Resurgence of modified Bessel functions of second kind

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1 Modified Bessel function of second kind

The modified Bessel function of the second kind $K_\mu(z)$ is a solution of the equation

$$w'' + \frac{w'}{z} - w - \frac{\mu^2}{z^2} w = 0 \tag{1}$$

such that $K_\mu(z) \sim \sqrt{\pi/(2z)} e^{-z}$ as $z \to \infty$ in $|\arg z| < \frac{3\pi}{2}$[1]. It has a branch point at $z = 0$ for every $\mu \in \mathbb{C}$ and the principal branch is analytic in $\mathbb{C} \setminus (-\infty, 0]$.

1.1 Differential equation

From the general theory of ODE, the formal integral solution of (1) is two parameters family

$$\tilde{w}(z) = U_1 e^{-z} z^{-1/2} \tilde{w}_{\mu,+}(z) + U_2 e^{z} z^{-1/2} \tilde{w}_{\mu,-}(z) \tag{4}$$

where $\tilde{w}_{\mu,\pm} = \sum_{j\geq 0} a_{\pm,j} z^{-j} \in \mathbb{C}[\![z^{-1}]\!]$ are unique formal solutions of

$$\begin{aligned}
\tilde{w}''_{\mu,+} - 2\tilde{w}'_{\nu,+} + \frac{\tilde{w}_{\mu,+}}{4z^2} - \frac{\mu^2}{z^2}\tilde{w}_{\mu,+} = 0 \\
\tilde{w}''_{\mu,-} + 2\tilde{w}'_{\mu,-} + \frac{\tilde{w}_{\mu,-}}{4z^2} - \frac{\mu^2}{z^2}\tilde{w}_{\mu,-} = 0
\end{aligned}$$

In particular, $\tilde{K}_\mu(z) = \sqrt{\frac{\pi}{2}} U_1 e^{-z} z^{-1/2} \tilde{w}_{\mu,+}(z)$ and $\tilde{I}_\mu(z) = \frac{1}{\sqrt{2\pi}} U_2 e^{z} z^{-1/2} \tilde{w}_{\mu,-}(z)$ (we assume $a_{\pm,0} = 1$) for some constants U_1, U_2. We now compute the Borel transform of $\tilde{w}_+(z)$[2]:

[1] A system of solution of Bessel equation is given by $I_\mu(z)$ and $K_\mu(z)$. In particular, their asymptotic behaviour as $z \to \infty$ is given by

$$\tilde{I}_\mu(z) = \frac{1}{\sqrt{2\pi}} e^{z} z^{-1/2} \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\mu\right)_k \left(\frac{1}{2}+\mu\right)_k}{2^k k!} z^{-k} \tag{2}$$

$$\tilde{K}_\mu(z) = \sqrt{\frac{\pi}{2}} e^{-z} z^{-1/2} \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\mu\right)_k \left(\frac{1}{2}+\mu\right)_k}{(-2)^k k!} z^{-k} \tag{3}$$

[2] We do not consider constant term of $\tilde{w}_{\mu,\pm}$, i.e. $\mathcal{B} : \mathbb{C}[\![z^{-1}]\!] \to \mathbb{C}[\zeta]$.

it is a solution of

$$\zeta^2\hat{w}_{\mu,+} + 2t\hat{w}_{\mu,+} + \left(\frac{1}{4} - \nu^2\right)\int_0^\zeta (\zeta - s)\hat{w}_{\nu,+}(s)ds = 0$$

$$\zeta^2\hat{w}''_{\mu,+} + 2\zeta\hat{w}''_{+} + 4\zeta\hat{w}'_{\mu,+} + \left(\frac{9}{4} - \mu^2\right)\hat{w}_{\mu,+} = 0$$

$$t(1-t)\hat{w}''_{\mu,+} + (2-4t)\hat{w}'_{\mu,+} - \left(\frac{9}{4} - \mu^2\right)\hat{w}_{mu,+} = 0 \qquad t = -\frac{\zeta}{2}$$

therefore $\hat{w}_{\mu,+}(\zeta)$ is an hypergeometric function

$$\hat{w}_{\mu,+}(\zeta) = c_{\mu,+2}F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; -\frac{\zeta}{2}\right) \tag{5}$$

and it has a branch point singularities at $\zeta = -2$. By the same reasoning,

$$\hat{w}_{\mu,-}(\zeta) = c_{\mu,-2}F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; \frac{\zeta}{2}\right) \tag{6}$$

and it has branch point at $\zeta = 2$.

1.2 Exponential integral

Let $X = \mathbb{C}^*$, $f(x) = x + \frac{1}{x}$ and for every $\mu \in [0, +\infty)$ let $\nu = (x^\mu + x^{-\mu})\,\frac{dx}{x}$, then

$$I(z; m) := \int_0^\infty e^{-zf}\nu \tag{7}$$

In particular, on the universal cover $\pi\colon \tilde{C} \to \mathbb{C}^*$ setting $x = e^u$ **[DLMF, Identity 10.32.9]**

$$I(\tfrac{z}{2}; \mu) = 2\int_{-\infty}^{\infty} e^{-z\cosh(u)}\cosh(\mu u)\,du = 4K_\mu(z) \quad |\arg(z)| < \pi/2 \tag{8}$$

where $K_\mu(z)$ is the second kind modified Bessel function with parameter μ.

The critical points of $\pi^* f$ are at $u = ki\pi$, for $k \in \mathbb{Z}$ and we denote $\tilde{I}_1(z; \mu)$ the asymptotic expansion of $I(\frac{z}{2}; \mu)$ at $u = 0$ and $\tilde{I}_{-1}(z; \mu)$ the expansion at $u = i\pi$. They are respectively multiple of $\tilde{K}_\mu$ and $\tilde{I}_\mu$, because they solve (1) and they have the same leading order asymptotic of $\tilde{K}_\mu, \tilde{I}_\mu$ which are a basis.

Notice that $I(z; \mu)$ differs from $I(z; 0)$ only in $\pi^*(\nu)$ while $\pi^*(f)$ stays the same for every $\mu \in [0, \infty)$. Hence we can adapt part of the argument used in Bessel example (see), and apply the 3/2-derivative formula: let $\zeta = \cosh(u)$ and $\mathcal{C}_0(\zeta)\colon \theta \in \mathbb{R} \to \cosh(\theta) \in \mathbb{C}_\zeta$

$$
\begin{aligned}
\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu) &= \int_{\mathcal{C}_0(\zeta)} \cosh(\mu u) du \\
&= \frac{1}{\mu}\Big[\sinh(\mu u)\Big]_{\mathrm{start}\mathcal{C}_0(\zeta)}^{\mathrm{end}\mathcal{C}_0(\zeta)} \\
&= \frac{1}{\mu}\left(\sinh\left(\mu \operatorname{acosh}(\zeta)\right) - \sinh\left(-\mu \operatorname{acosh}(\zeta)\right)\right) \\
&= \frac{2}{\mu}\sinh\left(\mu \operatorname{acosh}(\zeta)\right)
\end{aligned}
$$

The we set $\xi = \frac{1}{2}(\zeta - 1)$, thanks to identity 15.4.16 **DLMF**

$$
\begin{aligned}
\sinh(\tau) {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; -\sinh^2(\tau)\right) &= \frac{1}{2\mu}\sinh(2\mu\tau) \\
\xi^{1/2} {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; -\xi\right) &= \frac{1}{2\mu}\sinh(2\mu\tau) \qquad\qquad \sinh^2(\tau) = \xi \\
\xi^{1/2} {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; -\xi\right) &= \frac{1}{2\mu}\sinh(\mu \ \operatorname{acosh}(\zeta)) \qquad \cosh(2\tau) = \zeta \\
&= \frac{1}{4}\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu)
\end{aligned}
$$

Thus we take 3/2-derivative based at $\zeta = 1$

$$
\begin{aligned}
\partial_\zeta^{3/2}\left(\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu)\right) &= \partial_\zeta^2\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^\zeta (\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_0(\zeta')} \pi^*(\nu)\right) d\zeta'\right) \\
&= 4\partial_\zeta^2\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^\xi \frac{1}{\sqrt{2}}(\xi-\xi')^{-1/2}(\xi')^{1/2} {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; -\xi'\right) 2 \ d\xi'\right] \\
&= \frac{8}{\sqrt{2}}\partial_\zeta^2\left[\Gamma\left(\frac{3}{2}\right) \xi \ {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; 2; -\xi\right)\right] \\
&= \frac{8}{\sqrt{2}}\frac{\sqrt{\pi}}{2}\frac{1}{4}\partial_\xi^2\left[\xi \ {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; 2; -\xi\right)\right] \\
&= -\frac{\sqrt{\pi}}{\sqrt{2}}\partial_\xi \ {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; 1; -\xi\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{2}}\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\mu\right) {}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; -\xi\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{2}}\frac{\pi}{\cos(\mu\pi)} {}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; \frac{1}{2}-\frac{\zeta}{2}\right)
\end{aligned}
$$

Let us now consider the integral whose asymptotic behvaior is given in terms of $\hat{w}_{\mu,-}(z)$:

I have to check the correct form of the integral I which correspond to the path C_π. I suspect a scaling factor of $\cos(\pi\mu)$ that will adjust the Stokes factor computations.

set $\zeta = -\cosh(u)$,

$$\begin{aligned}
\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu) &= \int_{\mathcal{C}_\pi(\zeta)} \cosh(\mu u) du \\
&= \frac{1}{\mu}\Big[\sinh(\mu u)\Big]_{\text{start}\mathcal{C}_\pi(\zeta)}^{\text{end}\mathcal{C}_\pi(\zeta)} \\
&= \frac{1}{\mu}\left(\sinh\left(\mu \,\text{acosh}\left(-\zeta\right)\right) - \sinh\left(-\mu\,\text{acosh}\left(-\zeta\right)\right)\right) \\
&= \frac{2}{\mu}\sinh\left(\mu\,\text{acosh}\left(-\zeta\right)\right)
\end{aligned}$$

The we set $\xi = \frac{1}{2}\left(\zeta + 1\right)$, thanks to identity 15.4.16 **DLMF**

$$\begin{aligned}
\sinh(\tau)_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; -\sinh^2(\tau)\right) &= \frac{1}{2\mu}\sinh(2\mu\tau) \\
(-\xi)^{1/2}{}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; \xi\right) &= \frac{1}{2\mu}\sinh(2\mu\tau) && \sinh^2(\tau) = -\xi \\
(-\xi)^{1/2}{}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; \xi\right) &= \frac{1}{2\mu}\sinh(\mu\ \text{acosh}\left(-\zeta\right)) && \cosh(2\tau) = -\zeta \\
&= \frac{1}{4}\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu)
\end{aligned}$$

Thus we take 3/2-derivative based at $\zeta = -1$

$$
\begin{aligned}
\partial_\zeta^{3/2}\left(\int_{\mathcal{C}_\pi(\zeta)}\pi^*(\nu)\right) &= \partial_\zeta^2\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{\zeta}(\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_\pi(\zeta')}\pi^*(\nu)\right)d\zeta'\right)\\
&= 4\partial_\zeta^2\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^{\xi}\frac{1}{\sqrt{2}}(\xi-\xi')^{-1/2}(-\xi')^{1/2}{}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;\frac{3}{2};\xi'\right)2\;d\xi'\right)\\
&= -i\frac{8}{\sqrt{2}}\partial_\zeta^2\left(\Gamma\left(\tfrac{3}{2}\right)(-\xi)\;{}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;2;\xi\right)\right)\\
&= i\frac{8}{\sqrt{2}}\frac{\sqrt{\pi}}{2}\frac{1}{4}\partial_\xi^2\left(\xi\;{}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;2;\xi\right)\right)\\
&= i\frac{\sqrt{\pi}}{\sqrt{2}}\partial_\xi\;{}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;1;\xi\right)\\
&= i\frac{\sqrt{\pi}}{\sqrt{2}}\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\mu\right){}_2F_1\left(\frac{3}{2}-\mu,\frac{3}{2}+\mu;2;\xi\right)\\
&= i\frac{\sqrt{\pi}}{\sqrt{2}}\frac{\pi}{\cos(\mu\pi)}{}_2F_1\left(\frac{3}{2}-\mu,\frac{3}{2}+\mu;2;\frac{1}{2}-\frac{\zeta}{2}\right)
\end{aligned}
$$

1.3 Stokes factors

Thanks to explicit formula for the analytic continuation of hypergeomtric functions (see [**?**] 15.2.3) and using the constants prescribed by the fractional derivative formula we are able ot compute the Stokes constants: set $\hat{w}_{+,\mu}(\zeta) = \frac{\pi\sqrt{\pi}}{\cos(\mu\pi)}{}_2F_1\left(\frac{3}{2}-\mu,\frac{3}{2}-\mu;2,1-\frac{\zeta}{2}\right)$ and $\hat{w}_{-,\mu}(\zeta) := i\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)}{}_2F_1\left(\frac{3}{2}-\mu,\frac{3}{2}-\mu;2,1+\frac{\zeta}{2}\right)$

$$
\begin{aligned}
\hat{w}_{\mu,+}(\zeta+i0)-\hat{w}_{\mu,+}(\zeta-i0) &= \frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)}\frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)}\big(-\frac{\zeta}{2}-1\big)^{-1}{}_2F_1\left(\frac{1}{2}+\mu,\frac{1}{2}-\mu;0;1+\frac{\zeta}{2}\right) \quad \zeta>-2\\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)}\frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)}\sum_{k\geq 0}\frac{\left(\frac{1}{2}-\mu\right)_k\left(\frac{1}{2}+\mu\right)_k}{\Gamma(k)k!}\left(1+\frac{\zeta}{2}\right)^{k-1}\\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)}\frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)}\sum_{k\geq 1}\frac{\left(\frac{1}{2}-\mu\right)_k\left(\frac{1}{2}+\mu\right)_k}{\Gamma(k)k!}\left(1+\frac{\zeta}{2}\right)^{k-1}\\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)}\frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)}\frac{1}{\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\mu\right)}\cdot\\
&\quad\cdot\sum_{k\geq 1}\frac{\Gamma\left(\frac{1}{2}-\mu+k\right)\Gamma\left(\frac{1}{2}+\mu+k\right)}{\Gamma(k)k!}\left(1+\frac{\zeta}{2}\right)^{k-1}\\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)}\frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)}\frac{1}{\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\mu\right)}\cdot\\
&\quad\cdot\sum_{k\geq 0}\frac{\Gamma\left(\frac{3}{2}-\mu+k\right)\Gamma\left(\frac{3}{2}+\mu+k\right)}{\Gamma(k+1)(k+1)!}\left(1+\frac{\zeta}{2}\right)^{k}\\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)}\frac{2\pi i}{\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\mu\right)}\;{}_2F_1\left(\frac{3}{2}-\mu,\frac{3}{2}+\mu;2;1+\frac{\zeta}{2}\right)\\
&= -2\pi i\frac{\sqrt{\pi}}{2}\;{}_2F_1\left(\frac{3}{2}-\mu,\frac{3}{2}+\mu;2;1+\frac{\zeta}{2}\right)\\
&= -\mathbf{2}\cos(\pi\mu)\hat{w}_{-,\mu}(\zeta+2)
\end{aligned}
$$

and for $\hat{w}_{\mu,-}(\zeta)$

$$
\begin{aligned}
\hat{w}_{\mu,-}(\zeta+i0)-\hat{w}_{\mu,-}(\zeta-i0) &= i\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)}\frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)}\big(\frac{\zeta}{2}-1\big)^{-1}{}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;0;1-\frac{\zeta}{2}\right) \quad \zeta<2\\
&= -i\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)}\frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)}\sum_{k\geq 0}\frac{\left(\frac{1}{2}-\mu\right)_k\left(\frac{1}{2}+\mu\right)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{2}\right)^{k-1}\\
&= -2i\cos(\mu\pi)\;\frac{i\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)}\;{}_2F_1\left(\frac{3}{2}-\mu,\frac{3}{2}+\mu;2;1-\frac{\zeta}{2}\right)\\
&= +\mathbf{2}\cos(\mu\pi)\hat{w}_{\mu,+}(\zeta-2)
\end{aligned}
$$

Therefore we have shown that Stokes constants are independent on μ and equal to ± 2.

1.4 Why the Stokes factors are non-integer

The infinite dihedral group $D_{2\infty}\cong\mathbb{Z}\ltimes\mathbb{Z}/2\mathbb{Z}$ has a nice action on $\tilde{C}$ that leaves $e^{-z\cosh(u)}$ invariant. The action of $1\in\mathbb{Z}$ adds $2\pi i$ to the value of u, and the actions of $\{\pm 1\}\cong\mathbb{Z}/2$ multiply the value of u by ± 1.

Consider exponential integrals of the form

$$\langle \Gamma, e^{-z\cosh(u)}\,\alpha\rangle := \int_\Gamma e^{-z\cosh(u)}\,\alpha,$$

where the variety $\tilde{C}$ and the twisting factor $e^{-z\cosh(u)}$ are fixed, while the integration path Γ and the twisted 1-form $e^{-z\cosh(u)}\,\alpha$ are allowed to vary. We can apply the action of $D_{2\infty}$ to these integrals by pushing forward the integration contour, or equivalently by pulling back the twisted 1-form. This makes the vector space of exponential integrals into a representation of $D_{2\infty}$.

Let's see what the action does to $I(\frac{z}{2};\mu) = \langle -\infty \to \infty, e^{-z\cosh(u)}\,\nu\rangle$, recalling that $\nu = \cosh(\mu u)\,du$. Pulling ν back along $1 \in \mathbb{Z}$ gives

$$\begin{aligned}\cosh(\mu u + 2\pi i\mu)\,du &= \big[\cosh(2\pi i\mu)\cosh(\mu u) + \sinh(2\pi i\mu)\sinh(\mu u)\big]\,du \\ &= \cos(2\pi\mu)\,\nu + i\sin(2\pi\mu)\sinh(\mu u)\,du.\end{aligned}$$

and pulling it back along $\pm 1 \in \mathbb{Z}/2\mathbb{Z}$ gives $\pm\nu$. Hence, $1 \in \mathbb{Z}$ sends

$$I(\tfrac{z}{2};\mu) = 2\int_{-\infty}^{\infty} e^{-z\cosh(u)}\,\nu$$

to

$$\begin{aligned}&2\int_{-\infty}^{\infty} e^{-z\cosh(u)}\big[\cos(2\pi\mu)\,\nu + i\sin(2\pi\mu)\sinh(\mu u)\,du\big] \\ &= 2\cos(2\pi\mu)\int_{-\infty}^{\infty} e^{-z\cosh(u)}\,\nu + 2i\sin(2\pi\mu)\int_{-\infty}^{\infty} e^{-z\cosh(u)}\sinh(\mu u)\,du \\ &= \cos(2\pi\mu)\,I(\tfrac{z}{2};\mu),\end{aligned}$$

with the second term vanishing in the last step because $\sinh(\mu u)$ is odd. Also, $\pm 1 \in \mathbb{Z}/2\mathbb{Z}$ sends $I(\frac{z}{2};\mu)$ to $\pm I(\frac{z}{2};\mu)$. Now we see that in the space of exponential integrals, considered as a representation of $D_{2\infty}$, the span of $I(\frac{z}{2};\mu)$ is a one-dimensional subrepresentation, giving the character **[correct word?]**

$$\begin{aligned}D_{2\infty} &\to \mathbb{C}^\times \\ 1 \in \mathbb{Z} &\mapsto \cos(2\pi\mu) \\ \pm 1 \in \mathbb{Z}/2\mathbb{Z} &\mapsto \pm 1.\end{aligned}$$

[The other invariant subspace should be generated by the path $-\infty + \pi i \to \infty + \pi i$.]

Another way to think of this: if we choose bases for the space of (relative homology classes of) integration paths and the space of (relative cohomology classes of) twisted 1-forms, we can write a period matrix for $\langle\ ,\ \rangle$. The entries of the period matrix form a basis for the space of exponential integrals. The action of $D_{2\infty}$ on exponential integrals can be seen as an action on the period matrix. When $D_{2\infty}$ acts on integration paths, it acts on the period matrix by left multiplication; when it acts on 1-forms, it acts on the period matrix by right multiplication. The the left and right action matrices are adjoints with respect to $\langle\ ,\ \rangle$. The argument above shows that for well-chosen bases, we can make the action matrices block-diagonal, with two blocks.

When $\mu = 1/n$ for $n \in \mathbb{N}$, the space of integration paths and the space of twisted 1-forms are both n-dimensional, and **[...]**