 1\right) \left(\tfrac{n}{2}\right)^{1-4/n} \eta^{4/n} z^{1-4/n} e^{\pm z} \\
&= \dots
\end{aligned}
$$

The corresponding holomorphic solutions should have the same exponential decay. The $e^z$ and $e^{-z}$ solutions are usually linearly independent, but they can become linearly dependent when $\lambda\eta$ is in the spectrum of the rest of the operator acting on $L^2(\mathbb{R})$. In this case, we should get a solution in $O_{z\to-\infty}(e^z) \cap O_{z\to\infty}(e^{-z})$, which decays exponentially in both directions. You can see this explicitly for the $n = 2$ case: each eigenvector of the harmonic oscillator Hamiltonian is $e^{-\eta x^2/2}$ times a polynomial in $x$. The exponential decay of the eigenvectors sets a natural length scale, and $z$ measures distance with respect to it.

When $f = 0$ and $\lambda = 0$, rescalings of $x$ and $\eta$ that keep $z$ fixed have no effect on the equation. Thus, any rescaling of $x$ has the same effect on the equation as a corresponding rescaling of $\eta$. This scale-invariance is why taking the Borel transforms with respect to $\frac{2}{n}x^{n/2}$ and $\eta$ give the same result.

The $|x| \to \infty$ limit, $f$ is very small by assumption, so we're approximately in the scale-invariant situation described above. In the $\eta \to \infty$ limit, the eigenvectors of the Hamiltonian get very narrow, so it's similar to making $|x|$ large whie keeping $\eta$ fixed. This should mean once again that we can neglect $f$. The upshot is that both the $|x| \to \infty$ limit and the $\eta \to \infty$ limit take us toward the scale-invariant situation where the two Borel transforms give the same result.