

Resurgence of a parabolic cylinder function

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January 18, 2022

1 Hypergeometric functions in trigonometry

1.1 Chebyshev sine formulas

By definition, $\cos(n\phi) = T_n(\cos(\phi))$. Hence

$$\begin{aligned}\cos(n\phi - \tfrac{n}{2}\pi) &= T_n(\cos(\phi - \tfrac{\pi}{2})) \\ \cos(n\phi) \cos(\tfrac{n}{2}\pi) + \sin(n\phi) \sin(\tfrac{n}{2}\pi) &= T_n(\sin(\phi)).\end{aligned}$$

For $n = 2k + 1$, we have

$$(-1)^k \sin(n\phi) = T_n(\sin(\phi)),$$

implying also that

$$\begin{aligned}(-1)^k n \cos(n\phi) &= T'_n(\sin(\phi)) \cos(\phi) \\ (-1)^k n T_n(\cos(\phi)) &= n U_{n-1}(\sin(\phi)) \cos(\phi) \\ (-1)^k T_n(\cos(\phi)) &= U_{n-1}(\sin(\phi)) \cos(\phi).\end{aligned}$$

1.2 A hypergeometric identity

Let $a = \frac{1}{2} - \frac{1}{2n}$, giving $1 - 2a = \frac{1}{n}$.

1.2.1 All orders

From [?, equation 15.4.16],

$$F(a, 1 - a, \tfrac{3}{2}; \sin(\theta)^2) = \frac{\sin((1 - 2a)\theta)}{(1 - 2a) \sin(\theta)}.$$

so we have

$$\begin{aligned}F\left(\tfrac{1}{2} - \tfrac{1}{2n}, \tfrac{1}{2} + \tfrac{1}{2n}, \tfrac{3}{2}; \sin(\theta)^2\right) &= \frac{\sin(\theta/n)}{\sin(\theta)} \\ &= \frac{1}{U_{n-1}(\cos(\theta/n))}.\end{aligned}$$

Letting $u = \cos(\theta/n)$, we get the identity

$$F\left(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}, \frac{3}{2}; 1 - T_n(u)^2\right) = \frac{1}{U_{n-1}(u)}.$$

Identity 15.10.17 from [?] splits the left-hand side above into

$$\begin{aligned} & \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(1 - \frac{1}{2n})\Gamma(1 + \frac{1}{2n})} F\left(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}, \frac{1}{2}; T_n(u)^2\right) \\ & + \frac{\Gamma(-\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2} - \frac{1}{2n})\Gamma(\frac{1}{2} + \frac{1}{2n})} F\left(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}, \frac{3}{2}; T_n(u)^2\right). \end{aligned}$$

which simplifies¹ to

$$\begin{aligned} & n \sin\left(\frac{\pi}{2n}\right) F\left(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}, \frac{1}{2}; T_n(u)^2\right) \\ & - \cos\left(\frac{\pi}{2n}\right) F\left(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}, \frac{3}{2}; T_n(u)^2\right). \end{aligned}$$

1.2.2 Odd order

From [?, equation 15.4.14],

$$F(a, 1 - a, \frac{1}{2}; \sin(\theta)^2) = \frac{\cos((1 - 2a)\theta)}{\cos(\theta)}.$$

so we have

$$\begin{aligned} F\left(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}, \frac{1}{2}; \sin(\theta)^2\right) &= \frac{\cos(\theta/n)}{\cos(\theta)} \\ &= \frac{\cos(\theta/n)}{T_n(\cos(\theta/n))}. \end{aligned}$$

Let $u = \sin(\theta/n)$. If $n = 2k + 1$, we get the identity

$$F\left(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}, \frac{1}{2}; T_n(u)^2\right) = \frac{(-1)^k}{U_{n-1}(u)}.$$

2 The Weber equation

2.1 Basics

The Weber equation is

$$\left[\left(\frac{\partial}{\partial y}\right)^2 - \left(\frac{1}{4}y^2 + a\right)\right]\psi = 0.$$

Setting a to zero, we get

$$\left[\left(\frac{\partial}{\partial y}\right)^2 - \frac{1}{4}y^2\right]\psi = 0. \tag{1}$$

¹Calculate $\Gamma(-\frac{1}{2})\Gamma(\frac{3}{2}) = -\pi$ and $[\Gamma(\frac{1}{2} - \frac{1}{2n})\Gamma(\frac{1}{2} + \frac{1}{2n})]^{-1} = \frac{1}{\pi} \sin((\frac{1}{2} + \frac{1}{2n})\pi) = \frac{1}{\pi} \cos(\frac{\pi}{2n})$. Similarly, $\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}) = \frac{\pi}{2}$ and $[\Gamma(1 - \frac{1}{2n})\Gamma(1 + \frac{1}{2n})]^{-1} = [\Gamma(1 - \frac{1}{2n})\frac{1}{2n}\Gamma(\frac{1}{2n})]^{-1} = \frac{2n}{\pi} \sin(\frac{\pi}{2n})$.

One solution is given by the parabolic cylinder function [?, equation 12.7.10]

$$U(0, y) = \frac{1}{\sqrt{2\pi}} y^{1/2} K_{1/4}\left(\frac{1}{4}y^2\right).$$

Combining the identity [?, equation 10.27.4]

$$K_\nu(z) = \frac{\pi}{2 \sin(\nu\pi)} (I_{-\nu}(z) - I_\nu(z))$$

and the integral [?, equation 10.32.12]

$$I_\nu(z) = \frac{1}{2\pi i} \int_{\mathcal{H}} e^{z \cosh(\phi)} e^{-\nu\phi} d\phi,$$

where \mathcal{H} runs clockwise around the rectangle

$$0 < \operatorname{Re}(\phi) \qquad |\operatorname{Im}(\phi)| < \pi,$$

we learn that

$$\begin{aligned} I_{-\nu}(z) - I_\nu(z) &= \frac{1}{\pi i} \int_{\mathcal{H}} e^{z \cosh(\phi)} \sinh(\nu\phi) d\phi \\ K_\nu(z) &= \frac{1}{2i \sin(\nu\pi)} \int_{\mathcal{H}} e^{z \cosh(\phi)} \sinh(\nu\phi) d\phi. \end{aligned}$$

In particular, we have

$$K_{1/4}(z) = \frac{1}{i\sqrt{2}} \int_{\mathcal{H}} e^{z \cosh(\phi)} \sinh(\phi/4) d\phi.$$

Setting $u = \cosh(\phi/4)$ and getting $\cosh(\phi) = 8u^4 - 8u^2 + 1$ from a table of Chebyshev polynomials, we can write

$$K_{1/4}(z) = \frac{2\sqrt{2}}{i} \int_{\Gamma} \exp[z(8u^4 - 8u^2 + 1)] du, \quad (2)$$

where Γ is a path that comes from ∞ at -45° and goes to ∞ at 45° .

2.2 Contour argument

We can recast integral 2 into $\hat{\mathbb{C}}$ by setting $-\zeta = 8u^4 - 8u^2 + 1$. Projecting Γ to a contour γ in $\hat{\mathbb{C}}$ and choosing the branch of u that lifts γ back to Γ , we have

$$K_{1/4} = \frac{i}{\sqrt{2}} \int_{\gamma} e^{-z\zeta} \frac{d\zeta}{8u^3 - 4u}. \quad (3)$$

The integrand has poles at $u = \pm \frac{1}{\sqrt{2}}$, and γ runs counterclockwise around $[\frac{1}{\sqrt{2}}, \infty)$.

Using the identity from Section 1.2.1, we can rewrite integral 3 as

$$K_{1/4} = \frac{i}{\sqrt{2}} \int_{\gamma_z} e^{-z\zeta} F\left(\frac{3}{8}, \frac{5}{8}; \frac{3}{2}; 1 - \zeta^2\right) d\zeta.$$