Borel regularity for ODEs and exponential integrals

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1 Introduction

1.1 The unreasonable effectiveness of Borel summation

You can often find a formal power series

$$\tilde{\Phi} = \frac{c_0}{z^\tau} + \frac{c_1}{z^{\tau+1}} + \frac{c_2}{z^{\tau+2}} + \frac{c_3}{z^{\tau+3}} + \dots,$$

with $\tau \in (0,1]$, that looks or acts like a solution to a problem whose actual solutions are holomorphic functions of z. For example, if you want to understand how the solutions of the holomorphic ordinary differential equation (ODE)

$$\left[\left[\left(\tfrac{\partial}{\partial z}\right)^2 - 1\right] + z^{-1}\tfrac{\partial}{\partial z} - \left(\tfrac{1}{3}\right)^2 z^{-2}\right] \Phi = 0. \tag{1}$$

behave near $z = \infty$, you might start by looking for formal *transmonomial* solutions $e^{-\alpha z}\,\tilde{\Phi}$, where $\tilde{\Phi}$ is a formal power series of the kind above. Setting $\alpha = -\frac{1}{12}$ and $\tau = \frac{1}{2}$ gives a well-behaved recurrence relation for $\tilde{\Phi}$, which produces the solution [check]

$$e^{z/12}\left[\frac{(-1)!!}{z^{1/2}} + \frac{5}{6}\cdot\frac{1!!}{z^{3/2}} + \frac{385}{216}\cdot\frac{3!!}{z^{5/2}} + \frac{17017}{3888}\cdot\frac{5!!}{z^{7/2}} + \dots\right] \tag{2}$$

and its constant multiples. As another example, you might rewrite the integral

$$\Phi(z) = \int_\Lambda \exp\left[-\tfrac{1}{12} z\left(4u^3 - 3u\right)\right]\, du$$

$$\Phi(z) = \int_\Lambda \exp\left[-z\left(\tfrac{1}{3}u^3 - \tfrac{1}{4}u\right)\right]\, du$$

as

$$e^{z/12}\int_{-\infty}^{\infty} e^{-z\tau^2/2}\left[1 - \frac{2}{3}\tau + \frac{5}{6}\tau^2 - \frac{32}{27}\tau^3 + \frac{385}{216}\tau^4 - \frac{224}{81}\tau^5 + \frac{17017}{3888}\tau^6 - \dots\right] d\tau$$

using the substitution $\frac{1}{2}\tau^2 = \frac{1}{3}u^3 - \frac{1}{4}u + \frac{1}{12}$. Naïvely integrating term by term, you again get the transmonomial (2).

$$\begin{aligned} & e^{z/12} z^{-1/2} \left[(-1)!! + \frac{5}{6}\, 1!!\, z^{-1} + \frac{385}{216}\, 3!!\, z^{-2} + \dots \right] \\ &= e^{z/12} z^{1/2} \left[\frac{(-1)!!}{z} + \frac{5}{6} \cdot \frac{1!!}{z^2} + \frac{385}{216} \cdot \frac{3!!}{z^3} + \frac{17017}{3888} \cdot \frac{5!!}{z^4} + \dots \right] \\ &= e^{z/12} \left[\frac{(-1)!!}{z^{1/2}} + \frac{5}{6} \cdot \frac{1!!}{z^{3/2}} + \frac{385}{216} \cdot \frac{3!!}{z^{5/2}} + \frac{17017}{3888} \cdot \frac{5!!}{z^{7/2}} + \dots \right] \end{aligned}$$

Once you have the formal solution $\tilde{\Phi}$, you might try to get an actual solution by applying *Borel summation*, which turns a formal power series into a function asymptotic to it. Borel summation works in three steps.

1. Thinking of z as a "frequency variable," we take the Borel transform (also known as formal inverse Laplace transform) of $\tilde{\Phi}$, producing a formal power series $\tilde{\phi}$ in a new "position variable" ζ.

2. If $\tilde{\phi}$ has a positive radius of convergence we say $\tilde{\Phi}$ is 1-*Gevrey*. We sum $\tilde{\phi}$ to get a holomorphic function $\hat{\phi}$ on a neighborhood of $\zeta = 0$. Then, by analytic continuation, we expand the domain of $\hat{\phi}$ to a Riemann surface B with a distinguished 1-form λ—the continuation of $d\zeta$. [Nikita has a complementary picture where the Borel plane is the cotangent fiber? Ask more about this.]

3. Furthermore, if $\hat{\phi}$ grows slowly enough along an infinite ray $b + e^{i\theta}[0, \infty)$ [change, explain, or link to notation] its Laplace transform $\mathcal{L}_b^\theta \hat{\phi}$ [link to definition] turns out to be a holomorphic function of z, well-defined on some sector of the frequency plane. In this case, we say $\tilde{\Phi}$ is *Borel-summable*, and we call $\mathcal{L}_b^\theta \hat{\phi}$ its *Borel sum* at b.

The series $\tilde{\Phi}$ and its Borel sum $\hat{\Phi}$ have a special relationship, which is best described in the language of *Gevrey asymptoticity*. [Do we need to extend these definitions, and the associated theorems, to cover slight series? Hopefully it wouldn't be too hard, but it would be a pain.]

Definition 1. On an open sector Ω around ∞, a power series

$$\frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \frac{a_4}{z^4} + \dots \tag{3}$$

is *asymptotic* to a holomorphic function F if

$$\left| F - \left(\frac{a_1}{z} + \frac{a_2}{z^2} + \dots + \frac{a_n}{z^n} \right) \right| \in o_{z \to \infty}\big(|z|^{-n}\big)$$

over all orders n.

The *asymptotic expansion* map æ^θ sends a holomorphic function that vanishes at ∞ to the unique power series of the form (3) which is asymptotic to it on the half-plane $\operatorname{Re} e^{i\theta} z > 0$ [Theorem C.11, Nikolaev, "Existence and uniqueness…"].

Definition 2. On an open sector Ω around ∞, a power series of the form (3) is *uniformly* 1-*Gevrey-asymptotic* to a holomorphic function F if there's a constant $A \in (0, \infty)$ for which

$$\left| F - \left(\frac{a_1}{z} + \frac{a_2}{z^2} + \ldots + \frac{a_{n-1}}{z^{n-1}} \right) \right| \lesssim \frac{A^n n!}{|z|^n}$$

over all orders n and all $z \in \Omega$. We'll use $\lesssim$ throughout the paper to mean "bounded by a constant multiple of."

To compare the Borel sum $\hat{\Phi}$ with the original series $\tilde{\Phi}$, let's take it one more step, sending it back into the world of formal power series by taking its asymptotic expansion.

4. By construction, $\text{æ}^\theta \hat{\Phi} = \tilde{\Phi}$. It turns out that $\tilde{\Phi}$ is not only asymptotic to $\hat{\Phi}$, but *uniformly* 1-*Gevrey*-asymptotic [**?**, Corollary 5.23].

The Borel summation process is summarized in the following diagram:

$$\begin{tikzcd}
& \text{problem} \arrow[dl, dotted, mapsto] \arrow[dr, mapsto, "\text{formally solves}"] & \\
\hat{\Phi} \arrow[rr, mapsto, "\text{æ}^\theta", "\text{(1-Gevrey)}"'] & & \tilde{\Phi} \arrow[d, mapsto, "\mathcal{B}"] \\
\hat{\phi} \arrow[u, mapsto, "\mathcal{L}_b^\theta"] & & \tilde{\phi} \arrow[ll, mapsto, "\text{sum}"]
\end{tikzcd}$$

You can't be sure *a priori* that $\hat{\Phi}$ solves your original problem, even if you know that $\tilde{\Phi}$ is asymptotic to an actual solution. After all, $\tilde{\Phi}$ is asymptotic to lots of functions; Borel summation just picks one of them. In many cases, however, Borel summation picks correctly, delivering an actual solution to your problem. The question of how that happens is the starting point for this paper.

1.2 A new perspective: Borel regularity

1.2.1 Introducing Borel regularity

- Start instead with a holomorphic solution Φ.
- Taking its asymptotic expansion along the half-plane $\operatorname{Re} e^{i\theta} z > 0$, we get a formal power series $\tilde{\Phi}$.

We consider a new diagram: we start reading it from the upper-left corner, namely we assume there exists a holomorphic function Φ which is asymptotic to a formal power series $\tilde{\Phi}$ as $\operatorname{Re} e^{i\theta} z$ goes to ∞ in the direction of θ

$$\begin{tikzcd}
& \text{problem} \arrow[dl, dash, "\text{solves}"'] \arrow[ddr, dash, "\text{formally solves}"] & \\
\Phi \arrow[drr, mapsto, "æ^{\theta}"] & & \\
\hat{\Phi} \arrow[rr, "æ^{\theta}", "\text{(1-Gevrey)}"'] & & \tilde{\Phi} \arrow[d, mapsto, "\mathcal{B}"] \\
\hat{\phi} \arrow[u, mapsto, "\mathcal{L}_b^{\theta}"] & & \tilde{\phi} \arrow[ll, mapsto, "\text{sum}"]
\end{tikzcd}$$

If $\tilde{\Phi}$ is Borel-summable, as described in Section 1.1, its Borel sum $\hat{\Phi}$ is a new holomorphic function. Since different functions can be asymptotic to the same power series, taking the Borel sum of the asymptotic series of Φ must smooth away some details. We'll therefore call this process *Borel regularization*. Explicitly, Borel regularization works in four steps.

1. Take the asymptotic expansion $\tilde{\Phi} = æ^{\theta}\Phi$.
2. Take the Borel transform $\tilde{\phi} = \mathcal{B}\tilde{\Phi}$.
3. Take the sum $\hat{\phi}$ of $\tilde{\phi}$, and expand its domain to a Riemann surface with a distinguished 1-form, as before. This is possible, by definition, if $\tilde{\Phi}$ is 1-Gevrey.
4. Take the Laplace transform $\hat{\Phi} = \mathcal{L}^{\theta}_{\zeta,0}\hat{\phi}$. This is possible, by definition, if $\tilde{\Phi}$ is Borel-summable. **[Should we be allowed to take the Laplace transform at another points $b \in B$ instead?]**

We'll say a function is *Borel-regularizable* if its asymptotic expansion is Borel-summable, ensuring that we can carry out the last two steps.

Defining a regularization process picks out a class of regular functions: the ones that are invariant under regularization. We'll say a function is *Borel regular* if it's Borel-regularizable and Borel regularization leaves it unchanged. In other words, Φ is Borel regular if $\hat{\Phi} = \Phi$.

1.2.2 Borel regularity as a good approximation condition

Borel regular functions can be characterized as functions that are approximated well, asymptotically, by polynomials. Watson showed a century ago [Watson] that a function F that vanishes at ∞ is Borel regular whenever there's an obtuse-angled sector around ∞ and a pair of constants $C, A \in (0, \infty)$ for which

$$\left| F - \left(\frac{a_1}{z} + \frac{a_2}{z^2} + \ldots + \frac{a_{n-1}}{z^{n-1}} \right) \right| \leq \frac{CA^n n!}{|z|^n}$$

over all orders n and all z in the sector.

Nevanlinna's improvement of Watson's theorem (Sokal, "An improvement of Watson's theorem on Borel summability", see a proof in Nikita, "Exact Solutions for the Singularly Perturbed Riccati Equation and Exact WKB Analysis" Theorem B.15) tells us that a function Φ holomorphic in a sectorial neighbourhood $\mathsf{A}_\theta := \{z \in \mathbb{C} | \operatorname{Re}(e^{i\theta} z)\}$ is Borel regular.

Indeed let $\tilde{\Phi} \in \mathbb{C}[\![z^{-1}]\!]$ be the "Gevrey-asymptotics" of Φ as $\mathrm{Re}(e^{i\theta}z) \to \infty$, then its Borel transform $\hat{\phi} := \mathcal{B}\tilde{\Phi}$ is a germ of holomorphic function at the origin and it can be analytically continued in a tubular neighbourhood of the ray $e^{i\theta}[0,\infty)$ with at most exponential growth at infinity. Hence Φ is Borel-regularizable. In addition, $\tilde{\phi}$ converges uniformly to a holomorphic function $\phi(\zeta)$ whose Laplace transform coincides with $\Phi(z)$. In particular, we learn that being "Gevrey-asymptotic" to a formal series is a sufficient condition for Borel regularity.

1.2.3 Borel regularity sometimes explains why Borel summation works

The idea of Borel regularity can help us understand why Borel summation is so effective in some situations. Roughly speaking, Borel summation works well for problems with Borel regular solutions that stand out asymptotically [this may need refining].

The central goal of this paper is to explain, from this perspective, why Borel summation works well for the two kinds of problems exemplified in Section 1.1.

1. Solving a linear ODE as the one in example Equation (1). The general example is the equation $\mathcal{P}\Phi = 0$ for a differential operator of the form

$$\mathcal{P} = P(\partial_z) + \frac{1}{z}Q(\partial_z) + \frac{1}{z^2}\left[R_0(z^{-1}) + R_1(z^{-1})\,\partial_z + \ldots + R_d(z^{-1})\,\partial_z^d\right],$$

   where P is a monic degree-d polynomial, Q is a degree-$(d-1)$ polynomial, and $R_0, \ldots, R_d$ are entire functions of exponential type.

   We'll restrict our attention to the case where the roots of P are distinct.

   The form of $\mathcal{P}$ derives from Equation 2.2.3 (p. 105) of [?], which encompasses both linear and nonlinear ODEs of Poincaré rank 1. Borel summation still works well for the nonlinear ones, raising the question of whether our analysis generalizes.

2. the second problem is evaluating a certain kind of exponential integral: a one-dimensional *thimble integral*

$$I(z) := \int_\Lambda \exp\left[-zf\right]\nu \tag{4}$$

   where $f\colon X \to \mathbb{C}$ is a Morse function, X is 1-dimensional complex algebraic variety, $\nu \in \Omega^1(X)$ is holomorphic 1-form in X and Λ is a suitable contour such that $I(z)$ is a holomorphic function of z.

These two problems are closely linked. By playing with derivatives of an exponential integral, you can often find a linear ODE that the integral satisfies. Conversely, for many classical ODEs, there are useful bases of exponential integral solutions.

1.3 Goals and Results

First of all we want to clearly separate the parts of the theory that deal with holomorphic functions and formal series. Indeed, for ODEs, we can build an analytic frame of solutions with an explicit growth behaviour (see Theorem 1.1), while for thimble integrals we can write an explicit analytic function whose Laplace transform is the thimble integral itself (see part 4 of Theorem 3.1).

Second of all, we want to explain why it is useful to work in the Borel plane (the spatial domain): on the hand integral equations are more regular than differential equations. On the other hand, a thimble integral in the frequency domain can be recast as the Laplace transform of a function in the spatial domain.

1.3.1 Why does Borel resummation work for solutions of irregular singular ODEs?

We restrict to ODEs with irregular singularity at ∞ and of Poincaré rank 1:

$$P(\partial_z) + \frac{1}{z}Q_1(\partial_z) + \frac{1}{z^2}\left[R_0(z^{-1}) + R_1(z^{-1})\,\partial_z + \ldots + R_d(z^{-1})\,\partial_z^d\right] \tag{5}$$

where $P(\lambda)$ is a (monic) degree d polynomial, $Q(\lambda)$ is a degree $d-1$ polynomial and $R_0, \ldots, R_d$ are holomorphic functions on $\mathbb{C}$ [1]. Furthermore we assume P has simple zeros $P(-\alpha_j) = 0$, $j = 1, ..., d$ and $Q(-\alpha_j) \neq 0$.

Under these assumptions, (5) admits a basis of formal solutions [**?**] [**?**, Proposition 2.2.7, p. 111]: let $\tau_j := Q(-\alpha_j)/P'(-\alpha_j) \in \mathbb{Q}^*$, then the formal solutions $\tilde{\Psi}_1, ..., \tilde{\Psi}_d$ are of the form

$$\tilde{\Psi}_j(z) = e^{-\alpha_j z} z^{-\tau_j} \tilde{F}_j(z) \in e^{-\alpha_j z} z^{-\tau_j}\, \mathbb{C}[\![z^{-\tau_j}]\!] \tag{6}$$

Notice that this basis is distinguished only up to scaling, so we have a distinguished frame in the space of formal solutions.

Classically it has been proved that the formal solutions $\tilde{\Psi}_j$ (6) are Borel-Laplace summable and their Borel-Laplace sum $\hat{\Psi}_j$ satisfies the original equation (5) [**?**][Check references]. Therefore, the distinguished frame of formal solutions become a distinguished frame of analytic solutions $\hat{\Psi}_1, ..., \hat{\Psi}_d$.

However, can we find a distinguished basis of solutions in a purely analytic way? This should be possible as the coefficients of the ODE are regular enough. In fact, the Main Asymptotic Existence Theorem (M.A.E.T.) guarantees the existence of an analytic frame of solutions asymptotics to a formal frame [reference-see Sauzin-Mitchi or Delabere book], but it is not a constructive method (see Balser chap. 14). Conversely, we can explicitly built a distinguished basis of analytic solutions:

Theorem 1.1. *In the previous set-up, for every α_j there exists a unique solution in the ζ-plane which blows-up as $\zeta^{\tau_j - 1}$ at α_j: set $\zeta_j = \zeta - \alpha_j$,*

$$\psi_j(\zeta_j) = \zeta_j^{\tau_j - 1} + f_j$$

.

[verify the exponent, is it $\tau_j - 1$ or τ_j? What are the conditions for f_j?]

At this point, we may ask what is the relationship between the two frame of analytic solutions, namely the Borel sum $\hat{\Psi}_j$ of the formal solutions $\tilde{\Psi}_j$ and the Laplace transform

[1] Looking at the existence theorem in `airy-resurgence` 2.1.3, we could apply this reasoning on the analytic side for more general equations, but this particular case makes it easier to talk about the formal side as well.]

of ψ_j. Our main result says they are actually the same, hence proving Borel summation works.

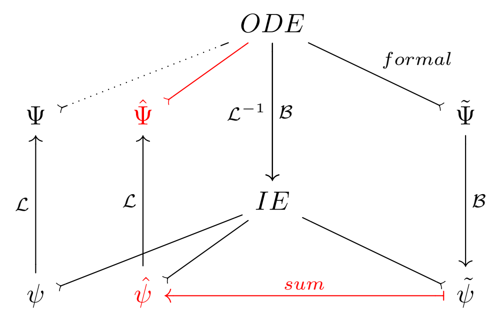
Theorem 1.2. Let $\Psi_j := \mathcal{L}^{\theta}_{\zeta_j}\psi_j$ be the Laplace transform of ψ_j solutions constructed in Theorem 1.1. The following results hold true

1. Ψ_j is asymptotic to $\tilde{\Psi}_j$;

2. the Borel sum of $\tilde{\Psi}_j$ is the same as Ψ_j

In fact, Ψ_j is a Borel regular solution of (5).

In the following diagram we sketch the main step of the proof of Theorem 1.2: on the right hand side of the diagram, we have the formal solution $\tilde{\Psi}_j$ and its Borel transform $\tilde{\psi}_j$. Then, the theory of multi-summability allows us to draw the "square" with corners $\hat{\psi}_j$ and its Laplace transform $\hat{\Psi}_j$.



Conversely, on the left hand side, we have the unique (up to scaling) analytic function ψ_j of Theorem 1.1 which solves the integral equation (IE) obtained as the inverse Laplace transform of the (ODE).

The key step, is then comparing ψ_j and $\hat{\psi}_j$. First, using properties of the Borel transform, it is easy to show that $\tilde{\psi}_j$ is a formal solution of the integral equation (IE), hence its sum $\hat{\psi}_j$ must be a solution too. In addition, from the explicit formula for $\tilde{\Psi}_j$, we can show that $\hat{\psi}_j$ grows as $\zeta^{-\tau_j+1}$ near $\zeta = \alpha_j$ therefore, by our uniqueness result 1.1, we conclude that ψ_j and $\hat{\psi}_j$ must coincide up to scaling by a constant.

As a consequence, by properties of the Laplace transform, $\Psi_j := \mathcal{L}^{\theta}_{\zeta_j}\psi_j$ is an analytic solution of the (ODE) and it is asymptotic to $\tilde{\Psi}_j$. Furthermore, Ψ_j is proportional to $\hat{\Psi}_j$.

Remark 1.1. There are many ways to built a frame of solutions for these kind of ODE: formally in the z-plane Poincaré gave an explicit solution [**?**]. Formally in the ζ-plane Écalle figured out how to find a frame of formal solutions of the Borel transformed ODE [add reference]. It can indeed been seen as a consequence of "resurgence" of the ODE. Our results show how to built a frame of solutions, analytically in the Borel plane. Furthermore, it shows we are picking the same frame: from the properties of the Laplace transform we get an analytic frame asymptotic to Poincaré frame $\tilde{\Psi}_j$. From the uniqueness result of Theorem 1.1, we get a frame of analytic solutions equivalent to Écalle's ones.

Remark 1.2. Although, in our proof we use multi-summability to ensure that the Borel sum of the formal solution $\tilde{\Psi}_j$ exists and it solves the ODE, the approach of Borel regularity is different as it is based on the analytic solution in the frequency domain (z-plane). [Is it possible to prove Borel regularity without proving Borel summability? For instance, can we argue that ψ_j being the unique solution of (IE) with a certain behaviour is enough to say that it is the sum of $\tilde{\psi}_j$? Check carefully the argument of multi-summability]

- Can we see there exists a distinguished basis in a purely analytic way? YES. [Thm 1]
- why Borel summations of $\tilde{\Psi}_1, ..., \tilde{\Psi}_d$ finds this solutions? because they are an analytic frame of Borel regular functions.
- [Thm 1]: for every α_j there exists a unique solution in the ζ-plane which blows-up in a certain way at α_j: $\phi_j(\zeta_j) = \zeta_j^{-\tau_j} + \tilde{f}_j$, $\zeta_j = \zeta - \alpha_j$. (proof based on existence theorem A.1.3)
- [Thm 2]: The Borel sum of the formal solutions $\tilde{\Psi}_j$ are the same as the Laplace transform of the solutions of Thm 1.
- Cor of Thm 2: the Laplace transform of solutions of Thm 1 are Borel regular.
- Say there's a unique solution (up to scaling) that shrinks as you go right; everything else blows up exponentially. Then this is the only solution that can be expressed as a Laplace transform. [Follows from Aaron's argument in Airy resurgence, even if Aaron works with more general ODEs]
- Draw diagram showing formal vs. holomorphic solutions in time vs. frequency domains.

$$
\begin{tikzcd}
 & & ODE \arrow[dll, dotted, mapsto] \arrow[dl, mapsto, red] \arrow[dd, "\mathcal{L}^{-1}"', "\mathcal{B}"] \arrow[dr, mapsto, "formal"] & \\
\Phi & \hat{\Phi} & & \tilde{\Phi} \arrow[dd, "\mathcal{B}"] \\
 & & IE \arrow[dll, mapsto] \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\phi \arrow[uu, "\mathcal{L}"] & \hat{\phi} \arrow[uu, "\mathcal{L}"] & & \tilde{\phi} \arrow[ll, mapsto, red, "sum"']
\end{tikzcd}
$$

where the arrow in red are a consequence of multisummability. In addition, we can distinguish on the right hand side of the diagram the formal solutions and on the left hand side the holomorphic ones. On the upper part of the diagram the functions in the z-plane while on the lower part the functions in the Borel plane ζ-plane.

- there are many ways to see this problem have a distinguished base of solutions, Poincaré see it formally in the z-plane, Ecalle figured it out how to see it formally in the Borel plane, our results shows how to see it analytically in the Borel plane.

- M.A.E.T. says you can start in formal z-plane but it is not really constructive (see Balser chap 14); going from formal ζ to analytic is constructive and it's essentially Borel-Laplace summation. Our method uses just Laplace transform.
- How do we know we are picking the same frame? from properties of Laplace transform we get solution asymptotics to Poincaré frame. From uniqueness result we get a frame equivalent to Ecalle's frame.
- multi-summability is a regularity result starting form the formal solution in the z-plane. Borel regularity is instead based on the analytic solution in the z-plane. The argument we gave about getting the same frame is what proves Borel regularity of Ψ_j.

1.3.2 Why does Borel resummation work for thimbles integrals?

- This is the setup. Let X be an algebraic variety of dimension N, $f\colon X \to \mathbb{C}$ be a holomorphic Morse function with only simple critical points and $\nu \in \Gamma(X, \Omega^N)$.
  - recall theory of homology and cohomology to define the thimbles integrals. Pham has briefly discussed it, but apparently was introduced by Malgrange, Milnor and ?.
- background: Pham prove Borel regularity for exponential integral in N-dimension and without assuming Morse critical points (but only isolated). His proof is geometric [it may be useful to add it in Appendix].
- We give an analytic proof analogous of Pham.
- I think we should separate what can we do in general for N-dimensional integral and what is special for the 1-dimensional one.
- [Thm 4] 1-dim thimbles integrals are Borel regular, namely let

$$I_\alpha(z) := \int_{\Lambda_\alpha} e^{-zf}\nu \tag{7}$$

then the following diagram is commutative

$$\begin{tikzcd} I_\alpha(z) \arrow[r, "æ^\theta"] & \tilde{I}_\alpha(z) \arrow[d, "\mathcal{B}"] \\ \hat{\imath}_\alpha(\zeta) \arrow[u, "\mathcal{L}^\theta_\alpha"] & \tilde{\imath}_\alpha(\zeta) \arrow[l, "\text{sum}"] \end{tikzcd} \tag{8}$$

  - on the one hand take asymptotic expansion of integral via saddle point approximation and then study the Borel Laplace sum of the divergent series [from left upper corners clockwise]
  - on the other hand see that the thimble integral is a generalized Laplace transform (which in certain cases can be rewritten as a usual Laplace transform).

- A priori, the Laplace transform of $\hat{\iota}_\alpha(\zeta)$ and $I_\alpha(z)$ have the same asymptotic behaviour in a given sector (indeed taking the asymptotic of $I_\alpha(z)$ we *loose* information); however Borel regularity guarantees that $I_\alpha(z) = \mathcal{L}^\theta \hat{\iota}_\alpha$ in a given sector.
  - A corollary of [Thm 4] is the fractional derivative formula.
  - Conjecturally, we expect $\hat{\varphi}_\alpha(\zeta)$ to have simple singularities.

- From the result of Pham we can deduce Stokes constants are always integers as they are intersection numbers (use same argument of Maxim). See also the argument of KS21 sec 6.2 in the framework of analytic WCFs. Prove integrality of Stokes constants using the fractional integral formula vs differential equation (see example in Appendix C)

1.3.3 Other results

As part of the treatment, we have made use of some new perspectives on the Laplace transform (see Section 2). Indeed, the Laplace transform is often used to solve ODEs on the frequency domain by relating them to ODEs on the spatial domain. We find, however, that it is much easier and more natural to relate ODEs on the frequency domain to integral equations on the spatial domain. In particular, working with integral equations in the spatial domain will be our main strategy to prove Borel regularity for ODEs (see Theorem 1.1).

Furthermore, we introduce a geometric picture where the spatial domain B is a translation surface. If $b \in B$ is non-singular, the frequency domain for $\mathcal{L}_b^\theta$ is T^*B_b. If b is a conical singularity, the frequency domain is more interesting, as we'll see in our main example. [where do we do that?] Explain why this new perspective is useful

We will illustrate our main results with detailed treatments of several examples: we will mostly focus on degree two ODEs that admit a frame of solutions expressed as thimble integrals (the Airy–Lucas functions, modified Bessel function, Airy function, the anharmonic oscillator...). Then, on the one hand we explicitly solves the integral equation associated to the ODE, building a frame of analytic solutions as Laplace transform of the solutions of the integral equation. On the other hand, our *contour argument* shows how to rewrite explicitly thimble integrals as a Laplace transform. Borel regularity will be evident from the explicit computations. Some examples have been discussed many times, using different approaches and conventions. We try to give an idea of how all these different treatments fit together. For instance, for the Airy function we will make a comparison with (Marino, Sauzin, Takei). The anharmonic oscillator was also discussed by (Bender–Wu, Schiappa). Other example haven't been discussed much, as for Airy–Lucas functions.

Recently, resurgence theory (first developed by Écalle in the '80) has attracted interests in math and physics as a powerful alternative to Borel summability. Resurgence of lienar ODEs have been studied (see Costin slides for ReNewQuantum, [**?**]). Many results are also known for non linear ODEs (see Schiappa PI, Costin PI). For algebraic thimble integrals of the type we studied in this paper, resurgence of their asymptotic expansion can be understood geometrically (see Maxim's slides ReNewQuantum, KS paper analytic WCS), however for more general exponential integrals (see examples in Maxim's talk) resurgence remains a conjecture. Despite their simplicity, our examples of linear ODEs and of 1-dimensional integrals show some features of resurgence and they are toy model to get a feeling on Écalle formalism.

1.4 Results

- what does it mean being Borel regular?
- when does it happen?
  - State new Borel regularity results
    - Linear, homogeneous ODE with regular singularity at 0 and irregular singularity at infinity [big idea in `airy-resurgence`]
      - Contextualize with previous work of Braaksma ("Multisummability and Stokes multipliers of linear meromorphic differential equations")
      - Also contextualize with Balser, Braaksma, Sibuya, and Ramis ("Multisummability of formal power series solutions of linear ordinary differential equations")
      - Also contextualize with Loday-Richaud
    - *Borel regularity* for **thimbles integrals** can be stated a the commutativity of the following diagram:

$$\begin{tikzcd} I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\nu \arrow[r, "\sim"] & \tilde{I}_\alpha(z) \arrow[d, "\mathcal{B}"] \\ \hat{\imath}_\alpha(\zeta) \arrow[u, leftarrow, "\mathcal{L}^\theta"'] \arrow[r, equal, "\text{sum}"'] & \tilde{\imath}_\alpha(\zeta) \end{tikzcd} \tag{9}$$

    - A priori, the Laplace transform of $\hat{\imath}_\alpha(\zeta)$ and $I_\alpha(z)$ have the same asymptotic behaviour in a given sector (indeed taking the asymptotic of $I_\alpha(z)$ we *loose* information); however Borel regularity guarantees that $I_\alpha(z) = \mathcal{L}^\theta \hat{\imath}_\alpha$ in a given sector.
    - fractional derivative formula
    - Conjecturally, we expect $\hat{\varphi}_\alpha(\zeta)$ to have simple singularities.
    - in the examples, $\hat{\varphi}_\alpha(\zeta)$ turn out be an hypergeometric function of type ${}_pF_{p-1}$ where p is the number of critical values.
    - We expect that hypergeometric functions play a special role in resurgence theory as they may always appear when there are only finitely many singularities.
    - maybe we can say more about algebraic hypergeomtric functions
  - Recall Watson condition (old): Let R_N be the difference between a function and the partial sum

$$\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \ldots + \frac{\varphi_{N-2}}{z^{N-1}}$$

of its asymptotic series. Watson showed a century ago that the function is Borel regular whenever there's a constant $c \in (0, \infty)$ with

$$|R_N| \leq \frac{c^{N+1} N!}{|z|^N}$$

over all orders N and all z in a wide enough wedge around infinity.

Let X be a translation surface—a Riemann surface carrying a holomorphic 1-form ν. Suppose X is of *meromorphic type*, meaning that we got it by puncturing a compact Riemann surface $\overline{X}$ at finitely many points, and ν has a pole at each puncture. A *translation coordinate* on X is a local coordinate whose derivative is ν.

Take another meromorphic-type translation surface B and a holomorphic Morse[2] map $f \colon \overline{X} \to \overline{B}$ that sends punctures to punctures [actually, don't require this; the Orr–Sommerfeld integrals, for example, don't satisfy it]. Suppose every singularity of B is a critical value of f. [**Typical usage of "Borel plane" seems ambiguous, so maybe we can use "Borel plane" for B and "Borel cover" for the Riemann surface of the Borel-transformed series. How to handle the OrrSommerfeld functions (DLMF §9.13)? We know $f = 4u^3 - 3u$ is the pullback of a translation coordinate, but we also need a puncture at $f(0)$...**] For each critical point p, let Γ_p be the ray going rightward from $f(p)$, and let ζ_p be the translation coordinate around Γ_p which vanishes at $f(p)$. These are well-defined as long as Γ_p misses the critical values of f. The preimage $f^{-1}(\Gamma_p)$ is a bunch of disjoint curves, as long as Γ_p misses the other critical values of f. The *Lefschetz thimble* Λ_p is the component of $f^{-1}(\Gamma_p)$ that goes through p, oriented so that shifting it to its left would make its projection run clockwise around Γ_p. The *thimble integral*

$$I_p = \int_{\Lambda_p} e^{-z f^* \zeta_p} \nu$$

is a holomorphic function on the right half-plane parametrized by z, and it turns out [**we hope**] to be Borel regular.

[**Talk about exponential integrals and their decomposition into thimble integrals.**]

In higher-dimensional complex manifolds, integrals over Lefschetz thimbles are still Borel regular [**"Exponential integrals, Lefschetz thimbles and linear resurgence"**][**"Exponential Integral" lectures?**]. This fact plays an important technical role in quantum mechanics, where infinite-dimensional exponential integrals are supposed to give the expectation values of observable quantities. Physicists often use Borel summation and related techniques to assign values to these integrals [**Costin & Kruskal, "On optimal truncation..."**].

Choose a path $\gamma \colon \mathbb{R} \to X$ whose projection $f \circ \gamma$ starts out going leftward out of a puncture, ends up going rightward into a puncture, and never touches a critical value of f. Choose a translation coordinate ζ on B and continue it along $f \circ \gamma$, noting that it may become multi-valued if $f \circ \gamma$ intersects itself. This data defines the *exponential integral*

$$I = \int_{\gamma} e^{-z f^* \zeta} \nu,$$

[2] This condition means that the critical points of f are isolated (the compactness of $\overline{X}$ guarantees this) and the 2-jet of f is non-zero at every critical point.

a holomorphic function on the right half-plane parametrized by z. It turns out [**we hope**] that we can get I by summing $e^{-\alpha_p z} I_p$ over various critical points—as long as none of the Γ_p run into each other. [**We get jumps at phases where the Γ_p do hit each other.**] The constants α_p are values of ζ, continued to the critical points along certain paths.

- Each resummation method for asymptotic series makes some implicit assumption that allows us to reconstruct a holomorphic function from its asymptotic behaviour.
- The resummation method works correctly for functions which satisfy that assumption.
- For the modified Bessel function $K_{1/3}$, Borel resummation works because the asymptotic series encodes a second-order differential equation.
  - Different aspects of this example appear in various places (Mariño, Kawai–Takei, Sauzin). We give a detailed, unified treatment.
- We can generalize this argument to all K_ν with $\nu \in \mathbb{Q}$.
- We can also generalize to all third-order exponential integrals.
  - Most of them are equivalent to the $K_{1/3}$ integral, but there's also an interesting degeneration.

1.5 Fractional derivative formula

- Theorem **??** says that for a certain class of exponential integrals

$$I(z) = \int_\Gamma e^{-zf}\ \nu,$$

  the inverse Laplace [**better to say Borel?**] transform is the $\frac{3}{2}$ derivative of $d\zeta/df$, where $f^* d\zeta = \nu$ [**check**].
- the asymptotic expansion of $I(z)$ is a resurgent function.
- Is it always a *simple* resurgent function?
  - **Maxim belies it is in general, and indeed in our examples we get simple resurgent functions. But how to prove it in general?**

1.6 Stokes phenomenon

- For Bessel functions, we can see explicitly how solutions jump when the Laplace transform angle crosses a critical value.
- The jump comes from the branch cut difference identity for hypergeometric functions.
- Possible interpretation of the Stokes factors as intersections numbers in Morse–Novikov theory [**ask Maxim**]

2 The Laplace and Borel transforms

2.1 The geometry of the Laplace transform

Classically, the Laplace transform turns functions on the position domain into functions on the frequency domain. In the study of Borel summation and resurgence, it's useful to see the position domain as a *translation surface* B, and the frequency domain as one of its cotangent spaces. Roughly speaking, the Laplace transform lifts holomorphic functions on B to holomorphic functions on T^*B.

2.1.1 Translation surfaces, briefly

A translation surface is a Riemann surface B carrying a holomorphic 1-form λ [**Zorich, "Flat Surfaces"?**]. A translation chart is a local coordinate ζ with $d\zeta = \lambda$. The standard metric on $\mathbb{C}$ pulls back along translation charts to a flat metric on B, with a conical singularity of angle $2\pi n$ wherever λ has a zero of order $n-1 > 0$. We'll require B to be finite-type and λ to have a pole at each puncture. This kind of translation surface has a "cylindrical end" **(figure)** at each puncture where λ has order -1, and a "$|2n|$-planar end" **(figure)** at each puncture where λ has order $n-1 < -1$ [**Gupta, "Meromorphic quadratic differentials with half-plane structures," §2.5**] **(or cite Aaron's article, which will hopefully present the same background in the translation surface context)**.

2.1.2 Direction

The translation structure gives B a notion of direction as well as distance. Away from the zeros of λ, which we'll call *branch points*, we can talk about moving upward, rightward, or at any angle, just as we would on $\mathbb{C}$. At a branch point of cone angle $2\pi n$, we can also talk about moving upward, rightward, or at any angle in $\mathbb{R}/2\pi\mathbb{Z}$, but here there are n directions that fit each description. To make this more concrete, note that around any point $b \in B$, there's a unique holomorphic function ζ_b that vanishes at b and has $d\zeta_b = \lambda$. [If we define "translation parameter" earlier, we can say:] there's a unique translation parameter ζ_b that vanishes at b. This function is a translation chart when b is an ordinary point, and an n-fold branched covering when b is a branch point of cone angle $2\pi n$. In either case, $\zeta_b \in e^{i\theta}[0, \infty)$ is a ray or a set of rays leaving b at angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Near each branch point b, let's fix a coordinate ω_b with $\zeta_b = \frac{1}{n}\omega_b^n$, where $2\pi n$ is the cone angle at b. This lets us label each direction at b with an "extended angle" in $\mathbb{R}/2\pi n\mathbb{Z}$. Of course, there are n different choices for ω_b.

2.1.3 Frequency

The translation structure also gives us an isomorphism $z\colon T^*B_b \to \mathbb{C}$ when $b \in B$ is an ordinary point, and an isomorphism $z\colon T^*B_b^{\otimes n} \to \mathbb{C}$ when b is a branch point of cone angle $2\pi n$. At an ordinary point, we can define z simply as the map

$$\begin{aligned} z\colon T^*B_b &\to \mathbb{C} \\ \lambda\big|_b &\mapsto 1. \end{aligned}$$

To get a definition that generalizes to branch points, though, it's worth taking a fancier point of view. Recall that $T^*B_b = \mathfrak{m}_b/\mathfrak{m}_b^2$, where $\mathfrak{m}_b$ is the ideal of holomorphic functions that vanish at b. Observing that $(f + \mathfrak{m}_b)^n$ lies within $f^n + \mathfrak{m}_b^{n+1}$ for any $f \in \mathfrak{m}_b$, we can identify $T^*B_b^{\otimes n}$ with $\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}$ for $n \geq 1$. When b is an ordinary point, the function ζ_b defined in Section 2.1.2 represents a nonzero element of $\mathfrak{m}_b/\mathfrak{m}_b^2$: the cotangent vector $\lambda\big|_b$. In general, ζ_b represents a nonzero element of $\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}$, where $2\pi n$ is the cone angle at b. We define z as the isomorphism

$$\begin{aligned} z\colon \mathfrak{m}_b^n/\mathfrak{m}_b^{n+1} &\to \mathbb{C} \\ \zeta_b + \mathfrak{m}^{n+1} &\mapsto 1. \end{aligned}$$

When b is a branch point, the coordinate ω_b we chose in Section 2.1.2 gives us an isomorphism

$$\begin{aligned} w_b\colon T^*B_b &\to \mathbb{C} \\ \omega_b + \mathfrak{m}^2 &\mapsto 1 \end{aligned}$$

that makes the diagram

$$\begin{tikzcd} T^*B_b^{\otimes n} \arrow[r, "z"] & \mathbb{C} \\ T^*B_b \arrow[u, "\blacksquare^n"] \arrow[r, "w"'] & \mathbb{C} \arrow[u, "\blacksquare^n"'] \end{tikzcd}$$

commute.

2.1.4 Boundary

Discuss the visual boundary, citing Lemma 3.1 of Dankwart's thesis *On the large-scale geometry of flat surfaces* for the description of geodesics.

2.1.5 The Laplace transform over an ordinary point

Pick a local holomorphic function ζ on B with $d\zeta = \lambda$, and an extended angle $\theta \in \mathbb{R}$. [If we define "translation parameter" earlier, we can say:] Pick a translation parameter ζ. The *Laplace transform* $\mathcal{L}_\zeta^\theta$ turns a local holomorphic function f on B into a local holomorphic function on T^*B. When $b \in B$ is an ordinary point, $\mathcal{L}_\zeta^\theta f$ is defined on T^*B_b by the formula

$$\mathcal{L}_\zeta^\theta f\big|_b = \int_{\Gamma_b^\theta} e^{-z\zeta} f \, d\zeta, \tag{10}$$

where z is the frequency function defined in Section **??** and Γ_b^θ is the ray that leaves b at angle θ.

To make sense of this formula, we ask for the following conditions.

- The base point b is in the domain of ζ. Once we have this, we can continue ζ along the whole ray Γ_b^θ.

- The ray Γ_b^θ avoids the branch points after leaving b.
- The integral converges. We guarantee this by asking for a pair of simpler conditions.
  - With respect to the flat metric, f grows subexponentially along Γ_b^θ **[define]**, and is locally integrable throughout.
  - The value of z is in the half-plane $H_{-\theta}$ centered around the ray $e^{-i\theta}[0, \infty)$.

2.1.6 The Laplace transform over a branch point

When b is a branch point, we can still use formula 10 to define $\mathcal{L}_\zeta^\theta f$ on T^*B_b, as long as we take care of a few subtleties. Thanks to the labeling choices we made at the end of Section 2.1.2, the extended angle $\theta \in \mathbb{R}$ still picks out a ray Γ_b^θ. The function z is defined on $T^*B_b^{\otimes n}$, where $2\pi n$ is cone angle at b, so we pull it back to T^*B_b along the nth-power map. This amounts to substituting w_b^n for z in formula 10. The half-plane $z \in H_{-\theta}$ in $T^*B_b^{\otimes n}$ pulls back to n sectors of angle π/n in T^*B_b. We only define $\mathcal{L}_\zeta^\theta f$ on one of them: the one centered around the ray $w_b \in e^{-i\theta/n}[0, \infty)$.

2.2 The Laplace transform: analytic version

2.2.1 Regularity and decay properties

where $\Gamma_{\zeta,\alpha}$ is the rightward ray starting at $\zeta = \alpha$ (compare `draft2`). We'll use the shorthand $\mathcal{L}_{\zeta,\alpha} f := \mathcal{L}_\zeta f\big|_{\zeta=\alpha}$ throughout this document [but maybe we should get rid of it].

Let's say a function f is in $O_{\zeta,\alpha\leftarrow}(g)$ or $O_{\zeta,\alpha\rightarrow}(g)$, respectively, if $|f| \lesssim g$ on some neighborhood of the starting point or infinite end of $\Gamma_{\zeta,\alpha}$. A function is *subexponential* along $\Gamma_{\zeta,\alpha}$ if it's in $O_{\zeta,\alpha\rightarrow}(e^{c\zeta})$ for all $c > 0$. Let $\mathcal{E}_{\zeta,\alpha}$ be the space of functions which are subexponential on $\Gamma_{\zeta,\alpha}$, integrable at the starting point, and locally integrable throughout. If f is in $\mathcal{E}_{\zeta,\alpha}$, then $\mathcal{L}_{\zeta,\alpha} f$ is well-defined and holomorphic for $\mathrm{Re}(z) > 0$ on the part of $T^*\mathbb{C}$ that lies over $\Gamma_{\zeta,\alpha}$ [**?**, §5.6].

The asymptotics of f at the starting point of $\Gamma_{\zeta,\alpha}$ control the asymptotics of $\mathcal{L}_{\zeta,\alpha} f$ at the infinite end of $\Gamma_{z,0}$. Once we see how this works for $\alpha = 0$, Section 2.2.3 will do the rest. Let $F = \mathcal{L}_{\zeta,0} f$. Equation 1.8 of **??** shows[3] that

$$f \in O_{\zeta,0\leftarrow}(1) \quad \Longrightarrow \quad F \in O_{z,0\rightarrow}\left(\frac{1}{z}\right).$$

More generally, for $\tau > -1$ [prove or cite],

$$f \in O_{\zeta,0\leftarrow}(\zeta^\tau) \quad \Longrightarrow \quad F \in O_{z,0\rightarrow}\left(\frac{1}{z^{1+\tau}}\right).$$

Exact power law asymptotics relate similarly [prove or cite]:

$$f \sim \zeta^\tau \text{ at the start of } \Gamma_{\zeta,0} \quad \Longrightarrow \quad F \sim \frac{\Gamma(1+\tau)}{z^{1+\tau}} \text{ at the end of } \Gamma_{z,0}.$$

[The big-O asymptotics dictionary is interesting, but we might not need it. Consider dropping.]

[3] The argument cited still works in our generality. For holomorphic f, one can also use Equation 1.5 of *Borel-Laplace Transform and Asymptotic Theory* (Sternin & Shatalov).

2.2.2 Action on integral operators

When $\varphi \in \mathcal{E}_\zeta$, you can use differentiation under the integral to show that [**?**, Theorem 1.34]

$$\mathcal{L}_\zeta(\zeta^n \varphi) = \left(-\tfrac{\partial}{\partial z}\right)^n \mathcal{L}_\zeta \varphi. \tag{11}$$

for all integers $n \geq 0$. You can also use a 2d integration argument, akin to the one in [**?**, Theorem 2.39], to show that $\partial_{\zeta,a}^{-\lambda} \varphi \in \mathcal{E}_\zeta$ and

$$\mathcal{L}_{\zeta,a}\, \partial_{\zeta,a}^{-\lambda} \varphi = z^{-\lambda} \mathcal{L}_{\zeta,a} \varphi$$

for all $\lambda \in (0, \infty)$.

2.2.3 Change of translation chart

Define a new coordinate ζ_α on $\mathbb{C}$ so that $\zeta = \alpha + \zeta_\alpha$. From the calculation

$$\begin{aligned}
\mathcal{L}_{\zeta,a}\varphi &= \int_{\Gamma_{\zeta,\alpha}} e^{-z\zeta}\, \varphi\, d\zeta \\
&= \int_{\Gamma_{\zeta_\alpha,0}} e^{-z(\alpha+\zeta_\alpha)}\, \varphi\, d\zeta_\alpha \\
&= e^{-\alpha z} \int_{\Gamma_{\zeta_\alpha,0}} e^{-z\zeta_\alpha}\, \varphi\, d\zeta_\alpha \\
&= e^{-\alpha z} \mathcal{L}_{\zeta_\alpha,0}\varphi,
\end{aligned}$$

we learn that

$$\mathcal{L}_{\zeta_\alpha,0}\varphi = e^{\alpha z} \mathcal{L}_{\zeta,\alpha}\varphi.$$

2.2.4 Rescaling of translation structure

Let's rescale the translation structure of $\mathbb{C}$, expanding displacements by a factor of $\mu \in (0, \infty)$. The coordinate $\xi = \mu\zeta$ is a chart for the new translation structure. The corresponding frequency coordinate $x \colon T^*\mathbb{C} \to \mathbb{C}$ is given by $d\xi \mapsto 1$, so $x = \mu^{-1} z$. From the calculation

$$\begin{aligned}
\mathcal{L}_{\xi,0}\varphi &= \int_{\Gamma_{\xi,0}} e^{-x\xi}\, \varphi\, d\xi \\
&= \int_{\Gamma_{\zeta,0}} e^{-z\zeta}\, \varphi\, \mu\, d\zeta \\
&= \mu\, \mathcal{L}_{\zeta,0}\varphi
\end{aligned}$$

we learn that

$$\mathcal{L}_{\xi,0}\varphi = \mu\, \mathcal{L}_{\zeta,0}\varphi.$$

Note that $\mathcal{L}_{\xi,0}$ is defined in the new translation structure on $\mathbb{C}$, while $\mathcal{L}_{\zeta,0}$ is defined in the old translation structure. We can still compare them, because they both turn complex-valued functions on $\mathbb{C}$ into holomorphic functions on $T^*\mathbb{C}$.

2.3 The Laplace transform

- Action on differential equations.
  - Can we find a way to prove this when the differential operator spits out a function that's not integrable around zero?
- Global picture?

2.4 The Borel transform

- Action on differential equations.
  - No inhomogeneous terms! How is this consistent with the Laplace transform's action? Is there always an inhomogeneous solution with subexponential asymptotics?

The Borel transform is a linear map from formal power series in the *frequency variable* z to formal power series in the *position variable* ζ, defined by

$$\begin{aligned} \mathcal{B}\colon \mathbb{C}[\![z^{-1}]\!] &\to \mathbb{C}\delta \oplus \mathbb{C}[\![\zeta]\!] \\ 1 &\mapsto \delta \\ z^{-n-1} &\mapsto \frac{\zeta^n}{n!} \qquad \text{when } n \geq 0 \end{aligned}$$

In general, by linearity,

$$\mathcal{B}\big[a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4} + \dots\big] = a_0\delta + a_1 + a_2\zeta + a_3\frac{\zeta^2}{2!} + a_4\frac{\zeta^3}{3!} + \dots.$$

The motivation for adding the formal "convolution unit" δ to the codomain is discussed [in the convolution product section].

Notice that we are at the level of formal series, so there are no convergence assumptions. However, there is a special class of formal series that behaves well under Borel transform, meaning that their Borel transform gives a germ of holomorphic fucntions at the origin in $\mathbb{C}_\zeta$. These formal series are the so called Gevrey-1 series:

$$\tilde{\Phi}(z) = \sum_{n\geq 0} a_n z^{-n} \in \mathbb{C}[\![z^{-1}]\!]_1$$

is Gevrey-1 if there exists $A > 0$ such that $|a_n| \leq A^n n!$ for all $n \geq 0$.[4]

Notation. We want to distinghuish between formal series and holomorphic functions, as well as between the Borel plane (time domain) and the z-plane (spatial domain). Therefore we adopt the following notation:

[4] In asymptotic analysis Gevrey-k series $\sum_{n\geq 0} a_n z^{-n}$ have coefficients that grow as $(n!)^k$, i.e. there exists $A > 0$ such that $|a_n| \leq A^n (n!)^k$ for every $n \geq 0$. The Borel transform can be generalized in order to obatin germs of holomorphic function for higher Gevrey series.

- $\Phi(z)$: upper-case letters are holomorphic functions in the z-plane;
- $\tilde{\Phi}(z)$: *tilde* stands for formal series, so an upper-case letter with *tilde* is a formal series in the z-plane;
- $\phi(\zeta)$: lower-case letter are holomorphic functions in the Borel plane. We follow the convention for Laplace transform of holomorphic functions, namely $\mathcal{L}(\phi(\zeta)((z) = \Phi(z)$.
- $\tilde{\phi}(\zeta)$: lower-case letter with *tilde* are formal series in the Borel plane. As we will see, the Borel transform of $\tilde{\Phi}(z)$ is $\mathcal{B}(\tilde{\Phi})(\zeta) =: \tilde{\phi}(\zeta)$;
- $\hat{\phi}(\zeta)$: lower-case letters with *hat* are the sum of the formal series $\tilde{\phi}(\zeta)$, when it exists.

Lemma 2.1. Let $\tilde{\Phi}(z) \in \mathbb{C}[\![z^{-1}]\!]$. Then, $\tilde{\Phi} \in \mathbb{C}[\![z^{-1}]\!]_1$ is Gevrey-1 if and only if $\mathcal{B}(\tilde{\Phi}) \in \mathbb{C}\{\zeta\}$ is a germ of holomorphic functions.

In particular, we deduce from the lemma that the Borel transform is an isomorphism between Gevrey-1 series and $\mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$.

The inverse of $\mathcal{B}$ is what we call a **formal Laplace transform**.

[add definition of convolution product and $\mathcal{B}$ being an algebra isomorphisms]

2.4.1 Generalization of the Borel transform

The definition of $\mathcal{B}$ can be easily extended to fractional power of z, replacing the factorial with the Gamma function:

$$\mathcal{B}(z^{-\alpha}) := \frac{\zeta^{\alpha-1}}{\Gamma(\alpha)} \qquad \text{if } \alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$$

and then extending by linearity.

If $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ is not an integer, then $\mathcal{B}$ is an isomorphisms from Gevrey-1 formal power series $\mathbb{C}[\![z^{-\alpha}]\!]_1$ to the germs of holomorphic functions $\mathbb{C}\{\zeta^{\alpha-1}\}$, with inverse given by the formal Laplace transform. Indeed, let $\tilde{\Phi}(z) = \sum_{n\geq 0} a_n z^{-n} \in \mathbb{C}[\![z^{-\alpha}]\!]_1$ with $|a_n| \leq A^n n!$ and $A > 0$ then

$$\mathcal{B}(z^{-\alpha}\tilde{\Phi})(\zeta) = \frac{\zeta^{\alpha-1}}{\Gamma(\alpha)} * \left(a_0\delta + \sum_{n\geq 0} a_{n+1}\frac{\zeta^n}{n!} \right) = a_0\frac{\zeta^{\alpha-1}}{\Gamma(\alpha)} + \sum_{n\geq 0} a_{n+1}\frac{\zeta^{n+\alpha}}{\Gamma(1+n+\alpha)}$$

and

$$\left| \frac{a_{n+1}}{\Gamma(1+n+\alpha)} \right| \leq A^{n+1}\frac{\Gamma(n+2)}{|\Gamma(n+1+\alpha)|}$$

Under which conditions it doesn't grow faster than factorial?

2.4.2 Properties of the Borel transform

We now recall some properties of the Borel transform: let $\tilde{\Phi}(z) = \sum_{n\geq 0} a_n z^{-n}$ and denote by $\tilde{\phi}(\zeta) := \mathcal{B}(\tilde{\Phi}(z))$

[check notation]

(i) *[fractional derivative]* if $\alpha \in \mathbb{R}_{\geq 0}$, $\mathcal{B}\left(z^\alpha \tilde{\Phi}(z)\right) = \partial_\zeta^\alpha \tilde{\phi}(\zeta)$

(ii) *[fractional integral]* if $\alpha \in \mathbb{R}_{<0} \setminus \mathbb{Z}_{\geq 0}$, $\mathcal{B}\left(z^\alpha \tilde{\Phi}(z)\right) = \partial_\zeta^\alpha \tilde{\phi}(\zeta)$

(iii) if $n \in \mathbb{Z}$ and $z^n f(z) \in z^{-1}\mathbb{C}[\![z^{-1}]\!]$, then $\mathcal{B}(z^n \tilde{\Phi}(z)) = \partial_\zeta^n \tilde{\phi}(\zeta)$

(iv) $\mathcal{B}\left(\partial_z^n \tilde{\Phi}(z)\right) = (-\zeta)^n \tilde{\phi}(\zeta)$, for every $n \geq 0$

(v) $\mathcal{B}(\tilde{\Phi}(z - c)) = e^{-c\zeta} \tilde{\phi}(\zeta)$

(vi) if $\tilde{\Psi}(z) \in z^{-1}\mathbb{C}[\![z^{-1}]\!]$, then $\mathcal{B}(\tilde{\Phi}(z)\tilde{\Psi}(z)) = \int_0^\zeta d\zeta' \tilde{\phi}(\zeta - \zeta')\tilde{\psi}(\zeta') =: \tilde{\phi} * \tilde{\psi}$, where $\tilde{\psi}(\zeta) = \mathcal{B}(\tilde{\Psi}(z))$.

Lemma 2.2. For any non-integer $\mu \in (0, \infty)$ and any integer $k \geq 0$,

$$\partial^\mu_{\zeta \text{ from } 0}\left[\mathcal{B}\left(z^{-(k+1)}\right)(\zeta)\right] = \mathcal{B}\left(z^\mu z^{-(k+1)}\right)(\zeta).$$

Proof. We'll show that for any $\alpha \in (0,1)$ and any integer $n \geq 0$, the claim holds with $\mu = n + \alpha$. First, evaluate

$$\begin{aligned}
\partial^{\alpha-1}_{\zeta \text{ from } 0}\left[\mathcal{B}\left(z^{-(k+1)}\right)(\zeta)\right] &= \frac{1}{\Gamma(1-\alpha)} \int_0^\zeta (\zeta - \zeta')^{-\alpha} \frac{\zeta'^k}{\Gamma(k+1)}\, d\zeta' \\
&= \frac{1}{\Gamma(1-\alpha)\,\Gamma(k+1)} \int_0^1 (\zeta - \zeta t)^{-\alpha} (\zeta t)^k\, \zeta\, dt \\
&= \frac{\zeta^{k-(\alpha-1)}}{\Gamma(1-\alpha)\,\Gamma(k+1)} \int_0^1 (1-t)^{-\alpha} t^k\, dt \\
&= \frac{\zeta^{k-(\alpha-1)}}{\Gamma\big(k - (\alpha-1) + 1\big)}
\end{aligned}$$

by reducing the integral to Euler's beta function [**DLMF 5.12.1**]. This establishes that

$$\left(\tfrac{\partial}{\partial\zeta}\right)^{n+1} \partial^{\alpha-1}_{\zeta \text{ from } 0}\left[\mathcal{B}\left(z^{-(k+1)}\right)(\zeta)\right] = \frac{\zeta^{k-(n+\alpha)}}{\Gamma\big(k - (n+\alpha) + 1\big)} \tag{12}$$

for $n = -1$. If (12) holds for $n = m$, it also holds for $n = m + 1$, because

$$\begin{aligned}
\tfrac{\partial}{\partial\zeta}\left(\tfrac{\partial}{\partial\zeta}\right)^{m+1} \partial^{\alpha-1}_{\zeta \text{ from } 0}\left[\mathcal{B}\left(z^{-(k+1)}\right)(\zeta)\right] &= \tfrac{\partial}{\partial\zeta}\left(\frac{\zeta^{k-(m+\alpha)}}{\big(k-(m+\alpha)\big)\,\Gamma\big(k-(m+\alpha)\big)}\right) \\
&= \frac{\zeta^{k-(m+1+\alpha)}}{\Gamma\big(k-(m+\alpha)\big)}
\end{aligned}$$

Hence, (12) holds for all $n \geq -1$, and the desired result quickly follows. The condition $\alpha \in (0,1)$ saves us from the trouble we'd run into if $k - (m + \alpha)$ were in $\mathbb{Z}_{\leq 0}$. This is how we avoid the initial value corrections that appear in ordinary derivatives of Borel transforms. $\square$

Proof. We are going to prove properties (i)–(vi).

(i) follows from Lemma 2.2.

(ii) Notice that for $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$ the fractional integral $\partial_\zeta^\alpha \zeta^k = \zeta^{k-\alpha} \frac{k!}{\Gamma(k-\alpha+1)}$, hence

$$\mathcal{B}(z^\alpha \tilde{\Phi}(z)) = \sum_{k\geq 0} a_k \tfrac{\zeta^{k-\alpha}}{\Gamma(k-\alpha+1)} = \sum_{k\geq 0} a_k \tfrac{\zeta^{k-\alpha}}{k!} \tfrac{k!}{\Gamma(k-\alpha+1)} = \sum_{k\geq 0} a_k \tfrac{1}{k!} \partial_\zeta^\alpha \zeta^k = \partial_\zeta^\alpha \tilde{\phi}(\zeta).$$

(iii) $z^n \tilde{\Phi}(z) = \sum_{k\geq 0} a_k z^{-(k-n)-1}$ and by assumption $k - n \geq 0$, hence by defintion

$$\mathcal{B}(z^n \tilde{\Phi}(z)) = \sum_{k\geq n} a_k \tfrac{\zeta^{k-n}}{(k-n)!} = \sum_{k\geq n} a_k \tfrac{1}{k!} \tfrac{k!\zeta^{k-n}}{(k-n)!} = \sum_{k\geq n} a_k \tfrac{1}{k!} \partial_\zeta^n \zeta^k = \partial_\zeta^n \tilde{\phi}(\zeta).$$

(vi) $\partial_z^n \tilde{\Phi}(z) = \sum_{k\geq 0} a_k (-1)^n z^{-k-n-1} \frac{\Gamma(k+n+1)}{k!}$, hence

$$\mathcal{B}(z^\alpha \tilde{\Phi}(z)) = \sum_{k\geq 0} a_k (-1)^n \tfrac{\zeta^{k+n}}{\Gamma(k+n+1)} \tfrac{\Gamma(k+n+1)}{k!} = (-\zeta)^n \tilde{\phi}(\zeta).$$

(v) see Lemma 5.10 [?].

(vi) see Definition 5.12 and Lemma 5.14 [?]. □

Remark 2.3. We notice that properties (i) and (ii) are special cases of property (vi), indeed we can use the convolution product

$$\begin{aligned}
\mathcal{B}(z^\alpha \tilde{\Phi}(z)) &= \mathcal{B}(z^\alpha) * \tilde{\phi}(\zeta) \\
&= \frac{\zeta^{-\alpha-1}}{\Gamma(-\alpha)} * \tilde{\phi}(\zeta) \\
&= \int_0^\zeta \frac{(\zeta')^{-\alpha-1}}{\Gamma(-\alpha)} \sum_{k\geq 0} a_k \frac{(\zeta - \zeta')^k}{k!} d\zeta' \\
&= \sum_{k\geq 0} \frac{a_k}{k!} \frac{1}{\Gamma(-\alpha)} \int_0^1 (t\zeta)^{-\alpha-1} (\zeta - t\zeta)^k \zeta dt \\
&= \sum_{k\geq 0} \frac{a_k}{k!} \frac{1}{\Gamma(-\alpha)} \zeta^{k-\alpha} \int_0^1 t^{-\alpha-1} (1-t)^k dt \\
&= \sum_{k\geq 0} \frac{a_k}{k!} \frac{1}{\Gamma(-\alpha)} \zeta^{k-\alpha} \frac{\Gamma(k+1)\Gamma(-\alpha)}{\Gamma(k-\alpha+1)} \\
&= \sum_{k\geq 0} \frac{a_k}{k!} \partial_\zeta^\alpha \zeta^k \\
&= \partial_\zeta^\alpha \tilde{\phi}(\zeta)
\end{aligned}$$

2.5 Differential equation

Should we keep this section? If ζ is a translation chart for B, the functions ζ, z are Darboux coordinates on the part of T^*B that lies over the domain of ζ. In these coordinates, we can write down the partial differential operator

$$\mathcal{D}_\zeta = \left(\frac{\partial}{\partial z} - \zeta\right)\frac{\partial}{\partial \zeta}. \tag{13}$$

The multiplication by ζ is the only part of $\mathcal{D}_\zeta$ that depends on which ζ we choose.

Now we can pose a boundary value problem for each angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$: find a local holomorphic function F on T^*B that satisfies the equation $\mathcal{D}_\zeta F = 0$ and approaches zero when continued at fixed z along rays of angle θ. The Laplace transform $\mathcal{L}^\theta_\zeta$ turns local holomorphic functions on B into solutions of this problem. In fact, it produces all the solutions [**check argument**].[Veronica: I don't understand–why it produces all the solutions?]

2.5.1 Order shifting

[why the name order shifting?] Consider holomorphic functions on a simply connected open set that touches but doesn't contain $\zeta = 0$. A function is *regular* at $\zeta = 0$ if it extends holomorphically over that point. We'll say a function is *slight* at $\zeta = 0$ if it can be written as [rephrase in terms of $\mathcal{H}L^{\infty,\rho}$]

$$\zeta^{\alpha_1} f_1 + \ldots + \zeta^{\alpha_r} f_r + g \tag{14}$$

where $f_1, \ldots, f_r$ are regular, $\alpha_1, \ldots, \alpha_r \in \mathbb{R} \smallsetminus \mathbb{Z}$, and $g, g', g'', \ldots$ go to zero at $\zeta = 0$.

Locally integrable slight functions play a special role in Laplace transform methods for linear differential equations. This is because differential equations in the frequency domain arise most naturally from integral equations in the spatial domain, but we'd like to work with differential equations in the spatial domain too. In the space of locally integrable slight functions, differential and integral equations enjoy their simplest equivalence.

Proposition 2.4. When ψ is slight and locally integrable at $\zeta = 0$,

$$\left[\sum_{k=0}^{n} \left(\tfrac{\partial}{\partial\zeta}\right)^k \circ h_k + \sum_{k=1}^{m} \partial_{\zeta,0}^{-k} \circ h_{-k}\right]\psi = 0,$$

if and only if

$$\left[\sum_{k=1}^{n} \left(\tfrac{\partial}{\partial\zeta}\right)^{k-1} \circ h_k + \sum_{k=0}^{m} \partial_{\zeta,0}^{-k-1} \circ h_{-k}\right]\psi = 0,$$

where $h_n, \ldots, h_{-m}$ are regular functions.

To prove this result, we'll need a little background.

Each slight function has a unique *normal form*: an expression of the form (14) where $\alpha_1, \ldots, \alpha_r$ are distinct modulo $\mathbb{Z}$ and $f_1, \ldots, f_r$ are non-vanishing at $\zeta = 0$. The *order* of a slight function is the smallest power α_k in its normal form. If there are no $\zeta^\alpha f$ terms, we say the order is ∞. A slight function vanishes at $\zeta = 0$ if and only if its order is positive, and it's locally integrable at $\zeta = 0$ if and only if its order is greater than -1.

Multiplication by a regular function and differentiation with respect to ζ both preserve the space of slight functions. Integration from $\zeta = 0$ preserves the space of locally integrable slight functions. Thus, in general, integro-differential operators with regular coefficients send locally integrable slight functions to slight functions. Each basic operation has a simple effect on a slight function's order. Multiplication by $\zeta^n f$, where f is regular and non-vanishing at $\zeta = 0$, adds n to the order. Differentiation and integration add -1 and 1, respectively.

Proof of Proposition 2.4. The reverse implication holds without any special condition on ψ, because $\frac{\partial}{\partial\zeta}\,\partial_{\zeta,0}^{-1}$ acts as the identity on all differentiable functions.

To prove the forward implication, rewrite the first equation in the statement as

$$\frac{\partial}{\partial\zeta}\left[\sum_{k=1}^{n}\left(\tfrac{\partial}{\partial\zeta}\right)^{k-1}\circ h_k\right]\psi = -\left[h_0 + \sum_{k=1}^{m}\partial_{\zeta,0}^{-k}\circ h_{-k}\right]\psi. \tag{15}$$

The function

$$\phi = \left[\sum_{k=1}^{n}\left(\tfrac{\partial}{\partial\zeta}\right)^{k-1}\circ h_k\right]\psi$$

is slight. Looking at the right-hand side of equation 15, we can see that ϕ' is slight and locally integrable, so its order is greater than -1. Hence, ϕ has positive order, which means it vanishes at $\zeta = 0$.

Integrating both sides of equation 15, we get

$$\partial_{\zeta,0}^{-1}\,\frac{\partial}{\partial\zeta}\,\phi = -\left[\partial_{\zeta,0}^{-1}\circ h_0 + \sum_{k=1}^{m}\partial_{\zeta,0}^{-k-1}\circ h_{-k}\right]\psi.$$

Since $\partial_{\zeta,0}^{-1}\,\frac{\partial}{\partial\zeta}$ acts as the identity on functions that vanish at $\zeta = 0$, this simplifies to

$$\phi = -\left[\sum_{k=0}^{m}\partial_{\zeta,0}^{-k-1}\circ h_{-k}\right]\psi,$$

which rearranges to the second equation in the statement. $\square$

3 Proof of main results

3.1 Borel regularity for ODEs

3.2 Borel regularity for thimble integrals

Let X be a $N-$dim manifold, $f\colon X \to \mathbb{C}$ be a holomorphic Morse function with only simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$I(z) := \int_{\mathcal{C}} e^{-zf}\nu \tag{16}$$

where $\mathcal{C}$ is a suitable countur such that the integral is well defined. For any Morse cirtial points x_α of f, the saddle point approximation gives the following formal series

$$I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)}(2\pi)^{N/2}z^{-N/2}\sum_{n\geq 0} a_{\alpha,n}z^{-n} \qquad \text{as } z \to \infty \tag{17}$$

where $\mathcal{C}_\alpha$ is a steepest descent path through the critical point x_α. Notice that $f \circ \mathcal{C}_\alpha$ lies in the ray $\zeta_\alpha + [0, \infty)$, where $\zeta_\alpha := f(x_\alpha)$.

Theorem 3.1. Let $N = 1$. Let $\tilde{\varphi}_\alpha(z) := e^{-zf(x_\alpha)}(2\pi)^{1/2}\sum_{n\geq 0} a_{\alpha,n}z^{-n}$ and assume $f''(x_\alpha) \neq 0$ for every critical point x_α. Then:

1. The series $\tilde{\varphi}_\alpha$ is Gevrey-1.

2. The series $\hat{\varphi}_\alpha(\zeta) := \mathcal{B}(\tilde{\varphi})$ converges near $\zeta = \zeta_\alpha$.

3. If you continue the sum of $\hat{\phi}_\alpha$ along the ray going rightward from ζ_α, and take its Laplace transform along that ray, you'll recover $z^{1/2}I_\alpha$.

4. For any ζ on the ray going rightward from ζ_α, we have

$$\hat{\varphi}_\alpha(\zeta) = \partial^{3/2}_{\zeta\text{ from }\zeta_\alpha}\left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right) = \left(\frac{\partial}{\partial\zeta}\right)^2 \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{\zeta_\alpha}^{\zeta} (\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_\alpha(\zeta')} \nu\right) d\zeta', \quad (18)$$

   where $\mathcal{C}_\alpha(\zeta)$ is the part of $\mathcal{C}_\alpha$ that goes through $f^{-1}([\zeta_\alpha, \zeta])$. Notice that $\mathcal{C}_\alpha(\zeta)$ starts and ends in $f^{-1}(\zeta)$. [**Be careful about the orientation of $\mathcal{C}_\alpha$.**]

Proof. Part (1): Let's write $\approx$ when two functions are asymptotic (at all orders around the base point [**is this the right condition?**]), and $\sim$ when a function is asymptotic to a formal power series (at the truncation order of each partial sum).

Since f is Morse, we can find a holomorphic chart τ around x_α with $\frac{1}{2}\tau^2 = f - \zeta_\alpha$. Let $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$ be the parts of $\mathcal{C}_\alpha$ that go from the past to x_α and from x_α to the future, respectively. We can arrange for τ to be valued in $(-\infty, 0]$ and $[0, \infty)$ on $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$, respectively. [**We should explicitly spell out and check the conditions that make this possible. I think we're implicitly orienting $\mathcal{C}_\alpha$ so that τ in the upper half-plane.**] Since ν is holomorphic, we can express it as a Taylor series

$$\nu = \sum_{n\geq 0} b_n^\alpha \tau^n \, d\tau$$

that converges in some disk $|\tau| < \varepsilon$.

By the steepest descent method,

$$e^{+z\zeta_\alpha} I_\alpha(z) \approx \int_{\tau\in[-\varepsilon,\varepsilon]} e^{-z\tau^2/2}\nu$$

as $z \to \infty$. [**I need to learn how this works! Do we get asymptoticity at all orders? —Aaron**] Plugging in the Taylor series above, we get

$$\begin{aligned} e^{-z\zeta_\alpha} I_\alpha(z) &\approx \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n\geq 0} b_n^\alpha \tau^n \, d\tau \\ &= \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n\geq 0} b_{2n}^\alpha \tau^{2n} \, d\tau. \end{aligned}$$

By the dominated convergence theorem,[5]

$$
\begin{aligned}
e^{-z\zeta_\alpha} I_\alpha(z) &\approx \sum_{n\geq 0} b_{2n}^\alpha \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \tau^{2n}\, d\tau \\
&= \sum_{n\geq 0} (2n-1)!!\, b_{2n}^\alpha \left[\sqrt{2\pi}\, z^{-(n+1/2)} \operatorname{erf}\left(\varepsilon\sqrt{z/2}\right) - 2e^{-z\varepsilon^2/2} \sum_{\substack{k\in\mathbb{N}_+ \\ k\leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{n-k+1} \right].
\end{aligned}
$$

The annoying $e^{-z\varepsilon^2/2}$ correction terms are dwarfed by their $z^{-(n+1/2)}$ counterparts when z is large. These terms are crucial, however, for the convergence of the sum. To see why, consider their absolute sum C_{exp}. When $z \in [0, \infty)$,

$$
\begin{aligned}
C_{\text{exp}} &= 2e^{-\operatorname{Re}(z)\varepsilon^2/2} \sum_{n\geq 1} (2n-1)!! \left| b_{2n}^\alpha \sum_{\substack{k\in\mathbb{N}_+ \\ k\leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{n-k+1} \right| \\
&= 2e^{-z\varepsilon^2/2} \sum_{n\geq 1} (2n-1)!!\, |b_{2n}^\alpha| \sum_{\substack{k\in\mathbb{N}_+ \\ k\leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{n-k+1} \\
&\geq -2\varepsilon e^{-z\varepsilon^2/2} \sum_{n\geq 1} (2n-1)!!\, |b_{2n}^\alpha|\, z^n,
\end{aligned}
$$

which diverges for typical f and ν. [**Does it? Veronica points out that we expect b_{2n} to shrink at least as fast as $(n!)^{-1}$.**]

This argument suggests that no matter how tiny the correction terms get, we can't expect to swat them all aside. We can, however, set aside any finite set of them. [**Use Miller's proof of Watson's lemma in place of the following argument, which has a few soft spots. See also Loday-Richaud, §5.1.5, Theorem 5.1.3**] For each cutoff N, the tail

$$
\sum_{n\geq N} b_{2n}^\alpha \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \tau^{2n}\, d\tau
$$

[5] Notice that the sum over k is empty when $n = 0$. Following convention, we extend the double factorial to all odd integers by its recurrence relation, giving $(-1)!! = 1$.

For each cutoff N, the tail error **[check]**

$$\begin{aligned}\left|\sum_{n\geq N} b_{2n}^{\alpha} \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2}\tau^{2n}\,d\tau\right| &\leq \sum_{n\geq N} |b_{2n}^{\alpha}| \int_{-\varepsilon}^{\varepsilon} e^{-|z|\tau^2/2}\tau^{2n}\,d\tau \\ &\leq \sum_{n\geq N} |b_{2n}^{\alpha}| \int_{-\infty}^{\infty} e^{-|z|\tau^2/2}\tau^{2n}\,d\tau \\ &= \sqrt{2\pi}\sum_{n\geq N}(2n-1)!!\;|b_{2n}^{\alpha}|\,|z|^{-(n+1/2)} \\ &\lesssim \sum_{n\geq N}(2n-1)!!\,\varepsilon^{-2n}|z|^{-(n+1/2)} \\ &= \varepsilon\sum_{n\geq N}(2n-1)!!\,\left(\varepsilon^{-1}\right)^{2n+1}\left(|z|^{-1/2}\right)^{2n+1} \\ &= \varepsilon\sum_{n\geq N}(2n-1)!!\,\left(\varepsilon^{-1}|z|^{-1/2}\right)^{2n+1} \\ &= \textbf{uh-oh!}\end{aligned}$$

is in $o_{z\to\infty}(z^{-N})$ **[check]**, and the absolute sum

$$\begin{aligned}C_{\exp}^{N} &= 2e^{-\operatorname{Re}(z)\varepsilon^2/2}\sum_{n=1}^{N-1}(2n-1)!!\,\left|b_{2n}^{\alpha}\sum_{\substack{k\in\mathbb{N}_+\\k\leq n}}\frac{\varepsilon^{2k-1}}{(2k-1)!!}z^{n-k+1}\right| \\ &\leq 2e^{-\operatorname{Re}(z)\varepsilon^2/2}\sum_{n=1}^{N-1}(2n-1)!!\;|b_{2n}^{\alpha}|\sum_{\substack{k\in\mathbb{N}_+\\k\leq n}}\frac{\varepsilon^{2k-1}}{(2k-1)!!}|z|^{n-k+1} \\ &\geq -2\varepsilon e^{-z\varepsilon^2/2}\sum_{n\geq 1}(2n-1)!!\;|b_{2n}^{\alpha}|\,z^{n},\end{aligned}$$

is in $o_{z\to\infty}(z^{-m})$ for every m **[check]**.Hence,

$$e^{-z\zeta_\alpha}I_\alpha(z)\sim\sqrt{2\pi}\sum_{n\geq 0}(2n-1)!!\,b_{2n}^{\alpha}\,z^{-(n+1/2)}\operatorname{erf}\left(\varepsilon\sqrt{z/2}\right).$$

The differences $1-\operatorname{erf}\left(\varepsilon\sqrt{z/2}\right)$ shrink exponentially as z grows, allowing the simpler estimate

$$e^{-z\zeta_\alpha}I_\alpha(z)\sim\sqrt{2\pi}\sum_{n\geq 0}(2n-1)!!\,b_{2n}^{\alpha}\,z^{-(n+1/2)}.$$

Call the right-hand side $\tilde{I}_\alpha$. We now see that $a_{\alpha,n}=(2n-1)!!\,b_{2n}^{\alpha}$ in the statement of the theorem. **[Resolve discrepancy with previous calculation.]** Note that **[explain**

formally what it means to center at ζ_α]

$$\begin{aligned}\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha &= \sqrt{2\pi}\sum_{n\geq 0}\frac{2^n}{\sqrt{\pi}}\Gamma\big(n+\tfrac{1}{2}\big)\,b^\alpha_{2n}\,\frac{(\zeta-\zeta_\alpha)^{n-1/2}}{\Gamma\big(n+\tfrac{1}{2}\big)}\\ &= \sum_{n\geq 0}2^{n+1/2}\,b^\alpha_{2n}\,(\zeta-\zeta_\alpha)^{n-1/2}.\end{aligned}$$

We know from the definition of ε that $|b^\alpha_n|\,\varepsilon^n \lesssim 1$. Recalling that $(2n-1)!! \approx (\pi n)^{-1/2}\,2^n\,n!$ as $n\to\infty$, we deduce that $|a_{\alpha,n}| \lesssim \big(\frac{2}{\varepsilon^2}\big)^n\,n!$, showing that $\tilde{\varphi}_\alpha$ is Gevrey-1.

Part (2):

$$\hat{\varphi}_\alpha(\zeta) = \mathcal{B}\left(e^{-zf(x_\alpha)}(2\pi)^{1/2}\sum_{n\geq 0}a_{\alpha,n}z^{-n}\right)(\zeta) = T_{f(x_\alpha)}(2\pi)^{1/2}\left(\delta a_0 + \sum_{n\geq 0}a_{n+1}\frac{\zeta^n}{n!}\right)$$
$$(2\pi)^{1/2}\left(\delta(f_{x_\alpha})a_0 + \sum_{n\geq 0}a_{n+1}\frac{(\zeta-f(x_\alpha))^n}{n!}\right)$$

Since $a_n \leq CA^n n!$, the series $\sum_{n\geq 0}a_{n+1}\frac{(\zeta-f(x_\alpha))^n}{n!}$ has a finite radius of convergence.

Part (3): Let's recast the integral I_α into the f plane. As ζ goes rightward from ζ_α, the start and end points of $\mathcal{C}_\alpha(\zeta)$ sweep backward along $\mathcal{C}^-_\alpha(\zeta)$ and forward along $\mathcal{C}^+_\alpha(\zeta)$, respectively. Hence, we have

$$\begin{aligned}I_\alpha(z) &= \int_{\mathcal{C}_\alpha} e^{-zf}\nu\\ &= \int_{\zeta_\alpha}^{\infty} e^{-z\zeta}\left[\frac{\nu}{df}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)}\,d\zeta.\end{aligned}$$

Noticing that the right-hand side is a Laplace transform, we learn that

$$\hat{I}_\alpha(\zeta) = \left[\frac{\nu}{df}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)}. \tag{19}$$

In Ecalle's formalism, $\frac{\nu}{df}$ and $\hat{I}_\alpha$ are a major and a minor of the singularity $\overset{\triangledown}{I}_\alpha$. [Can we say: $\frac{\nu}{df}$ is a major representing the singularity $\overset{\triangledown}{I}_\alpha$? Or something even simpler?]

We can rewrite our Taylor series for ν as

$$\begin{aligned}\nu &= \sum_{n\geq 0}b^\alpha_n[2(f-\zeta_\alpha)]^{n/2}\,\frac{df}{[2(f-\zeta_\alpha)]^{1/2}}\\ &= \sum_{n\geq 0}b^\alpha_n[2(f-\zeta_\alpha)]^{(n-1)/2}\,df,\end{aligned}$$

taking the positive branch of the square root on $\mathcal{C}^+_\alpha$ and the negative branch on $\mathcal{C}^-_\alpha$. Plugging this into our expression for $\hat{I}_\alpha$, we learn that

$$\begin{aligned}
\hat{I}_\alpha(\zeta) &= \left[\sum_{n\geq 0} b_n^\alpha [2(f-\zeta_\alpha)]^{(n-1)/2}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)} \\
&= \sum_{n\geq 0} b_n^\alpha \Big([2(\zeta-\zeta_\alpha)]^{(n-1)/2} - (-1)^{n-1}[2(\zeta-\zeta_\alpha)]^{(n-1)/2}\Big) \\
&= \sum_{n\geq 0} 2b_{2n}^\alpha [2(\zeta-\zeta_\alpha)]^{n-1/2} \\
&= \sum_{n\geq 0} 2^{n+1/2} b_{2n}^\alpha (\zeta-\zeta_\alpha)^{n-1/2} \\
&= \mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha.
\end{aligned}$$

We already knew, from the general theory of the Borel transform, that the sum of $\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha$ would be asymptotic to $\hat{I}_\alpha$. We've now shown that the sum of $\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha$ is actually equal to $\hat{I}_\alpha$.

Theorem **??** tells us that

$$\begin{aligned}
\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha &:= \mathcal{B}_{\zeta_\alpha} z^{-1/2}\tilde{\varphi}_\alpha \\
&= \partial_{\zeta \text{ from } \zeta_\alpha}^{-1/2} \mathcal{B}\tilde{\varphi}_\alpha \\
&= \partial_{\zeta \text{ from } \zeta_\alpha}^{-1/2} \hat{\varphi}_\alpha.
\end{aligned}$$

It follows, from our conclusion above, that

$$\hat{I}_\alpha(\zeta) = \partial_{\zeta \text{ from } \zeta_\alpha}^{-1/2} \hat{\varphi}_\alpha. \tag{20}$$

Taking the Laplace transform of both sides and applying **the inverse of Theorem ?? that works for shifted analytic functions**, we see that

$$\begin{aligned}
I_\alpha(z) &= \mathcal{L}_{\zeta,\zeta_\alpha}\left[\partial_{\zeta \text{ from } \zeta_\alpha}^{-1/2} \hat{\varphi}_\alpha\right] \\
&= z^{-1/2}\mathcal{L}_{\zeta,\zeta_\alpha}\hat{\varphi}_\alpha,
\end{aligned}$$

as we claimed.

Part (4): Since fractional integrals form a semigroup, equation (20) implies that

$$\partial_{\zeta \text{ from } \zeta_\alpha}^{-1} \hat{I}_\alpha(\zeta) = \partial_{\zeta \text{ from } \zeta_\alpha}^{-3/2} \hat{\varphi}_\alpha.$$

Rewriting equation (19) as

$$\hat{I}_\alpha(\zeta) = \partial_\zeta \left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right),$$

we can see that

$$\partial_{\zeta \text{ from } \zeta_\alpha}^{-1} \hat{I}_\alpha(\zeta) = \int_{\mathcal{C}_\alpha(\zeta)} \nu - \int_{\mathcal{C}_\alpha(0)} \nu.$$

The initial value term vanishes, because the path $\mathcal{C}_\alpha(0)$ is a point. Hence,

$$\int_{\mathcal{C}_\alpha(\zeta)} \nu = \partial^{-3/2}_{\zeta \text{ from } \zeta_\alpha} \hat{\varphi}_\alpha(\zeta).$$

Recalling that the Riemann-Liouville fractional derivative is a left inverse of the fractional integral, we conclude that

$$\partial^{3/2}_{\zeta \text{ from } \zeta_\alpha} \left(\int_{\mathcal{C}_p(\zeta)} \nu \right) = \hat{\varphi}_p(\zeta).$$

□

4 Examples

4.1 Airy

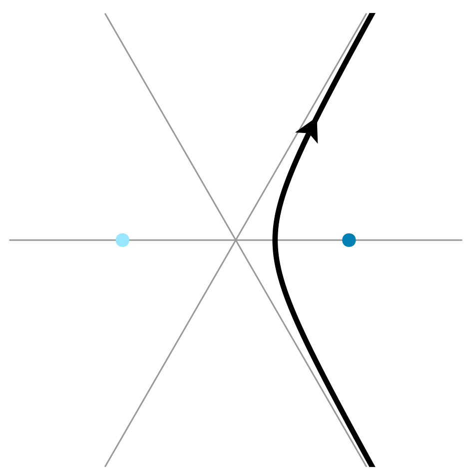
The Airy equation is

$$\left[\left(\tfrac{\partial}{\partial y} \right)^2 - y \right] \psi = 0. \tag{21}$$

One solution is given by the Airy function,

$$\mathrm{Ai}(y) = \frac{1}{2\pi i} \int_\Gamma \exp\left(\tfrac{1}{3} t^3 - yt \right) \, dt,$$

where Γ is a path that comes from ∞ at $-60°$ and goes to ∞ at $60°$.



The contour Γ in the u plane.

With the substitution $t = 2uy^{1/2}$, we can rewrite the Airy integral as

$$\mathrm{Ai}(y) = y^{1/2} \, \frac{1}{\pi i} \int_{y^{-1/2}\Gamma} \exp\left[\tfrac{2}{3} y^{3/2} \left(4u^3 - 3u \right) \right] \, du.$$

We have rescaled the contour by a factor of two, but it still approaches ∞ in the desired way. Note that $4u^3 - 3u$ is the third Chebyshev polynomial.

By considering other Chebyshev polynomials, we can situate the Airy function within the family of *Airy-Lucas functions*, so we will go straight to the general case. However, since the Airy function is a classic example in the study of Borel summation and resurgence, it may be useful to see it on its own. For this reason, in Appendix [**?**] we give a detailed treatment of the Airy function, specializing the general argument of all Airy-Lucas functions.

4.2 Airy–Lucas

The Airy-Lucas equation is

$$\left[\left(\tfrac{\partial}{\partial y}\right)^2 - (m-1)y^{-1}\tfrac{\partial}{\partial y} - y^{n-2}\right]\psi = 0 \tag{22}$$

with $n \in \{3, 4, 5, \ldots\}$ and $m \in \{1, 2, \ldots, r-1\}$. A few solutions are given by the Airy-Lucas functions [Charbonnier et al., equation 3.6]

$$\widehat{\mathrm{Ai}}^{(k)}_{n,m-1}(y) = \left\{\begin{array}{ll} 1 & j \text{ even} \\ i & j \text{ odd} \end{array}\right\} \frac{y^{m/2}}{\pi} \int_{\Lambda^{(j)}} \exp\left[\tfrac{2}{n} y^{n/2}\, T_n(u)\right] U_{m-1}(u)\, du,$$

where $\Lambda^{(k)}$ is the Lefschetz thimble through $u = \cos\left(\frac{k}{n}\pi\right)$.

4.2.1 Rewriting as a modified Bessel $\frac{m}{n}$ equation

We can distill the most interesting parts of the Airy-Lucas function by writing

$$\widehat{\mathrm{Ai}}^{(k)}_{n,m-1}(y) = \text{const.}\, y^{m/2}\, K\left(\tfrac{2}{n} y^{n/2}\right),$$

where

$$K(z) = \text{const.} \int_{z^{-1/n}\Lambda^{(k)}} \exp\left[zT_n(u)\right] U_{m-1}(u)\, du. \tag{23}$$

Saying that $\widehat{\mathrm{Ai}}^{(k)}_{n,m-1}$ satisfies the Airy-Lucas equation is equivalent to saying that K satisfies the modified Bessel $\frac{m}{n}$ equation

$$\left[z^2\left(\tfrac{\partial}{\partial z}\right)^2 + z\tfrac{\partial}{\partial z} - \left[\left(\tfrac{m}{n}\right)^2 + z^2\right]\right] K = 0. \tag{24}$$

In fact, as we'll see in Section ?, K is the modified Bessel function $K_{m/n}$.

Let's put equation (24) in the form (5):

$$\left[\left[\left(\tfrac{\partial}{\partial z}\right)^2 - 1\right] + z^{-1}\tfrac{\partial}{\partial z} - \left(\tfrac{m}{n}\right)^2 z^{-2}\right] K = 0. \tag{25}$$

4.2.2 Asymptotic analysis

From the general theory of ODE of Poincaré rank 1, we know that the space of trans-series solutions of (25) has a basis of trans-monomials

$$\{e^{-\alpha z} z^{-\tau_\alpha}\, \tilde{W}_\alpha \mid \alpha^2 - 1 = 0\}$$

where the $\tilde{W}_\alpha \in \mathbb{C}[\![z^{-1}]\!]$ are formal power series in z^{-1} and $\tau_\alpha = 1/2$. From equations 10.40.2 and 10.17.1 of [?], we learn that $K(z) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-z} z^{-1/2}\, \tilde{W}_1$, with

$$\tilde{W}_1 = 1 - \frac{(\frac{1}{2} - \frac{m}{n})_1(\frac{1}{2} + \frac{m}{n})_1}{2^1 \cdot 1!}\, z^{-1} + \frac{(\frac{1}{2} - \frac{m}{n})_2(\frac{1}{2} + \frac{m}{n})_2}{2^2 \cdot 2!}\, z^{-2} - \frac{(\frac{1}{2} + \frac{m}{n})_3(\frac{1}{2} + \frac{m}{n})_3}{2^3 \cdot 3!}\, z^{-3} + \dots \tag{26}$$

The holomorphic analysis in Section 4.2.3 will give us holomorphic solutions

$$\{e^{-\alpha z} z^{-\tau_\alpha}\, W_\alpha \mid \alpha^2 - 1 = 0\},$$

which seem analogous to the trans-monomials above. Borel summation makes the analogy precise. We will see in Section 4.2.6 that each $z^{-\tau_\alpha}\, W_\alpha$ is proportional to the Borel sum of $z^{-\tau_\alpha}\, \tilde{W}_\alpha$. This is an evidence of Theorem **??**.

4.2.3 The big idea

We're going to look for functions v_α whose Laplace transforms $\mathcal{L}_{\zeta,\alpha} v_\alpha$ satisfy equation (25). We'll succeed when $\alpha^2 - 1 = 0$, and we'll see that K is a scalar multiple of $\mathcal{L}_{\zeta,1} v_1$.

We can see from Section 2.2.2 that $\mathcal{L}_{\zeta,\alpha} v$ satisfies the differential equation (25) if and only if v satisfies the integral equation

$$\left[\left[\zeta^2 - 1\right] - \partial_{\zeta,\alpha}^{-1} \circ \zeta - \left(\tfrac{m}{n}\right)^2 \partial_{\zeta,\alpha}^{-2}\right] v = 0. \tag{27}$$

It's tempting to differentiate both sides of this equation until we get

$$\left[\left(\tfrac{\partial}{\partial \zeta}\right)^2 \circ \left[\zeta^2 - 1\right] - \tfrac{\partial}{\partial \zeta} \circ \zeta - \left(\tfrac{m}{n}\right)^2\right] v = 0, \tag{28}$$

which is easier to solve. Unfortunately, a solution of equation (28) won't satisfy equation (27) in general. However, as we learned in Section 2.5.1, a solution of equation (28) *will* satisfy equation (27) if it's slight and locally integrable at $\zeta = \alpha$.

This is great news, because equation 28 has a regular singularity at each root of $\zeta^2 - 1$, and the Frobenius method often gives a slight solution at each regular singular point. We can see the regular singularities by moving the derivatives to the right:

$$\left[(\zeta^2 - 1)\left(\tfrac{\partial}{\partial \zeta}\right)^2 + 3\zeta \tfrac{\partial}{\partial \zeta} + \left[1 - \left(\tfrac{m}{n}\right)^2\right]\right] v = 0.$$

In Sections 4.2.4–4.2.5, we'll see this approach succeed. For each root α, we'll find a solution v_α of equation (28) which is slight and locally integrable at $\zeta = \alpha$. We know the function $\mathcal{L}_{\zeta,\alpha} v_\alpha$ will satisfy equation (25), and we can even find its asymptotics from the order τ_α of v_α. We learned in Section 2.2.3 that

$$\mathcal{L}_{\zeta,\alpha} v_\alpha = e^{-\alpha z} V_\alpha$$

where $V_\alpha = \mathcal{L}_{\zeta_\alpha,0} v_\alpha$ and $\zeta = \alpha + \zeta_\alpha$. We can see from Section 2.2.1 that V_α is asymptotic to a scalar multiple of $z^{-1-\tau_\alpha}$ at $z = \infty$, so the further decomposition

$$\mathcal{L}_{\zeta,\alpha} v_\alpha = e^{-\alpha z} z^{-\tau_\alpha} W_\alpha,$$

makes W_α is asymptotic to a scalar multiple of z^{-1} at $z = \infty$.

4.2.4 Focus on $\zeta = 1$

Let's find a solution of equation 28 which is slight and locally integrable at $\zeta = 1$. Define a new coordinate ζ_1 on $\mathbb{C}$ so that $\zeta = 1 + \zeta_1$. In this coordinate, equation 28 looks like

$$\left[\zeta_1(2+\zeta_1)\left(\tfrac{\partial}{\partial \zeta_1}\right)^2 + 3(1+\zeta_1)\tfrac{\partial}{\partial \zeta_1} + \left[1 - \left(\tfrac{m}{n}\right)^2\right]\right] v = 0. \tag{29}$$

With another change of coordinate, given by $\zeta_1 = -2\xi_1$, we can rewrite equation 28 as the hypergeometric equation

$$\left[\xi_1(1-\xi_1)\left(\tfrac{\partial}{\partial \xi_1}\right)^2 + 3(\tfrac{1}{2}-\xi_1)\tfrac{\partial}{\partial \xi_1} - \left[1 - \left(\tfrac{m}{n}\right)^2\right]\right] v = 0. \tag{30}$$

Looking through the twenty-four expressions for Kummer's six solutions, we find one [**?**, formula 15.10.12] which is manifestly slight and locally integrable at $\xi_1 = 0$:

$$\begin{aligned} v_1 &= \qquad \xi_1^{-1/2}\,{}_2F_1\left(\tfrac{1}{2} - \tfrac{m}{n}, \tfrac{1}{2} + \tfrac{m}{n}; \tfrac{1}{2}; \xi_1\right) \\ &= -i\sqrt{2}\,\zeta_1^{-1/2}\,{}_2F_1\left(\tfrac{1}{2} - \tfrac{m}{n}, \tfrac{1}{2} + \tfrac{m}{n}; \tfrac{1}{2}; -\tfrac{1}{2}\zeta_1\right) \end{aligned}$$

From the argument in Section 4.2.3, we know that $\mathcal{L}_{\zeta,1} v_1$ satisfies equation (24), and can be written as $e^{-z} V_1$, where $V_1 = \mathcal{L}_{\zeta_1,0} v_1$. Since v_1 has order $-1/2$, the decomposition $V_1 = z^{-1/2} W_1$ makes W_1 asymptotic to a scalar multiple of $1 + O(z^{-1})$ at $z = \infty$.

4.2.5 Focus on $\zeta = -1$

Let's find a solution of equation (28) which is slight and locally integrable at $\zeta = -1$. In the rescaled coordinate from Section 4.2.4, this is the point $\xi_1 = 1$. Looking again through Kummer's table of solutions, we find another expression [**?**, formula 15.10.14] which is manifestly slight and locally integrable at $\xi_1 = 1$:

$$\begin{aligned} v_{-1} &= (1-\xi_1)^{-1/2}\,{}_2F_1\left(\tfrac{1}{2} - \tfrac{m}{n}, \tfrac{1}{2} + \tfrac{m}{n}; \tfrac{1}{2}; 1-\xi_1\right) \\ &= \qquad \sqrt{2}\,\zeta_{-1}^{-1/2}\,{}_2F_1\left(\tfrac{1}{2} - \tfrac{m}{n}, \tfrac{1}{2} + \tfrac{m}{n}; \tfrac{1}{2}; \tfrac{1}{2}\zeta_{-1}\right) \end{aligned}$$

where ζ_{-1} is the coordinate with $\zeta = -1+\zeta_{-1}$. From the argument in Section 4.2.3, we know that $\mathcal{L}_{\zeta,-1} v_{-1}$ satisfies equation (24), and can be written as $e^{z} V_{-1}$, where $V_{-1} = \mathcal{L}_{\zeta_{-1},0} v_{-1}$. Since v_{-1}, like our other solution, has order $-1/2$, the same decomposition $V_{-1} = z^{-1/2} W_{-1}$ makes W_{-1} asymptotic to a scalar multiple of $1 + O(z^{-1})$ at $z = \infty$.

In this example, v_1 and v_{-1} happen to be related by a symmetry: the Möbius transformation that pulls ζ back to $-\zeta$. Kummer's solutions typically come from six different hypergeometric equations, which are related by the Möbius transformations that permute their singularities. In our case, though, exchanging 1 with -1 keeps equation (28) the same.

4.2.6 Abstract argument for Borel regularity

The analysis in Sections 4.2.3–4.2.5 picks out a frame in the space of analytic solutions of (25). The frame is generated by solutions of the form $\mathcal{L}_{\zeta,1} v_1$ and $\mathcal{L}_{\zeta,-1} v_{-1}$, with $v_\alpha \in \zeta_\alpha^{-1/2} + o\big(\zeta_\alpha^{-1/2}\big)$.

We get v_α from the existence result in Section A.1.3, which gives $v_\alpha = \zeta^{\tau-1} + v'_\alpha$ with $\tau \in (0, \infty)$ and $v'_\alpha \in \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega)$ for some $\epsilon \in (0, 1]$. Looking back at the definitions of our weighted L^∞ spaces, we can see that

$$\begin{aligned} v'_\alpha \in \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega) &\Longrightarrow \zeta_\alpha^{1-\tau-\epsilon}\, v'_\alpha \text{ is bounded near } \zeta_\alpha = 0 \\ &\Longrightarrow \zeta_\alpha^{\epsilon}\, \zeta_\alpha^{1-\tau-\epsilon}\, v'_\alpha \text{ goes to 0 as } \zeta_\alpha \text{ goes to 0} \\ &\Longrightarrow \zeta_\alpha^{1-\tau}\, v'_\alpha \text{ goes to 0 as } \zeta_\alpha \text{ goes to 0} \\ &\Longrightarrow v'_\alpha / \zeta_\alpha^{\tau-1} \text{ goes to 0 as } \zeta_\alpha \text{ goes to 0} \\ &\Longleftrightarrow v'_\alpha \in o\big(\zeta_\alpha^{\tau-1}\big). \end{aligned}$$

In this case, $\tau = \frac{1}{2}$.

The Poincaré algorithm, as we saw in Section [**asymptotic analysis**], picks out a frame in the space of formal trans-series solutions of equation (25). The frame is generated by trans-monomial solutions of the form $e^{-z} z^{-1/2}\, \tilde{W}_1$ and $e^{z} z^{-1/2}\, \tilde{W}_{-1}$, with $\tilde{W}_\alpha \in \mathbb{C}[\![z^{-1}]\!]$. We'll now show that if these solutions are Borel-summable, their Borel sums generate the same frame as $\mathcal{L}_{\zeta,1} v_1$ and $\mathcal{L}_{\zeta,-1} v_{-1}$.

The Borel transform [if we define it for trans-monomials, to take care of exponential tilt] maps $e^{-\alpha z} z^{-1/2}\, \mathbb{C}[\![z^{-1}]\!]$ into $\zeta_\alpha^{-1/2}\, \mathbb{C}[\![\zeta_\alpha]\!]$, and it turns formal trans-monomial solutions of equation (25) into formal power series solutions of equation (27) [make sure we include the relevant correspondences in our Borel transform dictionary]. Summation sends convergent elements of $\zeta_\alpha^{-1/2}\, \mathbb{C}[\![\zeta_\alpha]\!]$ into $\zeta_\alpha^{-1/2} + o\big(\zeta_\alpha^{-1/2}\big)$, and it turns convergent formal solutions of a fractional integral equation into analytic solutions [prove or cite]. Thus, if $e^{-z} z^{-1/2}\, \tilde{W}_1$ and $e^{z} z^{-1/2}\, \tilde{W}_{-1}$ are 1-Gevrey, their Borel transforms sum to solutions of equation (27), which lie in $\zeta_\alpha^{-1/2} + o\big(\zeta_\alpha^{-1/2}\big)$ for $\alpha = 1$ and $\alpha = -1$, respectively.

When α is 1 or -1, equation (27) becomes the kind of singular integral equation discussed in Section A.1.3. It therefore has exactly one solution in $\zeta_\alpha^{-1/2} + o\big(\zeta_\alpha^{-1/2}\big)$, up to scaling. That means $\mathcal{B}\big[e^{-\alpha z} z^{-1/2}\, \tilde{W}_\alpha\big]$ must sum to a scalar multiple of v_α. Thus, if $e^{-\alpha z} z^{-1/2}\, \tilde{W}_\alpha$ is Borel-summable, its Borel sum is a scalar multiple of $\mathcal{L}_{\zeta,\alpha} v_\alpha$.

4.2.7 Confirmation of Borel regularity

We can confirm the conclusion of Section 4.2.6 using our explicit expressions for the formal power series $\tilde{W}_\alpha$ and the functions v_α. We found in Section [**asymptotic analysis**] that

$$\begin{aligned} \tilde{W}_1 &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{m}{n}\right)_k \left(\frac{1}{2} + \frac{m}{n}\right)_k}{k!} \left(-\frac{1}{2}\right)^k z^{-k} \\ \tilde{W}_{-1} &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{m}{n}\right)_k \left(\frac{1}{2} + \frac{m}{n}\right)_k}{k!} \left(\frac{1}{2}\right)^k z^{-k}. \end{aligned}$$

Computing

$$\begin{aligned}
\mathcal{B}\big[e^{-z} z^{-1/2}\,\tilde{W}_1\big] &= \mathcal{B}\left[\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}-\frac{m}{n}\right)_k \left(\frac{1}{2}+\frac{m}{n}\right)_k}{k!} \left(-\frac{1}{2}\right)^k z^{-k-\frac{1}{2}}\right] \\
&= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}-\frac{m}{n}\right)_k \left(\frac{1}{2}+\frac{m}{n}\right)_k}{k!} \left(-\frac{1}{2}\right)^k \frac{\zeta_1^{k-\frac{1}{2}}}{\Gamma\big(k+\frac{1}{2}\big)} \\
&= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}-\frac{m}{n}\right)_k \left(\frac{1}{2}+\frac{m}{n}\right)_k}{k!} \left(-\frac{1}{2}\right)^k \frac{\zeta_1^{k-\frac{1}{2}}}{\Gamma\big(\frac{1}{2}\big)\left(\frac{1}{2}\right)_k} \\
&= \frac{\zeta_1^{-\frac{1}{2}}}{\Gamma\big(\frac{1}{2}\big)} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}-\frac{m}{n}\right)_k \left(\frac{1}{2}+\frac{m}{n}\right)_k}{\left(\frac{1}{2}\right)_k} \left(-\frac{1}{2}\right)^k \frac{\zeta_1^k}{k!},
\end{aligned}$$

we see that $\mathcal{B}\big[e^{-z} z^{-1/2}\,\tilde{W}_1\big]$ sums to

$$\tfrac{1}{\Gamma(1/2)}\,\zeta_1^{-1/2}\,{}_2F_1\big(\tfrac{1}{2}-\tfrac{m}{n}, \tfrac{1}{2}+\tfrac{m}{n}; \tfrac{1}{2}; -\tfrac{1}{2}\zeta_1\big).$$

Looking back at Section 4.2.4, we recognize this as a scalar multiple of v_1.

Through a similar calculation, we see that $\mathcal{B}\big[e^{z} z^{-1/2}\,\tilde{W}_{-1}\big]$ sums to

$$\tfrac{1}{\Gamma(1/2)}\,\zeta_{-1}^{-1/2}\,{}_2F_1\big(\tfrac{1}{2}-\tfrac{m}{n}, \tfrac{1}{2}+\tfrac{m}{n}; \tfrac{1}{2}; \tfrac{1}{2}\zeta_{-1}\big).$$

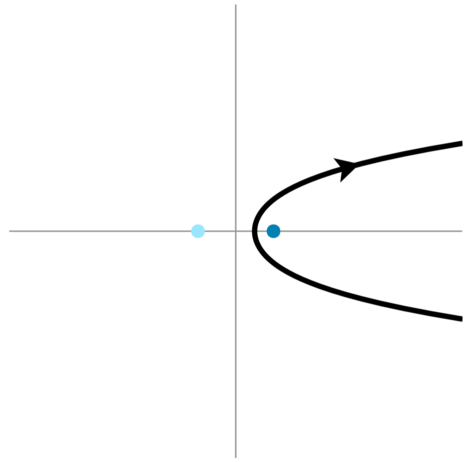
Looking back at Section 4.2.5, we recognize this as a scalar multiple of v_{-1}.

4.2.8 Contour argument for Airy-Lucas functions

We can recast integral **??** into the ζ plane by setting $-\zeta = T_n(u)$, which implies that $-d\zeta = nU_{n-1}(u)\,du$. Projecting $z^{-1/n}\Lambda^{(k)}$ to a contour γ_z in the ζ plane and choosing the branch of u that lifts γ_z back to $z^{-1/n}\Lambda^{(k)}$, we get [sign flipped] [see identity in `cyl-resurgence.tex`]

$$\begin{aligned}
K_{m/n}(z) &= \frac{n}{2\sinh\left(\frac{m}{n}\,i\pi\right)} \int_{z^{-1/n}\Lambda^{(k)}} \exp\left[zT_n(u)\right]\,U_{m-1}(u)\,du \\
&= -\frac{1}{2\sinh\left(\frac{m}{n}\,i\pi\right)} \int_{\gamma_z} e^{-z\zeta}\,\frac{U_{m-1}(u)}{U_{n-1}(u)}\,d\zeta \\
&= -\frac{1}{2\,\frac{n}{m}\sinh\left(\frac{m}{n}\,i\pi\right)} \int_{\gamma_z} e^{-z\zeta}\,F(1-\tfrac{m}{n}, 1+\tfrac{m}{n}, \tfrac{3}{2}, \tfrac{1}{2} \pm \tfrac{1}{2}\zeta)\,d\zeta.
\end{aligned}$$

where the sign must be chosen so that $\pm\zeta$ stays in the left half-plane over the whole integration path (?). For $z \in (0,\infty)$, the contour γ_z runs counterclockwise around $[1,\infty)$, as shown below, so we have to choose the negative sign above (?). Let's assume $z \in (0,\infty)$ for the rest of the section. [Our conclusions should probably hold whenever $\mathrm{Re}(z) > 0$.]



The contour γ_1 [reversed] in the ζ plane.

The integrand is non-meromorphic at $\zeta = 1$. Along the branch cut $\zeta \in [1, \infty)$, its above-minus-below difference is

$$\begin{aligned} & -(2\pi i)\tfrac{n}{2\pi m}\sin(\tfrac{m}{n}\pi)\,(\pm\tfrac{1}{2}\zeta - \tfrac{1}{2})^{-1/2}{}_2F_1(\tfrac{1}{2}-\tfrac{m}{n}, \tfrac{1}{2}+\tfrac{m}{n}, \tfrac{1}{2}, \tfrac{1}{2} \mp \tfrac{1}{2}\zeta) \\ = & -\tfrac{n}{m}\sinh(\tfrac{m}{n}\,i\pi)\,(\pm\tfrac{1}{2}\zeta - \tfrac{1}{2})^{-1/2}{}_2F_1(\tfrac{1}{2}-\tfrac{m}{n}, \tfrac{1}{2}+\tfrac{m}{n}, \tfrac{1}{2}, \tfrac{1}{2} \mp \tfrac{1}{2}\zeta), \end{aligned}$$

as given[6] by equation 15.2.3 from [?]. Hence, $K_{m/n}$ turns out to be the Laplace transform along $(1, \infty)$ of

$$\tfrac{1}{2}\,(-\tfrac{1}{2}\zeta - \tfrac{1}{2})^{-1/2}F(\tfrac{1}{2}-\tfrac{m}{n}, \tfrac{1}{2}+\tfrac{m}{n}, \tfrac{1}{2}, \tfrac{1}{2}+\tfrac{1}{2}\zeta).$$

Guessing the branch of the square root for consistency with the numerically checked result in Section B.4, we get

$$\tfrac{i}{2}\,(\tfrac{1}{2}+\tfrac{1}{2}\zeta)^{-1/2}F(\tfrac{1}{2}-\tfrac{m}{n}, \tfrac{1}{2}+\tfrac{m}{n}, \tfrac{1}{2}, \tfrac{1}{2}+\tfrac{1}{2}\zeta).$$

In other words, $K_{m/n} = \frac{i}{2}\mathcal{L}_{\zeta,-1}v_{-1}$ with

$$v_{-1} = \sqrt{2}\,\zeta_{-1}^{-1/2}F(\tfrac{1}{2}-\tfrac{m}{n}, \tfrac{1}{2}+\tfrac{m}{n}, \tfrac{1}{2}, \tfrac{1}{2}\zeta_{-1}),$$

where $\zeta = -1 + \zeta_{-1}$.

4.3 Modified Bessel

The modified Bessel equation is a generalization of equation (24) where $\frac{m}{n}$ is replaced by a complex parameter μ

$$\left[z^2\left(\tfrac{\partial}{\partial z}\right)^2 + z\tfrac{\partial}{\partial z} - \left[\mu^2 + z^2\right]\right]\varphi = 0 \tag{31}$$

and a basis of solutions is given by the modified Bessel functions (see formulas 10.27.4 and 10.32.12 from [?])

[6] Note that $\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})^{-1} = \frac{1}{2}$ and $\left[\Gamma(1-\frac{m}{n})\Gamma(1+\frac{m}{n})\right]^{-1} = \left[\Gamma(1-\frac{m}{n})\,\frac{m}{n}\Gamma(\frac{m}{n})\right]^{-1} = \frac{n}{m\pi}\sin(\frac{m}{n}\pi)$.

$$I_\mu(z) = \frac{1}{2\pi i}\int_\Omega e^{z\cosh t - \mu t}dt$$
$$K_\mu(z) = \frac{\pi}{\sin(\mu\pi)}\cdot\frac{1}{2}[I_{-\mu}(z) - I_\mu(z)]$$

where Ω is a path that comes from ∞ along $-i\pi+(0,\infty)$ and goes to ∞ along $i\pi+(0,\infty)$.

Figure 1: Add figure path Ω

On the one hand, this new condition does not really affect the argument we present in Section 4.2.6, so we briefly state the main results:

- **Asymptotic analysis**: equation (63) admits a basis of formal solutions $K_\mu \sim e^{-z}z^{-1/2}\tilde{W}_{\mu,1}$ and $I_\mu \sim e^{z}z^{-1/2}\tilde{W}_{\mu,2}$ with

$$\tilde{W}_{\mu,1} = 1 - \frac{\left(\frac{1}{2}-\mu\right)\left(\frac{1}{2}+\mu\right)}{2\cdot 1!}z^{-1} + \frac{\left(\frac{1}{2}-\mu\right)_2\left(\frac{1}{2}+\mu\right)_2}{2^2\cdot 2!}z^{-2} - \frac{\left(\frac{1}{2}-\mu\right)_3\left(\frac{1}{2}+\mu\right)_3}{2^3\cdot 3!}z^{-3} + \ldots$$
$$\tilde{W}_{\mu,2} = 1 + \frac{\left(\frac{1}{2}-\mu\right)\left(\frac{1}{2}+\mu\right)}{2\cdot 1!}z^{-1} + \frac{\left(\frac{1}{2}-\mu\right)_2\left(\frac{1}{2}+\mu\right)_2}{2^2\cdot 2!}z^{-2} + \frac{\left(\frac{1}{2}-\mu\right)_3\left(\frac{1}{2}+\mu\right)_3}{2^3\cdot 3!}z^{-3} + \ldots$$

- **Frame of analytic solutions**: there exist two functions $v_{\mu,1}, v_{\mu,-1}$ such that $\mathcal{L}_\alpha v_{\mu,\alpha}$ satisfies equation (63) and they are explicitly

$$v_{\mu,1} = -i\sqrt{2}\,\zeta_1^{-1/2}{}_2F_1\left(\tfrac{1}{2}-\mu, \tfrac{1}{2}+\mu; \tfrac{1}{2}; \tfrac{1}{2}\zeta_1\right)$$
$$v_{\mu,-1} = \sqrt{2}\,\zeta_{-1}^{-1/2}{}_2F_1\left(\tfrac{1}{2}-\mu, \tfrac{1}{2}+\mu; \tfrac{1}{2}; \tfrac{1}{2}\zeta_{-1}\right)$$

- **Borel regularity** [Check with Aaron how to phrase it]

On the other hand, the contour argument we describe in Section 4.2.8 has to be generalized, as we will discuss in the following Section 4.3.1.

4.3.1 Lifting to a countable cover

Formula (**??**) expresses the modified Bessel function $K_{m/n}$ as an exponential integral on a finite cover of $\mathbb{C}$. Lifting to a countable cover reveals this formula as a special case of a general integral formula for modified Bessel functions.

Setting $u = \cosh(t/n)$ and recalling that

$$\cosh(n\tau) := T_n(\cosh(\tau))$$
$$\sinh(m\tau) := U_{m-1}(\cosh(\tau))\sinh(\tau),$$

we can rewrite formula **??** as [switching to the conventional sign for the projection map, so $\Lambda^{(3)}$ now comes from ∞ at $-60°$ and goes to ∞ at $60°$]

$$\begin{aligned}K_{m/n}(z) &= \frac{n}{2\sinh\left(\frac{m}{n}\,i\pi\right)}\int_{z^{-1/n}\Lambda^{(k)}} \exp\left[zT_n(u)\right]\,U_{m-1}(u)\,du \\ &= \frac{n}{2\sinh\left(\frac{m}{n}\,i\pi\right)}\int_{\Omega} \exp\left[z\cosh(t)\right]\,U_{m-1}(\cosh(t/n))\,\sinh(t/n)\,d(t/n) \\ &= \frac{1}{2\sinh\left(\frac{m}{n}\,i\pi\right)}\int_{\Omega} \exp\left[z\cosh(t)\right]\,\sinh\left(\tfrac{m}{n}\,t\right)dt. \end{aligned} \tag{32}$$

For any $\mu \in \mathbb{C} \setminus \mathbb{Z}$,

$$\begin{aligned}K_\mu(z) &= \frac{\pi}{\sin(\mu\pi)}\cdot\frac{1}{2}\left[I_{-\mu}(z) - I_\mu(z)\right] \\ &= \frac{1}{2i\sin(\mu\pi)}\int_\Omega \exp\left[z\cosh(t)\right]\,\frac{1}{2}\left[e^{\mu t} - e^{-\mu t}\right]\,dt \\ &= \frac{1}{2\sinh(\mu\, i\pi)}\int_\Omega \exp\left[z\cosh(t)\right]\,\sinh(\mu t)\,dt, \end{aligned} \tag{33}$$

where Ω is a path that comes from ∞ along $-i\pi + (0,\infty)$ and goes to ∞ along $i\pi + (0,\infty)$. The integral converges when z is in the right half-plane. We get formula (32) when we choose a rational parameter $\mu = m/n$.

When μ goes to 0, formula (32) becomes

$$K_0(z) = \frac{1}{2\pi i}\int_\Omega \exp\left[z\cosh(t)\right]\,t\,dt.$$

Choosing Ω to be the unit-speed path that runs from ∞ leftward to $-i\pi$, upward to $i\pi$, and rightward back to ∞, we can rewrite this formula as

$$\begin{aligned}K_0(z) &= \frac{1}{2\pi i}\int_0^\infty \exp\left[-z\cosh(t)\right]\,2\pi i\,dt \\ &= \int_0^\infty \exp\left[-z\cosh(t)\right]\,dt \\ &= \int_1^\infty \exp\left[-z\,\tfrac{1}{2}\left(s + \tfrac{1}{s}\right)\right]\,\frac{ds}{s}, \end{aligned}$$

with $s = e^t$. This is a special case of formula 10.32.9 from [**?**].

4.4 Higher Airy

The higher Airy equation is

$$\left[\left(-\tfrac{\partial}{\partial y}\right)^{n-1} - y\right]\psi = 0 \tag{34}$$

with $n \in \{3,4,5,\ldots\}$. A few solutions are given by the hyper-Airy functions [Charbonnier et al., equation 3.6]

With

$$z = (-1)^{n-1}\tfrac{n-1}{n} y^{n/(n-1)} \qquad\qquad w = (-1)^n (n-1) y^{1/(n-1)} u,$$

we have

$$\begin{aligned}
\widetilde{\mathrm{Ai}}_n^{(k)}(y) &= \frac{\exp\left(\pi i k \frac{n-2}{n-1}\right)}{2\pi i} \int_{\Lambda^{(j)}} \exp\left[\tfrac{1}{n} w^n - yw\right] dw \\
&= \frac{\exp\left(\pi i k \frac{n-2}{n-1}\right)}{2\pi i} \int_{\Lambda^{(j)}} \exp\left[\tfrac{1}{n} w \left(w^{n-1} - ny\right)\right] dw \\
&= \frac{\exp\left(\pi i k \frac{n-2}{n-1}\right)}{2\pi i} \int_{\Lambda^{(j)}} \exp\left[\tfrac{1}{n} w \big((n-1)^{n-1} y u^{n-1} - ny\big)\right] dw \\
&= \frac{\exp\left(\pi i k \frac{n-2}{n-1}\right)}{2\pi i} \int_{\Lambda^{(j)}} \exp\left[\tfrac{1}{n} y w \big((n-1)^{n-1} u^{n-1} - n\big)\right] dw \\
&= \frac{\exp\left(\pi i k \frac{n-2}{n-1}\right)}{2\pi i} \int_{\Lambda^{(j)}} \exp\left[(-1)^n \tfrac{n-1}{n} y^{n/(n-1)} u \big((n-1)^{n-1} u^{n-1} - n\big)\right] dw \\
&= \frac{\exp\left(\pi i k \frac{n-2}{n-1}\right)}{2\pi i} \int_{\Lambda^{(j)}} \exp\left[-z \big((n-1)^{n-1} u^n - nu\big)\right] dw \\
&= (-1)^n (n-1) \frac{\exp\left(\pi i k \frac{n-2}{n-1}\right)}{2\pi i} y^{1/(n-1)} \int_{\Lambda^{(j)}} \exp\left[-z \big((n-1)^{n-1} u^n - nu\big)\right] du \\
&= (-1)^n (n-1) \frac{\exp\left(\pi i k \big(1 - \frac{1}{n-1}\big)\right)}{2\pi i} y^{1/(n-1)} \int_{\Lambda^{(j)}} \exp\left[-z \big((n-1)^{n-1} u^n - nu\big)\right] du \\
&= (-1)^{n+k} (n-1) \frac{\exp\left(-\pi i \frac{k}{n-1}\right)}{2\pi i} y^{1/(n-1)} \int_{\Lambda^{(j)}} \exp\left[-z \big((n-1)^{n-1} u^n - nu\big)\right] du
\end{aligned}$$

$$\widetilde{\mathrm{Ai}}_n^{(k)}(y) = (-1)^{n+k} (n-1) \frac{\exp\left(-\pi i \frac{k}{n-1}\right)}{2\pi i} y^{1/(n-1)} \int_{\Lambda^{(j)}} \exp\left[-z \big((n-1)^{n-1} u^n - nu\big)\right] du,$$

where $\Lambda^{(k)}$ is the Lefschetz thimble through $u = \cos\left(\frac{k}{n}\pi\right)$.

4.4.1 Rewriting as a (???) equation

We can distill the most interesting parts of the hyper-Airy function by writing

$$\widetilde{\mathrm{Ai}}_n^{(k)}(y) = \text{const.}\, y^{1/(n-1)}\, K^{(k)}\left((-1)^n\, \tfrac{n-1}{n}\, y^{n/(n-1)}\right),$$

where

$$K^{(k)}(z) = \text{const.} \int_{\text{const.}(n) z^{-1/n} \Lambda^{(k)}} \exp\left[-z \big((n-1)^{n-1} u^n - nu\big)\right] du. \tag{35}$$

Saying that $\widetilde{\mathrm{Ai}}_n^{(k)}$ satisfies the higher Airy equation is equivalent to saying that K satisfies an equation of the form

$$\left[\left[\left(-\tfrac{\partial}{\partial z}\right)^{n-1} - 1\right] - c_n^{(1)} z^{-1}\left(-\tfrac{\partial}{\partial z}\right)^{n-2} - c_n^{(2)} z^{-2}\left(-\tfrac{\partial}{\partial z}\right)^{n-3} - \ldots - c_n^{(n-1)} z^{-(n-1)}\right] K^{(k)} = 0. \tag{36}$$

The sub-leading coefficients are

$$c_n^{(1)} = \frac{n-1}{2}.$$

The later coefficients can be written as[7]

$$c_n^{(k)} = \frac{b_n^{(k)}}{n^k}\,\frac{\Gamma(n+2)}{\Gamma(n-k)}$$

in terms of the polynomials

$$\begin{array}{lllllllllll}
b_n^{(2)} & = & \frac{1}{24} \\
b_n^{(3)} & = & \frac{1}{48}n \\
b_n^{(4)} & = & \frac{73}{5760}n^2 & + & \frac{1}{1152}n & - & \frac{1}{2880} \\
b_n^{(5)} & = & \frac{11}{1280}n^3 & + & \frac{1}{768}n^2 & - & \frac{1}{1920}n \\
b_n^{(6)} & = & \frac{3625}{580608}n^4 & + & \frac{61}{41472}n^3 & - & \frac{181}{322560}n^2 & - & \frac{1}{41472}n & + & \frac{1}{181440}.
\end{array}$$

Searching for 580608 in the OEIS turns up the leading coefficients $\beta^{(k)}$ of these polynomials, which are listed as `A249276` and `A249277`. They're defined by the identity [Yang, "Approximations for Constant e and Their Applications"]

$$\frac{1}{e}\left(\frac{n}{n-1}\right)^{n-1} = 1 - \frac{1/2}{n} - \frac{\beta^{(2)}}{n^2} - \frac{\beta^{(3)}}{n^3} - \frac{\beta^{(4)}}{n^4} - \frac{\beta^{(5)}}{n^5} - \ldots,$$

which tells us that

$$\frac{1}{e}\left(\frac{n}{n-1}\right)^{n-1} = 1 - \frac{1/2}{n} - \frac{b_n^{(2)}}{n^2} - \frac{b_n^{(3)}}{n^4} - \left[\frac{b_n^{(4)}}{n^6} + o\left(\frac{1}{n^5}\right)\right] - \left[\frac{b_n^{(5)}}{n^8} + o\left(\frac{1}{n^6}\right)\right] - \ldots.$$

The last coefficient can be written as

$$c_n^{(n-1)} = \left(\frac{n-1}{n}\right)^{n-1}\left(\frac{1}{n-1}\right)^{\overline{n-1}},$$

giving

$$b_n^{(n-1)} = (n-1)^{n-1}\left(\frac{1}{n-1}\right)^{\overline{n-1}}\Big/\Gamma(n+2).$$

4.4.2 Building a frame of analytic solutions

We're going to look for functions v_α whose Laplace transforms $\mathcal{L}_{\zeta,\alpha} v_\alpha$ satisfy equation (36). We'll succeed when $\alpha^{n-1} - 1 = 0$, and we'll see that $K^{(k)}$ is a scalar multiple of $\mathcal{L}_{\zeta,\alpha_k} v_k$ with $\alpha_k = e^{2\pi i \frac{k}{n-1}}$.

[7] Many thanks to Peter Taylor for noticing this [`https://mathoverflow.net/q/422337/1096`].

We can see from Section 2.2.2 that $\mathcal{L}_{\zeta,\alpha_k} v$ satisfies the differential equation (36) if and only if v satisfies the integral equation

$$\Big[\big[\zeta^{n-1}-1\big] - c_n^{(1)}\partial_{\zeta,\alpha_k}^{-1}\circ\zeta^{n-2} - c_n^{(2)}\partial_{\zeta,\alpha_k}^{-2}\circ\zeta^{n-3} - \ldots - c_n^{(n-1)}\partial_{\zeta,\alpha_k}^{-(n-1)}\Big]\, v = 0. \tag{37}$$

Since equation (36) is written in form (5) and the coefficient $c_n^{(1)}$ is explicitly known, we deduce that equation (37) admits a slight solution at $\zeta = \alpha_k$ of order $\tau_{\alpha_k} = \frac{1}{2}$. In particular, v is a solution of (37) if and only if it solves the following differential equation

$$\Big[\big(\tfrac{\partial}{\partial\zeta}\big)^{n-1}\circ\big[\zeta^{n-1}-1\big] - c_n^{(1)}\big(\tfrac{\partial}{\partial\zeta}\big)^{n-2}\circ\zeta^{n-2} - c_n^{(2)}\big(\tfrac{\partial}{\partial\zeta}\big)^{n-3}\circ\zeta^{n-3} - \ldots - c_n^{(n-1)}\Big]\, v = 0. \tag{38}$$

Using the Liebniz rule for the differential, we get

$$\begin{aligned}
\big(\tfrac{\partial}{\partial\zeta}\big)^{n-1}\circ\big[\zeta^{n-1}-1\big]v &= \sum_{j=0}^{n-1}\binom{n-1}{j} v^{(n-j-1)}\big(\tfrac{\partial}{\partial\zeta}\big)^{j}\big[\zeta^{n-1}-1\big]\\
&= (\zeta^{n-1}-1)\partial_\zeta^{n-1}v + (n-1)^2\zeta^{n-2}\partial_\zeta^{n-2}v +\\
&\qquad \sum_{j=2}^{n-1}\binom{n-2}{j} v^{(n-j-1)}\big(\tfrac{\partial}{\partial\zeta}\big)^{j}\big[\zeta^{n-1}-1\big] + (n-1)!v\\
\big(\tfrac{\partial}{\partial\zeta}\big)^{n-2}\circ\zeta^{n-2}v &= \zeta^{n-2}\partial_\zeta^{n-2}v + \sum_{j=1}^{n-3}\binom{n-2}{j} v^{(n-j-2)}\big(\tfrac{\partial}{\partial\zeta}\big)^{j}\big[\zeta^{n-1}-1\big] + (n-2)!v\\
\big(\tfrac{\partial}{\partial\zeta}\big)^{n-3}\circ\zeta^{n-3}v &= \sum_{j=0}^{n-4}\binom{n-3}{j} v^{(n-j-3)}\big(\tfrac{\partial}{\partial\zeta}\big)^{j}\big[\zeta^{n-1}-1\big] + (n-3)!v\\
\tfrac{\partial}{\partial\zeta}\circ\zeta v &= \zeta\partial_\zeta v + v
\end{aligned}$$

$$\Big[(\zeta^{n-1}-1)\partial_\zeta^{n-1} + \big[(n-1)^2 - c_n^{(1)}\big]\zeta^{n-2}\partial_\zeta^{n-2} + \ldots + \big[(n-1)! - c_n^{(1)}(n-2)! - \ldots - c_n^{(n-1)}\big]\Big]\, v = 0. \tag{39}$$

Let's find a solution of equation 39 which is slight and locally integrable at $\zeta = 1$. Define a new coordinate ζ_1 on $\mathbb{C}$ so that $\zeta = 1 + \zeta_1$. In this coordinate, equation 39 looks like

$$\begin{aligned}
\Big[\zeta_1(n-1+\ldots+(n-1)\zeta_1^{n-3}+\zeta_1^{n-2})\big(\tfrac{\partial}{\partial\zeta_1}\big)^{n-1} + \big[(n-1)^2 - c_n^{(1)}\big](1+\zeta_1)^{n-2}\tfrac{\partial}{\partial\zeta_1}^{n-2} + \ldots\\
+\big[(n-1)! - c_n^{(1)}(n-2)! - \ldots - c_n^{(n-1)}\big]\Big]\, v = 0.
\end{aligned} \tag{40}$$

4.4.3 Higher Airy of degree 3

Setting $n = 4$, equation 36 turns into

$$\left[\frac{\partial^3}{\partial z^3} + 1 + \frac{3}{2z}\frac{\partial^2}{\partial z^2} - \frac{5}{16}\frac{1}{z^2}\frac{\partial}{\partial z} + \frac{5}{32}\frac{1}{z^3}\right]\varphi = 0 \tag{41}$$

and going to the spatial domain, $\mathcal{L}_{\zeta,\alpha_k}$ satisfies equation (41) if and only if v satisfies

$$\left[(\zeta^3-1)\partial_\zeta^3 - \frac{15}{2}\zeta^2\partial_\zeta^2 - \frac{187}{16}\zeta\partial_\zeta + \frac{81}{32}\right]\varphi = 0 \tag{42}$$

Let's find a solution of equation (42) which is slight and locally integrable at $\zeta = 1$. Define a new coordinate ζ_1 on $\mathbb{C}$ so that $\zeta = 1+\zeta_1$. In this coordinate, equation (42) looks like

$$\left[\zeta_1(3+3\zeta_1+\zeta_1^2)\big(\tfrac{\partial}{\partial\zeta_1}\big)^3 - \frac{15}{2}(1+2\zeta_1+\zeta_1^2)\big(\tfrac{\partial}{\partial\zeta_1}\big)^2 - \frac{187}{16}(1+\zeta_1)\tfrac{\partial}{\partial\zeta_1} + \frac{81}{32}\right] v = 0. \tag{43}$$

[Find solutions using hypergeometric functions]

4.4.4 Contour argument for higher Airy of degree 3

We can recast integral (35) into the ζ-plane by setting $\zeta = 27u^4 - 4u$, which implies that $d\zeta = 4(27u^3-1)du$. Projecting $z^{-1/4}\Lambda^{(k)}$ to a contour γ_z in the ζ plane and choosing a branch of that lifts γ_z back to $z^{-1/4}\Lambda^{(k)}$, we get

$$\begin{aligned}
K^{(k)}(z) &= const. \int_{const. z^{-1/4}\Lambda^{(k)}} \exp\big[-z\big(27u^4-4u\big)\big]du \\
&= const. \int_{\gamma_z} e^{-z\zeta}\frac{d\zeta}{4(27u^3-1)} \\
&= const. \frac{1}{4}\int_{\gamma_z} e^{-z\zeta}\frac{1}{3}\left[\frac{1}{3u-1} + \frac{1}{3e^{\frac{2\pi i}{3}}u-1} + \frac{1}{3e^{\frac{4\pi i}{3}}u-1}\right] d\zeta \\
&= -const. \frac{1}{4}\int_{\gamma_z} e^{-z\zeta}{}_3F_2\left(\frac{1}{3},\frac{2}{3},1;\frac{1}{3},\frac{2}{3};27u^3\right)d\zeta
\end{aligned}$$

$$\frac{1}{4}\frac{1}{27u^3-1} = \frac{1}{4}\left(\frac{A}{3u-1} + \frac{B}{3u-\omega} + \frac{C}{3u-\omega^2}\right)$$

4.5 Generalized Airy

In [?] [?] the authors introduce generalized Airy functions $A_k(z), B_0(z), B_k(z)$, $k = 1,2,3$ as approximate solutions of the Orr–Sommerfield fluid equation. They are defined as countour integral ([?] §9.13(ii))

$$\begin{aligned}
\mathrm{A}_k(z,p) &= \frac{1}{2\pi i}\int_{\Gamma_k} e^{zt-\frac{t^3}{3}}\frac{dt}{t^p} \qquad k=1,2,3\ p\in\mathbb{C} \\
\mathrm{B}_0(z,p) &= \frac{1}{2\pi i}\int_{\Gamma_0} e^{zt-\frac{t^3}{3}}\frac{dt}{t^p} \qquad p\in\mathbb{Z} \\
\mathrm{B}_k(z,p) &= \int_{\gamma_k} e^{zt-\frac{t^3}{3}}\frac{dt}{t^p} \qquad k=1,2,3\ p\in\mathbb{Z}
\end{aligned}$$

where the countours $\Gamma_k, \Gamma_0, \gamma_k$ are represented in Figure ??

Each generalize Airy functions is a solution of

$$\left[\partial_z^3 - z\partial_z + (p-1)\right] f(z,p) = 0 \tag{44}$$

and if $p = 0$ they reduced to the classical Airy functions: $\mathrm{Ai}(z) = \mathrm{A}_1(z,0)$ and $\mathrm{Bi}(z) = \mathrm{B}_1(z,0)$.

We define the following exponential integral: let $p \geq 0$, $f(t) = 4t^3 - 3t$ and $\nu_p = \frac{dt}{t^p}$. The critical points of f are $\pm\frac{1}{2}$ (as for the classical Airy case), however the volume form ν_p is meromorphic

$$I_\alpha(z,p) := \frac{1}{2\pi i}\int_{\mathcal{C}_\alpha} e^{-z(4t^3-3t)}\frac{dt}{t^p} \tag{45}$$

where $\mathcal{C}_\alpha$ is the path through the point α, starting and ending at infinity (as in Figure **??**). In particular, $I_+(z,p) = (12z)^{\frac{p-1}{3}}\mathrm{A}_1((\frac{3}{2}z)^{2/3}, p)$:

$$\begin{aligned}
I_+(z,p) &= \frac{1}{2\pi i}\int_{\mathcal{C}_+} e^{-z(4t^3-3t)}\frac{dt}{t^p} \\
&= \frac{1}{2\pi i}(12z)^{(p-1)/3}\int_{z^{-1/3}\mathcal{C}_+} e^{-z(z^{-1}\frac{s^3}{3}-3(12z)^{-1/3}s)}\frac{ds}{s^p} \qquad\qquad t = (12z)^{-\frac{1}{3}}s \\
&= \frac{1}{2\pi i}(12z)^{(p-1)/3}\int_{z^{-1/3}\mathcal{C}_+} e^{-\left(\frac{s^3}{3}-(\frac{3}{2}z)^{2/3}s\right)}\frac{ds}{s^p} \\
&= (12z)^{(p-1)/3}\mathrm{A}_1((\tfrac{3}{2}z)^{2/3}, p)
\end{aligned}$$

It follows that $I_+(z)$ is a solution of

$$\left[\partial_z^3 - \partial_z + \frac{2-p}{z}\partial_z^2 + \frac{p-1}{z} + \frac{-1-3p+3p^2}{9}\frac{\partial_z}{z^2} + \frac{3+p-3p^2-p^3}{27z^3}\right] I_+(z,p) = 0 \tag{46}$$

From the general theory of linear ODE, the formal integral solution of (46) is a linear combinations of three generators

$$\tilde{I}_+(z,p) = U_1 z^{p-1}\tilde{W}_1(z) + U_2 e^{-z}z^{-1/2}\tilde{W}_2(z) + U_3 e^{z}z^{-1/2}\tilde{W}_3(z) \tag{47}$$

where $\tilde{W}_1, \tilde{W}_2$ and $\tilde{W}_3$ are formal power seirs $\tilde{W}_{\mathbf{k}}(z) = \sum_{j\geq 0} a_{\mathbf{k},j}z^{-j}$, which are the unique (we fix $a_{k,0} = 1$ for $k = 1,2,3$) solution of

$$\left[\partial_z^3 - \partial_z - \frac{1-2p}{z}\partial_z^2 + \frac{17-30p+12p^2}{9z^2}\partial_z + \frac{8}{27}\frac{p^3-6p^2+11p-6}{z^3}\right]\tilde{W}_1(z) \quad = \quad 0 \tag{48}$$

$$\begin{aligned}
&\left[\partial_z^3 - 3\partial_z^2 + 2\partial_z + \frac{1-2p}{2z}\partial_z^2 - \frac{1-2p}{z}\partial_z + \frac{5+24p+12p^2}{36z^2}\partial_z + \right. \\
&\qquad\qquad \left. - \frac{5+24p+12p^2}{36z^2} - \frac{1}{z^3}\left(\frac{5}{24} + \frac{59}{108}p + \frac{5}{18}p^2 + \frac{1}{27}p^3\right)\right]\tilde{W}_2(z) = 0
\end{aligned} \tag{49}$$

$$\Big[\partial_z^3 + 3\partial_z^2 + 2\partial_z + \frac{1-2p}{2z}\partial_z^2 + \frac{1-2p}{z}\partial_z + \frac{5+24p+12p^2}{36z^2}\partial_z + \\ + \frac{5+24p+12p^2}{36z^2} - \frac{1}{z^3}\left(\frac{5}{24} + \frac{59}{108}p + \frac{5}{18}p^2 + \frac{1}{27}p^3\right)\Big]\tilde{W}_3(z) = 0 \quad (50)$$

Let us first study equation (48): its Borel transform is

$$-\zeta^3\tilde{w}_1 + \zeta\tilde{w}_1 - (1-2p)\int_0^\zeta \tilde{w}_1(t)t^2dt + \frac{(17-30p+12p^2)}{9}\int_0^\zeta(-t\tilde{w}_1(t))(\zeta-t)dt + \\ + \frac{8}{27}(p^3-6p^2+11p-6)\int_0^\zeta \tilde{w}_1(t)\frac{(\zeta-t)^2}{2}dt = 0$$

We differentiate three times, getting

$$\left(\zeta-\zeta^3\right)\tilde{w}_1^{(3)}(\zeta) + \left(3-2(5-p)\zeta^2\right)\tilde{w}_1''(\zeta) - \left(\frac{215}{9} - \frac{34}{3}p + \frac{4}{3}p^2\right)\zeta\tilde{w}_1'(\zeta) + \\ + \frac{8}{27}\left(p-\frac{9}{2}\right)\left(p-\frac{7}{2}\right)\left(p-\frac{5}{2}\right)\tilde{w}_1(\zeta) = 0$$

we set $y = \zeta^2$

$$y^2(1-y)\tilde{w}_1^{(3)}(y) + y\left((p-\frac{13}{2})y + 3\right)\tilde{w}_1''(y) - \left(y\left(\frac{305}{36} - \frac{10}{3}p + \frac{p^2}{3}\right) - \frac{3}{4}\right)\tilde{w}_1'(y) + \\ + \frac{1}{27}\left(p-\frac{5}{2}\right)\left(p-\frac{9}{2}\right)\left(p-\frac{7}{2}\right)\tilde{w}_1(y) = 0 \quad (51)$$

which is a generalized hypergeomtric equation (16.8.5 [**?**]) with parameters

$$\mathbf{a}_0 = \left(\frac{3}{2} - \frac{p}{3}; \frac{7}{6} - \frac{p}{3}; \frac{5}{6} - \frac{p}{3}\right) \qquad \mathbf{b}_0 = \left(\frac{1}{2}; \frac{3}{2}\right)$$

therefore, $\tilde{w}_1(y) = c_1\ {}_3F_2\left(\mathbf{a}_0; \mathbf{b}_0; y\right)$.
Then we look at equation (49): its Borel transform is

$$-\zeta^3\tilde{w}_2(\zeta) - 3\zeta^2\tilde{w}_2(\zeta) - 2\zeta\tilde{w}_2(\zeta) + \frac{1-2p}{2}\int_0^\zeta t^2\tilde{w}_2(t)dt - (1-2p)\int_0^\zeta(-t\tilde{w}_2(t))dt + \\ + \frac{5+24p+12p^2}{36}\int_0^\zeta(\zeta-t)(-t\tilde{w}_2(t))dt - \frac{5+24p+12p^2}{36}\int_0^\zeta(\zeta-t)\tilde{w}_2(t)dt + \\ - \left(\frac{5}{54} + \frac{59}{108}p + \frac{5}{18}p^2 + \frac{1}{27}p^3\right)\int_0^\zeta \frac{(\zeta-t)^2}{2}\tilde{w}_2(t)dt = 0 \quad (52)$$

and taking three derivatives it simplifies to

$$\left[\left(\zeta^3+3\zeta^2+2\zeta\right)\partial_\zeta^3+6\partial_\zeta^2+\left(p-\frac{17}{2}\right)\zeta\left(\zeta+\frac{1}{2}\right)\partial_\zeta^2+\right.$$
$$\left.\left(\frac{p^2}{3}+\frac{14}{3}p+\frac{581}{36}\right)(\zeta+1)\,\partial_\zeta+\frac{1}{27}\left(p+\frac{7}{2}\right)\left(p+\frac{11}{2}\right)\left(p+\frac{15}{2}\right)\right]\tilde{w}_2(\zeta)=0 \quad (53)$$

define $y=\zeta\left(\zeta+\frac{4}{3}\right)$

$$\left[y\left(y+\frac{4}{9}\right)^2\partial_y^3+2\left(\frac{3}{2}y+\frac{4}{3}+\frac{y}{2}\left(\frac{17}{2}+p\right)\right)\left(y+\frac{4}{9}\right)\partial_y^2+\left(\frac{2}{3}+\frac{y}{4}\left(\frac{17}{2}+p\right)\right)\partial_y\right.$$
$$\left.+\frac{1}{4}\left(\frac{p^3}{3}+\frac{14}{3}p+\frac{581}{36}\right)\left(y+\frac{4}{9}\right)\partial_y+\frac{1}{8\cdot 27}\left(p+\frac{7}{2}\right)\left(p+\frac{11}{2}\right)\left(p+\frac{15}{2}\right)\right]\tilde{w}_2(y)=0 \quad (54)$$

which now looks like a generalized hypergeometric equation. Indeed setting $t=\frac{9}{4}y+1$, (54) reads

$$\left[t^2(1-t)\partial_t^3-t\left(\left(\frac{23}{4}+\frac{p}{2}\right)t-\frac{p}{2}-\frac{11}{4}\right)\partial_t^2-\left(-\frac{5}{8}-\frac{p}{4}+t\left(\frac{887}{144}+\frac{17}{12}p+\frac{p^2}{12}\right)\right)\partial_t+\right.$$
$$\left.-\frac{1}{8\cdot 27}\left(p+\frac{7}{2}\right)\left(p+\frac{11}{2}\right)\left(p+\frac{15}{2}\right)\right]\tilde{w}_2(t)=0 \quad (55)$$

hence a solution is given by the generalized hypergeometric function ${}_3F_2\left(\mathbf{a};\mathbf{b};y\right)$ with parameters

$$\mathbf{a}=\left(\frac{5}{4}+\frac{p}{6};\frac{7}{12}+\frac{p}{6};\frac{11}{12}+\frac{p}{6}\right) \qquad \mathbf{b}=\left(\frac{1}{2};\frac{5}{4}+\frac{p}{2}\right)$$

and we denote $\hat{w}_2(\zeta)=c_2\;{}_3F_2\left(\mathbf{a};\mathbf{b};\left(\frac{3}{2}\zeta+1\right)^2\right)$. The equation (50) differes from (49) for few signs: we find that its Borel transform is

$$-\zeta^3\tilde{w}_2(\zeta)+2\zeta^2\tilde{w}_2(\zeta)-\frac{8}{9}\zeta\tilde{w}_2(\zeta)+\frac{1-2p}{2}\int_0^\zeta t^2\tilde{w}_2(t)dt+\frac{2-4p}{3}\int_0^\zeta(-t\tilde{w}_2(t))dt+$$
$$+\frac{5+24p+12p^2}{36}\int_0^\zeta(\zeta-t)(-t\tilde{w}_2(t))dt+\frac{5+24p+12p^2}{54}\int_0^\zeta(\zeta-t)\tilde{w}_2(t)dt+$$
$$-\left(\frac{5}{54}+\frac{59}{108}p+\frac{5}{18}p^2+\frac{1}{27}p^3\right)\int_0^\zeta\frac{(\zeta-t)^2}{2}\tilde{w}_2(t)dt=0 \quad (56)$$

and differentiating three times we find a generalized hypergeometric equation with parameters $\mathbf{a},\mathbf{b}$ in the variable $y=\left(\frac{3}{2}\zeta-1\right)^2$, i.e. $\hat{w}_3(\zeta)=c_3\;{}_3F_2\left(\mathbf{a};\mathbf{b};\left(\frac{3}{2}\zeta-1\right)^2\right)$. In particular,

$$\hat{w}_3(\zeta) = C_{23}\hat{w}_2(\zeta - \tfrac{4}{3}) \tag{57}$$

5 Third-order exponential integrals

- Reduce to
$$I(z) = \int \exp\left[-z\big(u^3 + pu + q\big)\right]\, du$$
using change of coordinate.
- When $p \neq 0$, can reduce further to
$$I(z) = p^{1/2} e^{-qz} K_{1/3}(p^{3/2} z).$$
- As p goes to zero, $I(z)$ degenerates to
$$\left(\tfrac{1}{2}\right)^{2/3} e^{-qz}\Gamma(\tfrac{1}{3}) z^{-1/3} = \left(\tfrac{1}{2}\right)^{2/3} e^{-qz}\mathcal{L}_{\zeta,0}(\zeta^{-2/3}) = \left(\tfrac{1}{2}\right)^{2/3}\mathcal{L}_{\zeta_{-q},q}(\zeta^{-2/3}).$$

Outline

Title: Borel regularity and Resurgence of Exponential Integrals

1. introduction
   - Exponential integrals
     - they are function of z and they are defined from the data of (X, f) and $[\mathcal{C}], [\nu]$
     - the choice of the path $\mathcal{C}$:
       * $\mathcal{C} \in H_n^{B,z}(X, f)$
       * Witten's formalism, $\mathcal{C}$ is a Lefschetz thimbles (or steepest descendent path)
     - they define a paring between the relative homology (rapid decaying homology) $H_\bullet^{B,z}(X, f)$ and the twisted de Rham cohomology $H^\bullet_{dR,z}(X, f)$
       * there is a comparison isomorphism (Maxim)
     - varying z we have the Stokes phenomena
     - as $z \to \infty$, the asymptotic expansion of I is a divergent series $\tilde{I}$, usually of Gevrey-class
       * *exact resurgence relation* (Berry–Howls): divergence encodes contributions from other critical values
       * it is an example of resurgent series (Écalle)
       * $\tilde{I}$ is resurgent in $\mathbb{C} \setminus \{\text{poles of } \nu\,, \text{cirtical values of } f\}$
       * it is a toy example of resurgent series because there are only finitely many singularities in the Borel plane

            * we have to compute the **Stokes constants** relative to the singular points in B to fully understand B. There are two methods to compute Stokes constants: • geometric: using intersection theory of thimbles (Picard–Lefschtez, Witten, Maxim), • analytic: using Écalle formalism

  - what are exponential integrals? has to be done
      - motivation
          * In the classical theory of special functions, exponential integrals are often used to express solutions of linear differential and difference equations.
          * In physics ??
          * Geometrically they represent a Poincaré pairing (as explained by Kontsevich in **IHES lectures**).
  - What is the class of ODEs that we study? has to be done
  - State results about resurgence of exponential integrals and Stokes phenomena
      - Thimbles integrals [Kontsevich]: geometric computation of Stokes constants has to be done
      - ODE and fractional derivative formula [`draft2`]
      - if hypergeometric functions appear in a large class of examples: integral formulas for hypergeometric functions has to be done

2. Formalism for Laplace transform [`draft2`, “The geometry of the Laplace transform”]
    (a) Analytic
        i. Introduction
        ii. Brief revew of translation surfaces (we can refer to this from the introduction if we need to)
        iii. The Laplace transform of a holomorphic function
            A. Over an ordinary point
            B. Over a branch point
            C. Differential equation
        iv. Relating differential equations in the frequency domain to integral equations in the position domain
    (b) Formal
        i. Laplace transform of a formal series
        ii. Borel transform
        iii. Relating differential equations in the frequency variable to integral equations in the position variable

3. Review of integral equations
    - Existence of solutions
    - Fractional integrals and derivatives
    - Going between integral and differential equations (slight functions)

4. General cases

   (a) Borel regularity

      - General ODE of the form

        $$\left[P\left(\tfrac{\partial}{\partial z}\right) + z^{-1}Q\left(\tfrac{\partial}{\partial z}\right) + z^{-2}R(z^{-1})\right]\Phi = 0,$$

        where P is a polynomial, Q is a polynomial of one degree lower, and R is an entire function [see `airy-resurgence` and written notes]

        - More generally, for P of degree n, we should be able to handle

          $$\left[P\left(\tfrac{\partial}{\partial z}\right) + z^{-1}Q\left(\tfrac{\partial}{\partial z}\right) + z^{-2}R_2\left(\tfrac{\partial}{\partial z}\right) + \ldots + z^{-(n-1)}R_{n-1}\left(\tfrac{\partial}{\partial z}\right) + z^{-n}R(z^{-1})\right]\Phi = 0,$$

          where R_k has degree $n-k$. has to be done

          * We want the most general ODE with a regular singularity at $z = 0$ and its only other singularity, typically irregular, at $z = \infty$. has to be done
          * The singularity at ∞ should only be regular for an Euler equation. has to be done
        - Show that we can find a slight solution at each critical value.
        - Show that $\hat{\iota} = \tilde{\iota}$, where:
          * $I = \mathcal{L}\iota$
          * $\hat{\iota}$ is the Taylor expansion of ι
          * $\tilde{I}$ is the asymptotic series of I
          * $\tilde{\iota} = \mathcal{B}\tilde{I}$
          * Idea: Show that $\hat{\iota}$ and $\tilde{\iota}$ have matching asymptotics at $\zeta = 0$. Since they both satisfy the position-domain integral equation, they must coincide.
      - General thimble integral (conditions?)
        - Proof of Borel regularity
        - 3/2-derivative formula
        - Contour argument

   (b) Resurgence

      - Explain how Borel regularity relates resurgence of formal series to resurgence of holomorphic functions in the position domain. think more about what we're trying to say here
      - Relate to Ecalle's formalism and the alien derivative
      - Stokes factors
        - For ODEs
        - For thimble integrals

5. Examples make sure each example contains a computation of the Borel transform, so we can see it matches

(a) The Airy example

- $I(z)$ is a solution of a linear ODE. We explicitly find its Borel transform, knowing the nature of singularities and the asymptotic behaviour of a basis of solution for the ODE [`airy-resurgence`]
- Compute Stokes constants
  - Using fractional derivative formula and Borel transform computation [`draft2`]
  - Using Picard-Lefschetz theory (Pham, Kontsevich, etc.)
- Comparison with the literature has to be done
  - Mariño
  - Sauzin
  - Kontsevich slides
  - Kawai–Takei? [might take too long to understand well enough]

(b) The Airy–Lucas examples

- Compute Borel transform [`airy-resurgence`]
- Compute Stokes constants has to be done

(c) Bessel 0 (it is different because we have infinite cover)

- Compute Stokes constants [`draft2`]

(d) Bessel μ (follows from Bessel 0)

- Compute Stokes constants [`modified Bessel`]

(e) The generalized Airy example

(f) The vibrating beam example

- In addition to the simple example, maybe we can do an example where the equation on the spatial domain includes fractional integrals, since Andy is interested in that sort of thing

A Integro-differential equations

A.1 Existence of solutions

A.1.1 Algebraic integral operators

Take a simply connected open set $\Omega \subset \mathbb{C}$ that touches but doesn't contain $\zeta = 0$. Let $\mathcal{H}L^\infty(\Omega)$ be the space of bounded holomorphic functions on Ω with the supremum norm $\|\cdot\|_\infty$. For any $\sigma \in \mathbb{R}$, multiplying by $\zeta^{-\sigma}$ maps $\mathcal{H}L^\infty(\Omega)$ isomorphically onto another space of holomorphic functions on Ω. We'll call this space $\mathcal{H}L^{\infty,\sigma}(\Omega)$ and give it the norm $\|f\|_{\infty,\sigma} = \|\zeta^\sigma f\|_\infty$, so that

$$\begin{aligned} \mathcal{H}L^\infty(\Omega) &\to \mathcal{H}L^{\infty,\sigma}(\Omega) \\ \phi &\mapsto \zeta^{-\sigma}\phi \end{aligned}$$

is an isometry. More generally,

$$\mathcal{H}L^{\infty,\rho}(\Omega) \to \mathcal{H}L^{\infty,\rho+\delta}(\Omega)$$
$$f \mapsto \zeta^{-\delta} f$$

is an isometry for all $\rho \in \mathbb{R}$ and $\delta \in [0, \infty)$. This reduces to the previous statement when $\rho = 0$. For each $\delta \in [0, \infty)$, the functions in $\mathcal{H}L^{\infty,\rho}(\Omega)$ belong to $\mathcal{H}L^{\infty,\rho+\delta}(\Omega)$ too, and the inclusion map $\mathcal{H}L^{\infty,\rho}(\Omega) \hookrightarrow \mathcal{H}L^{\infty,\rho+\delta}(\Omega)$ has norm $\|\zeta^\delta\|_\infty$. Conceptually, $\|\zeta^\delta\|_\infty$ measures of the size of Ω, so let's write it as M^δ with $M = \|\zeta\|_\infty$.

Since $\mathcal{H}L^\infty(\Omega)$ is a Banach algebra, the function space $\mathcal{H}L^{\infty,\infty}(\Omega) := \bigcup_{\sigma\in\mathbb{R}} \mathcal{H}L^{\infty,\sigma}(\Omega)$ is a graded algebra, with a different norm on each grade. For each $\rho, \delta \in \mathbb{R}$, multiplication by a function $m \in \mathcal{H}L^{\infty,\delta}(\Omega)$ gives a map $\mathcal{H}L^{\infty,\rho}(\Omega) \to \mathcal{H}L^{\infty,\rho+\delta}(\Omega)$ with norm $\|m\|_{\infty,\delta}$.

We'll study integral operators $\mathcal{G} \colon \mathcal{H}L^{\infty,\rho}(\Omega) \to \mathcal{H}L^{\infty,\sigma}(\Omega)$ of the form

$$[\mathcal{G}f](a) = \int_{\zeta=0}^{a} g(a,\cdot)\, f\, d\zeta,$$

where the kernel g is an algebraic function over $\mathbb{C}^2$ which can be singular on Δ, the diagonal.[8] To avoid ambiguity, we fix a branch of g to use at the start of the integration path. The domain of g is a covering of $\mathbb{C}^2$ which can be branched over Δ. Continuing g around Δ changes its phase by a root of unity, leaving its absolute value the same **[check]**. That makes $|g|$ a well-defined function on $\mathbb{C}^2 \smallsetminus \Delta$, which we can use to bound $\|\mathcal{G}\|$.

For each $a \in \mathbb{C}$, the expression $|g(a,\cdot)\, d\zeta|$ defines a *density* on $\Omega \smallsetminus \{a\}$—a norm on the tangent bundle which is compatible with the conformal structure. The square of a density is a Riemannian metric. Let $\ell^{\sigma,\rho}_{g,\Omega}(a)$ be the distance from $\zeta = 0$ to a with respect to the density $|\zeta(a)^\sigma\, g(a,\cdot)\, \zeta^{-\rho}\, d\zeta|$ on $\Omega \smallsetminus \{a\}$. The bound

$$\begin{aligned} \big|[\zeta^\sigma \mathcal{G} f](a)\big| &\le \left|\zeta(a)^\sigma \int_{\zeta=0}^{a} g(a,\cdot)\, f\, d\zeta\right| \\ &\le \int_{\zeta=0}^{a} |\zeta^\sigma f|\, |\zeta(a)^\sigma\, g(a,\cdot)\, \zeta^{-\rho}\, d\zeta| \\ &\le \|f\|_{\infty,\sigma} \int_{\zeta=0}^{a} |\zeta(a)^\sigma\, g(a,\cdot)\, \zeta^{-\rho}\, d\zeta| \end{aligned}$$

holds for any integration path. Taking the infimum over all paths, we see that

$$\big|[\mathcal{G}f](a)\big| \le \ell^{\sigma,\rho}_{g,\Omega}(a)\, \|f\|_\infty.$$

so $\|\mathcal{G}\| \le \sup_{a\in\Omega} \ell^{\sigma,\rho}_{g,\Omega}(a)$. Crucially, we can always make $\|\mathcal{G}\|$ a contraction by restricting Ω.

A.1.2 The example of fractional integrals

Setting $g(a, a') = (\zeta(a) - \zeta(a'))^{-\lambda-1}$ with $\lambda \in (-\infty, 0)$, we get the fractional integral $\partial^\lambda_{\zeta \text{ from } 0}$. The shortest path from $\zeta = 0$ to a with respect to $|\zeta(a)^{\rho+\lambda}\, g(a,\cdot)\, \zeta^{-\rho}\, d\zeta|$ is the

[8] Thanks to Alex Takeda for suggesting this.

same as the shortest path with respect to $|d\zeta|$ [check]. It follows that

$$\begin{aligned}
\ell^{\sigma,\rho}_{g,\Omega}(a) &= \int_0^{|\zeta(a)|} |\zeta(a)|^{\rho+\lambda}\,(|\zeta(a)|-r)^{-\lambda-1}\,r^{-\rho}\,dr \\
&= |\zeta(a)|^{\rho+\lambda}\int_0^1 (|\zeta(a)|-|\zeta(a)|t)^{-\lambda-1}\,(|\zeta(a)|t)^{-\rho}\,|\zeta(a)|\,dt \\
&= |\zeta(a)|^{\rho+\lambda-\lambda-1-\rho+1}\int_0^1 (1-t)^{-\lambda-1}\,t^{-\rho}\,dt \\
&= \int_0^1 (1-t)^{-\lambda-1}\,t^{-\rho}\,dt \\
&= B(-\lambda,\,1-\rho).
\end{aligned}$$

The beta function B can be written more explicitly as

$$B(-\lambda,\,1-\rho) = \frac{\Gamma(-\lambda)\,\Gamma(1-\rho)}{\Gamma(1-\lambda-\rho)}.$$

Now we can see that for each $\lambda \in (-\infty, 0)$ and $\rho \in \mathbb{R}$, the fractional integral $\partial^\lambda_{\zeta \text{ from } 0}$ maps $\mathcal{H}L^{\infty,\rho}(\Omega)$ into $\mathcal{H}L^{\infty,\rho+\lambda}(\Omega)$, with norm $\|\partial^\lambda_{\zeta \text{ from } 0}\| \leq B(-\lambda,\,1-\rho)$.

A.1.3 Fractional integral equations near a regular singular point

Angeliki: Maybe Kato–Rellich perturbation theory can give existence immediately. It might not give uniqueness, though.

Consider an integral operator $\mathcal{J}$ of the form

$$p + \partial^{-1}_{\zeta \text{ from } 0} \circ q + \sum_{\lambda\in\Lambda} \partial^\lambda_{\zeta \text{ from } 0} \circ r_\lambda,$$

where:

- p is a function in $\mathcal{H}L^{\infty,-1}(\Omega)$ that extends holomorphically over $\zeta = 0$, and its derivative at $\zeta = 0$ is non-zero.
- q is a function in $\mathcal{H}L^\infty(\Omega)$ that extends holomorphically over $\zeta = 0$.
- r_λ are functions in $\mathcal{H}L^\infty(\Omega)$.
- Λ is a countable subset of $(-\infty, -1)$ whose supremum is less than -1.

Our demand that p and q have convergent power series at $\zeta = 0$ can probably be relaxed; having convergent Novikov series, for example, should be enough. We could also probably replace $\partial^{-1}_{\zeta \text{ from } 0}$ with $\partial^{-1+\delta}_{\zeta \text{ from } 0} \circ \zeta^\delta$ for some $\delta \in [0, 1)$, or adjust the $\partial^\lambda_{\zeta \text{ from } 0}$ similarly.

We want to solve the equation $\mathcal{J}f = 0$. Let's look for a solution of the form $f = \zeta^{\tau-1} + \tilde{f}$ with $\tau \in (0, \infty)$ and $\tilde{f} \in \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega)$ for some $\epsilon \in (0, 1]$. When ϵ is small enough that $\Lambda \subset (-\infty, -1-\epsilon]$, we'll see that we can always find such a solution, as long as we're willing to shrink Ω. In fact, there's exactly one such solution. [Add convergence conditions for ∂^λ terms.]

Let p'_0 and q_0 be the values of $\frac{\partial}{\partial\zeta}p$ and q, respectively, at $\zeta = 0$. We're assuming that p and q extend holomorphically over $\zeta = 0$, and the additional assumption that $p \in \mathcal{H}L^{\infty,-1}(\Omega)$ implies that p has a first-order zero at $\zeta = 0$.

Since p and q extend holomorphically over $\zeta = 0$, and p vanishes at $\zeta = 0$, we can write

$$\begin{aligned} p &= p'_0\zeta + \tilde{p} \\ q &= q_0 \quad + \tilde{q} \end{aligned}$$

with $\tilde{p} \in \mathcal{H}L^{\infty,-2}(\Omega)$ and $\tilde{q} \in \mathcal{H}L^{\infty,-1}(\Omega)$. Then we have

$$\mathcal{J} = p'_0\zeta + q_0\,\partial^{-1}_{\zeta\text{ from }0} + \tilde{\mathcal{J}}$$

with

$$\tilde{\mathcal{J}} = \tilde{p} + \partial^{-1}_{\zeta\text{ from }0} \circ \tilde{q} + \sum_{\lambda\in\Lambda} \partial^{\lambda}_{\zeta\text{ from }0} \circ r_\lambda$$

For any $\tau \in (0, \infty)$,

$$\mathcal{J}\zeta^{\tau-1} = (p'_0 + q_0/\tau)\,\zeta^\tau + \tilde{\mathcal{J}}\zeta^{\tau-1}.$$

Setting $\tau = -q_0/p'_0$ makes the first term vanish, leaving

$$\mathcal{J}\zeta^{\tau-1} = \tilde{\mathcal{J}}\zeta^{\tau-1}.$$

Then the equation $\mathcal{J}f = 0$ becomes

$$\begin{aligned} 0 &= \tilde{\mathcal{J}}\zeta^{\tau-1} + \mathcal{J}\tilde{f} \\ 0 &= \tilde{\mathcal{J}}\zeta^{\tau-1} + \left[p'_0\zeta + q_0\,\partial^{-1}_{\zeta\text{ from }0} + \tilde{\mathcal{J}}\right]\tilde{f} \\ -p'_0\zeta\tilde{f} &= \tilde{\mathcal{J}}\zeta^{\tau-1} + \left[q_0\,\partial^{-1}_{\zeta\text{ from }0} + \tilde{\mathcal{J}}\right]\tilde{f} \\ \tilde{f} &= \left[-\tfrac{1}{p'_0}\,\zeta^{-1} \circ \tilde{\mathcal{J}}\right]\zeta^{\tau-1} + \left[\tau\,\zeta^{-1} \circ \partial^{-1}_{\zeta\text{ from }0} - \tfrac{1}{p'_0}\,\zeta^{-1} \circ \tilde{\mathcal{J}}\right]\tilde{f}. \end{aligned} \tag{58}$$

$$\begin{tikzcd}
 & & \mathcal{H}L^{\infty,\rho-1}(\Omega) \\
\mathcal{H}L^{\infty,\rho}(\Omega) \arrow[r, "\tilde{p}", "\|\tilde{p}\|_{\infty,-2}"'] & \mathcal{H}L^{\infty,\rho-2}(\Omega) \arrow[ur, hook, "\|\zeta\|_\infty"'] \arrow[dr, hook, "\|\zeta\|_\infty^{1-\epsilon}"'] & \\
 & & \mathcal{H}L^{\infty,\rho-1-\epsilon}(\Omega)
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & \mathcal{H}L^{\infty,\rho-1}(\Omega) \\
\mathcal{H}L^{\infty,\rho}(\Omega) \arrow[r, "\tilde{q}", "\|\tilde{q}\|_{\infty,-1}"'] & \mathcal{H}L^{\infty,\rho-1}(\Omega) \arrow[r, "\partial^{-1}", "B(1,2-\rho)=\frac{1}{2-\rho}"'] & \mathcal{H}L^{\infty,\rho-2}(\Omega) \arrow[ur, hook, "\|\zeta\|_\infty"'] \arrow[dr, hook, "\|\zeta\|_\infty^{2}"'] & \\
 & & & \mathcal{H}L^{\infty,\rho}(\Omega)
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & \mathcal{H}L^{\infty,\rho-1}(\Omega) \\
\mathcal{H}L^{\infty,\rho}(\Omega) \arrow[r, "r_\lambda", "\|r_\lambda\|_\infty"'] & \mathcal{H}L^{\infty,\rho}(\Omega) \arrow[r, "\partial^{\lambda}", "B(-\lambda,1-\rho)"'] & \mathcal{H}L^{\infty,\rho+\lambda}(\Omega) \arrow[ur, hook, "\|\zeta\|_\infty^{-1-\lambda}"'] \arrow[dr, hook, "\|\zeta\|_\infty^{-1-\epsilon-\lambda}"'] & \\
 & & & \mathcal{H}L^{\infty,\rho-1-\epsilon}(\Omega)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{H}L^{\infty,\rho}(\Omega) \arrow[r, "\partial^{-1}", "B(1,1-\rho)=\frac{1}{1-\rho}"'] & \mathcal{H}L^{\infty,\rho-1}(\Omega) \arrow[r, "\zeta^{-1}", "\|\zeta^{-1}\|_{\infty,1}=1"'] & \mathcal{H}L^{\infty,\rho}(\Omega)
\end{tikzcd}$$

From [**our previous discussion**], we can work out that

$$\tilde{\mathcal{J}}\colon \mathcal{H}L^{\infty,1-\tau}(\Omega) \to \mathcal{H}L^{\infty,-\tau-\epsilon}(\Omega) \text{ with}$$
$$\|\tilde{\mathcal{J}}\| \le \left(\|\tilde{p}\|_{\infty,-2} + \tfrac{1}{1+\tau}\,\|\tilde{q}\|_{\infty,-1}\right) M^{1-\epsilon} + \sum_{\lambda\in\Lambda} B(-\lambda,\tau)\,\|r_\lambda\|_\infty\, M^{-1-\epsilon-\lambda}$$

and

$$\tilde{\mathcal{J}}\colon \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega) \to \mathcal{H}L^{\infty,-\tau-\epsilon}(\Omega) \text{ with}$$
$$\|\tilde{\mathcal{J}}\| \le \left(\|\tilde{p}\|_{\infty,-2} + \tfrac{1}{1+\tau+\epsilon}\,\|\tilde{q}\|_{\infty,-1}\right) M + \sum_{\lambda\in\Lambda} B(-\lambda,\tau+\epsilon)\,\|r_\lambda\|_\infty\, M^{-1-\lambda}.$$

Since $\|\zeta^{-1}\|_{\infty,1} = 1$, it follows that

$$\zeta^{-1}\circ\tilde{\mathcal{J}}\colon \mathcal{H}L^{\infty,1-\tau}(\Omega) \to \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega) \text{ with}$$
$$\|\zeta^{-1}\circ\tilde{\mathcal{J}}\| \le \left(\|\tilde{p}\|_{\infty,-2} + \tfrac{1}{1+\tau}\,\|\tilde{q}\|_{\infty,-1}\right) M^{1-\epsilon} + \sum_{\lambda\in\Lambda} B(-\lambda,\tau)\,\|r_\lambda\|_\infty\, M^{-1-\epsilon-\lambda}$$

and

$$\zeta^{-1}\circ\tilde{\mathcal{J}} \circlearrowright \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega) \text{ with} \tag{59}$$
$$\|\zeta^{-1}\circ\tilde{\mathcal{J}}\| \le \left(\|\tilde{p}\|_{\infty,-2} + \tfrac{1}{1+\tau+\epsilon}\,\|\tilde{q}\|_{\infty,-1}\right) M + \sum_{\lambda\in\Lambda} B(-\lambda,\tau+\epsilon)\,\|r_\lambda\|_\infty\, M^{-1-\lambda}. \tag{60}$$

We can also see that

$$\tau\,\zeta^{-1}\circ\partial^{-1}_{\zeta\text{ from }0} \circlearrowright \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega) \text{ with}$$
$$\|\tau\,\zeta^{-1}\circ\partial^{-1}_{\zeta\text{ from }0}\| = \tfrac{\tau}{\tau+\epsilon} < 1. \tag{61}$$

Now, let's return to equation 58, which tells us that $f = \zeta^{\tau-1} + \tilde{f}$ satisfies $\mathcal{J} f = 0$ when [and only when?] $\tilde{f}$ is a fixed point of the affine map $\mathcal{A}(\cdot) + b$, where

$$\mathcal{A} = \tau\,\zeta^{-1} \circ \partial^{-1}_{\zeta \text{ from } 0} - \tfrac{1}{p_0'}\,\zeta^{-1} \circ \tilde{\mathcal{J}}$$
$$b = \left[-\tfrac{1}{p_0'}\,\zeta^{-1} \circ \tilde{\mathcal{J}}\right]\,\zeta^{\tau-1}$$

Choosing $\epsilon \in (0, 1]$ so that $\Lambda \subset (-\infty, -1-\epsilon]$ has given us the domain and codomain statements in bounds 59 and 60, which tell us that $\mathcal{A}(\cdot) + b$ sends $\mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega)$ into itself. We'll show that when Ω is small enough, $\mathcal{A}(\cdot) + b$ contracts $\mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega)$, and thus—by the contraction mapping theorem—has a unique fixed point.

An affine map is a contraction if and only if its linear part is a contraction. We know from bound 61 that $\tau\,\zeta^{-1} \circ \partial^{-1}_{\zeta \text{ from } 0}$ contracts $\mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega)$. Since the supremum of Λ is less than -1, all the powers of $M = \|\zeta\|_\infty$ in bound 60 are positive. Thus, by shrinking Ω, we can make the norm of $\zeta^{-1} \circ \tilde{\mathcal{J}}$ on $\mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega)$ as small as we want—small enough to make $\mathcal{A}$ a contraction.

B The Airy equation

B.0.1 Rewriting as a modified Bessel equation

We can distill the most interesting part of the Airy function by writing

$$\operatorname{Ai}(y) = \tfrac{1}{\pi\sqrt{3}}\,y^{1/2}\,K\big(\tfrac{2}{3}y^{3/2}\big),$$

where

$$K(z) = i\sqrt{3}\int_{z^{-1/3}\Gamma} \exp\left[-z\left(4u^3 - 3u\right)\right]\,du. \tag{62}$$

Saying that Ai satisfies the Airy equation is equivalent to saying that K satisfies the modified Bessel equation

$$\left[z^2\big(\tfrac{\partial}{\partial z}\big)^2 + z\tfrac{\partial}{\partial z} - \left[\big(\tfrac{1}{3}\big)^2 + z^2\right]\right] K = 0. \tag{63}$$

In fact, K is the modified Bessel function $K_{1/3}$ [**?**, equation 9.6.1].

Like we did in equation 25, we can rewrite the modified Bessel equation above as

$$\left[\left[\big(\tfrac{\partial}{\partial z}\big)^2 - 1\right] + z^{-1}\tfrac{\partial}{\partial z} - \big(\tfrac{1}{3}\big)^2 z^{-2}\right] K = 0. \tag{64}$$

B.1 Asymptotic analysis

From the general theory of ODE of Poincaré rank 1, we know that the space of trans-series solutions of (64) has a basis of trans-monomials

$$\{e^{-\alpha z} z^{-\tau_\alpha}\,\tilde{W}_\alpha \mid \alpha^2 - 1 = 0\}$$

where the $\tilde{W}_\alpha \in \mathbb{C}[\![z^{-1}]\!]$ are formal power series in z^{-1}. From equations 10.40.2 and 10.17.1 of [**?**], we learn that $K \sim \big(\tfrac{\pi}{2}\big)^{1/2} e^{-z} z^{-1/2}\,\tilde{W}_1$, with

$$\tilde{W}_1 = 1 - \frac{(\frac{1}{6})_1(\frac{5}{6})_1}{2^1 \cdot 1!}\,z^{-1} + \frac{(\frac{1}{6})_2(\frac{5}{6})_2}{2^2 \cdot 2!}\,z^{-2} - \frac{(\frac{1}{6})_3(\frac{5}{6})_3}{2^3 \cdot 3!}\,z^{-3} + \dots \tag{65}$$

The holomorphic analysis in Section B.2.1 will give us holomorphic solutions

$$\{e^{-\alpha z} z^{-\tau_\alpha}\, W_\alpha \mid \alpha^2 - 1 = 0\},$$

which seem analogous to the trans-monomials above. Borel summation makes the analogy precise. We will see in Section B.3 that each $z^{-\tau_\alpha}\, W_\alpha$ is proportional to the Borel sum of $z^{-\tau_\alpha}\, \tilde{W}_\alpha$. This is an evidence of Theorem **??**.

B.2 Building a distinguished frame of analytic solutions

B.2.1 Going to the spatial domain

We are going to look for functions v_α whose Laplace transforms $\mathcal{L}_{\zeta,\alpha} v_\alpha$ satisfy equation (64). We will succeed when $\alpha^2 - 1 = 0$, and we will see that K is a scalar multiple of $\mathcal{L}_{\zeta,1} v_1$.

We can see from Section 2.2.2 that $\mathcal{L}_{\zeta,\alpha} v$ satisfies the differential equation (64) if and only if v satisfies the integral equation

$$\Big[\big[\zeta^2 - 1\big] - \partial_{\zeta,\alpha}^{-1} \circ \zeta - \big(\tfrac{1}{3}\big)^2 \partial_{\zeta,\alpha}^{-2}\Big]\, v = 0. \tag{66}$$

It's tempting to differentiate both sides of this equation until we get

$$\Big[\big(\tfrac{\partial}{\partial \zeta}\big)^2 \circ \big[\zeta^2 - 1\big] - \tfrac{\partial}{\partial \zeta} \circ \zeta - \big(\tfrac{1}{3}\big)^2\Big]\, v = 0, \tag{67}$$

which is easier to solve. However, as we learned in Section 2.5.1, a solution of equation (67) satisfies equation (66) if it's slight and locally integrable at $\zeta = \alpha$.

This is great news, because equation (67) has a regular singularity at each root of $\zeta^2 - 1$, and the Frobenius method often gives a slight solution at each regular singular point. We can see the regular singularities by moving the derivatives to the right:

$$\Big[(\zeta^2 - 1)\big(\tfrac{\partial}{\partial \zeta}\big)^2 + 3\zeta\, \tfrac{\partial}{\partial \zeta} + \big[1 - \big(\tfrac{1}{3}\big)^2\big]\Big]\, v = 0.$$

In Sections B.2.2 – B.2.3, we will see this approach succeed. For each root α, we'll find a solution v_α of equation (67) which is slight and locally integrable at $\zeta = \alpha$. We know the function $\mathcal{L}_{\zeta,\alpha} v_\alpha$ will satisfy equation (64), and we can even find its asymptotics from the order τ_α of v_α. We learned in Section 2.2.3 that

$$\mathcal{L}_{\zeta,\alpha} v_\alpha = e^{-\alpha z} V_\alpha$$

where $V_\alpha = \mathcal{L}_{\zeta_\alpha,0} v_\alpha$ and $\zeta = \alpha + \zeta_\alpha$. We can see from Section 2.2.1 that V_α is asymptotic to a scalar multiple of $z^{-\tau_\alpha}$ at $z = \infty$, so the further decomposition

$$\mathcal{L}_{\zeta,\alpha} v_\alpha = e^{-\alpha z} z^{-\tau_\alpha} W_\alpha,$$

makes W_α is asymptotic to a scalar multiple of $1 + O(z^{-1})$ at $z = \infty$.

B.2.2 Focus on $\zeta = 1$

Let's find a solution of equation (67) which is slight and locally integrable at $\zeta = 1$. Define a new coordinate ζ_1 on $\mathbb{C}$ so that $\zeta = 1 + \zeta_1$. In this coordinate, equation (67) looks like

$$\left[\zeta_1(2+\zeta_1)\left(\tfrac{\partial}{\partial\zeta_1}\right)^2 + 3(1+\zeta_1)\tfrac{\partial}{\partial\zeta_1} + \left[1 - \left(\tfrac{1}{3}\right)^2\right]\right] v = 0. \tag{68}$$

With another change of coordinate, given by $\zeta_1 = -2\xi_1$, we can rewrite equation (67) as the hypergeometric equation

$$\left[\xi_1(1-\xi_1)\left(\tfrac{\partial}{\partial\xi_1}\right)^2 + 3(\tfrac{1}{2}-\xi_1)\tfrac{\partial}{\partial\xi_1} - \left[1 - \left(\tfrac{1}{3}\right)^2\right]\right] v = 0. \tag{69}$$

Looking through the twenty-four expressions for Kummer's six solutions, we find one [**?**, formula 15.10.12] which is manifestly slight and locally integrable at $\xi_1 = 0$:

$$\begin{aligned} v_1 &= \xi_1^{-1/2}\,{}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; \xi_1\right) \\ &= -i\sqrt{2}\,\zeta_1^{-1/2}\,{}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; -\tfrac{1}{2}\zeta_1\right) \end{aligned}$$

From the argument in Section B.2.1, we know that $\mathcal{L}_{\zeta,1}v_1$ satisfies equation (63), and can be written as $e^{-z}V_1$, where $V_1 = \mathcal{L}_{\zeta_1,0}v_1$. Since v_1 has order $-1/2$, the decomposition $V_1 = z^{1/2}W_1$ makes W_1 asymptotic to a scalar multiple of z^{-1} at $z = \infty$.

B.2.3 Focus on $\zeta = -1$

Let's find a solution of equation (67) which is slight and locally integrable at $\zeta = -1$. In the rescaled coordinate from Section B.2.2, this is the point $\xi_1 = 1$. Looking again through Kummer's table of solutions, we find another expression [**?**, formula 15.10.14] which is manifestly slight and locally integrable at $\xi_1 = 1$:

$$\begin{aligned} v_{-1} &= (1-\xi_1)^{-1/2} F\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; 1-\xi_1\right) \\ &= \sqrt{2}\,\zeta_{-1}^{-1/2} F\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; \tfrac{1}{2}\zeta_{-1}\right) \end{aligned}$$

where ζ_{-1} is the coordinate with $\zeta = -1+\zeta_{-1}$. From the argument in Section 4.2.3, we know that $\mathcal{L}_{\zeta,-1}v_{-1}$ satisfies equation (63), and can be written as $e^{z}V_{-1}$, where $V_{-1} = \mathcal{L}_{\zeta_{-1},0}v_{-1}$. Since v_{-1}, like our other solution, has order $-1/2$, the same decomposition $V_{-1} = z^{1/2}W_{-1}$ makes W_{-1} asymptotic to a scalar multiple of z^{-1} at $z = \infty$.

In this example, v_1 and v_{-1} happen to be related by a symmetry: the Möbius transformation that pulls ζ back to $-\zeta$. Kummer's solutions typically come from six different hypergeometric equations, which are related by the Möbius transformations that permute their singularities. In our case, though, exchanging 1 with -1 keeps equation (67) the same.

B.3 Borel regularity

Recall that $\tilde{W}_1$ is a formal solution of (64)

$$\tilde{W}_1(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!} \frac{(-1)^n}{2^n} z^{-n}. \tag{70}$$

Our goal is to prove that the Borel sum of $z^{-1/2}\tilde{W}_1$ is proportional to $V_1 = \mathcal{L}_{\zeta_1,0} v_1$.

Let us compute the Borel transform of $z^{-1/2}\tilde{W}_1$: notice that the appropriate coordinate is ζ_1

$$\begin{aligned}\mathcal{B}\left[z^{-1/2}\tilde{W}_1(z)\right](\zeta_1) &= \mathcal{B}\left[\sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!} \frac{(-1)^n}{2^n} z^{-n-\frac{1}{2}}\right](\zeta_1) \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!} \frac{(-1)^n}{2^n} \frac{\zeta_1^{n-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})} \\ &= \zeta_1^{-1/2} \, {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; -\frac{\zeta_1}{2}\right)\end{aligned}$$

Comparing with the solution v_1 computed in Section B.2.2 we notice that

$$v_1(\zeta_1) \propto \mathcal{B}\left[z^{-1/2}\tilde{W}_1\right](\zeta_1)$$

therefore the Borel Laplace sum of $z^{-1/2}\tilde{W}_1$ is proportional to V_1. In particular, both K and $e^{-z}V_1$ are analytic solutions of the Airy equation (**??**) and they have the same asymptotics at ∞ up to a multiplicative constant.

Similarly, if we consider the formal power series

$$\tilde{W}_{-1}(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!} \frac{1}{2^n} z^{-n} \tag{71}$$

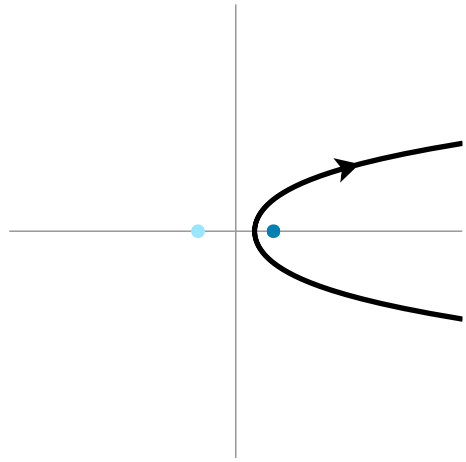
analogous computations give that $v_{-1}(\zeta_{-1}) \propto \mathcal{B}\left[z^{-1/2}\tilde{W}_{-1}\right](\zeta_{-1})$.

B.4 Contour argument for the Airy function

Generalizing from Section 4.2.8, we can recast integral (62) into the ζ plane by setting $-\zeta = 4u^3 - 3u$. Projecting $z^{-1/3}\Gamma$ to a contour γ_z in the ζ plane and choosing the branch of u that lifts γ_z back to $z^{-1/3}\Gamma$, we have

$$K = \frac{i}{\sqrt{3}} \int_{\gamma_z} e^{-z\zeta} \frac{d\zeta}{4u^2 - 1}. \tag{72}$$

For $z \in (0, \infty)$, the contour γ_z runs clockwise around $[1, \infty)$, as shown below. Let's assume $z \in (0, \infty)$ for the rest of the section. [Our conclusions should probably hold whenever $\mathrm{Re}(z) > 0$.]



The contour γ_1 in the ζ plane.

From formula 15.4.14 in [**?**], we learn that for our desired branch of u,

$$\frac{1}{4u^2-1} = -F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\right),$$

so we can rewrite integral (72) as

$$K = \frac{1}{i\sqrt{3}} \int_{\gamma_z} e^{-z\zeta} F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\right)\, d\zeta.$$

This gives us an alternate route to the conclusion of Section B.2.1, which we'll follow below.

In addition to the solutions v_1 and v_{-1} from Section B.2.2– B.2.3, equation (67) has the solutions

$$\begin{aligned} g_1 &= {}_2F_1\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; \xi_1\right) \\ g_{-1} &= {}_2F_1\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; 1-\xi_1\right), \end{aligned}$$

given by formulas 15.10.11 and 15.10.13 from [**?**].

The quadratic transformation identity 15.8.27 from [**?**] shows [verified numerically] that[9]

$$F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\right) = \tfrac{1}{3}(g_1 + g_{-1}),$$

so we have

$$K = \frac{1}{i\,3\sqrt{3}} \int_{\gamma_z} e^{-z\zeta}(g_1 + g_{-1})\, d\zeta.$$

The solution g_{-1} is holomorphic on $\zeta \in [1,\infty)$, so it integrates to zero. The solution g_1, in contrast, is non-meromorphic at $\zeta = 1$. Along the branch cut $\zeta \in [1,\infty)$, its above-minus-

[9]Note that $2\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}) = 2\Gamma(\frac{1}{2})\,\frac{1}{2}\Gamma(\frac{1}{2}) = \pi$ and $\left[\Gamma(\frac{5}{6})\Gamma(\frac{7}{6})\right]^{-1} = \left[\Gamma(\frac{5}{6})\,\frac{1}{6}\Gamma(\frac{1}{6})\right]^{-1} = \frac{6\sin(\frac{1}{6}\pi)}{\pi} = \frac{3}{\pi}$.

below difference is $-\frac{3\sqrt{3}}{2}\,v_1$, as given[10] by equation 15.2.3 from [?]. Hence,

$$\begin{aligned} K &= \frac{i}{2}\int_1^\infty e^{-z\zeta} v_1 \, d\zeta \\ e^z K &= \frac{i}{2}\int_1^\infty e^{-z(\zeta-1)} v_1 \, d\zeta \\ e^z K &= \tfrac{i}{2}\mathcal{L}_{\zeta_1} v_1, \end{aligned}$$

just as we found in Section B.3.

B.5 Another solution

Section B.4 associates the solution K of equation (63) with the solution g_1 of equation (69), which contributes the pole at $\zeta = 1$ of

$$\frac{du}{d\zeta} = \frac{1}{4u^2 - 1} = \tfrac{1}{3}(g_1 + g_{-1}).$$

The solution g_{-1}, which contributes the pole at $\zeta = -1$, is associated with another solution of equation 63.

To express this other solution as a Laplace transform, following the method of Section B.4, we would use the solution

$$v_{-1} = (1-\xi)^{-1/2}\ {}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; 1-\xi\big)$$

of equation ??, given by formula 15.10.14 from [?]. This is the only solution, up to scale, which has a fractional power singularity at $\zeta = -1$.

In summary, the contour integration method of solving equation (63) is associated with the basis

$$\begin{aligned} g_1 &= {}_2F_1\big(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; \xi\big) \\ g_{-1} &= {}_2F_1\big(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; 1-\xi\big) \end{aligned}$$

of solutions for equation (67), given by formulas 15.10.11 and 15.10.13 from [?]. These solutions contribute the poles at $\xi = 1$ and $\xi = 0$, respectively, of a generic solution.

The Laplace transformation method of solving equation (63), on the other hand, is associated with the basis

$$\begin{aligned} v_{-1} &= (1-\xi)^{-1/2}\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; 1-\xi\big) \\ v_1 &= \xi^{-1/2}\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; \xi\big) \end{aligned}$$

given by formulas 15.10.14 and 15.10.12 from [?]. These solutions, up to scale, are the only ones with fractional power singularities.

Identities 15.10.18, and 15.10.22 from [?] give the change of basis

$$\begin{aligned} v_{-1} &= \tfrac{1}{\sqrt{3}}\,g_{-1} + \tfrac{1}{2}\,v_1 \\ v_1 &= \tfrac{1}{\sqrt{3}}\,g_1 + \tfrac{1}{2}\,v_{-1}. \end{aligned}$$

[10]Note that $\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})^{-1} = \frac{1}{2}$ and $\big[\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})\big]^{-1} = \big[\Gamma(\frac{2}{3})\,\frac{1}{3}\Gamma(\frac{1}{3})\big]^{-1} = \frac{3\sin(\frac{1}{3}\pi)}{\pi} = \frac{3\sqrt{3}}{2\pi}$.

Summing these identities, we see that

$$g_1 + g_{-1} = \tfrac{\sqrt{3}}{2}\,(f_1 + f_{-1}),$$

giving the alternate decomposition

$$\frac{du}{d\zeta} = \tfrac{1}{2\sqrt{3}}\,(f_1 + f_{-1}).$$

B.6 Comparison with other treatments of the Airy equation

B.6.1 Different Borel transform convention

Physicists often use a different version of the Borel transform:

$$\begin{aligned} \mathcal{B}_{phys} \colon \mathbb{C}[\![z^{-1}]\!] &\to \mathbb{C}[\![\zeta]\!] \\ z^{-n} &\mapsto \frac{\zeta^n}{n!}. \end{aligned}$$

This version avoids sending 1 to the convolution unit δ, at the cost of no longer mapping multiplication to convolution or inverting the formal Laplace transform. It's sometimes convenient to $\mathcal{B}_{phys}(f) = \mathcal{B}(z^{-1}f)$.

For problems involving a small parameter $\hbar$ rather than a large parameter z, physicists also define

$$\begin{aligned} \mathcal{B}_{phys} \colon \mathbb{C}[\![\hbar]\!] &\to \mathbb{C}[\![\zeta]\!] \\ \hbar^n &\mapsto \frac{\zeta^n}{n!}. \end{aligned}$$

From a combinatorial perspective, this is just the transformation that sends an ordinary generating function to the corresponding exponential generating function.

In [**?**], the author studies the Airy functions as example of resurgent functions. He starts with the formal trans-monomial solutions of Airy equation:

$$\begin{aligned} \tilde{\Phi}_{\mathrm{Ai}}(x) &= \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} \tilde{W}_1(x^{-3/2}) \\ \tilde{\Phi}_{\mathrm{Bi}}(x) &= \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{\frac{2}{3}x^{3/2}} \tilde{W}_2(x^{-3/2}) \end{aligned}$$

where

$$\tilde{W}_{1,2}(\hbar) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(\mp\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} \hbar^n.$$

Then, by applying the two definitions of the Borel transform $\mathcal{B}_{phys}$ and $\mathcal{B}$, on the one hand we have

$$\begin{aligned}
w_{1,2}(\zeta) &:= \mathcal{B}_{phys}(\tilde{W}_{1,2})(\zeta) \\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi}\left(\mp\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}\frac{\zeta^n}{n!} \\
&= {}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\mp\frac{3}{4}\zeta\right)
\end{aligned}$$

on the other hand, we find

$$\begin{aligned}
\mathcal{B}(\tilde{W}_{1,2})(\zeta) &= \frac{1}{2\pi}\delta + \sum_{n=1}^{\infty} \frac{1}{2\pi}\left(\mp\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}\frac{\zeta^{n-1}}{(n-1)!} \\
&= \frac{1}{2\pi}\delta + \sum_{n=0}^{\infty} \frac{1}{2\pi}\left(\mp\frac{3}{4}\right)^{n+1} \frac{\Gamma(n+1+\frac{5}{6})\Gamma(n+1+\frac{1}{6})}{(n+1)!}\frac{\zeta^n}{n!} \\
&= \frac{1}{2\pi}\delta \mp \frac{3}{4}\sum_{n=0}^{\infty} \frac{1}{2\pi}\left(\mp\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{11}{6})\Gamma(n+\frac{7}{6})}{\Gamma(n+2)}\frac{\zeta^n}{n!} \\
&= \frac{1}{2\pi}\delta \mp \frac{5}{48}{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\mp\frac{3}{4}\zeta\right)
\end{aligned}$$

and comparing the two solutions we notice that up to the factor of δ

$$\mathcal{B}(\tilde{W}_{1,2})(\zeta) - \frac{1}{2\pi}\delta = \mp\frac{5}{48}{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\mp\frac{3}{4}\zeta\right) = \frac{d}{d\zeta}\mathcal{B}_{phys}(\tilde{W}_{1,2})(\zeta) \tag{73}$$

More generally, if $\tilde{\Phi}(z) \in z^{-1}\mathbb{C}[\![z^{-1}]\!]$, i.e. it has no constant term, then

$$\frac{d}{d\zeta} \circ \mathcal{B}_{phys}\tilde{\Phi} = \mathcal{B}\tilde{\Phi}. \tag{74}$$

In particular, $\frac{d}{d\zeta} \circ \mathcal{B}_{phys}\left[z^{-1/2}\tilde{W}_1\right]\left(\frac{2}{3}\zeta\right) = v_1(\zeta)$.

B.6.2 Integral formula for hypergeometric functions

In [**?**] the author studies summability and resurgent properties of solutions of the Airy equation.

He considers the formal power series

$$\tilde{\Phi}_\pm(z) := \sum_{n=0}^{\infty} \frac{1}{2\pi}\left(\mp\frac{1}{2}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} z^{-n}$$

such that

$$\begin{aligned}
\tilde{\Phi}_{\mathrm{Ai}}(y) &= \frac{1}{2\sqrt{\pi}} y^{-1/4} e^{-\frac{2}{3}y^{3/2}} \tilde{\Phi}_+\left(\tfrac{2}{3}y^{3/2}\right) \\
\tilde{\Phi}_{\mathrm{Bi}}(y) &= \frac{1}{2\sqrt{\pi}} y^{-1/4} e^{\frac{2}{3}y^{3/2}} \tilde{\Phi}_-\left(\tfrac{2}{3}y^{3/2}\right)
\end{aligned}$$

are formal solutions of the Airy equation. Notice that compared to Marino's formal solutions, Sauzin adopts a different change of coordinates $z = \frac{2}{3}y^{3/2}$.

By looking for solutions of the Borel transformed equation, he wrote the Borel transform of $\tilde{\Phi}_\pm$ as a convolution product:

$$\hat{\phi}_+(\zeta) := \mathcal{B}\tilde{\Phi}_+ = \delta + \frac{d}{d\zeta}\hat{\chi}(\zeta) \qquad \hat{\phi}_-(\zeta) := \mathcal{B}\tilde{\Phi}_- = \delta - \frac{d}{d\zeta}\hat{\chi}(-\zeta)$$

where $\hat{\chi}(\zeta) = \frac{2^{1/6}}{\Gamma(1/6)\Gamma(5/6)}(2\zeta + \zeta^2)^{-1/6} * \zeta^{-5/6}$.

Notice that the function $\hat{\chi}(\zeta)$ is an hypergeometric function:

$$\begin{aligned}
\hat{\chi}(\zeta) &= \frac{2^{1/6}}{\Gamma(1/6)\Gamma(5/6)}(2\zeta + \zeta^2)^{-1/6} * \zeta^{-5/6} \\
&= \frac{2^{1/6}}{\Gamma(1/6)\Gamma(5/6)} \int_0^\zeta (2\zeta' + \zeta'^2)^{-1/6}(\zeta - \zeta')^{-5/6} d\zeta' \\
&= \frac{2^{1/6}}{\Gamma(1/6)\Gamma(5/6)} \int_0^1 (\zeta t)^{-1/6}(2 + \zeta t)^{-1/6}(\zeta - \zeta t)^{-5/6} \zeta dt \\
&= \frac{2^{1/6}}{\Gamma(1/6)\Gamma(5/6)} \int_0^1 t^{-1/6} 2^{-1/6}(1 + \frac{\zeta}{2}t)^{-1/6}(1 - t)^{-5/6} d\zeta' \\
&= \frac{1}{\Gamma(1/6)\Gamma(5/6)} \int_0^1 t^{-1/6}(1 + \frac{\zeta}{2}t)^{-1/6}(1 - t)^{-5/6} d\zeta' \\
&= {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; -\frac{\zeta}{2}\right)
\end{aligned}$$

where in the last step we use the Euler formula for hypergeometric functions [11]. Then, by taking derivatives we recover $\hat{\phi}_\pm(\zeta)$:

$$\begin{aligned}
\hat{\phi}_+(\zeta) &= \delta - \frac{1}{2}\frac{5}{36}\ {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; -\frac{\zeta}{2}\right) = \delta - \frac{2}{3}\frac{5}{48}{}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; -\frac{\zeta}{2}\right) \\
\hat{\phi}_-(\zeta) &= \delta + \frac{1}{2}\frac{5}{36}\ {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; \frac{\zeta}{2}\right) = \delta + \frac{2}{3}\frac{5}{48}{}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; \frac{\zeta}{2}\right)
\end{aligned}$$

and up to a multiplicative constant they match our computations for $\mathcal{B}(\tilde{W}_{1,2})$ (see equation (73)). The main advantage of writing Gauss hypergeometric functions as a convolution product relies on Ecalle's singularity theory. Indeed $(2\zeta + \zeta^2)^{-1/6}$ extends analytically to the universal cover of $\mathbb{C} \setminus \{0, -2\}$ and the convolution with $\zeta^{-5/6}$ does not change the set of singularities (see **Sauzin notes**). Furthermore, the author proves that $\hat{\phi}_\pm(\zeta)$ are simple resurgent functions (see Lemma 6.106 [**?**]).

[11] The Euler formula is

$$ {}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - xt)^{-a} dt. \tag{75}$$

B.6.3 Comparison with exact WKB

Kawai and Takei study the WKB analysis of the Airy–type Schrodinger equation

$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) = 0 \tag{76}$$

as $\eta \to \infty$. They define $\psi_B(x,y)$ as the inverse Laplace transform of $\psi(x,\eta)$ with respect to η. In the coordinate $t = \frac{3}{2}yx^{-3/2}$ they find an explicit formula for $\psi_B(x,y)$ in terms of Gauss hypergeometric functions:

$$\psi_{+,B}(x,y) = \frac{1}{x}\phi_+(t) = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{1}{x}s^{-1/2}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};s\right)$$
$$\psi_{-,B}(x,y) = \frac{1}{x}\phi_-(t) = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{1}{x}(1-s)^{-1/2}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};1-s\right)$$

where $s = t/2 + 1/2$. The same hypergeometric functions have been computed in Section 77 as the Borel transform of the formal solutions of the Airy equation

$$\left[\left(\frac{d}{dw}\right)^2 - w\right] f(w) = 0. \tag{77}$$

Although the two equations look closely related (they are equivalent by the change of coordinates $w = x\eta^{2/3}$), the Borel transform of ψ is computed with respect to $\frac{2}{3}\eta x^{3/2}$ (which is the conjugate variable of t) while the Borel transform of $f(w)$ is computed with respect to w. So we need to find a different change of coordinates to explain why the Borel transforms of $\psi(x,\eta)$ and $f(w)$ are given by the same hypergeometric function.

First of all notice that if η and y are conjugate variables under Borel transform, meaning

$$\sum_{n\geq 0} a_n \eta^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n\geq 0}\frac{a_n}{n!}y^n$$

then $t = \frac{3}{2}yx^{-3/2}$ is the conjugate variable of $q = \frac{2}{3}\eta x^{3/2}$ up to correction by a factor of $\frac{3}{2}x^{-3/2}$

$$\sum_{n\geq 0} a_n q^{-n-1} = \sum_{n\geq 0} a_n x^{-3/2(n+1)}\left(\frac{2}{3}\eta\right)^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n\geq 0}\frac{a_n x^{-3/2(n+1)}}{n!}\left(\frac{3}{2}\right)^{n+1} y^n = \frac{3}{2}x^{-3/2}\sum_{n\geq 0}\frac{a_n}{n!}t^n.$$

In addition, $\psi_{B,\pm}(x,y) = \frac{1}{x}\phi_\pm(t)$, therefore we expect that $\psi(x,\eta) = x^{1/2}\Phi(q)$. Assume that $\psi(x,y)$ is a solution of (76), then $\Phi(q)$ solves

$$\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi(q) = 0 \tag{78}$$

Proof.

$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) = 0$$
$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]x^{1/2}\Phi(q) = 0$$
$$\frac{d}{dx}\left[\frac{1}{2}x^{-1/2}\Phi + x^{1/2}\frac{d}{dx}\Phi\right] - \eta^2 x^{3/2}\Phi = 0$$
$$-\frac{1}{4}x^{-3/2}\Phi + \frac{1}{2}x^{-1/2}\frac{d}{dx}\Phi + \frac{1}{2}x^{-1/2}\frac{d}{dx}\Phi + x^{1/2}\left(\frac{d}{dx}\right)^2\Phi - \eta^2 x^{3/2}\Phi = 0$$
$$\left[x^{1/2}\left(\frac{d}{dx}\right)^2 + x^{-1/2}\frac{d}{dx} - \frac{1}{4}x^{-3/2} - \eta^2 x^{3/2}\right]\Phi = 0$$
$$\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0$$

□

Now rewrite (78) in the coordinates $q = \frac{2}{3}\eta x^{3/2}$:

$$\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0$$
$$\left[\eta^2 x\left(\frac{d}{dq}\right)^2 + \frac{1}{2}\eta x^{-1/2}\frac{d}{dq} + x^{-1}\cdot\eta\, x^{1/2}\frac{d}{dq} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0$$
$$\left[\eta^2 x\left(\frac{d}{dq}\right)^2 + \frac{3}{2}\eta x^{-1/2}\frac{d}{dq} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0$$
$$\left[\left(\frac{d}{dq}\right)^2 + \frac{3}{2}\eta^{-1}x^{-3/2}\frac{d}{dq} - \frac{1}{4}\eta^{-2}x^{-3} - 1\right]\Phi = 0$$
$$\left[\left(\frac{d}{dq}\right)^2 + q^{-1}\frac{d}{dq} - \frac{1}{9}q^{-2} - 1\right]\Phi = 0$$

therefore $\Phi(q)$ is a solution of the transformed Airy equation (64).

Remark B.1. The change of coordinate $q = \frac{2}{3}\eta x^{3/2}$ is not casual: recall that the WKB ansatz for a Schrodinger type equation is

$$\psi(x,\eta) = \exp\left(\int_{x_0}^{x} S(\eta, x')dx'\right) \tag{79}$$

and $S(\eta, x) = \sum_{k\geq -1} S_k(x)\eta^{-k}$. In addition, for the Airy–type Schrodinger equation

$$S_{-1}^2 = x$$

hence, up to a choice of sign for the square root

$$q = \frac{2}{3}\eta x^{3/2} = \eta \int_0^x \sqrt{x'}dx' = \eta \int_0^x S_{-1}(x')dx'.$$

We expect that the change of coordinates $q = \eta \int_0^x S_{-1}(x')dx'$ would explain the analogies between the Borel transform of the WKB solution of a Schrodinger equation and the Borel transform of the associated ODE.

C Application of the integral formula

In Theorem (3.1) we prove that the Borel transform of $\tilde{\varphi}_\alpha$ can be computed by the fractional derivative formula (4).

By general theory it is expected that Stokes constants are integers, and numerically this can be verified only when a suitable normalization is chosen. Indeed as we have seen, if we consider exponential integrals the Borel transform can be computed in different ways: from the ODE, from the contour argument and from the integral formula. When computed from the ODE, there is an ambiguity in the choice of the constant, while with the contour argument and the fractional integral formula the ambiguity is solved. We will see this for the modified Bessel equation.

This method allows to uniquely determined the constants U_1, U_2 that appear in the formal integral solution $\tilde{I}(z)$. Indeed, $\tilde{I}$ is a solution of an homogeneous linear ODE and $\tilde{I}_+(z) = U_1 e^{-2/3z} z^{-1/2} \tilde{W}_+(z)$ for a particular choice of U_1.

Lemma C.1. The following formula holds true

$$\partial^{3/2}_{\zeta \text{ from } 2/3}\left(\int_{\mathcal{C}_+(\zeta)} \nu\right) = -i\frac{\sqrt{\pi}}{2}\,\hat{w}_+(\zeta - 2/3),$$

for any $\zeta \in [2/3, +\infty)$.

Proof. In our general picture of exponential integrals, for $I_+(z)$ the thimble $\mathcal{C}_+$ is parametrized as the path $\theta \mapsto \cosh(\theta - \frac{2}{3}\pi i)$, $f = \frac{2}{3}(4u^3 - 3u)$ and $\nu = du$. Hence,

$$\begin{aligned}\int_{\mathcal{C}_+(\zeta)} \nu &= \int_{\mathcal{C}_+(\zeta)} du \\ &= u\Big|_{\text{start}\,\mathcal{C}_+(\zeta)}^{\text{end}\,\mathcal{C}_+(\zeta)}.\end{aligned}$$

Since $4u^3 - 3u$ is the third Chebyshev polynomial, and cosh is 2π-periodic in the imaginary direction, the start and end points of $\mathcal{C}_+(\zeta)$ are characterized by

$$\begin{aligned}u &= \cosh(\mp\theta - \tfrac{2}{3}\pi i) \\ \zeta &= \tfrac{2}{3}\cosh(3\theta),\end{aligned}$$

so

$$\begin{aligned}
\int_{\mathcal{C}_+(\zeta)} \nu &= \cosh(\theta - \tfrac{2}{3}\pi i) - \cosh(-\theta - \tfrac{2}{3}\pi i) \\
&= \big[\cosh(\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(\theta)\sinh(-\tfrac{2}{3}\pi i)\big] \\
&\quad - \big[\cosh(-\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(-\theta)\sinh(-\tfrac{2}{3}\pi i)\big] \\
&= 2\sinh(\theta)\sinh(-\tfrac{2}{3}\pi i) \\
&= -i\sqrt{3}\,\sinh(\theta)
\end{aligned}$$

with $\frac{3}{2}\zeta = \cosh(3\theta)$. Let $\xi = \frac{1}{2}(1 - \frac{3}{2}\zeta)$, and notice that $\xi = -\sinh(\frac{3}{2}\theta)^2$ at the start and end points. The identity **[DLMF 15.4.16]**

$$\sinh(\theta) = \tfrac{2}{3}\sinh(\tfrac{3}{2}\theta)\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};-\sinh(\tfrac{3}{2}\theta)^2\big)$$

then shows us that

$$\frac{i}{\sqrt{3}}\int_{\mathcal{C}_+(\zeta)} \nu = \tfrac{2}{3}(-\xi)^{1/2}\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};\xi\big).$$

Now we can evaluate the half-integral of $\int_{\mathcal{C}_+}\nu$ using Bateman's fractional integral formula for hypergeometric functions **[Koornwinder, §4.1]**.

$$\begin{aligned}
\partial^{-1/2}_{\zeta \text{ from } 2/3}\left(\int_{\mathcal{C}_+(\zeta)} \nu\right) &= \frac{1}{\Gamma(\frac{1}{2})}\int_{2/3}^{\zeta}(\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_+(\zeta')}\nu\right) d\zeta' \\
&= \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\xi} \tfrac{\sqrt{3}}{2}(\xi'-\xi)^{-1/2}\Big[-i\sqrt{3}\,\tfrac{2}{3}(-\xi)^{1/2}\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};\xi\big)\Big]\,\big(-\tfrac{4}{3}\,d\xi'\big) \\
&= -i\frac{4}{3}\frac{\Gamma(\frac{3}{2})}{\Gamma(2)}(-\xi)\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big) \\
&= i\frac{2}{3}\sqrt{\pi}\,\xi\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big).
\end{aligned}$$

Finally, we differentiate twice using **[DLMF 15.5.4]** and **[DLMF 15.5.1]**.

$$\begin{aligned}
\partial^{3/2}_{\zeta \text{ from } 2/3}\left(\int_{\mathcal{C}_+(\zeta)} \nu\right) &= \Big(-\tfrac{3}{4}\tfrac{\partial}{\partial\xi}\Big)^2\left[i\frac{2}{3}\sqrt{\pi}\,\xi\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big)\right] \\
&= i\tfrac{3\sqrt{\pi}}{8}\,\Big(\tfrac{\partial}{\partial\xi}\Big)^2\big[\xi\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big)\big] \\
&= i\tfrac{3\sqrt{\pi}}{8}\,\tfrac{\partial}{\partial\xi}\big[{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};1;\xi\big)\big] \\
&= i\tfrac{\sqrt{\pi}}{8}\,\tfrac{5}{12}\,{}_2F_1\big(\tfrac{7}{6},\tfrac{11}{6};2;\xi\big).
\end{aligned}$$

□

Analogously, we can compute the correct constants for $\hat{w}_-(\zeta)$:

Lemma C.2. For any $\zeta \in (-\infty, -2/3]$

$$\partial^{3/2}_{\zeta \text{ from } -2/3}\left(\int_{\mathcal{C}_-(\zeta)} \nu\right) = -\frac{\sqrt{\pi}}{2}\,\hat{w}_-(\zeta + 2/3).$$

Proof. Let $\mathcal{C}_-$ is the path $\theta \mapsto -\cosh(\theta - \frac{2}{3}\pi i)$, when $x \in [0, \infty)$

$$I_-(z) = \int_{\mathcal{C}_-} \exp\left[-\tfrac{2}{3} z \left(4u^3 - 3u\right)\right] du.$$

In our general picture of exponential integrals, $f = \frac{2}{3}(4u^3 - 3u)$ and $\nu = du$. Hence,

$$\begin{aligned}\int_{\mathcal{C}_-(\zeta)} \nu &= \int_{\mathcal{C}_-(\zeta)} du \\ &= u\Big|_{\text{start}\,\mathcal{C}_-(\zeta)}^{\text{end}\,\mathcal{C}_-(\zeta)}.\end{aligned}$$

The start and end points of $\mathcal{C}_-(\zeta)$ are characterized by

$$\begin{aligned}u &= -\cosh(\mp\theta - \tfrac{2}{3}\pi i) \\ \zeta &= -\tfrac{2}{3}\cosh(3\theta),\end{aligned}$$

so

$$\begin{aligned}\int_{\mathcal{C}_-(\zeta)} \nu &= -\cosh(\theta - \tfrac{2}{3}\pi i) + \cosh(-\theta - \tfrac{2}{3}\pi i) \\ &= -\big[\cosh(\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(\theta)\sinh(-\tfrac{2}{3}\pi i)\big] \\ &\quad + \big[\cosh(-\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(-\theta)\sinh(-\tfrac{2}{3}\pi i)\big] \\ &= 2\sinh(\theta)\sinh(\tfrac{2}{3}\pi i) \\ &= i\sqrt{3}\,\sinh(\theta)\end{aligned}$$

with $\frac{3}{2}\zeta = -\cosh(3\theta)$. Let $\xi = \frac{1}{2}(1 + \frac{3}{2}\zeta)$, and notice that $\xi = -\sinh(\frac{3}{2}\theta)^2$ at the start and end points. The identity **[DLMF 15.4.16]**

$$\sinh(\theta) = \tfrac{2}{3}\sinh(\tfrac{3}{2}\theta)\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{3}{2}; -\sinh(\tfrac{3}{2}\theta)^2\big)$$

then shows us that

$$-\frac{i}{\sqrt{3}} \int_{\mathcal{C}_-(\zeta)} \nu = \tfrac{2}{3}(-\xi)^{1/2}\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{3}{2}; \xi\big).$$

Now we can evaluate the half-integral of $\int_{\mathcal{C}_-} \nu$ using Bateman's fractional integral formula for hypergeometric functions **[Koornwinder, §4.1]**.

$$\begin{aligned}\partial_{\zeta\text{ from }-2/3}^{-1/2}\left(\int_{\mathcal{C}_-(\zeta)} \nu\right) &= \frac{1}{\Gamma\big(\frac{1}{2}\big)} \int_{-2/3}^{\zeta} (\zeta - \zeta')^{-1/2} \left(\int_{\mathcal{C}_-(\zeta')} \nu\right) d\zeta' \\ &= \frac{1}{\Gamma\big(\frac{1}{2}\big)} \int_0^{\xi} \tfrac{\sqrt{3}}{2}(\xi - \xi')^{-1/2}\Big[i\sqrt{3}\,\tfrac{2}{3}(-\xi')^{1/2}\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{3}{2}; \xi'\big)\Big]\big(\tfrac{4}{3}\,d\xi'\big) \\ &= \frac{4}{3}\frac{\Gamma\big(\frac{3}{2}\big)}{\Gamma(2)}(-\xi)\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; 2; \xi\big) \\ &= -\frac{2}{3}\sqrt{\pi}\,\xi\,{}_2F_1\big(\tfrac{1}{6}, \tfrac{5}{6}; 2; \xi\big).\end{aligned}$$

Finally, we differentiate twice using [**DLMF 15.5.4**] and [**DLMF 15.5.1**].

$$\begin{aligned}
\partial_{\zeta \text{ from } 2/3}^{3/2}\left(\int_{\mathcal{C}_+(\zeta)} \nu\right) &= \left(\tfrac{3}{4}\tfrac{\partial}{\partial\xi}\right)^2\left[-\frac{2}{3}\sqrt{\pi}\,\xi\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big)\right] \\
&= -\tfrac{3\sqrt{\pi}}{8}\left(\tfrac{\partial}{\partial\xi}\right)^2\left[\xi\,{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big)\right] \\
&= -\tfrac{3\sqrt{\pi}}{8}\tfrac{\partial}{\partial\xi}\left[{}_2F_1\big(\tfrac{1}{6},\tfrac{5}{6};1;\xi\big)\right] \\
&= -\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\,{}_2F_1\big(\tfrac{7}{6},\tfrac{11}{6};2;\xi\big).
\end{aligned}$$

□