GENERAL THIMBLES INTEGRALS

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1. Proof of Borel regularity

We are going to prove Theorem 5.1 `draft2`. Let X be a $N-$dim manifold, $f: X \to \mathbb{C}$ be a holomorphic Morse function with simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$I(z) := \int_{\mathcal{C}} e^{-zf}\, \nu \tag{1.1}$$

where $\mathcal{C}$ is a suitable contour such that the integral is well defined. Indeed, $I(z)$ represents a pairing between a relative homology class $\mathcal{C} \in H_N^B(X, zf)$ and a cohomology class $\nu \in H^N_{dR}(X, zf)$ (see Section 1.3.1 Thimble integrals in the introduction). Let us restrict to one dimensional X. For any Morse critical points x_α[1] of f, the saddle point approximation allows to compute the asymptotic expansion of $I_\alpha(z)$

$$I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\, \nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)} \sqrt{2\pi} z^{-1/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n} \qquad \text{as } \operatorname{Re}(ze^{i\theta}) \to \infty \tag{1.2}$$

where $\mathcal{C}_\alpha$ is a steepest descent path through the critical point x_α and θ is chosen such that $f(x_\beta) \notin f(x_\alpha) + [0, e^{i\theta}\infty)$ for $\beta \neq \alpha$[2]. Notice that $f \circ \mathcal{C}_\alpha$ lies in the ray $\zeta_\alpha + [0, e^{i\theta}\infty)$, where $\zeta_\alpha := f(x_\alpha)$.

Theorem 1.1 (Theorem ??)**.** Let $N = 1$. Let $I_\alpha(z)$ defined as in (1.2) for every critical point x_α. Then $\tilde{I}_\alpha$ is Borel regular for $\operatorname{Re}(ze^{i\theta}) > 0$:

(1) The series $\tilde{I}_\alpha(z) = e^{-zf(x_\alpha)} \sqrt{2\pi} z^{-1/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n}$ is Gevrey-1.
(2) The series $\tilde{\iota}_\alpha(\zeta) := \mathcal{B}(\tilde{I}_\alpha)$ converges near $\zeta = \zeta_\alpha$.
(3) If you continue the sum of $\tilde{\iota}_\alpha$ along the ray going rightward from ζ_α in the direction θ, and take its Laplace transform along that ray, you'll recover I_α.

Remark 1.2. (1) We may drop the assumption of non degenerate critical points for f, however the asymptotic expansion of $I_\alpha(z)$ will depend on the order m such that $f^{(m)}(x_\alpha) \neq 0$ and $f^{(j)}(x_\alpha) = 0$ for every $j = 1, \ldots, m-1$ (see [Zorich] Theorem 1 Section 19.2.5).

[1] By Morse critical points we mean non–degenerate isolated critical points.
[2] Such a θ exists because f has a finite number of critical points.

(2) in [Malgrange74] (see also Chapter 5 of [Mistergard Phd thesis] for a general review), the author computes the asymptotic expansion of exponential integrals for $N > 1$ which get logarithmic terms like

$$\tilde{I}(z) = \sum_{j \in A} \sum_{n \geq 0} \sum_{q=0}^{N-1} a_{n,q,j} z^{-n-j} (\log z)^q,$$

for $A \subset \mathbb{Q}_{\geq 0}$ finite. Due to the presence of logarithmic terms, the definition of Borel transform has to be further extended (see [Mistergard phd] Definition pag 5) and the study of Borel regularity becomes more involved.

(3) in the proof of Theorem 1.1 we will derive formula (1.2) using Watson's lemma. However, the same result can be computed from geometric arguments as in Theorem 5.3.3 [Mistergard phd].

Proof. Part (1): Since f is Morse, we can find a holomorphic chart τ around x_α with $\frac{1}{2}\tau^2 = f - \zeta_\alpha$. Let $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$ be the parts of $\mathcal{C}_\alpha$ that go from the past to x_α and from x_α to the future, respectively. We can arrange for τ to be valued in $(-\infty e^{i\theta}, 0]$ and $[0, e^{i\theta}\infty)$ on $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$, respectively. [**We should explicitly spell out and check the conditions that make this possible. I think we're implicitly orienting $\mathcal{C}_\alpha$ so that τ in the upper half-plane.**] Since ν is holomorphic, we can express it as a Taylor series

$$\nu = \sum_{n \geq 0} b_n^\alpha \tau^n \, d\tau$$

that converges in some disk $|\tau| < \varepsilon$.

In coordinates τ the integral $I_\alpha(z)$ can be approximated as

$$I_\alpha(z) \sim e^{-z\zeta_\alpha} \int_{\tau \in [-\varepsilon, \varepsilon]} e^{-z\tau^2/2} \nu$$

as $\mathrm{Re}(ze^{i\theta}) \to \infty$ (see Lemma 1 in Section 19.2.2 Zorich). [**I need to learn how this works! Do we get asymptoticity at all orders? —Aaron**] Plugging in the Taylor series above, we get

$$\begin{aligned} I_\alpha(z) &\sim e^{-z\zeta_\alpha} \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n \geq 0} b_n^\alpha \tau^n \, d\tau \\ &= e^{-z\zeta_\alpha} \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n \geq 0} b_{2n}^\alpha \tau^{2n} \, d\tau \\ &= 2e^{-z\zeta_\alpha} \int_{0}^{\varepsilon} e^{-z\tau^2/2} \sum_{n \geq 0} b_{2n}^\alpha \tau^{2n} \, d\tau. \end{aligned}$$

By Watson's Lemma (see Lemma 4 Section 19.2.2 Zorich)

FIGURE 1. The contour $\mathcal{C}_\alpha$, its image under f which is the Hankel contour $\mathcal{H}_\alpha = f(\mathcal{C}_\alpha)$ and the ray $[\zeta_\alpha, +\infty]$.

$$
\begin{aligned}
I_\alpha(z) &\sim e^{-z\zeta_\alpha} \sum_{n\geq 0} b_{2n}^\alpha \Gamma\left(n+\tfrac{1}{2}\right) 2^{n+1/2} z^{-n-1/2} \\
&= e^{-z\zeta_\alpha} \sqrt{2\pi} \sum_{n\geq 0} b_{2n}^\alpha (2n-1)!! z^{-n-1/2}
\end{aligned}
$$

Call the right-hand side $\tilde{I}_\alpha$. We now see that $a_{\alpha,n} = (2n-1)!!\, b_{2n}^\alpha$ in the statement of the theorem. We know from the definition of ε that $\left|b_n^\alpha\right| \varepsilon^n \lesssim 1$. Recalling that $(2n-1)!! \sim (\pi n)^{-1/2} 4^n\, n!$ as $n \to \infty$, we deduce that $|a_{\alpha,n}| \lesssim \left(\frac{4}{\varepsilon^2}\right)^n n!$, showing that $\tilde{I}_\alpha$ is Gevrey-1.

Part (2): note that **[explain formally what it means to center at ζ_α]**

$$
\tilde{\imath}_\alpha := \mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha = \sqrt{2\pi} \sum_{n\geq 0} (2n-1)!!\, b_{2n}^\alpha \frac{(\zeta-\zeta_\alpha)^{n-1/2}}{\Gamma\left(n+\frac{1}{2}\right)}
$$

Since $(2n-1)!! = \pi^{-1/2} 2^n \Gamma\left(n+\frac{1}{2}\right)$ and $|b_n^\alpha|\epsilon^n \lesssim 1$, then $\tilde{\imath}_\alpha(\zeta)$ has a finite radius of convergence.

Part (3): Let's recast the integral I_α into the f plane. As ζ goes rightward from ζ_α, the start and end points of $\mathcal{C}_\alpha(\zeta)$ sweep backward along $\mathcal{C}_\alpha^-(\zeta)$ and forward along $\mathcal{C}_\alpha^+(\zeta)$, respectively. Hence, we have

$$
\begin{aligned}
I_\alpha(z) &= \int_{\mathcal{C}_\alpha} e^{-zf}\, \nu \\
&= \int_{\mathcal{H}_\alpha} e^{-z\zeta} \left(\int_{f^{-1}(\zeta)} \frac{\nu}{df} \right) d\zeta \\
&= \int_{\zeta_\alpha}^{e^{i\theta}\infty} e^{-z\zeta} \left[\frac{\nu}{df} \right]_{\operatorname{start}\mathcal{C}_\alpha(\zeta)}^{\operatorname{end}\mathcal{C}_\alpha(\zeta)} d\zeta.
\end{aligned}
$$

where $\mathcal{H}_\alpha$ is the Hanckel contour through the point ζ_α (see Figure [**?**]) with ends in the θ direction. Noticing that the last integral is a Laplace transform for the initial choice of θ, we learn that

$$
\hat{\imath}_\alpha(\zeta) = \left[\frac{\nu}{df} \right]_{\operatorname{start}\mathcal{C}_\alpha(\zeta)}^{\operatorname{end}\mathcal{C}_\alpha(\zeta)}. \tag{1.3}
$$

In Ecalle's formalism, $\overset{\triangledown}{\imath}_\alpha := \int_{f^{-1}(\zeta)} \frac{\nu}{df}$ and $\hat{\imath}_\alpha$ are respectively a major and a minor of the singularity and they differ by an holomorphic function (we will see this in the examples Section Airy, Bessel).

We can rewrite our Taylor series for ν as

$$\begin{aligned}\nu &= \sum_{n\geq 0} b_n^\alpha [2(f-\zeta_\alpha)]^{n/2} \frac{df}{[2(f-\zeta_\alpha)]^{1/2}} \\ &= \sum_{n\geq 0} b_n^\alpha [2(f-\zeta_\alpha)]^{(n-1)/2}\, df,\end{aligned}$$

taking the positive branch of the square root on $\mathcal{C}_\alpha^+$ and the negative branch on $\mathcal{C}_\alpha^-$. Plugging this into our expression for $\hat{\iota}_\alpha$, we learn that

$$\begin{aligned}\hat{\iota}_\alpha(\zeta) &= \left[\sum_{n\geq 0} b_n^\alpha [2(f-\zeta_\alpha)]^{(n-1)/2}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)} \\ &= \sum_{n\geq 0} b_n^\alpha \Big([2(\zeta-\zeta_\alpha)]^{(n-1)/2} - (-1)^{n-1} [2(\zeta-\zeta_\alpha)]^{(n-1)/2} \Big) \\ &= \sum_{n\geq 0} 2 b_{2n}^\alpha [2(\zeta-\zeta_\alpha)]^{n-1/2} \\ &= \sum_{n\geq 0} 2^{n+1/2} b_{2n}^\alpha (\zeta-\zeta_\alpha)^{n-1/2} \\ &= \mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha.\end{aligned}$$

We have now shown that the sum of $\mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha$ is actually equal to $\hat{\iota}_\alpha$ as $\zeta \in \zeta_\alpha + [0, e^{i\theta}\infty)$.

□

Remark 1.3. Different choices of admissible θ correspond to different choices of thimbles $[\mathcal{C}_\alpha] \in H_N^B(X, zf)$, but the Borel transform of $\tilde{I}_\alpha$ does not depend on θ. However, if $\theta_* := \arg(\zeta_\alpha - \zeta_\beta)$ and $\theta_\pm := \theta_* \pm \delta$ for small δ, then $I_\alpha(z)$ jumps on the intersection between $\mathrm{Re}(e^{i\theta_+} z) > 0$ and $\mathrm{Re}(e^{i\theta_-} z) > 0$. This is known as the Stokes phenomenon (see Section resurgence thimbles integrals).

2. 3/2 DERIVATIVE FORMULA

In Theorem 1.1 we have seen that the asymptotic behaviour of $I_\alpha(z)$ has a fractional power contribution namely $\tilde{I}_\alpha(z) = e^{-z\zeta_\alpha} z^{-1/2} \sqrt{2\pi} \sum_{n\geq 0} a_{\alpha,n} z^{-n}$, hence we have used the extended notion of Borel transform to deal with fractional powers. Now we will focus on the formal series $\tilde{\Phi}_\alpha(z) := e^{-z\zeta_\alpha} \sqrt{2\pi} \sum_{n\geq 0} a_{\alpha,n} z^{-n} = z^{1/2} \tilde{I}_\alpha(z)$ which does not contain any fractional power and we prove a fractional derivative formula which relates the Borel transforms $\hat{\varphi}_\alpha(\zeta)$ and $\hat{\iota}_\alpha(\zeta)$. Moreover we show that the $\hat{\varphi}_\alpha(\zeta)$ depends on ν and df as well as $\hat{\iota}_\alpha(\zeta)$ does.

Corollary 2.1. Under the same assumptions of Theorem 1.1, for any ζ on the ray going rightward from ζ_α in the direction of θ, we have

$$\hat{\varphi}_\alpha(\zeta) = \partial^{3/2}_{\zeta \text{ from } \zeta_\alpha}\left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right) = \left(\frac{\partial}{\partial \zeta}\right)^2 \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{\zeta_\alpha}^{\zeta} (\zeta - \zeta')^{-1/2} \left(\int_{\mathcal{C}_\alpha(\zeta')} \nu\right) d\zeta', \tag{2.1}$$

where $\mathcal{C}_\alpha(\zeta)$ is the part of $\mathcal{C}_\alpha$ that goes through $e^{-i\theta} f^{-1}([\zeta_\alpha, \zeta])$. Notice that $\mathcal{C}_\alpha(\zeta)$ starts and ends in $e^{-i\theta} f^{-1}(\zeta)$. **[Be careful about the orientation of $\mathcal{C}_\alpha$.]**

Proof. Theorem **??** tells us that

$$\begin{aligned} \mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha &= \mathcal{B}_{\zeta_\alpha} z^{-1/2} \tilde{\varphi}_\alpha \\ &= \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha} \mathcal{B} \tilde{\varphi}_\alpha \\ &= \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha} \hat{\varphi}_\alpha. \end{aligned}$$

It follows, from the proof of part 3 of Theorem 1.1, that

$$\hat{\iota}_\alpha(\zeta) = \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha} \hat{\varphi}_\alpha. \tag{2.2}$$

Since fractional integrals form a semigroup, equation (2.2) implies that

$$\partial^{-1}_{\zeta \text{ from } \zeta_\alpha} \hat{\iota}_\alpha(\zeta) = \partial^{-3/2}_{\zeta \text{ from } \zeta_\alpha} \hat{\varphi}_\alpha.$$

Rewriting equation (1.3) as

$$\hat{\iota}_\alpha(\zeta) = \partial_\zeta \left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right),$$

we can see that

$$\partial^{-1}_{\zeta \text{ from } \zeta_\alpha} \hat{\iota}_\alpha(\zeta) = \int_{\mathcal{C}_\alpha(\zeta)} \nu - \int_{\mathcal{C}_\alpha(0)} \nu.$$

The initial value term vanishes, because the path $\mathcal{C}_\alpha(0)$ is a point. Hence,

$$\int_{\mathcal{C}_\alpha(\zeta)} \nu = \partial^{-3/2}_{\zeta \text{ from } \zeta_\alpha} \hat{\varphi}_\alpha(\zeta).$$

Recalling that the Riemann-Liouville fractional derivative is a left inverse of the fractional integral, we conclude that

$$\partial^{3/2}_{\zeta \text{ from } \zeta_\alpha} \left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right) = \hat{\varphi}_\alpha(\zeta).$$

□

2.1. **Singularities.** From equation (2.2) we see that singularities of $\hat{\iota}_\alpha(\zeta)$ in the Borel plane comes from either poles of ν or zeros of df. Instead, the fractional derivatives formula tells that singularities of $\hat{\varphi}_\alpha$ are given by convolutions of $\zeta^{-1/2}/\Gamma(1/2)$ with $\hat{\iota}_\alpha$. Since $\zeta^{-1/2}/\Gamma(1/2)$ is singular at $\zeta = 0$ the set of singularities of $\hat{\varphi}_\alpha(\zeta)$ is exactly the same as the one of $\hat{\imath}_\alpha(\zeta)$. However, the type of singularities will change and we expect $\hat{\varphi}_\alpha(\zeta)$ to have only simple singularities.

In the examples we noticed that $\hat{\varphi}_\alpha(\zeta)$ is always an hypergeometric function. In particular when there are only two critical values (see Airy, Bessel) the $\hat{\varphi}_\alpha(\zeta)$ is a Gaussian hypergeometric function ${}_2F_1\left(a,b;c;\frac{\zeta}{\zeta_\alpha}\right)$ with $c=2$ and $a+b=c+1$. Whereas, in the generalized Airy example (see Section ??) we get generalized hypergeometric functions ${}_3F_2\left(\mathbf{a};\mathbf{b};(\frac{\zeta}{\zeta_\alpha}-1)^2\right)$ and ${}_3F_2\left(\mathbf{a}_0;\mathbf{b}_0;(\frac{\zeta}{\zeta_\alpha})^2\right)$ with $|\mathbf{a}|=|\mathbf{b}|+1$. This behaviour reflects the resurgence properties of $\hat{\varphi}_\alpha$ (as well as the one of $\hat{\imath}_\alpha$), meaning the analytic continuation of $\hat{\varphi}_\alpha(\zeta)$ at ζ_α is given in terms of $\hat{\varphi}_\beta(\zeta)$, $\zeta_\beta \neq \zeta_\alpha$ when $\hat{\varphi}_\alpha(\zeta), \hat{\varphi}_\beta(\zeta)$ are hypergeometric functions of the previous type.

Lemma 2.2. Let us assume f has only two critical values $\zeta_\alpha = -\zeta_\beta$ and let $\hat{\varphi}_\alpha(\zeta) = {}_2F_1(a,b;2;\frac{\zeta}{\zeta_\alpha})$ with $a+b=c+1$, then across the branch cut

$$\hat{\varphi}_\alpha(\zeta+i0)-\hat{\varphi}_\alpha(\zeta-i0)=C\ {}_2F_1\left(a,b;2;1+\frac{\zeta}{\zeta_\beta}\right) \tag{2.3}$$

$$\hat{\varphi}_\beta(\zeta+i0)-\hat{\varphi}_\beta(\zeta-i0)=-C\ {}_2F_1\left(a,b;2;1+\frac{\zeta}{\zeta_\alpha}\right) \tag{2.4}$$

Proof. It follows from DLMF eq. 15.2.2. □

It would be interesting to further investigate the relationship between the properties of resurgent functions (with finitely many singularities in the Borel plane) and hypergeometric functions.

3. Contour argument

As noticed in proof of Theorem 1.1, the integral $I_\alpha(z)$ can be written as

(i) the Laplace transform of $\hat{\imath}_\alpha(\zeta)$

(ii) the Hankel contour integral of the major $\overset{\triangledown}{\imath}_\alpha(\zeta)$

and $\overset{\triangledown}{\imath}_\alpha(\zeta)=\hat{\imath}_\alpha(\zeta)+\text{hol.fct.}$. In the applications we have evidence that $\overset{\triangledown}{\imath}_\alpha(\zeta)$ is an algebraic hypergeometric function and when there are only two critical values, it decomposes as a sum of two germs of holomorphic functions at each critical values respectively (see `airy-resurgence` Section 6.1, 6.3).