REGULAR SINGULAR VOLTERRA EQUATIONS ON COMPLEX DOMAINS

VERONICA FANTINI AND AARON FENYES

Abstract. The inverse Laplace transform can turn a linear differential equation on a complex domain into an equivalent Volterra integral equation on a real domain. This can make things simpler: for example, a differential equation with irregular singularities can become a Volterra equation with regular singularities. It can also reveal hidden structure, especially when the Volterra equation extends to a complex domain. Our main result is to show that for a certain kind of regular singular Volterra equation on a complex domain, there is always a unique solution of a certain form. As a motivating example, this kind of Volterra equation arises when using Laplace transform methods to solve a *level 1* differential equation.

1. Introduction.

1.1. Motivation. In its most basic form, the Laplace transform $\mathcal{L}$ turns exponential-type functions of a real "position" variable ζ into holomorphic functions of a complex "frequency" variable z. Through identities like

$$\begin{aligned} \frac{\partial}{\partial z}\mathcal{L}\varphi &= \mathcal{L}(-\zeta\varphi) \\ \mathcal{L}k\;\mathcal{L}\varphi &= \mathcal{L}(k * \varphi) \\ z^{-\nu}\mathcal{L}\varphi &= \mathcal{L}\,\partial^{-\nu}\varphi, \end{aligned}$$

where $\partial^{-\nu}$ is the Riemann-Liouville fractional integral of order $\nu \in (0, \infty)$, the Laplace transform pulls differential operators on the frequency domain back to Volterra integral operators on the position domain. The favorable regularity properties and comprehensive theory of Volterra equations can thus be brought to bear on linear differential equations.

Some differential equations pull back to Volterra equations with real-analytic kernels, which extend to holomorphic Volterra equations on complex extensions of the position domain. Solutions that look unrelated in the frequency domain may turn out to be linked by analytic continuation along the complex position domain, as seen in the phenomenon of *resurgence* [3][8][13, Section 2.4].

Differential equations with irregular singularities in the frequency domain can pull back to Volterra equations with regular singularities in the position domain. In Section 4, we'll see this helpful behavior in a certain subclass of what Ecalle calls *level 1* differential equations, which includes classical examples like the modified Bessel equation, the equation describing the vibration modes of a solid triangular cantilever [5, from Equation 12.58], and the Airy equation after a change of coordinate. This last example generalizes to the Airy-Lucas equations, a family of level 1 equations that feature in current research [1, Equations 3.2].

From our experience with solving level 1 differential equations by Borel summation, we expect each of the corresponding integral equations to have a special kind of solution when the integration base point coincides with a regular singularity. Let's say we've chosen the position variable ζ so that the integration base point is at $\zeta = 0$, and our position-domain Volterra operator has a holomorphic kernel $k(a, a')$ with a $\tau/\zeta(a)$ singularity. Assuming the residue τ is real and positive, we expect to find a solution with the following features:

- It has an $O\big(\zeta^{\tau-1}\big)$ singularity at $\zeta = 0$.
- It's of exponential type, meaning that it's $O\big(e^{\Lambda|\zeta|}\big)$ for some $\Lambda \in \mathbb{R}$ as ζ grows.

These conditions are just strong enough to ensure that the solution has a well-defined Laplace transform, which by construction will satisfy the differential equation we started with in the frequency domain. The first condition also tells us that the Laplace-transformed solution is $O(z^{-\tau})$ as the frequency variable z grows.

Our goal is to justify the expectations above with an existence and uniqueness result, stated in Section 1.4. We'll achieve it by embodying our expectations in a Banach space of holomorphic functions—the space $\mathcal{H}L^{\infty}_{\tau-1,\Lambda}(\Omega)$ defined in Section 2. We'll apply our results to level 1 differential equations in Section 4.

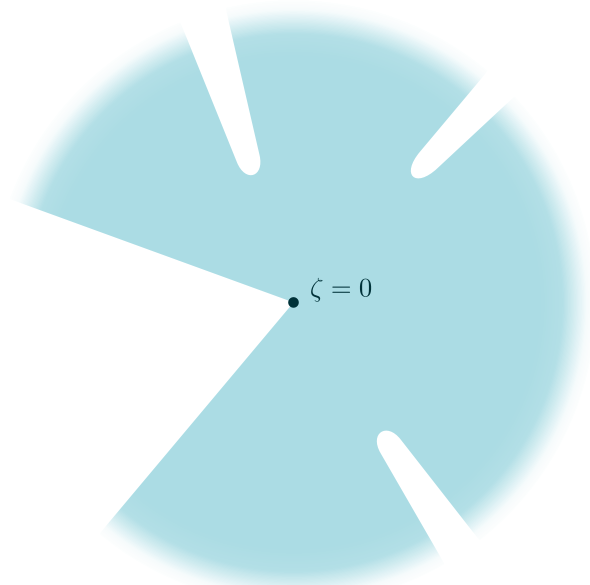
1.2. Notation for uniform bounds. Given a complex-valued function φ and a non-negative, real-valued function ω, we'll write $\varphi \lesssim \omega$ to say that $|\varphi|$ is bounded by a constant multiple of ω. Unless we say otherwise, the bound holds throughout the domain where both functions are defined.

1.3. Setting.

1.3.1. The domain. Throughout this paper, as described in Section 1.1, the position variable ζ will be the standard coordinate on $\mathbb{C}$. Take a simply connected open set $\Omega \subset \mathbb{C}$ that touches but doesn't contain $\zeta = 0$, and satisfies the following condition.

(STAR) The set Ω is star-shaped around $\zeta = 0$. In other words, for any $a \in \Omega$, a straight path from $\zeta = 0$ to a stays in Ω. Since Ω doesn't contain $\zeta = 0$, we'll always omit that starting point from the path.

For the applications we have in mind, Ω might look something like the set pictured below.



1.3.2. The prototype operator. The prototypical example of the kind of operator we'll be working with is a holomorphic Volterra operator $\mathcal{V}_0$ with a separable kernel and a regular singularity at $\zeta = 0$.

Being a holomorphic Volterra operator means that $\mathcal{V}_0$ sends each holomorphic function

φ on Ω to a new holomorphic function

$$[\mathcal{V}_0\, \varphi](a) = \int_{\zeta=0}^{a} k_0(a,\cdot)\, \varphi\, d\zeta.$$

Being separable means that the kernel $k_0(a,a')$ factors into a function of a times a function of a'. We'll suppose this product can be written as a ratio

$$k_0(a,a') = -\frac{q(a')}{p(a)},$$

where p and q are holomorphic functions on Ω. Having a regular singularity at $\zeta = 0$ means the following:

(SING | τ) For some non-zero constant τ—which we'll require, for simplicity, to be real and positive—the difference

$$k_0(a,a) - \frac{\tau}{\zeta(a)}$$

is bounded on a neighborhood of $\zeta(a) = 0$ in Ω. In addition, for each $\sigma > \tau$, the bound

$$|k_0(a,a')| < \frac{\sigma}{|\zeta(a)|}$$

holds over some neighborhood of $\big(\zeta(a),\zeta(a')\big) = (0,0)$ in Ω^2.

For most of our results, we'll need to make sure that Ω doesn't touch any singularities of $\mathcal{V}_0$ other than the one at $\zeta = 0$. We'll also need to control $k_0(a,a')$ when a is away from $\zeta = 0$, requiring k_0 to be bounded on the diagonal in Ω^2 and to grow at most exponentially as we move away from the diagonal. These requirements can be combined into one condition.

(DIAG$_0$ | λ_Δ) For some constant λ_Δ, we have

$$|k_0(a,a')| \lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega$.

This condition explains why Ω might have the sort of shape illustrated in Section 1.3.1. As Ω stretches out toward infinity, it has to part around the zeros of p, keeping well away from every zero except the one at $\zeta = 0$. We'll occasionally mention an optional condition on p that allows us to state our main results more explicitly.

(REG-P | B, ϵ) For some non-zero constant B and some $\epsilon > 0$,

$$p \in B\zeta + O\big(|\zeta|^{1+\epsilon}\big)$$

at $\zeta = 0$.

1.3.3. The perturbed operator. Now, let's perturb $\mathcal{V}_0$ to a more general operator $\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_\star$. The perturbation $\mathcal{V}_\star$ doesn't have a separable kernel,[1] but does have a

[1]Unless $\mathcal{V}_\star$ is zero, of course.

smoothing effect that counteracts the singularity of $\mathcal{V}_0$ (as we'll show in Proposition 3.4). To get the smoothing effect, we'll require the kernel $k_\star$ of $\mathcal{V}_\star$ to vanish to some order $\gamma > 0$ on the diagonal in Ω^2. This requirement, combined with two others, will be made precise in Condition ($\textsc{diag}_\star \mid \gamma, \lambda_\Delta$).

Since $\mathcal{V}_\star$ is a holomorphic Volterra operator, $k_\star$ is a holomorphic function on Ω^2. We'll allow $k_\star(a, a')$ to have a simple pole at $\zeta(a) = 0$, like $k_0(a, a')$ does, but we won't allow any sharper singularity. We'll also put an exponential bound on how fast $k_\star$ grows away from the diagonal, mimicking Condition ($\textsc{diag}_0 \mid \lambda_\Delta$) on k_0. Altogether, we'll require:

($\textsc{diag}_\star \mid \gamma, \lambda_\Delta$) For some constant $\gamma > 0$, we have

$$|k_\star(a, a')| \lesssim \frac{|\zeta(a) - \zeta(a')|^\gamma}{|\zeta(a)|}\, e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega$.

Notice that this condition prevents $k_\star$ from being separable—unless it's zero, of course.

Like k_0, the combined kernel $k = k_0 + k_\star$ of $\mathcal{V}$ grows in a controlled way when its arguments are near $\zeta = 0$, and when the difference between its arguments grows. We'll provide specific bounds in Section 3.4.2.

1.4. Main results. We want to show that when $\mathcal{V}$ is a regular singular Volterra operator of the kind described in Section 1.3.3, the equation $f = \mathcal{V} f$ has a unique solution of a certain form. For the prototypical operator $\mathcal{V}_0$ described in Section 1.3.2, this solution can be written explicitly.

Theorem 1.1. *Suppose $\mathcal{V}_0$ satisfies* Condition ($\textsc{sing} \mid \tau$). *Then the equation*

$$f = \mathcal{V}_0 f$$

has the prototype solution

$$f_0(a) = \frac{1}{p(a)} \exp\left(-\int_b^a \frac{q}{p}\, d\zeta \right). \tag{1.1}$$

Changing the base point $b \in \Omega$ just multiplies f_0 by a non-zero constant.

We'll prove this result in Section 3.2.1.

The solution f_0 from Theorem 1.1 has, at worst, a mild power-law singularity at $\zeta = 0$. With stronger constraints on $\mathcal{V}_0$, we can also ensure that f_0 grows at most exponentially as $|\zeta| \to \infty$. The function spaces defined in Section 2 express both of these regularity properties.

Theorem 1.2. *Suppose $\mathcal{V}_0$ satisfies* Condition ($\textsc{sing} \mid \tau$). *Then, on a small enough neighborhood of $\zeta = 0$, we have $|f_0| \lesssim |\zeta|^{\tau - 1}$.*

Suppose $\mathcal{V}_0$ also satisfies Condition ($\textsc{diag}_0 \mid \lambda_\Delta$). *Then f_0 belongs to the space $\mathcal{H}L^\infty_{\tau-1,\bullet}(\Omega)$ defined in Section 2.*

We'll prove this result in Section 3.2.2.

Remark 1.3. When $\mathcal{V}_0$ also satisfies Condition ($\textsc{reg-p} \mid B, \epsilon$), we can get a better estimate of the prototype solution near $\zeta = 0$, as described in Proposition 3.3 from Section 3.2.2.

When we perturb $\mathcal{V}_0$ to the more general operator $\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_\star$ from Section 1.3.3, the equation we're trying to solve gets more complicated, but its regular singularity at $\zeta = 0$

stays essentially the same. We might therefore expect to find a solution that looks like f_0 near the singularity, differing only by a less singular perturbation. If we strengthen the constraints on $\mathcal{V}_0$ a little more, this expectation is fulfilled.

THEOREM 1.4. *Suppose* $\mathcal{V}_0$ *satisfies* Conditions (SING | τ) *and* (DIAG$_0$ | λ_Δ), *and* $\mathcal{V}_\star$ *satisfies* Condition (DIAG$_\star$ | γ, λ_Δ). *Then the equation*

$$f = \mathcal{V} f$$

has a unique solution f *in the affine subspace*

$$f_0 + \mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$$

of the space $\mathcal{H}L^\infty_{\tau-1,\bullet}(\Omega)$ *defined in Section 2. Here,* f_0 *is the prototype solution* (1.1) *from Theorems 1.1–1.2, which belongs to the space* $\mathcal{H}L^\infty_{\tau-1,\bullet}(\Omega)$.

For any $\rho > \tau$, *the uniqueness of the solution still holds in* $f_0 + \mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$. *In other words, lowering* ρ *into* $(\tau, \tau+\gamma)$ *to allow a sharper singularity won't reveal any more solutions, and raising* ρ *too high to admit the solution found in* $f_0 + \mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$ *will leave no solution at all.*

This result will follow from a more general result about inhomogeneous equations.

LEMMA 1.5. *Suppose* $\mathcal{V}_0$ *satisfies* Conditions (SING | τ) *and* (DIAG$_0$ | λ_Δ), *and* $\mathcal{V}_\star$ *satisfies* Condition (DIAG$_\star$ | γ, λ_Δ). *Suppose we're also given a function* g, *which for some* $\rho > \tau$ *belongs to the space* $\mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$ *defined in Section 2. Then the inhomogeneous equation*

$$f = \mathcal{V} f + g,$$

has a unique solution f *in the space* $\mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$.

We'll prove Lemma 1.5 in Sections 3.4–3.5, using the contraction mapping theorem. The heart of the argument is Proposition 3.7, which shows us how to find relevant subspaces where $\mathcal{V}$ is a contraction.

We'll reduce Theorem 1.4 to Lemma 1.5 in Section 3.5, by rewriting the homogeneous equation we want to solve as an inhomogeneous equation in a more regular space. To show that the inhomogeneous term, $\mathcal{V}_\star f_0$, is regular enough, we'll use Proposition 3.4 from Section 3.3 proving that $\mathcal{V}_\star$ improves on the regularity of f_0 described by Theorem 1.2.

Remark 1.6. When $\mathcal{V}_0$ also satisfies Condition (REG-P | B, ϵ), Theorem 1.4 can be restated to give a unique solution in the affine subspace

$$\zeta^{\tau-1} + \mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$$

when $\rho > \tau$ is low enough, as described in Proposition 3.11 from Section 3.5.

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2. Function spaces for holomorphic Volterra operators.

2.1. Weighted holomorphic L^∞ spaces. Throughout this paper, as described in Section 1.1, the "position" variable ζ will be the standard coordinate on $\mathbb{C}$. Take a simply connected open set $\Omega \subset \mathbb{C}$ that touches but doesn't contain $\zeta = 0$. Let $\mathcal{C}(\Omega)$ be the space of continuous complex-valued functions on Ω. Give $\mathcal{C}(\Omega)$ the compact-open topology, recalling that this is the coarsest topology in which the seminorm $f \mapsto \sup_K |f|$ is continuous for every compact subset $K \subset \Omega$ [7, Example 2.6 and Section 4 notes]. The holomorphic functions form a closed subspace $\mathcal{H}(\Omega) \subset \mathcal{C}(\Omega)$ [7, Proposition 3.14].

Fixing a real constant Λ, let's restrict our attention to holomorphic functions on Ω which are bounded by constant multiples of $e^{\Lambda|\zeta|}$. One might describe these functions as being uniformly of exponential type Λ.[2] They form a space $\mathcal{H}L^\infty_{0,\Lambda}(\Omega)$, which we'll equip with the norm $\|f\|_{0,\Lambda} = \sup_\Omega e^{-\Lambda|\zeta|}\,|f|$. With respect to the seminorm on $\mathcal{H}(\Omega)$ given by a compact set $K \subset \Omega$, the inclusion map $\mathcal{H}L^\infty_{0,\Lambda}(\Omega) \hookrightarrow \mathcal{H}(\Omega)$ has norm $\sup_K e^{\Lambda|\zeta|}$. That means the inclusion is continuous.

Proposition 2.1. *The space $\mathcal{H}L^\infty_{0,\Lambda}(\Omega)$ is complete.*

Proof. Take a Cauchy sequence $f_1, f_2, f_3, \ldots \in \mathcal{H}L^\infty_{0,\Lambda}(\Omega)$. The inclusion map $\mathcal{H}L^\infty_{0,\Lambda}(\Omega) \hookrightarrow \mathcal{H}(\Omega)$ is bounded with respect to each of the seminorms on $\mathcal{H}(\Omega)$ given by $|f| \mapsto \sup_K |f|$ for compact subsets $K \subset \Omega$, so our sequence is Cauchy in $\mathcal{H}(\Omega)$ too. Since $\mathcal{H}(\Omega)$ is complete [7, Proposition 3.5],[3] our sequence converges to a function f there.

The Cauchy property in $\mathcal{H}L^\infty_{0,\Lambda}(\Omega)$ tells us that for any $r > 0$, we can find some n for which $e^{-\Lambda|\zeta|}\,|f_k - f_n| \le r$ whenever $k \ge n$. Since convergence in $\mathcal{H}(\Omega)$ implies pointwise convergence, we can see as k grows that $e^{-\Lambda|\zeta|}\,|f - f_n| \le r$. This shows that our sequence converges to f in the norm $\|\cdot\|_{0,\Lambda}$. We can also see from this argument that f is in $\mathcal{H}L^\infty_{0,\Lambda}(\Omega)$: picking some $r > 0$, we observe that

$$
\begin{aligned}
e^{-\Lambda|\zeta|}\,|f| &\le e^{-\Lambda|\zeta|}\,|f - f_n| + e^{-\Lambda|\zeta|}\,|f_n| \\
&\le r + \|f_n\|_{0,\Lambda}
\end{aligned}
$$

for the corresponding n, showing that $e^{-\Lambda|\zeta|}\,|f|$ is bounded. ∎

Now, let's relax our norm to allow both exponential growth at infinity and a power-law singularity at $\zeta = 0$. For any $\sigma \in \mathbb{R}$, let $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega)$ be the space of holomorphic functions on Ω which are bounded by constant multiples of $|\zeta|^\sigma e^{\Lambda|\zeta|}$. Give it the norm $\|f\|_{\sigma,\Lambda} = \sup_\Omega |\zeta|^{-\sigma} e^{-\Lambda|\zeta|}\,|f|$. Reprising the arguments from above, we can show that the inclusion $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega) \hookrightarrow \mathcal{H}(\Omega)$ is continuous, and we can generalize Proposition 2.1:

Proposition 2.2. *The space $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega)$ is complete.*

2.2. Continuous inclusions between different $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega)$.

Proposition 2.3. *If $\Lambda' \le \Lambda$, the inclusion map $\mathcal{H}L^\infty_{\sigma,\Lambda'}(\Omega) \hookrightarrow \mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega)$ is continuous.*

Proof. By definition,

$$
\|f\|_{\sigma,\Lambda} = \sup_\Omega |\zeta|^{-\sigma}\, e^{-\Lambda|\zeta|}\,|f|.
$$

[2] Recall that a function f is of exponential type Λ if for every $\varepsilon > 0$, there's a constant A_ε (which may depends on ε) such that $|f| \le A_\varepsilon e^{(\Lambda+\varepsilon)|\zeta|}$. We instead require a uniform constant A such that $|f| \le A e^{\Lambda|\zeta|}$.

[3] That is, a sequence which is Cauchy in each of the seminorms on $\mathcal{H}(\Omega)$ will always converge in the topology of $\mathcal{H}(\Omega)$, which is the coarsest topology in which all of the seminorms are continuous.

The norm $\|\cdot\|_{\sigma,\Lambda'}$ is designed to give $|f| \leq |\zeta|^{\sigma}\, e^{\Lambda'|\zeta|}\, \|f\|_{\sigma,\Lambda'}$, which tells us that

$$
\begin{aligned}
\|f\|_{\sigma,\Lambda} &\leq \sup_{\Omega} |\zeta|^{-\sigma}\, e^{-\Lambda|\zeta|}\, |\zeta|^{\sigma}\, e^{\Lambda'|\zeta|}\, \|f\|_{\sigma,\Lambda'} \\
&= \sup_{\Omega} e^{-(\Lambda-\Lambda')|\zeta|}\, \|f\|_{\sigma,\Lambda'} \\
&\leq \|f\|_{\sigma,\Lambda'}.
\end{aligned}
$$

In the last step, we use the assumption that $\Lambda' \leq \Lambda$. ∎

As we mentioned in Section 2.1, one might describe $\mathcal{H}L^{\infty}_{0,\Lambda}(\Omega)$ as the space of functions on Ω which are uniformly of exponential type Λ. Taking the union of these spaces over all $\Lambda \in \mathbb{R}$, we get the space $\mathcal{H}L^{\infty}_{0,\bullet}(\Omega)$ that contains all functions of exponential type. Having continuous inclusions between the subspaces $\mathcal{H}L^{\infty}_{0,\Lambda}(\Omega)$ as Λ increases, we can give $\mathcal{H}L^{\infty}_{0,\bullet}(\Omega)$ a meaningful topology: the finest topology that makes all the inclusions $\mathcal{H}L^{\infty}_{0,\Lambda}(\Omega) \hookrightarrow \mathcal{H}L^{\infty}_{0,\bullet}(\Omega)$ continuous.[4] For any $\sigma \in \mathbb{R}$, we define $\mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$ similarly.

Proposition 2.4. *If $\sigma' > \sigma$ and $\Lambda' < \Lambda$, the inclusion map $\mathcal{H}L^{\infty}_{\sigma',\Lambda'}(\Omega) \hookrightarrow \mathcal{H}L^{\infty}_{\sigma,\Lambda}(\Omega)$ is continuous.*

Proof. By definition,

$$
\begin{aligned}
\|f\|_{\sigma,\Lambda} &= \sup_{\Omega} |\zeta|^{-\sigma} e^{-\Lambda|\zeta|} |f| \\
&= \sup_{\Omega} |\zeta|^{\sigma'-\sigma}\, |\zeta|^{-\sigma'}\, e^{-\Lambda'|\zeta|}\, e^{-(\Lambda-\Lambda')|\zeta|}\, |f|.
\end{aligned}
$$

The function $|\zeta|^{\sigma'-\sigma}\, e^{-(\Lambda-\Lambda')|\zeta|}$ is bounded near $\zeta = 0$ because the power of $|\zeta|$ is positive, and it's bounded far from $\zeta = 0$ thanks to the decaying exponential. Hence,

$$
\begin{aligned}
\|f\|_{\sigma,\Lambda} &\leq C \sup_{\Omega} |\zeta|^{-\sigma'}\, e^{-\Lambda'|\zeta|}\, |f| \\
&= C\|f\|_{\sigma',\Lambda'}
\end{aligned}
$$

for $C = \sup_{\Omega} |\zeta|^{\sigma'-\sigma}\, e^{-(\Lambda-\Lambda')|\zeta|}$. ∎

Proposition 2.5. *When $\sigma' > \sigma$, there's a continuous inclusion $\mathcal{H}L^{\infty}_{\sigma',\bullet}(\Omega) \hookrightarrow \mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$.*

Proof. For each Λ', we can get a continuous inclusion $\mathcal{H}L^{\infty}_{\sigma',\Lambda'}(\Omega) \hookrightarrow \mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$ by choosing some $\Lambda > \Lambda'$ and composing the continuous inclusion $\mathcal{H}L^{\infty}_{\sigma',\Lambda'}(\Omega) \hookrightarrow \mathcal{H}L^{\infty}_{\sigma,\Lambda}(\Omega)$ given by Proposition 2.4 with the continuous inclusion $\mathcal{H}L^{\infty}_{\sigma,\Lambda}(\Omega) \hookrightarrow \mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$ that we get by definition. For any $\Lambda'' < \Lambda'$, the inclusions $\mathcal{H}L^{\infty}_{\sigma',\Lambda''}(\Omega) \hookrightarrow \mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$ and $\mathcal{H}L^{\infty}_{\sigma',\Lambda'}(\Omega) \hookrightarrow \mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$ constructed above automatically commute with the inclusion $\mathcal{H}L^{\infty}_{\sigma',\Lambda''}(\Omega) \hookrightarrow \mathcal{H}L^{\infty}_{\sigma',\Lambda'}(\Omega)$ given by Proposition 2.3, because we're ultimately working in the vector space of holomorphic functions on Ω. Because of how the topology on $\mathcal{H}L^{\infty}_{\sigma',\bullet}(\Omega)$ is defined, this gives us the desired continuous inclusion $\mathcal{H}L^{\infty}_{\sigma',\bullet}(\Omega) \hookrightarrow \mathcal{H}L^{\infty}_{\sigma,\bullet}(\Omega)$. ∎

3. Solving holomorphic Volterra equations.

[4] In category-theoretic language, $\mathcal{H}L^{\infty}_{0,\bullet}(\Omega)$ is the limit of the family $\mathcal{H}L^{\infty}_{0,\Lambda}(\Omega)$.

3.1. Overview.

The results stated in Section 1.4 lay out a method for solving the regular singular Volterra equation $f = \mathcal{V}f$. We'll now show that the method works by proving those results.

We start by constructing the prototype solution f_0 from the kernel of $\mathcal{V}_0$, which is the separable operator that gives $\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_\star$ its regular singularity. We'll show in Section 3.2 that f_0 satisfies the equation $f_0 = \mathcal{V}_0 f_0$ and belongs to the space $\mathcal{H}L^\infty_{\tau-1,\lambda_0}(\Omega)$.

To see what the perturbation $\mathcal{V}_\star$ does to f_0, we'll show in Section 3.3 that $\mathcal{V}_\star$ has a smoothing effect, reducing the sharpness of any power-law singularity at $\zeta = 0$. In particular, it sends f_0 into $\mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$.

Now we know that $\mathcal{V}$ sends f_0 into the affine subspace $f_0 + \mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$, suggesting that the equation $f = \mathcal{V}f$ has a solution there. To confirm this, we'll show in Section 3.4 that $\mathcal{V}$ is a contraction of $\mathcal{H}L^\infty_{\tau-1+\gamma,\Lambda}(\Omega)$ when Λ is large enough. This tells us that $\mathcal{V}$ has a unique fixed point in $f_0 + \mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$, as we'll see in Section 3.5.

3.2. Construction and regularity of the prototype solution *(proof of Theorems 1.1–1.2)*.

3.2.1. Construction.

Proof of Theorem 1.1. Rewrite f_0 as $(1/p)\,\chi$, where

$$\chi(a) = \exp\left(-\int_b^a \frac{q}{p}\,d\zeta\right).$$

Observing that $d\chi = -(q/p)\,\chi\,d\zeta$ greatly simplifies the calculation of $\mathcal{V}_0 f_0$. For each $a \in \Omega$,

$$\begin{aligned}
\big[\mathcal{V}_0 f_0\big](a) &= -\int_{\zeta=0}^a \frac{q}{p(a)}\,f_0\,d\zeta \\
&= -\int_{\zeta=0}^a \frac{q}{p(a)}\,\frac{1}{p}\,\chi\,d\zeta \\
&= -\frac{1}{p(a)}\int_{\zeta=0}^a \frac{q}{p}\,\chi\,d\zeta \\
&= \frac{1}{p(a)}\int_{\zeta=0}^a d\chi \\
&= \frac{1}{p(a)}\left[\chi(a) - \lim_{\zeta\to 0}\chi\right] \\
&= f_0(a) - \frac{1}{p(a)}\lim_{\zeta\to 0}\chi.
\end{aligned}$$

Now, to prove that $\mathcal{V}_0 f_0 = f_0$, we just need to show that $\lim_{\zeta\to 0}\chi = 0$.

By Condition ($\textsc{sing} \mid \tau$), we can find a radius $\delta > 0$ and a constant C such that

$$\left|\frac{q}{p} - \frac{\tau}{\zeta}\right| < C \tag{3.1}$$

whenever $|\zeta| < \delta$. We can take advantage of this bound by rewriting the integral in the definition of χ:

$$-\int_b^a \frac{q}{p}\,d\zeta = \int_b^a \frac{\tau}{\zeta}\,d\zeta - \int_b^a \left(\frac{q}{p}+\frac{\tau}{\zeta}\right)\,d\zeta$$
$$= \tau\log\left(\frac{\zeta(a)}{\zeta(b)}\right) - \int_b^a \left(\frac{q}{p}+\frac{\tau}{\zeta}\right)\,d\zeta$$

Exponentiating both sides, we see that

$$\chi(a) = \left(\frac{\zeta(a)}{\zeta(b)}\right)^{\tau} \exp\left[-\int_b^a \left(\frac{q}{p}+\frac{\tau}{\zeta}\right)\,d\zeta\right].$$

Recalling that a change of base point just multiplies f_0 by a non-zero constant, choose the base point $b \in \Omega$ so that $|\zeta(b)| = \delta/2$. The exponential factor in the formula above is then bounded between $\exp\left(-\frac{3}{2}C\delta\right)$ and $\exp\left(\frac{3}{2}C\delta\right)$ whenever $|\zeta(a)| < \delta$. Since τ is positive, this is enough to show that $\lim_{\zeta\to 0}\chi = 0$. It follows, as discussed above, that $\mathcal{V}_0 f_0 = f_0$. ∎

3.2.2. Regularity. Theorem 1.2 comprises two results with different conditions. We'll prove them separately as Propositions 3.1 and 3.2.

PROPOSITION 3.1. *Suppose* $\mathcal{V}_0$ *satisfies* Condition (SING | τ). *Then, on a small enough neighborhood of* $\zeta = 0$*, we have* $|f_0| \lesssim |\zeta|^{\tau-1}$.

Proof. Go back to the proof of Theorem 1.1, where we found a radius $\delta > 0$ and a constant C such that Inequality (3.1) holds whenever $|\zeta| < \delta$. The subsequent argument, which we used to show that $\lim_{\zeta\to 0}\chi = 0$, actually supports a stronger conclusion: it shows that $|\chi| \lesssim |\zeta|^{\tau}$ on the region $|\zeta| < \delta$. This tells us that

$$|f_0| \lesssim \left|\frac{1}{p}\right| |\zeta|^{\tau}$$

on the region $|\zeta| < \delta$.

Now, using Condition (SING | τ) again, choose some $\sigma > \tau$ and find a radius $r < \delta$ for which

$$\left|\frac{q(a')}{p(a)}\right| < \frac{\sigma}{|\zeta(a)|}$$

whenever $|\zeta(a)| < r$ and $|\zeta(a')| < r$. Choosing a point b' with $|\zeta(b')| < r$ and $q(b') \neq 0$, we can deduce that

$$|f_0| \lesssim \left|\frac{1}{q(b')}\right| \left|\frac{q(b')}{p}\right| |\zeta|^{\tau}$$
$$\lesssim \left|\frac{1}{q(b')}\right| \frac{\sigma}{|\zeta|} |\zeta|^{\tau}$$
$$\lesssim |\zeta|^{\tau-1}$$

on the region $|\zeta| < r$, as desired. ∎

PROPOSITION 3.2. *Suppose* $\mathcal{V}_0$ *satisfies* Conditions (SING | τ) *and* (DIAG_0 | λ_Δ). *Then* f_0 *belongs to the space* $\mathcal{H}L^{\infty}_{\tau-1,\bullet}(\Omega)$.

Proof. We want to show that $|f_0| \lesssim |\zeta|^{\tau-1} e^{\lambda_0|\zeta|}$ for some real constant λ_0.

By Proposition 3.1, we can find a radius $\delta > 0$ for which $|f_0| \lesssim |\zeta|^{\tau-1}$ over the region $|\zeta| < \delta$. No matter which value of λ_0 we pick, we know that $e^{\lambda_0|\zeta|}$ can't get arbitrarily close to zero on a bounded region, so we'll have $|f_0| \lesssim |\zeta|^{\tau-1} e^{\lambda_0|\zeta|}$ over the region $|\zeta| < \delta$.

By Condition ($\text{DIAG}_0 \mid \lambda_\Delta$), we have

$$\begin{aligned} |k_0(a, a')| &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|} \\ &\lesssim \delta^{-1} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|} \\ &\lesssim e^{\lambda_\Delta |\zeta(a) - \zeta(a')|} \end{aligned}$$

over all $a, a' \in \Omega$ with $|\zeta(a)| \geq \delta$. One consequence is that, for some $c_\Delta > 0$, we have

$$|k_0(a, a')| \leq c_\Delta e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

whenever $|\zeta(a)| \geq \delta$. Applying this bound along the diagonal in Ω^2, we learn that $|q/p| \leq c_\Delta$ on the region $|\zeta| \geq \delta$. On the other hand, by fixing some arbitrary $b' \in \Omega$, we see that

$$\begin{aligned} \left| \frac{q(b')}{p(a)} \right| &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(b')|} \\ &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta (|\zeta(a)| + |\zeta(b')|)} \\ &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a)|} e^{\lambda_\Delta |\zeta(b')|} \\ &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a)|} \end{aligned}$$

over all $a \in \Omega$ with $|\zeta(a)| \geq \delta$. Using both of the bounds we just found, we reason that

$$\begin{aligned} |f_0(a)| &= \left| \frac{1}{p(a)} \exp\left(-\int_b^a \frac{q}{p}\, d\zeta \right) \right| \\ &\leq \left| \frac{1}{p(a)} \right| \exp\left(\int_b^a \left| \frac{q}{p} \right| \, d\zeta \right) \\ &\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a)|}\, e^{c_\Delta (|\zeta(a)| + |\zeta(b)|)} \\ &\lesssim \frac{1}{|\zeta(a)|} e^{(\lambda_\Delta + c_\Delta)|\zeta(a)|}\, e^{c_\Delta |\zeta(b)|} \end{aligned}$$

over all $a \in \Omega$ with $|\zeta(a)| \geq \delta$. Setting $\lambda_0 = \lambda_\Delta + c_\Delta$, we have $|f_0| \lesssim |\zeta|^{-1} e^{\lambda_0|\zeta|}$ over the region $|\zeta| \geq \delta$. Since we're assuming τ is real and positive, $|\zeta|^\tau$ can't get arbitrarily close to zero on the region $|\zeta| > \delta$. It follows that $|f_0| \lesssim |\zeta|^{\tau-1} e^{\lambda_0|\zeta|}$ over the region $|\zeta| \geq \delta$. Combining this with our earlier argument on the region $|\zeta| < \delta$, we get the desired result. ∎

PROPOSITION 3.3. *Suppose* $\mathcal{V}_0$ *satisfies* Conditions ($\text{REG-P} \mid B, \epsilon$) *and* ($\text{SING} \mid \tau$). *Then, for some constant* M,

$$f_0 \in M\zeta^{\tau-1} + O(|\zeta|^{\tau-1+\epsilon'})$$

at $\zeta = 0$, *where* $\epsilon' = \min\{\epsilon, 1\}$.

Proof. It's enough to show that

$$f_0 \in M\zeta^{\tau-1}\Big[1 + O\big(|\zeta|^{\epsilon'}\big)\Big]$$

at $\zeta = 0$. Like we did in the proof of Theorem 1.1, we reason that

$$\begin{aligned}
f_0(a) &= \frac{1}{p(a)} \exp\left(-\int_b^a \frac{q}{p} d\zeta\right) \\
&= \frac{1}{p(a)} \frac{\zeta(a)^\tau}{\zeta(b)^\tau} \exp\left[-\int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta}\right) d\zeta\right] \\
&= \zeta(b)^{-\tau}\, \zeta(a)^{\tau-1} \frac{\zeta(a)}{p(a)} \exp\left[-\int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta}\right) d\zeta\right].
\end{aligned}$$

Since $\zeta(b)^{-\tau}$ is a constant, the $\zeta(b)^{-\tau}\, \zeta^{\tau-1}$ factor looks like what we want.

With a little work, the ζ/p factor also looks like what we want. Condition (REG-P $|$ B, ϵ) implies that

$$\frac{\zeta}{p} \in B^{-1} + O\big(|\zeta|^{\epsilon}\big),$$

at $\zeta = 0$.

Now, let's look at the exponential factor. By Condition (SING $|$ τ), we can find a constant C and a neighborhood Ω_{near} of $\zeta = 0$ for which

$$\left|\frac{q}{p} + \frac{\tau}{\zeta}\right| < C$$

in Ω_{near}. It follows that the improper integral

$$\eta(a) = \int_{\zeta=0}^a \left(\frac{q}{p} + \frac{\tau}{\zeta}\right) d\zeta$$

converges for all $a \in \Omega$, allowing us to write

$$\exp\left[-\int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta}\right) d\zeta\right] = e^{\eta(b)} e^{-\eta(a)}.$$

Observing that $|\eta| \le C|\zeta|$ in Ω_{near}, we can conclude that

$$\exp\left[-\int_b^a \left(\frac{q}{p} + \frac{\tau}{\zeta}\right) d\zeta\right] \in e^{\eta(b)}\Big[1 + O\big(|\zeta(a)|\big)\Big]$$

at $\zeta(a) = 0$.

Combining the arguments above, we learn that

$$\begin{aligned}
f_0 &\in \zeta(b)^{-\tau}\, \zeta^{\tau-1}\, \Big[B^{-1} + O\big(|\zeta|^{\epsilon}\big)\Big]\, e^{\eta(b)} \Big[1 + O\big(|\zeta|\big)\Big] \\
&= \zeta(b)^{-\tau} B^{-1} e^{\eta(b)}\, \zeta^{\tau-1} \Big[1 + O\big(|\zeta|^{\epsilon}\big)\Big]\Big[1 + O\big(|\zeta|\big)\Big] \\
&= M\zeta^{\tau-1}\Big[1 + O\big(|\zeta|^{\epsilon}\big) + O\big(|\zeta|\big)\Big]
\end{aligned}$$

at $\zeta = 0$, with $M = \zeta(b)^{-\tau} B^{-1} e^{\eta(b)}$. Observing that $O\big(|\zeta|^{\epsilon}\big) + O\big(|\zeta|\big) = O\big(|\zeta|^{\epsilon'}\big)$, we get the desired result. ∎

3.3. Showing that $\mathcal{V}_\star$ makes the prototype solution less singular *(toward Theorem 1.4)*. Now that we know the prototype solution f_0 belongs to $\mathcal{H}L^\infty_{\tau-1,\bullet}(\Omega)$, we'll show that $\mathcal{V}_\star$ reduces the sharpness of its singularity at $\zeta = 0$, mapping it into $\mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$. This is a consequence of the general smoothing effect of $\mathcal{V}_\star$, described in the following result.

Proposition 3.4. *Under* Condition ($\text{DIAG}_\star \mid \gamma, \lambda_\Delta$)*, the operator* $\mathcal{V}_\star$ *maps*

$$\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega) \to \mathcal{H}L^\infty_{\sigma+\gamma,\Lambda}(\Omega)$$

continuously for all $\Lambda \geq \lambda_\Delta$ *and* $\sigma > -1$.

Remark 3.5. We're assuming that $\gamma > 0$, but this result holds even under the weaker assumption that $\gamma > -1$. We'll take advantage of this in Section 4.4.2

Proof of Proposition 3.4. For any function $f \in \mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega)$,

$$\begin{aligned}
|\zeta(a)|^{-(\sigma+\gamma)}\, e^{-\Lambda|\zeta(a)|} \left| \big[\mathcal{V}_\star f\big](a) \right| &\leq |\zeta(a)|^{-(\sigma+\gamma)}\, e^{-\Lambda|\zeta(a)|} \int_{\zeta=0}^{a} |k_\star(a,\cdot)|\, |f|\, |d\zeta| \\
&\leq |\zeta(a)|^{-(\sigma+\gamma)}\, e^{-\Lambda|\zeta(a)|} \int_{\zeta=0}^{a} |k_\star(a,\cdot)|\, |\zeta|^\sigma\, e^{\Lambda|\zeta|} \|f\|_{\sigma,\Lambda}\, |d\zeta|
\end{aligned}$$

By Condition ($\text{DIAG}_\star \mid \gamma, \lambda_\Delta$),

$$\begin{aligned}
\int_{\zeta=0}^{a} |k_\star(a,\cdot)|\, |\zeta|^\sigma\, e^{\Lambda|\zeta|}\, |d\zeta| &\lesssim \int_{\zeta=0}^{a} \frac{|\zeta(a)-\zeta|^\gamma}{|\zeta(a)|} e^{\lambda_\Delta|\zeta(a)-\zeta|} |\zeta|^\sigma\, e^{\Lambda|\zeta|}\, |d\zeta| \\
&\lesssim |\zeta(a)|^{\gamma+\sigma} \int_0^1 (1-t)^\gamma e^{\lambda_\Delta|\zeta(a)|(1-t)} t^\sigma\, e^{\Lambda|\zeta(a)|t}\, dt \\
&\lesssim |\zeta(a)|^{\gamma+\sigma}\, e^{\lambda_\Delta|\zeta(a)|} \int_0^1 (1-t)^\gamma t^\sigma\, e^{(\Lambda-\lambda_\Delta)|\zeta(a)|t}\, dt \\
&\lesssim |\zeta(a)|^{\gamma+\sigma}\, e^{\lambda_\Delta|\zeta(a)|}\, e^{(\Lambda-\lambda_\Delta)|\zeta(a)|} \int_0^1 (1-t)^\gamma t^\sigma\, dt.
\end{aligned}$$

The last step takes advantage of the assumption that $\Lambda \geq \lambda_\Delta$. Recognizing the integral as an evaluation of the beta function, we can rewrite the bound as

$$\int_{\zeta=0}^{a} |k_\star(a,\cdot)|\, |\zeta|^\sigma\, e^{\Lambda|\zeta|}\, |d\zeta| \lesssim |\zeta(a)|^{\gamma+\sigma}\, e^{\Lambda|\zeta(a)|} \frac{\Gamma(\gamma+1)\Gamma(\sigma+1)}{\Gamma(\sigma+\gamma+2)}.$$

Our assumptions that $\gamma > 0$ and $\sigma > -1$ ensure that the gamma functions are well-defined.[5] Rearranging to get

$$|\zeta(a)|^{-(\gamma+\sigma)}\, e^{-\Lambda|\zeta(a)|} \int_{\zeta=0}^{a} |k_\star(a,\cdot)|\, |\zeta|^\sigma\, e^{\Lambda|\zeta|}\, |d\zeta| \lesssim \frac{\Gamma(\gamma+1)\Gamma(\sigma+1)}{\Gamma(\sigma+\gamma+2)},$$

we conclude that $|\zeta(a)|^{-\sigma-\gamma}\, e^{-\Lambda|\zeta(a)|} \left| \big[\mathcal{V}_\star f\big](a) \right|$ is uniformly bounded in Ω. ∎

[5] We could even weaken the constraint on γ, allowing any $\gamma > -1$.

Corollary 3.6. *Consider the prototype solution f_0 from Equation* (1.1). *If $\mathcal{V}_\star$ satisfies* Condition ($\text{DIAG}_\star \mid \gamma, \lambda_\Delta$), *then $\mathcal{V}_\star f_0$ belongs to $\mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$.*

Proof. We know from Theorem 1.2 that f_0 belongs to $\mathcal{H}L^\infty_{\tau-1,\bullet}(\Omega)$. Choose a constant $\Lambda \geq \lambda_\Delta$ big enough that f_0 is in $\mathcal{H}L^\infty_{\tau-1,\Lambda}(\Omega)$. Since we're assuming τ is positive, we can apply Proposition 3.4, concluding that $\mathcal{V}_\star f_0$ belongs to $\mathcal{H}L^\infty_{\tau-1+\gamma,\Lambda}(\Omega)$. ∎

3.4. Showing that $\mathcal{V}$ shrinks less singular functions *(toward Lemma 1.5)*.

3.4.1. Overview. In this section, we'll prove the following proposition.

Proposition 3.7. *Suppose that $\mathcal{V}_0$ satisfies* Conditions ($\text{SING} \mid \tau$) *and* ($\text{DIAG}_0 \mid \lambda_\Delta$), *and $\mathcal{V}_\star$ satisfies* Condition ($\text{DIAG}_\star \mid \gamma, \lambda_\Delta$). *Then, for each $\rho > \tau$, we can ensure that $\mathcal{V}$ is a contraction of $\mathcal{H}L^\infty_{\rho-1,\Lambda}(\Omega)$ by making Λ big enough.*

Remark 3.8. For the argument, we'll use, "big enough" always requires $\Lambda > \lambda_\Delta$, and may require $\Lambda \gg \lambda_\Delta$.

First, pick some $\sigma \in (\tau, \rho)$.[6] By Proposition 3.9, which we'll state and prove in Section 3.4.2, we can find a neighborhood $\Omega_{\text{near}} \subset \Omega$ of $\zeta = 0$ with the property that

$$|k(a, a')| \leq \frac{\sigma}{|\zeta(a)|} \tag{3.2}$$

for all $a, a' \in \Omega_{\text{near}}$. By choosing a small enough positive radius δ, we can take Ω_{near} to be the part of Ω where $|\zeta| < \delta$. Complementarily, let Ω_{far} be the part of Ω where $\delta \leq |\zeta|$.

Take any function $\varphi \in \mathcal{H}L^\infty_{\rho,\Lambda}(\Omega)$. In Section 3.4.4, we'll bound $|\zeta|^{-(\rho-1)}\, e^{-\Lambda|\zeta|}\, |\mathcal{V}\varphi|$ by $\frac{\sigma}{\rho}\|\varphi\|_{\rho-1,\Lambda}$ on Ω_{near}. In Section 3.4.5, we'll see that by making Λ big enough, we can bound $|\zeta|^{-(\rho-1)}\, e^{-\Lambda|\zeta|}\, |\mathcal{V}\varphi|$ by an arbitrarily small constant multiple of $\|\varphi\|_{\rho-1,\Lambda}$ on Ω_{far}. Together, these results show that $\|\mathcal{V}\varphi\|_{\rho-1,\Lambda} \leq \frac{\sigma}{\rho}\|\varphi\|_{\rho-1,\Lambda}$ when Λ is large enough. Since we set $\sigma < \rho$, this proves Proposition 3.7.

3.4.2. Bounds on the perturbed kernel. The conditions on k_0 and $k_\star$ described in Sections 1.3.2 and 1.3.3 can be combined into convenient bounds on the kernel $k = k_0 + k_\star$ of $\mathcal{V}$. One bound, which works when both arguments of k are close to $\zeta = 0$, will be used in Section 3.4.4.

Proposition 3.9. *Suppose $\mathcal{V}_0$ satisfies Condition* ($\text{SING} \mid \tau$), *and $\mathcal{V}_\star$ satisfies Condition* ($\text{DIAG}_\star \mid \gamma, \lambda_\Delta$). *Then, for any $\sigma > \tau$, we can ensure that*

$$|k(a, a')| \leq \frac{\sigma}{|\zeta(a)|}$$

by keeping a and a' close enough to $\zeta = 0$.

Proof. First, choose some $\sigma_0 \in (\tau, \sigma)$. Condition ($\text{SING} \mid \tau$) tells us that by making $\delta > 0$ small enough, we can ensure that

$$|k_0(a, a')| \leq \frac{\sigma_0}{|\zeta(a)|}$$

[6] If you'd like, you can fix a choice of σ—for example, taking $\sigma = \frac{1}{2}(\tau + \rho)$.

whenever $|\zeta(a)| < \delta$ and $|\zeta(a')| < \delta$. Now, we'll show that under Condition ($\text{DIAG}_\star \mid \gamma, \lambda_\Delta$), we can make $|k_\star(a, a')|$ as small as we want by keeping a and a' close to $\zeta = 0$. The condition tells us that

$$|k_\star(a, a')| \lesssim \frac{|\zeta(a) - \zeta(a')|^\gamma}{|\zeta(a)|}\, e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega$, which means that

$$|k_\star(a, a')| \lesssim \frac{(2\delta)^\gamma}{|\zeta(a)|}\, e^{\lambda_\Delta (2\delta)}$$

over all $a, a' \in \Omega$ with $|\zeta(a)| < \delta$ and $|\zeta(a')| < \delta$. Combining this conclusion with the previous one, we get the desired result. ∎

Another bound, which works when the first argument of k is kept away from $\zeta = 0$, will be used in Section 3.4.5.

Proposition 3.10. *Suppose $\mathcal{V}_0$ satisfies* Condition ($\text{DIAG}_0 \mid \lambda_\Delta$), *and $\mathcal{V}_\star$ satisfies* Condition ($\text{DIAG}_\star \mid \gamma, \lambda_\Delta$). *Choose a subset $\Omega_{far} \subset \Omega$ that doesn't touch $\zeta = 0$.*[7] *Then, for any $\lambda > \lambda_\Delta$, we have*

$$|k(a, a')| \lesssim e^{\lambda |\zeta(a) - \zeta(a')|}$$

over all $a \in \Omega_{far}$ and $a' \in \Omega$.

Proof. Find a radius $\delta > 0$ with $|\zeta| \geq \delta$ on Ω_{far}, and choose some $\lambda > \lambda_\Delta$. Conditions ($\text{DIAG}_0 \mid \lambda_\Delta$) and ($\text{DIAG}_\star \mid \gamma, \lambda_\Delta$) tell us that

$$\begin{aligned}
|k(a, a')| &\leq |k_0(a, a')| + |k_\star(a, a')| \\
&\lesssim \frac{1}{|\zeta(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|} + \frac{|\zeta(a) - \zeta(a')|^\gamma}{|\zeta(a)|}\, e^{\lambda_\Delta |\zeta(a) - \zeta(a')|} \\
&\lesssim \delta^{-1} \big(1 + |\zeta(a) - \zeta(a')|^\gamma\big)\, e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}
\end{aligned}$$

over all $a \in \Omega_{\text{far}}$ and $a' \in \Omega$. Since

$$1 + |\zeta(a) - \zeta(a')|^\gamma$$

grows polynomially with respect to $|\zeta(a) - \zeta(a')|$, we can bound it with any growing exponential function of $|\zeta(a) - \zeta(a')|$. In particular,

$$1 + |\zeta(a) - \zeta(a')|^\gamma \lesssim e^{(\lambda - \lambda_\Delta)|\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega$. It follows that

$$|k(a, a')| \lesssim e^{(\lambda - \lambda_\Delta)|\zeta(a) - \zeta(a')|}\, e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a \in \Omega_{\text{far}}$ and $a' \in \Omega$. This simplifies to the desired result. ∎

[7] A subset of a topological space *touches* the points in its closure [6, Chapter 5, Definition 2.11].

3.4.3. First steps toward showing that $\mathcal{V}$ is a contraction. The first steps of our calculation are the same throughout Ω. For each $a \in \Omega$, we have

$$|\zeta(a)|^{-(\rho-1)}\, e^{-\Lambda|\zeta(a)|}\, |[\mathcal{V}\varphi](a)| \le |\zeta(a)|^{-(\rho-1)}\, e^{-\Lambda|\zeta(a)|} \int_0^a |k(a,\cdot)\, \varphi\, d\zeta|$$

for any choice of integration path.[8] The norm on $\mathcal{H}L^\infty_{\rho-1,\Lambda}(\Omega)$ is designed to give the bound $|\varphi| \le |\zeta|^{\rho-1}\, e^{\Lambda|\zeta|}\, \|\varphi\|_{\rho-1,\Lambda}$, so

$$\begin{aligned}
|\zeta(a)|^{-(\rho-1)}\, e^{-\Lambda|\zeta(a)|}\, |[\mathcal{V}\varphi](a)| &\le |\zeta(a)|^{-(\rho-1)}\, e^{-\Lambda|\zeta(a)|} \int_0^a |k(a,\cdot)|\, |\zeta|^{\rho-1}\, e^{\Lambda|\zeta|}\, \|\varphi\|_{\rho-1,\Lambda}\, |d\zeta| \\
&= \|\varphi\|_{\rho-1,\Lambda} \int_0^a |k(a,\cdot)| \left|\frac{\zeta}{\zeta(a)}\right|^{\rho-1} e^{-\Lambda(|\zeta(a)|-|\zeta|)}\, |d\zeta|
\end{aligned}$$

What we do next depends on whether a is in Ω_{near} or Ω_{far}.

3.4.4. Near the origin. Suppose that $a \in \Omega_{\text{near}}$. Then inequality (3.2) tells us that

$$\begin{aligned}
|\zeta(a)|^{-(\rho-1)}\, e^{-\Lambda|\zeta(a)|}\, |[\mathcal{V}\varphi](a)| &\le \|\varphi\|_{\rho-1,\Lambda} \int_0^a \frac{\sigma}{|\zeta(a)|} \left|\frac{\zeta}{\zeta(a)}\right|^{\rho-1} e^{-\Lambda(|\zeta(a)|-|\zeta|)}\, |d\zeta| \\
&\le \sigma\|\varphi\|_{\rho-1,\Lambda} \int_0^a \left|\frac{\zeta}{\zeta(a)}\right|^{\rho-1} e^{-\Lambda(|\zeta(a)|-|\zeta|)} \left|\frac{d\zeta}{\zeta(a)}\right|
\end{aligned}$$

for any integration path that stays within Ω_{near}. Taking advantage of the fact that Ω_{near} is a sector of a disk, let's use the straight path $\zeta = t\zeta(a)$, with $t \in (0, 1]$.

$$\begin{aligned}
|\zeta(a)|^{-(\rho-1)}\, e^{-\Lambda|\zeta(a)|}\, |[\mathcal{V}\varphi](a)| &\le \sigma\|\varphi\|_{\rho-1,\Lambda} \int_0^1 t^{\rho-1}\, e^{-\Lambda|\zeta(a)|(1-t)}\, dt \\
&\le \sigma\|\varphi\|_{\rho-1,\Lambda} \int_0^1 t^{\rho-1}\, dt \\
&= \frac{\sigma}{\rho}\|\varphi\|_{\rho-1,\Lambda}.
\end{aligned}$$

Since we set $\sigma < \rho$, this brings us halfway to proving Proposition 3.7.

3.4.5. Away from the origin. Going back to the end of Section 3.4.3, suppose that $a \in \Omega_{\text{far}}$. Choose some $\lambda > \lambda_\Delta$. By Proposition 3.10 from Section 3.4.2, we can find a constant M for which

$$|k(a,\cdot)| \le M e^{\lambda|\zeta(a)-\zeta|}$$

for all $a \in \Omega_{\text{far}}$. Note that only the first argument of k has its domain restricted; this bound holds throughout Ω in the second argument. Applying this bound, we learn that

$$|\zeta(a)|^{-(\rho-1)}\, e^{-\Lambda|\zeta(a)|}\, |[\mathcal{V}\varphi](a)| \le \|\varphi\|_{\rho-1,\Lambda} \int_0^a M e^{\lambda|\zeta(a)-\zeta|} \left|\frac{\zeta}{\zeta(a)}\right|^{\rho-1} e^{-\Lambda(|\zeta(a)|-|\zeta|)}\, |d\zeta|.$$

[8]The absolute value of a 1-form, like $|k(a,\cdot)\, \varphi\, d\zeta|$, is a density on $\mathbb{C}$—a norm on tangent vectors.

Let's again use the straight integration path $\zeta = t\zeta(a)$, with $t \in (0, 1]$. Along this path, $|\zeta(a) - \zeta| = |\zeta(a)| - |\zeta|$, allowing us to combine the exponential factors in our bound:

$$\begin{aligned}
|\zeta(a)|^{-(\rho-1)}\, e^{-\Lambda|\zeta(a)|}\, |[\mathcal{V}\varphi](a)| &\leq \|\varphi\|_{\rho-1,\Lambda} \int_0^1 M e^{\lambda|\zeta(a)|(1-t)}\, t^{\rho-1}\, e^{-\Lambda|\zeta(a)|(1-t)}\, dt \\
&\leq M\|\varphi\|_{\rho-1,\Lambda} \int_0^1 t^{\rho-1}\, e^{-(\Lambda-\lambda)|\zeta(a)|(1-t)}\, dt.
\end{aligned}$$

Let's set $\Lambda > \lambda$, ensuring that the exponential factor shrinks as $|\zeta(a)|$ grows. Then we can make our bound uniform over all $a \in \Omega_{\text{far}}$, since $|\zeta(a)| \geq \delta$ for these points:

$$|\zeta(a)|^{-(\rho-1)}\, e^{-\Lambda|\zeta(a)|}\, |[\mathcal{V}\varphi](a)| \leq M\|\varphi\|_{\rho-1,\Lambda} \int_0^1 t^{\rho-1}\, e^{-(\Lambda-\lambda)\delta(1-t)}\, dt.$$

We can make this bound less than one, as required to show that $\mathcal{V}$ is a contraction, by increasing Λ. To see this in the trickiest case, where $\rho < 1$, it helps to look at the beginning and end of the integration path separately. At the beginning of the path—for $t \in \left(0, \frac{1}{5}\right]$, say—we can use a worst-case estimate on the exponential factor:

$$\begin{aligned}
\int_0^{1/5} t^{\rho-1}\, e^{-(\Lambda-\lambda)\delta(1-t)}\, dt &\leq e^{-\frac{4}{5}(\Lambda-\lambda)\delta} \int_0^{1/5} t^{\rho-1}\, dt \\
&= e^{-\frac{4}{5}(\Lambda-\lambda)\delta}\, \tfrac{1}{\rho}\left(\tfrac{1}{5}\right)^{\rho}.
\end{aligned}$$

At the end of the path, we instead use a worst-case estimate on $t^{\rho-1}$:

$$\begin{aligned}
\int_{1/5}^1 t^{\rho-1}\, e^{-(\Lambda-\lambda)\delta(1-t)}\, dt &\leq \max\{(1/5)^{\rho-1}, 1\} \int_{1/5}^1 e^{-(\Lambda-\lambda)\delta(1-t)}\, dt \\
&= \max\{(1/5)^{\rho-1}, 1\} \frac{1}{(\Lambda-\lambda)\delta}\left[1 - e^{-\frac{4}{5}(\Lambda-\lambda)\delta}\right] \\
&\leq \max\{(1/5)^{\rho-1}, 1\} \frac{1}{(\Lambda-\lambda)\delta}.
\end{aligned}$$

In summary, the beginning of the integral is bounded by a decaying exponential function of $(\Lambda - \lambda)\delta$, and the end of the integral is bounded by a reciprocal function of $(\Lambda - \lambda)\delta$. That means we can make $|\zeta(a)|^{-(\rho-1)}\, e^{-\Lambda|\zeta(a)|}\, |[\mathcal{V}\varphi](a)|$ as small as we want over all $a \in \Omega_{\text{far}}$. This completes our proof of Proposition 3.7.

3.5. Existence and uniqueness of a fixed point *(proof of Lemma 1.5 and Theorem 1.4)*.

Proof of Lemma 1.5. Choose Λ large enough to ensure that $\mathcal{V}$ is a contraction of $\mathcal{H}L^\infty_{\rho-1,\Lambda}(\Omega)$ and g belongs to $\mathcal{H}L^\infty_{\rho-1,\Lambda}(\Omega)$. Proposition 3.7 guarantees that we can do the former, given our assumptions about $\mathcal{V}$ and ρ, and the definition of $\mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$ guarantees that we can also do the latter. Our choice of Λ ensures that the affine map $f \mapsto \mathcal{V}f + g$ is also a contraction of $\mathcal{H}L^\infty_{\rho-1,\Lambda}(\Omega)$, and thus has a unique fixed point in $\mathcal{H}L^\infty_{\rho-1,\Lambda}(\Omega)$ by the contraction mapping theorem.

To see that the fixed point is still unique in the bigger space $\mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$, first recall from Proposition 2.3 that we have inclusions $\mathcal{H}L^\infty_{\rho-1,\Lambda'}(\Omega) \hookrightarrow \mathcal{H}L^\infty_{\rho-1,\Lambda}(\Omega)$ for all $\Lambda' \leq \Lambda$.

Any fixed point in $\mathcal{H}L^\infty_{\rho-1,\Lambda'}(\Omega)$ must map to the unique fixed point in $\mathcal{H}L^\infty_{\rho-1,\Lambda}(\Omega)$ under this inclusion. Next, observe that for any $\Lambda'' \geq \Lambda$, the map $f \mapsto \mathcal{V}f + g$ is also a contraction of $\mathcal{H}L^\infty_{\rho-1,\Lambda''}(\Omega)$. The inclusion $\mathcal{H}L^\infty_{\rho-1,\Lambda}(\Omega) \hookrightarrow \mathcal{H}L^\infty_{\rho-1,\Lambda''}(\Omega)$ must send the unique fixed point in the smaller space to the unique fixed point in the larger one. Together, these arguments show that the fixed point is unique in $\mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$. ∎

Proof of Theorem 1.4. Start with the prototype solution f_0 constructed in Theorem 1.1, which satisfies the equation

$$f_0 = \mathcal{V}_0 f_0.$$

The base point for the construction can be chosen arbitrarily. Under our assumptions about $\mathcal{V}$, Theorem 1.2 tells us that f_0 is in $\mathcal{H}L^\infty_{\tau-1,\bullet}(\Omega)$. Our goal is to find a perturbation $f_\star \in \mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$ that makes $f = f_0 + f_\star$ a solution of

$$f = \mathcal{V}f. \tag{3.3}$$

Observing that

$$\begin{aligned}\mathcal{V}f_0 &= \mathcal{V}_0 f_0 + \mathcal{V}_\star f_0\\ &= f_0 + \mathcal{V}_\star f_0,\end{aligned}$$

we can rewrite the homogeneous equation we're trying to solve as an inhomogeneous equation for $f_\star$:

$$\begin{aligned}f_0 + f_\star &= \mathcal{V}f_0 + \mathcal{V}f_\star\\ &= f_0 + \mathcal{V}_\star f_0 + \mathcal{V}f_\star\\ f_\star &= \mathcal{V}_\star f_0 + \mathcal{V}f_\star. \qquad (3.4)\end{aligned}$$

Since we know that f_0 is in $\mathcal{H}L^\infty_{\tau-1,\bullet}(\Omega)$, Proposition 3.4 tells us that the inhomogeneous term $\mathcal{V}_\star f_0$ is in $\mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$. Since $\tau + \gamma > \tau$, and we've made all the necessary assumptions about $\mathcal{V}$, Lemma 1.5 guarantees that Equation (3.4) has a unique solution $f_\star$ in $\mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$. Equivalently, Equation (3.3) has a unique solution f in $f_0 + \mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$.

Now, we just want to show that the uniqueness of the solution still holds in $f_0 + \mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$ for any $\rho > \tau$.

First, suppose $\rho \in (\tau, \tau+\gamma)$. In this case, we can use the inclusion $\mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega) \hookrightarrow \mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$ given by Proposition 2.5 to see that the inhomogeneous term $\mathcal{V}_\star f_0$ is in $\mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$. Lemma 1.5 then guarantees that Equation (3.4) has a unique solution in $\mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$—which must be the solution we already found in $\mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$.

On the other hand, suppose $\rho > \tau + \gamma$. In this case, Proposition 2.5 gives an inclusion $\mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega) \hookrightarrow \mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$. Under this inclusion, any solution of Equation (3.4) that we might find in $\mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$ must match the unique solution we found in $\mathcal{H}L^\infty_{\tau-1+\gamma,\bullet}(\Omega)$. ∎

Proposition 3.11. *Suppose $\mathcal{V}$ satisfies* Condition (REG-P | B, ϵ) *in addition to the other conditions we assume in* Theorem 1.4. *Then, as long as $\rho > \tau$ is low enough, the equation*

$$f = \mathcal{V}f$$

has a unique solution f in the affine subspace

$$\zeta^{\tau-1} + \mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$$

of the space $\mathcal{H}L^\infty_{\tau-1,\bullet}(\Omega)$. To be precise, ρ is low enough when it's in the interval

$$(\tau, \tau + \min\{\gamma, \epsilon, 1\}].$$

Proof. Propositions 3.2 and 3.3 imply that for some constant M,

$$\zeta^{\tau-1} + \mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega) = M^{-1} f_0 + \mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$$

whenever $\rho \leq \tau + \min\{\epsilon, 1\}$. Theorem 1.4 implies that our equation has a unique solution in

$$M^{-1} f_0 + \mathcal{H}L^\infty_{\rho-1,\bullet}(\Omega)$$

as long as $\rho \in (\tau, \tau + \gamma]$. Putting these facts together, we get the desired result. ∎

4. A motivating example.

4.1. Overview. We can use our understanding of regular singular Volterra equations to study so-called *level 1 differential equations*, which each have an irregular singularity at ∞. In particular, we can build a set of analytic solutions in the frequency domain by solving certain regular singular Volterra equations in the position domain. Our main results guarantee that these position domain solutions exist, are unique, and have well-defined Laplace transforms.

4.2. Level 1 differential equations. Let $\mathcal{P}$ be a linear differential operator of the form

$$\mathcal{P} = P(\partial_z) + \frac{1}{z} Q(\partial_z) + \frac{1}{z^2} R(z^{-1}),$$

where

- P is a monic degree-d polynomial whose roots are all simple;
- Q is a degree-$(d-1)$ polynomial that's non-zero at every root of P;
- $R(z^{-1})$ is holomorphic in some disk $|z| > A$ around $z = \infty$. In particular, the power series

  $$R(z^{-1}) = \sum_{j=0}^{\infty} R_j z^{-j}$$

  converges in the region $|z| > A$.

Equations of the form $\mathcal{P}\Phi = 0$ are a sub-class of what Ecalle calls *level 1* differential equations [3, Section 2.1][2, Section 5.2.2.1]. In an upcoming paper [4], we'll study various examples of such equations using Laplace transform methods, with the help of our existence and uniqueness result Theorem 1.4.

4.3. Notation. So far, we've studied a Volterra equation with a regular singularity at $\zeta = 0$. When we use Laplace transform methods to solve a level 1 differential equation $\mathcal{P}\Phi = 0$ in the frequency domain, we'll end up with several Volterra equations $\hat{\mathcal{P}}_\alpha \varphi = 0$ in the position domain, each with a regular singularity at a different root $\zeta = \alpha$ of the polynomial $P(-\zeta)$.

To adapt our previous reasoning to this more general situation, we'll reinterpret the language of Sections 1–3 whenever we refer to it Section 4. The role of the coordinate ζ in Sections 1–3 will now be played by $\zeta - \alpha$. This substitution leads to several other reinterpretations, summarized below.

References to the point $\zeta = 0$ become references to the point $\zeta - \alpha = 0$, which we'll rewrite as $\zeta = \alpha$. For example, we'll now work on a domain that touches but doesn't contain $\zeta = \alpha$, introducing the notation Ω_α as a reminder of this change. Condition (STAR) now says that Ω_α is star-shaped around $\zeta = \alpha$. The function space $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_\alpha)$ becomes a space of holomorphic functions on Ω_α which have a power-law singularity at $\zeta = \alpha$ and are uniformly of exponential type Λ. Explicitly, f is in $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_\alpha)$ if

$$|f| \lesssim |\zeta - \alpha|^\sigma e^{\Lambda|\zeta-\alpha|}$$

over Ω_α, making the norm

$$\|f\|_{\sigma,\Lambda} = \sup_{\Omega_\alpha} |\zeta - \alpha|^{-\sigma} e^{-\Lambda|\zeta-\alpha|} |f|$$

well-defined.

Our conditions on Volterra operators change to describe a regular singularity at $\zeta = \alpha$. For example, consider a Volterra operator $\mathcal{V}_0^\alpha$ of the kind described in Section 1.3.2, with kernel k_0^α. This operator now satisfies Condition (SING $|\ \tau$) if for some real, positive constant τ_α, the difference

$$k_0^\alpha(a,a) - \frac{\tau_\alpha}{\zeta(a) - \alpha}$$

is bounded on a neighborhood of $\zeta(a) = \alpha$ in Ω_α, and for each $\sigma > \tau_\alpha$, the bound

$$|k_0^\alpha(a,a')| < \frac{\sigma}{|\zeta(a) - \alpha|}$$

holds over some neighborhood of $\big(\zeta(a), \zeta(a')\big) = (\alpha, \alpha)$ in Ω_α^2. Condition (DIAG$_0$ $|\ \lambda_\Delta$) now requires that for some constant λ_Δ,

$$|k_0^\alpha(a,a')| \lesssim \frac{1}{|\zeta(a) - \alpha|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega_\alpha$. Similarly, an operator $\mathcal{V}_\star^\alpha$ with kernel $k_\star^\alpha$ now satisfies Condition (DIAG$_\star$ $|\ \gamma, \lambda_\Delta$) if for some constants $\gamma > 0$ and λ_Δ, we have

$$|k_\star^\alpha(a,a')| \lesssim \frac{|\zeta(a) - \zeta(a')|^\gamma}{|\zeta(a) - \alpha|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega_\alpha$.

4.4. The Laplace transform.

4.4.1. Definition. Let Ω_α be a simply connected open subset of $\mathbb{C}$ that touches but doesn't contain $\zeta = \alpha$, and let $\Gamma^\theta_{\zeta,\alpha}$ be the ray that leaves $\zeta = \alpha$ at angle θ. When Ω_α contains $\Gamma^\theta_{\zeta,\alpha}$, we can define the Laplace transform $\mathcal{L}^\theta_{\zeta,\alpha}$, which maps functions in $\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_\alpha)$ with $\sigma > -1$ and $\Lambda \in \mathbb{R}$ to holomorphic functions on the half-plane $\operatorname{Re}(ze^{-i\theta}) > \Lambda$ in the frequency domain [10, Section 5.6]. The Laplace transform of φ is defined by the formula

$$\mathcal{L}^\theta_{\zeta,\alpha}\varphi := \int_{\Gamma^\theta_{\zeta,\alpha}} e^{-z\zeta}\varphi \, d\zeta. \tag{4.1}$$

It's a function on the frequency domain because it depends on the frequency variable z.

4.4.2. Action on integral operators. For each $\nu \in (0, \infty)$, the fractional integral $\partial_{\zeta,\alpha}^{-\nu}$ is the Volterra operator defined by

$$[\partial_{\zeta,\alpha}^{-\nu}\varphi](a) := \frac{1}{\Gamma(\nu)} \int_{\zeta=\alpha}^{a} \big(\zeta(a) - \zeta\big)^{\nu-1} \varphi \, d\zeta$$

for each $a \in \Omega_\alpha$. It obeys the expected semigroup law [11, Section 1.3]

$$\partial_{\zeta,\alpha}^{-\mu}\,\partial_{\zeta,\alpha}^{-\nu} = \partial_{\zeta,\alpha}^{-\mu-\nu} \qquad\qquad \mu, \nu \in (0, \infty),$$

and agrees with ordinary repeated integration when ν is an integer [11, Equation 35].

Fractional integration has a smoothing effect: it reduces the sharpness of any power-law singularity at $\zeta = \alpha$.

Proposition 4.1. *For each $\nu \in (0, \infty)$, the fractional integral $\partial_{\zeta,\alpha}^{-\nu}$ maps*

$$\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_\alpha) \to \mathcal{H}L^\infty_{\sigma+\nu,\Lambda}(\Omega_\alpha)$$

continuously for all $\sigma > -1$ and $\Lambda \geq 0$.

Proof. Rewrite the fractional integral as

$$\begin{aligned}\left[\partial_{\zeta,\alpha}^{-\nu}\varphi\right](a) &= \frac{1}{\Gamma(\nu)} \int_{\zeta=\alpha}^{a} (\zeta(a) - \zeta)^{\nu-1} \varphi \, d\zeta \\ &= \frac{\zeta(a) - \alpha}{\Gamma(\nu)} \int_{\zeta=\alpha}^{a} \frac{(\zeta(a) - \zeta)^{\nu-1}}{\zeta(a) - \alpha} \varphi \, d\zeta.\end{aligned}$$

The Volterra operator with kernel

$$h(a, a') = \frac{1}{\Gamma(\nu)} \frac{(\zeta(a) - \zeta(a'))^{\nu-1}}{\zeta(a) - \alpha}$$

satisfies Condition ($\text{DIAG}_\star \mid \gamma, \lambda_\Delta$) with $\gamma = \nu - 1$ and $\lambda_\Delta = 0$, as long as we loosen the condition to allow any $\gamma > -1$. Hence, by Proposition 3.4, this operator maps

$$\mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_\alpha) \to \mathcal{H}L^\infty_{\sigma+\nu-1,\Lambda}(\Omega_\alpha)$$

continuously for all $\sigma > -1$ and $\Lambda \geq 0$. Multiplication by $\zeta - \alpha$ then maps

$$\mathcal{H}L^\infty_{\sigma+\nu-1,\Lambda}(\Omega_\alpha) \to \mathcal{H}L^\infty_{\sigma+\nu,\Lambda}(\Omega_\alpha)$$

continuously. ∎

From Proposition 4.1, we deduce that when φ belongs to $\mathcal{H}L^\infty_{\sigma,\bullet}(\Omega_\alpha)$ for some $\sigma > -1$, its fractional integral $\partial_{\zeta,\alpha}^{-\nu}\varphi$ has a well-defined Laplace transform along any ray $\Gamma^\theta_{\zeta,\alpha} \subset \Omega_\alpha$. Using a 2d integration argument, akin to the one in [12, Theorem 2.39], one can show that

$$\mathcal{L}^\theta_{\zeta,\alpha}\,\partial_{\zeta,\alpha}^{-\nu}\varphi = z^{-\nu}\mathcal{L}^\theta_{\zeta,\alpha}\varphi$$

for all $\nu \in (0, \infty)$ and $\varphi \in \mathcal{H}L^\infty_{\sigma,\Lambda}(\Omega_\alpha)$ with $\sigma > -1$. This mirrors the Laplace transform's action on multiplication operators: under the same conditions on φ, you can use differentiation under the integral to show that [12, Theorem 1.34]

$$\mathcal{L}^\theta_{\zeta,\alpha}(\zeta^n\varphi) = \left(-\tfrac{\partial}{\partial z}\right)^n \mathcal{L}^\theta_{\zeta,\alpha}\varphi$$

for all integers $n \geq 0$ and suitable directions θ.

Remark 4.2. In the relationship between $\partial_{\zeta,\alpha}^{-\nu}$ and multiplication by $z^{-\nu}$ described above, smoothing out the singularity at $\zeta = \alpha$ in the position domain corresponds to speeding up the decay as $|z| \to \infty$ in the frequency domain. This is a general feature of the Laplace transform.

4.5. Going to the position domain. Using the properties of the Laplace transform $\mathcal{L}_{\zeta,\alpha}^{\theta}$ we can turn differential operators in the frequency domain into Volterra operators in the position domain.

LEMMA 4.3. *For each root $-\alpha$ of P, let $\hat{\mathcal{P}}_\alpha$ be the Volterra operator*

$$\hat{\mathcal{P}}_\alpha := P(-\zeta) + \partial_{\zeta,\alpha}^{-1} \circ Q(-\zeta) + \partial_{\zeta,\alpha}^{-2} \circ R(\partial_{\zeta,\alpha}^{-1}).$$

If ψ_α satisfies the equation $\hat{\mathcal{P}}_\alpha \psi_\alpha = 0$, then its Laplace transform $\Psi_\alpha := \mathcal{L}_{\zeta,\alpha}^{\theta} \psi_\alpha$ satisfies the equation $\mathcal{P}\Psi_\alpha = 0$, as long as the Laplace transform is well-defined.

Remark 4.4. For the Laplace transform to have any hope of being well-defined, we have to choose the direction θ so that the ray $\Gamma_{\zeta,\alpha}^{\theta}$ stays within Ω_α. We'll see later that for our results to apply, Ω_α can't touch any zero of $P(-\zeta)$ other than $\zeta = \alpha$. As a result, Ω_α might look like the domain illustrated in Section 1.3.1, reinterpreting the origin as $\zeta = \alpha$.

Proof of Lemma 4.3. Comparing the definitions of $\mathcal{P}$ and $\hat{\mathcal{P}}_\alpha$, and using the properties of the Laplace transform discussed in Section 4.4.2, we can work out that $\mathcal{P}\mathcal{L}_{\zeta,\alpha}^{\theta} = \mathcal{L}_{\zeta,\alpha}^{\theta}\hat{\mathcal{P}}_\alpha$. Hence,

$$\begin{aligned} \mathcal{P}\Psi_\alpha &= \mathcal{P}\mathcal{L}_{\zeta,\alpha}^{\theta}\psi_\alpha \\ &= \mathcal{L}_{\zeta,\alpha}^{\theta}\hat{\mathcal{P}}_\alpha\psi_\alpha. \end{aligned}$$

□

We can now state and prove the main result of this section. Choose a simply connected open set Ω_α that touches $\zeta = \alpha$ and doesn't touch any other roots of $P(-\zeta)$, as illustrated in Section 1.3.1 under the reinterpretation from Section 4.3. We'll show that the Volterra equation $\hat{\mathcal{P}}_\alpha \varphi = 0$ has a unique solution of a certain form on Ω_α. By Lemma 4.3, the Laplace transform of this solution satisfies the equation $\mathcal{P}\Phi = 0$.

THEOREM 4.5. *Consider a root $-\alpha$ of P where $\tau_\alpha := Q(-\alpha)/P'(-\alpha)$ is real and positive. Choose a simply connected open set Ω_α that touches $\zeta = \alpha$, and doesn't touch any other root of $P(-\zeta)$. Then the equation*

$$\hat{\mathcal{P}}_\alpha \psi_\alpha = 0$$

has a unique solution ψ_α in the affine subspace

$$\zeta^{\tau_\alpha - 1} + \mathcal{H}L^{\infty}_{\tau_\alpha,\bullet}(\Omega_\alpha)$$

of the space $\mathcal{H}L^{\infty}_{\tau_\alpha - 1,\bullet}(\Omega_\alpha)$.

Proof of Theorem 4.5. Choose a root $-\alpha$ of P where $\tau_\alpha := Q(-\alpha)/P'(-\alpha)$ is real and positive. For convenience, let $p = P(-\zeta)$ and $q = Q(-\zeta)$, noting that $\zeta = \alpha$ is a root of p. The conditions laid out in Section 4.2 guarantee that $\frac{\partial}{\partial \zeta} p$ and q are both non-zero $\zeta = \alpha$.

To apply Proposition 3.11, we need to rewrite the equation $\hat{\mathcal{P}}_\alpha \varphi = 0$ in the form $\varphi = \mathcal{V}^\alpha \varphi$, where $\mathcal{V}^\alpha$ is a Volterra operator of the kind described in Section 1.3 under the reinterpretation from Section 4.3. When we write $\hat{\mathcal{P}}_\alpha$ explicitly as a Volterra operator, the equation $\hat{\mathcal{P}}_\alpha \varphi = 0$ becomes

$$\big[p\,\varphi\big](a) + \int_{\zeta=\alpha}^{a} q\,\varphi\,d\zeta + \int_{\zeta=\alpha}^{a} k_R(a,\cdot)\,\varphi\,d\zeta = 0,$$

where

$$k_R(a,a') = \sum_{j=0}^{\infty} \frac{R_j}{j!}\,\big(\zeta(a) - \zeta(a')\big)^{j+1}.$$

Isolating $p\,\varphi$ on the left and dividing both sides by p yields the equivalent equation

$$\varphi(a) = -\int_{\zeta=\alpha}^{a} \frac{q}{p(a)}\,\varphi\,d\zeta - \int_{\zeta=\alpha}^{a} \frac{k_R(a,\cdot)}{p(a)}\,\varphi\,d\zeta. \tag{4.2}$$

Let $\mathcal{V}_0^\alpha$ be the Volterra operator with kernel

$$k_0^\alpha(a,a') = -\frac{q(a')}{p(a)},$$

and let $\mathcal{V}_\star^\alpha$ be the Volterra operator with kernel

$$k_\star^\alpha(a,a') = -\frac{k_R(a,a')}{p(a)}.$$

Now we can write Equation (4.2) as $\varphi = \mathcal{V}^\alpha \varphi$, where $\mathcal{V}^\alpha = \mathcal{V}_0^\alpha + \mathcal{V}_\star^\alpha$.

The kernel of $\mathcal{V}^\alpha$ extends meromorphically over all of $\mathbb{C}^2$. One part of this observation demands some explanation: we need to confirm that the power series defining k_R converges everywhere. To do this, recall that $R(z^{-1})$ is holomorphic on the disk $|z| > A$. Consequently, for any $\lambda > A$, we have $|R_j| \lesssim \lambda^j$ over all $j \in \{0, 1, 2, \ldots\}$. This is enough to show that k_R is well-defined throughout $\mathbb{C}^2$. As we'll see later, it also leads to a useful bound on k_R.

Since $\mathcal{V}_0^\alpha$ has a separable kernel of the form described in Section 1.3.2, we're now working in the setting of Proposition 3.11. To apply the theorem, we just need to show that $\mathcal{V}_0^\alpha$ and $\mathcal{V}_\star^\alpha$ satisfy the required conditions from Section 1.3.

- Condition (SING | τ) *on* $\mathcal{V}_0^\alpha$ *is satisfied.*

  Since P is a monic polynomial whose roots are all simple, $1/P(-\zeta)$ has the nice partial fraction decomposition

  $$\frac{1}{P(-\zeta)} = \sum_{-\beta \in \mathfrak{B}} \frac{1}{-P'(-\beta)\,(\zeta - \beta)},$$

  where $\mathfrak{B}$ is the zero set of P [9, Section 1.4, Exercise 2]. It follows that

  $$\begin{aligned} -\frac{Q\big(-\zeta(a')\big)}{P\big(-\zeta(a)\big)} &= \sum_{-\beta \in \mathfrak{B}} \frac{Q\big(-\zeta(a')\big)}{P'(-\beta)\,\big(\zeta(a) - \beta\big)} \\ &= \sum_{\substack{-\beta \in \mathfrak{B} \\ \beta \neq \alpha}} \frac{Q\big(-\zeta(a')\big)}{P'(-\beta)\,(\zeta(a) - \beta)} + \frac{Q\big(-\zeta(a')\big)}{P'(-\alpha)\,(\zeta(a) - \alpha)}. \end{aligned}$$

Now we expand Q around $-\alpha$, using the decomposition $Q(\zeta) = Q(-\alpha) + (\zeta - \alpha)\,Q_\alpha(\zeta)$ to split up the last term of the expression above:

(4.3)
$$-\frac{Q\big(-\zeta(a')\big)}{P\big(-\zeta(a)\big)} = \sum_{\substack{-\beta\in\mathfrak{B}\\ \beta\neq\alpha}} \frac{Q\big(-\zeta(a')\big)}{P'(-\beta)\,\big(\zeta(a)-\beta\big)} + \frac{\big(\zeta(a')-\alpha\big)\,Q_\alpha\big(\zeta(a')\big)}{P'(-\alpha)\,\big(\zeta(a)-\alpha\big)} + \frac{Q(-\alpha)}{P'(-\alpha)\,\big(\zeta(a)-\alpha\big)}.$$

First, we want to show that $-q/p - \tau_\alpha/(\zeta-\alpha)$ is bounded on a neighborhood of $\zeta = \alpha$ in Ω_α. Equation (4.3) tells us that

$$\begin{aligned}
-\frac{Q(-\zeta)}{P(-\zeta)} &= \sum_{\substack{-\beta\in\mathfrak{B}\\ \beta\neq\alpha}} \frac{Q(-\zeta)}{P'(-\beta)\,(\zeta-\beta)} + \frac{(\zeta-\alpha)\,Q_\alpha(\zeta)}{P'(-\alpha)\,(\zeta-\alpha)} + \frac{Q(-\alpha)}{P'(-\alpha)\,(\zeta-\alpha)} \\
&= \sum_{\substack{-\beta\in\mathfrak{B}\\ \beta\neq\alpha}} \frac{Q(-\zeta)}{P'(-\beta)\,(\zeta-\beta)} + \frac{Q_\alpha(\zeta)}{P'(-\alpha)} + \frac{Q(-\alpha)}{P'(-\alpha)\,(\zeta-\alpha)}
\end{aligned}$$

Recalling that $\tau_\alpha = Q(-\alpha)/P'(-\alpha)$, we conclude that

$$-\frac{Q(-\zeta)}{P(-\zeta)} - \frac{\tau_\alpha}{\zeta-\alpha} = \sum_{\substack{-\beta\in\mathfrak{B}\\ \beta\neq\alpha}} \frac{Q(-\zeta)}{P'(-\beta)\,(\zeta-\beta)} + \frac{Q_\alpha(\zeta)}{P'(-\alpha)}$$

Since Q and Q_α are polynomials, the right-hand side is bounded on any neighborhood of $\zeta = \alpha$ that avoids the other roots of p and avoids infinity.

Next, we want to show that for any $\sigma > \tau_\alpha$, the bound

$$|\zeta(a)-\alpha|\left|\frac{q(a')}{p(a)}\right| < \sigma$$

holds when a and a' are close enough to $\zeta = \alpha$. Taking the absolute value of both sides of Equation (4.3), and using the assumption that τ_α is real and positive, we see that

$$\begin{aligned}
\left|\frac{Q\big(-\zeta(a')\big)}{P\big(-\zeta(a)\big)}\right| &\le \sum_{\substack{-\beta\in\mathfrak{B}\\ \beta\neq\alpha}} \left|\frac{Q\big(-\zeta(a')\big)}{P'(-\beta)\,\big(\zeta(a)-\beta\big)}\right| + \left|\frac{\big(\zeta(a')-\alpha\big)\,Q_\alpha\big(\zeta(a')\big)}{P'(-\alpha)\,\big(\zeta(a)-\alpha\big)}\right| + \left|\frac{\tau_\alpha}{\zeta(a)-\alpha}\right| \\
|\zeta(a)-\alpha|\left|\frac{Q\big(-\zeta(a')\big)}{P\big(-\zeta(a)\big)}\right| &\le \sum_{\substack{-\beta\in\mathfrak{B}\\ \beta\neq\alpha}} \left|\frac{Q\big(-\zeta(a')\big)\,\big(\zeta(a)-\alpha\big)}{P'(-\beta)\,\big(\zeta(a)-\beta\big)}\right| + \left|\frac{\big(\zeta(a')-\alpha\big)\,Q_\alpha\big(\zeta(a')\big)}{P'(-\alpha)}\right| + \tau_\alpha.
\end{aligned}$$

By bringing $\zeta(a')$ closer to α, we can make the middle term

$$\left|\frac{\big(\zeta(a')-\alpha\big)\,Q_\alpha\big(\zeta(a')\big)}{P'(-\alpha)}\right|$$

as small as we want. Similarly, by bringing $\zeta(a)$ closer to α, we can make the sum

$$\sum_{\substack{-\beta\in\mathfrak{B}\\ \beta\neq\alpha}} \left|\frac{Q\big(-\zeta(a')\big)\,\big(\zeta(a)-\alpha\big)}{P'(-\beta)\,\big(\zeta(a)-\beta\big)}\right|$$

as small as we want. Hence, for any $\sigma > \tau_\alpha$, the bound

$$|\zeta(a) - \alpha| \left| \frac{Q\big(-\zeta(a')\big)}{P\big(-\zeta(a)\big)} \right| \leq \sigma$$

holds over some neighborhood of $\big(\zeta(a), \zeta(a')\big) = (\alpha, \alpha)$ in Ω_α^2.

- Condition ($\text{DIAG}_0 \mid \lambda_\Delta$) *on* $\mathcal{V}_0^\alpha$ *is satisfied.*
  Since we're trying to bound $\big(\zeta(a) - \alpha\big)\, k_0^\alpha(a, a')$ with a function of the difference $\omega(a, a') := \zeta(a) - \zeta(a')$, let's rewrite it as a rational function of $\omega(a, a')$ and $\zeta(a)$. First, rewrite

$$\begin{aligned} q(a') &= Q\big(-\zeta(a')\big) \\ &= Q\big(\omega(a, a') - \zeta(a)\big) \end{aligned}$$

  in the form

$$q(a') = Q_{d-1}\big(\zeta(a)\big)\, \omega(a, a')^{d-1} + \ldots + Q_1\big(\zeta(a)\big)\, \omega(a, a') + Q_0\big(\zeta(a)\big),$$

  where $Q_0, \ldots Q_{d-1}$ are polynomials of degree at most $d-1$. Next, knowing that $-\alpha$ is a root of p, rewrite p in the form $(\zeta - \alpha)\, P_\alpha(\zeta)$, where P_α is a polynomial of degree $d - 1$. We can then write $-\big(\zeta(a) - \alpha\big)\, k_0^\alpha(a, a')$ in the form

$$\big(\zeta(a) - \alpha\big) \frac{q(a')}{p(a)} = \frac{Q_{d-1}\big(\zeta(a)\big)}{P_\alpha\big(\zeta(a)\big)}\, \omega(a, a')^{d-1} + \ldots + \frac{Q_1\big(\zeta(a)\big)}{P_\alpha\big(\zeta(a)\big)}\, \omega(a, a') + \frac{Q_0\big(\zeta(a)\big)}{P_\alpha\big(\zeta(a)\big)},$$

  viewing it as a polynomial in $\omega(a, a')$ whose coefficients are rational functions in $\zeta(a)$. If we keep a away from the roots of p other than $\zeta = \alpha$, each coefficient is bounded, so $|\zeta(a) - \alpha|\, |k_0^\alpha(a, a')|$ is bounded by a polynomial in $|\omega(a, a')|$. It follows that for any $\lambda_\Delta > 0$, and any domain Ω_α that avoids all the roots of p other than $\zeta = \alpha$, we have

$$|\zeta(a) - \alpha|\, |k_0^\alpha(a, a')| \lesssim e^{\lambda_\Delta |\omega(a, a')|}$$

  over all $a, a' \in \Omega_\alpha$.
  For consistency with Condition ($\text{DIAG}_\star \mid \gamma, \lambda_\Delta$), we choose $\lambda_\Delta > A$.
- Condition ($\text{REG-P} \mid B, \epsilon$) *on* $\mathcal{V}_0^\alpha$ *is satisfied.*
  This is true because p is a polynomial.
- Condition ($\text{DIAG}_\star \mid \gamma, \lambda_\Delta$) *on* $\mathcal{V}_\star^\alpha$ *is satisfied.*
  First, obseve that

$$\begin{aligned} |k_\star^\alpha(a, a')| &= \left| \frac{k_R(a, a')}{p(a)} \right| \\ &\leq \frac{1}{|p(a)|} \sum_{j=0}^{\infty} \frac{|R_j|}{j!}\, |\zeta(a) - \zeta(a')|^{j+1} \\ &= \frac{|\zeta(a) - \zeta(a')|}{|p(a)|} \sum_{j=0}^{\infty} \frac{|R_j|}{j!}\, |\zeta(a) - \zeta(a')|^{j}. \end{aligned}$$

Now, choose $\lambda_\Delta > A$. Since $R(z^{-1})$ is holomorphic on the disk $|z| > A$, we have $|R_j| \lesssim \lambda_\Delta^j$ over all and $j \in \{0, 1, 2, \ldots\}$, as mentioned earlier. It follows that

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{|R_j|}{j!} \, |\zeta(a) - \zeta(a')|^j &\lesssim \sum_{j=0}^{\infty} \frac{\lambda_\Delta^j}{j!} \, |\zeta(a) - \zeta(a')|^j \\ &\lesssim e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}\end{aligned}$$

over all $a, a' \in \Omega_\alpha^2$, so

$$|k_\star^\alpha(a, a')| \lesssim \frac{|\zeta(a) - \zeta(a')|}{|p(a)|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega_\alpha^2$. Like before, rewrite p in the form $(\zeta - \alpha)\, P_\alpha(\zeta)$, so we have

$$|k_\star^\alpha(a, a')| \lesssim \frac{|\zeta(a) - \zeta(a')|}{|\zeta(a) - \alpha| \, \big|P_\alpha\big(\zeta(a)\big)\big|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega_\alpha$. Since P_α is a polynomial with a finite number of roots, and Ω_α doesn't touch any of the roots, $|P_\alpha(\zeta)|^{-1}$ is bounded on Ω_α. We conclude that

$$|k_\star^\alpha(a, a')| \lesssim \frac{|\zeta(a) - \zeta(a')|}{|\zeta(a) - \alpha|} e^{\lambda_\Delta |\zeta(a) - \zeta(a')|}$$

over all $a, a' \in \Omega_\alpha$, so Condition ($\mathrm{DIAG}_\star \mid \gamma, \lambda_\Delta$) is satisfied with $\gamma = 1$.

We can now apply Proposition 3.11, yielding the desired result. ∎

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