Resurgence of a parabolic cylinder function

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1 Hypergeometric functions in trigonometry

1.1 Chebyshev sine formulas

By definition,

$$\begin{aligned}\cos(n\phi) &= T_n(\cos(\phi)) \\ \sin(n\phi) &= U_{n-1}(\cos(\phi))\,\sin(\phi).\end{aligned}$$

Furthermore,

$$\begin{aligned}\cos(n\phi - \tfrac{n}{2}\pi) &= T_n(\cos(\phi - \tfrac{\pi}{2})) \\ \cos(n\phi)\cos(\tfrac{n}{2}\pi) + \sin(n\phi)\sin(\tfrac{n}{2}\pi) &= T_n(\sin(\phi)).\end{aligned}$$

Thus, for $n = 2k+1$, we have

$$(-1)^k \sin(n\phi) = T_n(\sin(\phi)),$$

implying also that

$$\begin{aligned}(-1)^k n\cos(n\phi) &= T_n'(\sin(\phi))\cos(\phi) \\ (-1)^k nT_n(\cos(\phi)) &= nU_{n-1}(\sin(\phi))\cos(\phi) \\ (-1)^k T_n(\cos(\phi)) &= U_{n-1}(\sin(\phi))\cos(\phi).\end{aligned}$$

1.2 A hypergeometric identity

Let $a = \frac{1}{2} - \frac{1}{2n}$, giving $1 - 2a = \frac{1}{n}$.

1.2.1 All orders

From [**?**, equation 15.4.16],

$$F(a, 1-a, \tfrac{3}{2}; \sin(\theta)^2) = \frac{\sin((1-2a)\theta)}{(1-2a)\sin(\theta)}.$$

so we have

$$
\begin{aligned}
F\left(\tfrac{1}{2}-\tfrac{m}{2n},\tfrac{1}{2}+\tfrac{m}{2n},\tfrac{3}{2};\sin(\theta)^2\right) &= \frac{\sin(\frac{m}{n}\,\theta)}{\frac{m}{n}\sin(\theta)} \\
&= \frac{n}{m}\cdot\frac{\sin(m\,\theta/n)}{\sin(\theta/n)}\cdot\frac{\sin(\theta/n)}{\sin(\theta)} \\
&= \frac{n}{m}\cdot\frac{U_{m-1}(\cos(\theta/n))}{U_{n-1}(\cos(\theta/n))}.
\end{aligned}
$$

Letting $u=\cos(\theta/n)$, we get the identity

$$
F\left(\tfrac{1}{2}-\tfrac{m}{2n},\tfrac{1}{2}+\tfrac{m}{2n},\tfrac{3}{2};1-T_n(u)^2\right) = \frac{n}{m}\cdot\frac{U_{m-1}(u)}{U_{n-1}(u)}.
$$

From our reasoning so far, we can only conclude the identity holds when $\pm T_n(u)$ is in the right half-plane.

Identity 15.10.17 [or, better, 15.8.4] from [**?**] splits the left-hand side above into

$$
\begin{aligned}
&\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(1-\frac{m}{2n})\Gamma(1+\frac{m}{2n})}F(\tfrac{1}{2}-\tfrac{m}{2n},\tfrac{1}{2}+\tfrac{m}{2n},\tfrac{1}{2};T_n(u)^2) \\
\pm&\frac{\Gamma(-\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2}-\frac{m}{2n})\Gamma(\frac{1}{2}+\frac{m}{2n})}\,T_n(u)\,F(1-\tfrac{m}{2n},1+\tfrac{m}{2n},\tfrac{3}{2};T_n(u)^2),
\end{aligned}
$$

where the sign must be chosen so that $\pm T_n(u)$ is in the right half-plane (?). Applying identityies 15.8.27–28 from [**?**] turns this into

$$
\begin{aligned}
&\tfrac{1}{2}F(1-\tfrac{m}{n},1+\tfrac{m}{n},\tfrac{3}{2},\tfrac{1}{2}\mp\tfrac{1}{2}T_n(u))+\tfrac{1}{2}F(1-\tfrac{m}{n},1+\tfrac{m}{n},\tfrac{3}{2},\tfrac{1}{2}\pm\tfrac{1}{2}T_n(u)) \\
+&\tfrac{1}{2}F(1-\tfrac{m}{n},1+\tfrac{m}{n},\tfrac{3}{2},\tfrac{1}{2}\mp\tfrac{1}{2}T_n(u))-\tfrac{1}{2}F(1-\tfrac{m}{n},1+\tfrac{m}{n},\tfrac{3}{2},\tfrac{1}{2}\pm\tfrac{1}{2}T_n(u)).
\end{aligned}
$$

After cancellation, we conclude that

$$
F(1-\tfrac{m}{n},1+\tfrac{m}{n},\tfrac{3}{2},\tfrac{1}{2}\mp\tfrac{1}{2}T_n(u)) = \frac{n}{m}\cdot\frac{U_{m-1}(u)}{U_{n-1}(u)}
$$

when $\pm T_n(u)$ is in the right half-plane (?).

1.2.2 Odd order

From [**?**, equation 15.4.14],

$$
F(a,1-a,\tfrac{1}{2};\sin(\theta)^2) = \frac{\cos((1-2a)\theta)}{\cos(\theta)}.
$$

so we have

$$
\begin{aligned}
F\left(\tfrac{1}{2}-\tfrac{1}{2n},\tfrac{1}{2}+\tfrac{1}{2n},\tfrac{1}{2};\sin(\theta)^2\right) &= \frac{\cos(\theta/n)}{\cos(\theta)} \\
&= \frac{\cos(\theta/n)}{T_n(\cos(\theta/n))}.
\end{aligned}
$$

Let $u=\sin(\theta/n)$. If $n=2k+1$, we get the identity

$$
F\left(\tfrac{1}{2}-\tfrac{1}{2n},\tfrac{1}{2}+\tfrac{1}{2n},\tfrac{1}{2};T_n(u)^2\right) = \frac{(-1)^k}{U_{n-1}(u)}.
$$

2 The Weber equation

2.1 Basics

The Weber equation is

$$\left[\left(\tfrac{\partial}{\partial y}\right)^2 - \left(\tfrac{1}{4}y^2 + a\right)\right]\psi = 0.$$

Setting a to zero, we get

$$\left[\left(\tfrac{\partial}{\partial y}\right)^2 - \tfrac{1}{4}y^2\right]\psi = 0. \tag{1}$$

One solution is given by the parabolic cylinder function [**?**, equation 12.7.10]

$$U(0,y) = \tfrac{1}{\sqrt{2\pi}}\, y^{1/2}\, K_{1/4}\left(\tfrac{1}{4}y^2\right).$$

Combining the identity [**?**, equation 10.27.4]

$$K_\nu(z) = \frac{\pi}{2\sin(\nu\pi)}\big(I_{-\nu}(z) - I_\nu(z)\big)$$

and the integral [**?**, equation 10.32.12]

$$I_\nu(z) = \frac{1}{2\pi i}\int_{\mathcal{H}} e^{z\cosh(\phi)} e^{-\nu\phi}\, d\phi,$$

where $\mathcal{H}$ runs clockwise around the rectangle

$$0 < \mathrm{Re}(\phi) \qquad\qquad |\,\mathrm{Im}(\phi)| < \pi,$$

we learn that

$$\begin{aligned} I_{-\nu}(z) - I_\nu(z) &= \frac{1}{\pi i}\int_{\mathcal{H}} e^{z\cosh(\phi)}\sinh(\nu\phi)\, d\phi \\ K_\nu(z) &= \frac{1}{2i\,\sin(\nu\pi)}\int_{\mathcal{H}} e^{z\cosh(\phi)}\sinh(\nu\phi)\, d\phi. \end{aligned}$$

In particular, we have

$$K_{1/4}(z) = \frac{1}{i\sqrt{2}}\int_{\mathcal{H}} e^{z\cosh(\phi)}\sinh(\phi/4)\, d\phi.$$

Setting $u = \cosh(\phi/4)$ and getting $\cosh(\phi) = 8u^4 - 8u^2 + 1$ from a table of Chebyshev polynomials, we can write

$$K_{1/4}(z) = \frac{2\sqrt{2}}{i}\int_\Gamma \exp\left[z\left(8u^4 - 8u^2 + 1\right)\right]\, du, \tag{2}$$

where Γ is a path that comes from ∞ at $-45°$ and goes to ∞ at $45°$. **[This feels dubious; check using definition of $\mathcal{H}$.]**

2.2 Contour argument

We can recast integral 2 into $\hat{\mathbb{C}}$ by setting $-\zeta = 8u^4 - 8u^2 + 1$. Projecting Γ to a contour γ in $\hat{\mathbb{C}}$ and choosing the branch of u that lifts γ back to Γ, we have

$$K_{1/4} = \frac{i}{\sqrt{2}} \int_{\gamma} e^{-z\zeta} \frac{d\zeta}{8u^3 - 4u}. \tag{3}$$

The integrand has poles at $u = \pm\frac{1}{\sqrt{2}}$, and γ runs counterclockwise around $\left[\frac{1}{\sqrt{2}}, \infty\right)$.

Using the identity from Section 1.2.1, we can rewrite integral 3 as

$$K_{1/4} = \frac{i}{\sqrt{2}} \int_{\gamma_z} e^{-z\zeta} F\left(\tfrac{3}{8}, \tfrac{5}{8}; \tfrac{3}{2}; 1 - \zeta^2\right) d\zeta.$$