

Resurgence of the Airy function and other exponential integrals

Veronica Fantini and Aaron Fenyes

April 1, 2022

1 Introduction

1.1 Why does Borel resummation work?

Borel resummation is a way of turning a formal power series

$$\varphi_{\bullet} = z^{\sigma} \left(\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \frac{\varphi_3}{z^4} + \dots \right),$$

with $\sigma \in [0, 1)$, into a function which is asymptotic to φ_{\bullet} as $z \rightarrow \infty$. Different functions can be asymptotic to the same power series, and Borel resummation picks one of them, performing an implicit regularization [[arXiv:1705.03071](#), or maybe [arXiv:1412.6614](#)]. When a function matches the Borel sum of its asymptotic series, we'll say it's *Borel regular*. Several familiar kinds of regularity imply Borel regularity, and shed light on why it occurs.

- **Having a good asymptotic approximation**

Let R_N be the difference between a function and the partial sum

$$\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \dots + \frac{\varphi_{N-2}}{z^{N-1}}$$

of its asymptotic series. Watson showed a century ago that the function is Borel regular whenever there's a constant $c \in (0, \infty)$ with

$$|R_N| \leq \frac{c^{N+1} N!}{|z|^N}$$

over all orders N and all z in a wide enough wedge around infinity.

- **Satisfying a singular differential equation**

- Think about conditions where this works.
- Maybe the correct place is the setting of Ecalle's formal integral. See §5.2.2.1 of Delabaere's *Divergent Series, Summability and Resurgence III*.
- Say there's a unique solution (up to scaling) that shrinks as you go right; everything else blows up exponentially. Then this is the only solution that can be expressed as a Laplace transform.

- If the Borel-transformed equation has a subexponential solution \hat{f} which is “shifted holomorphic” (we called this having a “fractional power singularity” in **airy-resurgence**), then $\mathcal{L}\hat{f}$ satisfies the original equation, because there are no boundary terms.
- Draw diagram showing formal vs. holomorphic solutions in time vs. frequency domains.

• Being a thimble integral

Let X be a translation surface—a Riemann surface carrying a holomorphic 1-form ν . Suppose X is of *meromorphic type*, meaning that we got it by puncturing a compact Riemann surface \bar{X} at finitely many points, and ν has a pole at each puncture. A *translation coordinate* on X is a local coordinate whose derivative is ν .

Take another meromorphic-type translation surface B and a holomorphic Morse¹ map $f: \bar{X} \rightarrow \bar{B}$ that sends punctures to punctures. Suppose every singularity of B is a critical value of f . [Typical usage of “Borel plane” seems ambiguous, so maybe we can use “Borel plane” for B and “Borel cover” for the Riemann surface of the Borel-transformed series. How to handle the Orr–Sommerfeld functions (DLMF §9.13)? We know $f = 4u^3 - 3u$ is the pullback of a translation coordinate, but we also need a puncture at $f(0)\dots$] For each critical point p , let Γ_p be the ray going rightward from $f(p)$, and let ζ_p be the translation coordinate around Γ_p which vanishes at $f(p)$. These are well-defined as long as Γ_p misses the critical values of f . The preimage $f^{-1}(\Gamma_p)$ is a bunch of disjoint curves, as long as Γ_p misses the other critical values of f . The *Lefschetz thimble* Λ_p is the component of $f^{-1}(\Gamma_p)$ that goes through p , oriented so that shifting it to its left would make its projection run clockwise around Γ_p . The *thimble integral*

$$I_p = \int_{\Lambda_p} e^{-zf^*\zeta_p} \nu$$

is a holomorphic function on the right half-plane parametrized by z , and it turns out [we hope] to be Borel regular.

[Talk about exponential integrals and their decomposition into thimble integrals.]

In higher-dimensional complex manifolds, integrals over Lefschetz thimbles are still Borel regular [“Exponential integrals, Lefschetz thimbles and linear resurgence”][“Exponential Integral” lectures?]. This fact plays an important technical role in quantum mechanics, where infinite-dimensional exponential integrals are supposed to give the expectation values of observable quantities. Physicists often use Borel summation and related techniques to assign values to these integrals [Costin & Kruskal, “On optimal truncation...”].

Choose a path $\gamma: \mathbb{R} \rightarrow X$ whose projection $f \circ \gamma$ starts out going leftward out of a puncture, ends up going rightward into a puncture, and never touches a critical value of f . Choose a translation coordinate ζ on B and continue it along $f \circ \gamma$, noting that

¹This condition means that the critical points of f are isolated (the compactness of \bar{X} guarantees this) and the 2-jet of f is non-zero at every critical point.

it may become multi-valued if $f \circ \gamma$ intersects itself. This data defines the *exponential integral*

$$I = \int_{\gamma} e^{-zf^*\zeta} \nu,$$

a holomorphic function on the right half-plane parametrized by z . It turns out [**we hope**] that we can get I by summing $e^{-\alpha_p z} I_p$ over various critical points—as long as none of the Γ_p run into each other. [**We get jumps at phases where the Γ_p do hit each other.**] The constants α_p are values of ζ , continued to the critical points along certain paths.

- Each resummation method for asymptotic series makes some implicit assumption that allows us to reconstruct a holomorphic function from its asymptotic behaviour.
- The resummation method works correctly for functions which satisfy that assumption.
- For the modified Bessel function $K_{1/3}$, Borel resummation works because the asymptotic series encodes a second-order differential equation
 - Different aspects of this example appear in various places (Mariño, Kawai–Takei, Sauzin). We give a detailed, unified treatment.
- We can generalize this argument to all $K_{1/n}$ and their limit K_0 .
- We can also generalize to all third-order exponential integrals.
 - Most of them are equivalent to the $K_{1/3}$ integral, but there’s also an interesting degeneration.

1.2 Fractional derivative formula

- Theorem ?? says that for a certain class of exponential integrals

$$I(z) = \int_{\Gamma} e^{-zf} \nu,$$

the inverse Laplace [**better to say Borel?**] transform is the $\frac{3}{2}$ derivative of $d\zeta/df$, where $f^*d\zeta = \nu$ [**check**].

- the asymptotic expansion of $I(z)$ is a resurgent function.
- Is it always a *simple* resurgent function?
 - **Maxim believes it is in general, and indeed in our examples we get simple resurgent functions. But how to prove it in general?**

1.3 Stokes phenomenon

- For Bessel functions, we can see explicitly how solutions jump when the Laplace transform angle crosses a critical value.
- The jump comes from the branch cut difference identity for hypergeometric functions.
- Possible interpretation of the Stokes factors as intersection numbers in Morse–Novikov theory [[ask Maxim](#)]

2 The Laplace and Borel transforms

2.1 The Laplace transform

- Action on differential equations.
 - Can we find a way to prove this when the differential operator spits out a function that’s not integrable around zero?
- Global picture?

2.2 The Borel transform

- Action on differential equations.
 - No inhomogeneous terms! How is this consistent with the Laplace transform’s action? Is there always an inhomogeneous solution with subexponential asymptotics?

3 Third-order exponential integrals

- Reduce to

$$I(z) = \int \exp[-z(u^3 + pu + q)] du$$

using change of coordinate.

- When $p \neq 0$, can reduce further to

$$I(z) = p^{1/2} e^{-qz} K_{1/3}(p^{3/2}z).$$

- As p goes to zero, $I(z)$ degenerates to

$$\left(\frac{1}{2}\right)^{2/3} e^{-qz} \Gamma\left(\frac{1}{3}\right) z^{-1/3} = \left(\frac{1}{2}\right)^{2/3} e^{-qz} \mathcal{L}_{\zeta,0}(\zeta^{-2/3}) = \left(\frac{1}{2}\right)^{2/3} \mathcal{L}_{\zeta-q,q}(\zeta^{-2/3}).$$

Veronica's proposal

Title: Borel regularity and Resurgence of Exponential Integrals

1. introduction

- what are exponential integrals? has to be done
 - motivation
 - * In the classical theory of special functions, exponential integrals are often used to express solutions of linear differential and difference equations.
 - * In physics ??
 - * Geometrically they represent a Poincaré pairing (as explained by Kontsevich in **IHES lectures**).
- Borel regularity
 - what does it mean being Borel regular?
 - when does it happen?
 - * Recall Watson condition (old)
 - * State new Borel regularity results
 - Linear, homogeneous ODE with regular singularity at 0 and irregular singularity at infinity [P.Ramis ?, Loday-Richaud ?, big idea in **airy-resurgence**]
 - Thimble integral [**draft2**]
- State results about resurgence of exponential integrals and Stokes phenomena
 - Thimbles integrals [Kontsevich]: geometric computation of Stokes constants has to be done
 - ODE and fractional derivative formula [**draft2**]
 - if hypergeometric functions appear in a large class of examples: integral formulas for hypergeometric functions has to be done

2. Formalism for Laplace transform

(a) Introduction

i.

3. General cases

(a) Borel regularity

- General ODE of the form

$$\left[P\left(\frac{\partial}{\partial z}\right) + z^{-1}Q\left(\frac{\partial}{\partial z}\right) + z^{-2}R(z^{-1}) \right] \Phi = 0,$$

where P is a polynomial, Q is a polynomial of one degree lower, and R is an entire function [see **airy-resurgence** and written notes]

- General thimble integral (conditions?)

(b) Resurgence

4. The Airy example

- $I(z)$ is a solution of a linear ODE. We explicitly find its Borel transform, knowing the nature of singularities and the asymptotic behaviour of a basis of solution for the ODE [airy-resurgence]
- Stokes constants using fractional derivative formula and Borel transform computation [draft2]
- Comparison with the literature (Marino, Sauzin, Kontsevich slides) has to be done

5. Bessel $1/n$ (follows from Airy) we compute Borel transform and Stokes constants has to be done

6. Bessel 0 (it is different because we have infinite cover) we compute Stokes constants [draft2]

7. Bessel μ (follows from Bessel 0) we compute Stokes constants [modified Bessel]