RESURGENCE OF THE AIRY FUNCTION AND OTHER EXPONENTIAL INTEGRALS

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1. Hypergeometric functions as Borel transform of second order ODE (*series normales de* I*er ordre*)

Let us consider the following linear second order ODE

$$\left[P\left(\frac{\partial}{\partial z}\right)+\frac{1}{z}Q\left(\frac{\partial}{\partial z}\right)+\frac{1}{z}R\left(\frac{1}{z}\right)\right]f(z)=0 \tag{1.1}$$

with $\deg P = 2$, $\deg Q = 1$ and $R = O(\frac{1}{z})$. We denote by α_1, α_2 the roots of $P(-\lambda)$ and we assume they are distinct. Furthermore we assume $\tau_j := \frac{Q(\alpha_j)}{P'(\alpha_j)} \neq 0$. The latter assumption guarantees the formal solution $\tilde{f}$ being slight, while the former assumption implies there will be two independent solutions.

Under the previous assumptions we prove that the Borel transformed solution $\hat{f}(\zeta_j)$ is a Gauss hypergeometric function, $\zeta_j = \zeta - \alpha_j$.

Proposition 1.1. Let $P(\lambda) = \lambda^2 + a_1\lambda + a_0$, $Q(\lambda) = b_1\lambda + b_0$ and $R(\frac{1}{z}) = \frac{c_1}{z}$ satisfying the previous assumptions. Then

$$\hat{f}(\zeta_1)=\left(1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)^{c-1}{}_2F_1\left(a,b;c;1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right) \tag{1.2}$$

$$\hat{f}(\zeta_2)=\left(1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right)^{c-1}{}_2F_1\left(a,b;c;1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right) \tag{1.3}$$

where the coefficients a, b, c depend on the parameter of P, Q, R.

Proof. We start by taking the Borel transform of (1.1):

(1.4)

$$(\zeta^2 - a_1\zeta + a_0)\hat{f}(\zeta) + \int_0^\zeta b_1(-\zeta')\hat{f}(\zeta')d\zeta' + b_0\int_0^\zeta \hat{f}(\zeta')d\zeta' + c_1\int_0^\zeta (\zeta-\zeta')\hat{f}(\zeta')d\zeta' = 0$$

then we differentiate twice in order to have a differential equation which can be easier recognized as a hypergeometric equation. Since $\tilde{F}$ is slight and locally integrable at 0 by assumption, Proposition 1 `Resurgent Airy doc by Aaron` tells we are not loosing information taking derivatives, and that $\hat{f}(\zeta)$ is a solution of (1.4) if and only if it is a solution of (1.5)

$$\left[(\zeta^2-a_1\zeta+a_0)\partial_\zeta^2+(4\zeta-b_1\zeta-2a_1+b_0)\partial_\zeta+(c_1+2-b_1)\right]\hat{f}(\zeta)=0 \tag{1.5}$$

We introduce some notation to simplify the computations, we denote by $\beta_1=4-b_1$, $\beta_0=b_0-2a_1$, $\gamma=c_1+2-b_1$ so (1.5) turns into

$$\left[(\zeta-\alpha_1)(\zeta-\alpha_2)\partial_\zeta^2+(\beta_1\zeta+\beta_0)\partial_\zeta+\gamma\right]\hat{f}(\zeta)=0$$

We consider the following change of coordinates $\zeta=\alpha_2-(\alpha_2-\alpha_1)\xi$[1]

$$\left[(\alpha_2-(\alpha_2-\alpha_1)\xi-\alpha_1)(\alpha_2-(\alpha_2-\alpha_1)\xi-\alpha_2)(\alpha_1-\alpha_2)^{-2}\partial_\xi^2+(\beta_1(\alpha_2-(\alpha_2-\alpha_1)\xi)+\beta_0)(\alpha_1-\alpha_2)^{-1}\partial_\xi+\gamma\right]\hat{f}(\xi)=0$$

$$\left[(\alpha_2-\alpha_1)(1-\xi)(\alpha_1-\alpha_2)\xi(\alpha_1-\alpha_2)^{-2}\partial_\xi^2+(\beta_1\alpha_2-\beta_1(\alpha_2-\alpha_1)\xi+\beta_0)(\alpha_1-\alpha_2)^{-1}\partial_\xi+\gamma\right]\hat{f}(\xi)=0$$

$$\left[-(1-\xi)\xi\partial_\xi^2+((\beta_1\alpha_2+\beta_0)(\alpha_1-\alpha_2)^{-1}+\beta_1\xi)\partial_\xi+\gamma\right]\hat{f}(\xi)=0$$

$$\left[(1-\xi)\xi\partial_\xi^2-((\beta_1\alpha_2+\beta_0)(\alpha_1-\alpha_2)^{-1}+\beta_1\xi)\partial_\xi-\gamma\right]\hat{f}(\xi)=0$$

The latter equation is an hypergeometric equation of parameters

$$\begin{aligned}
&C=(\beta_1\alpha_2+\beta_0)(\alpha_2-\alpha_1)^{-1}\\
&A+B+1=\beta_1=4-b_1\Rightarrow A+B=3-b_1\\
&AB=\gamma=c_1+2-b_1
\end{aligned}$$

and a solution is given by

$$\begin{aligned}
\hat{f}(\xi)&=\xi^{1-C}{}_2F_1(A-C+1,B-C+1;2-C;\xi)\\
&=\left(\frac{\alpha_2-\zeta}{\alpha_2-\alpha_1}\right)^{1-C}{}_2F_1\left(A-C+1,B-C+1;2-C;\frac{\alpha_2-\zeta}{\alpha_2-\alpha_1}\right)\\
&=\left(1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)^{1-C}{}_2F_1\left(A-C+1,B-C+1;2-C;1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)
\end{aligned}$$

□

Proposition 1.2. We verify that

$$\hat{f}(\zeta_1)=\left(1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)^{1-C}{}_2F_1\left(A-C+1,B-C+1;2-C;1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right) \tag{1.6}$$

$$\hat{f}(\zeta_2)=\left(1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right)^{1-C}{}_2F_1\left(A-C+1,B-C+1;2-C;1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right) \tag{1.7}$$

are indeed resurgent.

[1] $\partial_\zeta=(\alpha_1-\alpha_2)^{-1}\partial_\xi$ and $\partial_\zeta^2=(\alpha_1-\alpha_2)^{-2}\partial_\xi^2$

Proof. Recall that the analytic continuation of Gauss hypergeometric functions is given in terms of other hypergeometric fucntions (see DLMF 15.2.3):

$$\hat{f}(\zeta_1+i0)-\hat{f}(\zeta_1-i0)\propto\left(\frac{\zeta_1}{\alpha_1-\alpha_2}\right)^{C-A-B}\left(1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)^{1-C}{}_2F_1\left(1-A,1-B;C+1-A-B;\frac{\zeta_1}{\alpha_2-\alpha_1}\right)$$
$$=(-1)^{C-A-B}\left(1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right)^{C-A-B}\left(\frac{\zeta_2}{\alpha_1-\alpha_2}\right)^{1-C}\cdot$$
$$\cdot{}_2F_1\left(1-A,1-B;C+1-A-B;1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right)$$

□

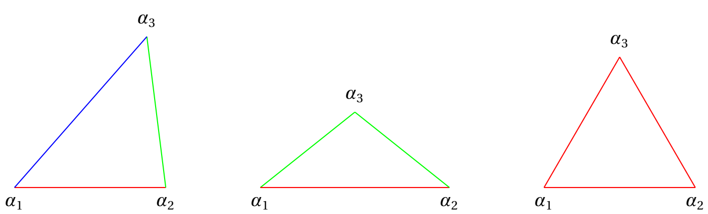
2. Hypergeometric functions as Borel transform of third order ODE

Let us consider the following linear third order ODE

$$\left[P(\tfrac{\partial}{\partial z})+\tfrac{1}{z}Q(\tfrac{\partial}{\partial z})+\tfrac{1}{z}R(\tfrac{1}{z})\right]f(z)=0 \tag{2.1}$$

with $\deg P=3$, $\deg Q=2$ and $R=O(\frac{1}{z})$. We denote by $\alpha_1,\alpha_2,\alpha_3$ the roots of $P(-\lambda)$ and we assume they are distinct. There are three possible scenarios illustrated in figure **??** below:

- $|\alpha_1-\alpha_2|<|\alpha_2-\alpha_3|<|\alpha_1-\alpha_3|$
- $|\alpha_2-\alpha_3|=|\alpha_1-\alpha_3|<|\alpha_1-\alpha_2|$
- $|\alpha_1-\alpha_2|=|\alpha_2-\alpha_3|=|\alpha_1-\alpha_3|$



Furthermore we assume $\tau_j:=\frac{Q(\alpha_j)}{P'(\alpha_j)}\in\mathbb{Q}$. The latter assumption guarantees the formal solution $\tilde{f}$ being slight, while the former assumption implies there will be three independent solutions.

Under the previous assumptions we prove that the Borel transformed solution $\hat{f}(\zeta_j)$ is hypergeometric, $\zeta_j=\zeta-\alpha_j$.

Proposition 2.1. Let $P(\lambda)=\lambda^3+a_2\lambda^2+a_1\lambda+a_0$, $Q(\lambda)=b_2\lambda^2+b_1\lambda+b_0$ and $R(\frac{1}{z})=\frac{c_1}{z}+\frac{c_2}{z^2}$ satisfying the previous assumptions. Then

- if $|\alpha_1-\alpha_2|<|\alpha_2-\alpha_3|<|\alpha_1-\alpha_3|$

$$\hat{f}(\zeta_1)=\left(1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)^{c-1}{}_2F_1\left(\mathbf{m};\mathbf{n};1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right) \tag{2.2}$$

$$\hat{f}(\zeta_2)=\left(1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right)^{c-1}{}_2F_1\left(\mathbf{m};\mathbf{n};1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right) \tag{2.3}$$

$$\hat{f}(\zeta_3)=\left(1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)^{c-1}{}_2F_1\left(\mathbf{m};\mathbf{n};1-\frac{\zeta_3}{\alpha_2-\alpha_1}\right) \tag{2.4}$$

- if $|\alpha_1-\alpha_2|<|\alpha_2-\alpha_3|=|\alpha_1-\alpha_3|$

$$\hat{f}(\zeta_1)=\left(1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)^{c-1}{}_2F_1\left(\mathbf{m};\mathbf{n};1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right) \tag{2.5}$$

$$\hat{f}(\zeta_2)=\left(1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right)^{c-1}{}_2F_1\left(\mathbf{m};\mathbf{n};1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right) \tag{2.6}$$

$$\hat{f}(\zeta_3)=? \tag{2.7}$$

- if $|\alpha_1-\alpha_2|=|\alpha_2-\alpha_3|=|\alpha_1-\alpha_3|$

$$\hat{f}(\zeta_1)=\left(1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)^{c-1}{}_3F_2\left(\mathbf{m};\mathbf{n};1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right) \tag{2.8}$$

$$\hat{f}(\zeta_2)=\left(1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right)^{c-1}{}_3F_2\left(\mathbf{m};\mathbf{n};1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right) \tag{2.9}$$

$$\hat{f}(\zeta_3)=\left(1-\frac{\zeta_3}{\alpha_2-\alpha_1}\right)^{c-1}{}_3F_2\left(\mathbf{m};\mathbf{n};1-\frac{\zeta_3}{\alpha_2-\alpha_1}\right) \tag{2.10}$$

where the coefficients $\mathbf{m}=(m_1,m_2,m_3),\mathbf{n}=(n_1,n_2)$ depend on the parameter of P,Q,R.

Proof. The strategy is analogous to the one for second order ODEs. Hence, we first consider the Borel transform of equation 2.1:

$$P(-\zeta)\hat{f}+\int_0^\zeta Q(-\zeta')\hat{f}(\zeta')d\zeta'+c_1\int_0^\zeta(\zeta-\zeta')\hat{f}(\zeta')d\zeta'+\frac{c_2}{2}\int_0^\zeta(\zeta-\zeta')^2\hat{f}(\zeta')d\zeta'=0 \tag{2.11}$$

Although equation (2.11) is an integral equation, since $\tilde{f}$ is slight, $\hat{f}$ is a solution of (2.11) if and only if it is a solution of the third order ODE obtained by differentiating (2.11) three times:

$$\left[P(-\zeta)\partial_\zeta^3+(-3P'(-\zeta)+Q(-\zeta))\partial_\zeta^2+(3P''(-\zeta)-2Q'(-\zeta)+c_1)\partial_\zeta+(-P'''(-\zeta)+Q''(-\zeta)+c_2)\right]\hat{f}=0 \tag{2.12}$$

Notice that

$$
\begin{aligned}
P(-\zeta) &= (\zeta-\alpha_1)(\zeta-\alpha_2)(\zeta-\alpha_3)\\
-P'(-\zeta) &= (\zeta-\alpha_2)(\zeta-\alpha_3)+(\zeta-\alpha_1)(\zeta-\alpha_3)+(\zeta-\alpha_2)(\zeta-\alpha_1)\\
P''(-\zeta) &= 6\zeta-2\alpha_1-2\alpha_2-2\alpha_3
\end{aligned}
$$

We now study separately the three scenarios:

- if $\ell = |\alpha_1-\alpha_2| < |\alpha_2-\alpha_3| < |\alpha_1-\alpha_3|$ we claim that (2.12) is ??
- if $\alpha_2 = \alpha_3+\ell$, $\alpha_1 = \alpha_3-\ell$ we claim that (2.12) is a ?? Change coordinates $y = \frac{\zeta-\alpha_1}{\ell}$:

$$
\begin{aligned}
P(-\zeta)\partial_\zeta^3 &\longrightarrow -\ell^3 y(\ell y+\alpha_1-\alpha_2)(\ell y+\alpha_1-\alpha_3)\frac{1}{\ell^3}\partial_y^3 = y(1-y^2)\partial_y^3\\
(-3P'(-\zeta)+Q(-\zeta))\partial_\zeta^2 &\longrightarrow (3\ell^2(3y^2-1)+Q(-\ell y-\alpha_1))\frac{1}{\ell^2}\partial_y^2\\
(3P''(-\zeta)-2Q'(-\zeta)+c_1)\partial_\zeta &\longrightarrow (-18\ell y-2Q'(-\ell y-\alpha_1)+c_1)\frac{1}{\ell}\partial_y\\
-P'''(-\zeta)+Q''(-\zeta)+c_2 &\longrightarrow -6+2b_2+c_2
\end{aligned}
$$

Change coordinates $s = y^2$:

$$
\begin{aligned}
y(1-y^2)\partial_y^3 &\longrightarrow 8(1-s)s^2\partial_s^3+12(1-s)s\partial_s^2\\
(3\ell^2(3y^2-1)+Q(-\ell y-\alpha_1))\frac{1}{\ell^2}\partial_y^2 &\longrightarrow \Big[6(3s-1)+2(b_2s+\frac{2b_2\alpha_1-b_1}{\ell}y+\frac{b_2\alpha_1^2-\alpha_1b_1+b_0}{\ell^2})\Big](\partial_s+2s\partial_s^2)\\
(-18\ell y-2Q'(-\ell y-\alpha_1)+c_1)\frac{1}{\ell}\partial_y &\longrightarrow 4(2b_2-9)s\partial_s+2\frac{4b_2\alpha_1-2b_1+c_1}{\ell}y\partial_s
\end{aligned}
$$

$$
\begin{aligned}
&\Big[8(1-s)s^2\partial_s^3+\Big[4(6+b_2)s+4\frac{2b_2\alpha_1-b_1}{\ell}y+4\frac{b_2\alpha_1^2-\alpha_1b_1+b_0}{\ell^2}\Big]s\partial_s^2\\
&\Big[-2(9-2b_2)s-2(3-2b_2)+2\frac{b_2\alpha_1^2-\alpha_1b_1+b_0}{\ell^2}+2\frac{6b_2\alpha_1-3b_1+c_1}{\ell}y\Big]\partial_s-6+2b_2+c_2\Big]\hat{f}=0
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
\frac{2b_2\alpha_1-b_1}{\ell}y+\frac{b_2\alpha_1^2-\alpha_1b_1+b_0}{\ell^2} &= 2n_1+2n_2-2\\
6+b_2 &= -2(3+m_1+m_2+m_3)\\
\Big[-(3-2b_2)+2(\frac{b_2\alpha_1^2-\alpha_1b_1+b_0}{\ell^2})+\frac{8b_2\alpha_1-4b_1+c_1}{\ell}y\Big] &= 4n_1n_2\\
(9-2b_2) &= 4(1+m_1+m_2+m_3+m_1m_2+m_1m_3+m_2m_3)\\
-6+2b_2+c_2 &= 8m_1m_2m_3
\end{aligned}
$$

- if $|\alpha_1-\alpha_2| = |\alpha_2-\alpha_3| = |\alpha_1-\alpha_3| = \ell$ we claim that (2.12) is a generalized hypergeometric equation. Without loss of generality we assume $\alpha_2 = \alpha_1+\ell$ and $\alpha_3 = \alpha_1 + \frac{\ell}{2} + i\frac{\sqrt{3}}{2}\ell$.

  Change coordinates $y = \frac{\zeta-\alpha_1}{\ell} + \frac{\sqrt{3}+i}{2}$, then

$$
\begin{aligned}
P(-\zeta)\partial_\zeta^3 &\longrightarrow -\ell y(\ell y+\alpha_1-\alpha_2)(\ell y+\alpha_1-\alpha_3)\frac{1}{\ell^3}\partial_y^3 = y(1-y^2)\partial_y^3\\
(-3P'(-\zeta)+Q(-\zeta))\partial_\zeta^2 &\longrightarrow (3\ell^2(3y^2-1)+Q(-\ell y-\alpha_1))\frac{1}{\ell^2}\partial_y^2\\
(3P''(-\zeta)-2Q'(-\zeta)+c_1)\partial_\zeta &\longrightarrow (-18\ell y-2Q'(-\ell y-\alpha_1)+c_1)\frac{1}{\ell}\partial_y\\
-P'''(-\zeta)+Q''(-\zeta)+c_2 &\longrightarrow -6+2b_2+c_2
\end{aligned}
$$

$\square$