

Resurgence of modified Bessel functions of second kind

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1 Modified Bessel function of second kind

The modified Bessel function of the second kind $K_\mu(z)$ is a solution of the equation

$$w'' + \frac{w'}{z} - w - \frac{\mu^2}{z^2}w = 0 \quad (1)$$

such that $K_\mu(z) \sim \sqrt{\pi/(2z)}e^{-z}$ as $z \rightarrow \infty$ in $|\arg z| < \frac{3\pi}{2}$ ¹. It has a branch point at $z = 0$ for every $\mu \in \mathbb{C}$ and the principal branch is analytic in $\mathbb{C} \setminus (-\infty, 0]$.

1.1 Differential equation

From the general theory of ODE, the formal integral solution of (??) is two parameters family

$$\tilde{w}(z) = U_1 e^{-z} z^{-1/2} \tilde{w}_{\mu,+}(z) + U_2 e^z z^{-1/2} \tilde{w}_{\mu,-}(z) \quad (4)$$

where $\tilde{w}_{\mu,\pm} = \sum_{j \geq 0} a_{\pm,j} z^{-j} \in \mathbb{C}[[z^{-1}]]$ are unique formal solutions of

$$\begin{aligned} \tilde{w}_{\mu,+}'' - 2\tilde{w}_{\mu,+}' + \frac{\tilde{w}_{\mu,+}}{4z^2} - \frac{\mu^2}{z^2}\tilde{w}_{\mu,+} &= 0 \\ \tilde{w}_{\mu,-}'' + 2\tilde{w}_{\mu,-}' + \frac{\tilde{w}_{\mu,-}}{4z^2} - \frac{\mu^2}{z^2}\tilde{w}_{\mu,-} &= 0 \end{aligned}$$

In particular, $\tilde{K}_\mu(z) = \sqrt{\frac{\pi}{2}} U_1 e^{-z} z^{-1/2} \tilde{w}_{\mu,+}(z)$ and $\tilde{I}_\mu(z) = \frac{1}{\sqrt{2\pi}} U_2 e^z z^{-1/2} \tilde{w}_{\mu,-}(z)$ (we assume $a_{\pm,0} = 1$) for some constants U_1, U_2 . We now compute the Borel transform of $\tilde{w}_+(z)$ ²:

¹A system of solution of Bessel equation is given by $I_\mu(z)$ and $K_\mu(z)$. In particular, their asymptotic behaviour as $z \rightarrow \infty$ is given by

$$\tilde{I}_\mu(z) = \frac{1}{\sqrt{2\pi}} e^z z^{-1/2} \sum_{k \geq 0} \frac{(\frac{1}{2} - \mu)_k (\frac{1}{2} + \mu)_k}{2^k k!} z^{-k} \quad (2)$$

$$\tilde{K}_\mu(z) = \sqrt{\frac{\pi}{2}} e^{-z} z^{-1/2} \sum_{k \geq 0} \frac{(\frac{1}{2} - \mu)_k (\frac{1}{2} + \mu)_k}{(-2)^k k!} z^{-k} \quad (3)$$

²We do not consider constant term of $\tilde{w}_{\mu,\pm}$, i.e. $\mathcal{B} : \mathbb{C}[[z^{-1}]] \rightarrow \mathbb{C}[\zeta]$.

it is a solution of

$$\begin{aligned}\zeta^2 \hat{w}_{\mu,+} + 2t \hat{w}_{\mu,+} + \left(\frac{1}{4} - \nu^2\right) \int_0^\zeta (\zeta - s) \hat{w}_{\nu,+}(s) ds &= 0 \\ \zeta^2 \hat{w}_{\mu,+}'' + 2\zeta \hat{w}_{\mu,+}' + 4\zeta \hat{w}_{\mu,+}' + \left(\frac{9}{4} - \mu^2\right) \hat{w}_{\mu,+} &= 0 \\ t(1-t) \hat{w}_{\mu,+}'' + (2-4t) \hat{w}_{\mu,+}' - \left(\frac{9}{4} - \mu^2\right) \hat{w}_{\mu,+} &= 0 \quad t = -\frac{\zeta}{2}\end{aligned}$$

therefore $\hat{w}_{\mu,+}(\zeta)$ is an hypergeometric function

$$\hat{w}_{\mu,+}(\zeta) = c_{\mu,+2} F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; -\frac{\zeta}{2}\right) \quad (5)$$

and it has a branch point singularities at $\zeta = -2$. By the same reasoning,

$$\hat{w}_{\mu,-}(\zeta) = c_{\mu,-2} F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; \frac{\zeta}{2}\right) \quad (6)$$

and it has branch point at $\zeta = 2$.

1.2 Exponential integral

Let $X = \mathbb{C}^*$, $f(x) = x + \frac{1}{x}$ and for every $\mu \in [0, +\infty)$ let $\nu = x^{\mu-1} + \frac{1}{x^{\mu+1}} dx$, then

$$I(z; m) := \int_0^\infty e^{-zf} \nu \quad (7)$$

In particular, on the universal cover $\pi: \tilde{C} \rightarrow \mathbb{C}^*$ setting $x = e^u$

$$I\left(\frac{z}{2}; \mu\right) = 2 \int_{-\infty}^\infty e^{-z \cosh(u)} \cosh(\mu u) du = 4K_\mu(z) \quad |\arg(z)| < \pi/2 \quad (8)$$

where $K_\mu(z)$ is the second kind modified Bessel function with parameter μ .

The critical points of $\pi^* f$ are at $u = ki\pi$, for $k \in \mathbb{Z}$ and we denote $\tilde{I}_1(z; \mu)$ the asymptotic expansion of $I(\frac{z}{2}; \mu)$ at $u = 0$ and $\tilde{I}_{-1}(z; \mu)$ the expansion at $u = i\pi$. They are respectively multiple of \tilde{K}_μ and \tilde{I}_μ , because they solve (??) and they have the same leading order asymptotic of $\tilde{K}_\mu, \tilde{I}_\mu$ which are a basis.

Notice that $I(z; \mu)$ differs from $I(z; 0)$ only in $\pi^*(\nu)$ while $\pi^*(f)$ stays the same for every $\mu \in [0, \infty)$. Hence we can adapt part of the argument used in Bessel example (see), and apply the 3/2-derivative formula: let $\zeta = \cosh(u)$ and $\mathcal{C}_0(\zeta): \theta \in \mathbb{R} \rightarrow \cosh(\theta) \in \mathbb{C}_\zeta$

$$\begin{aligned}\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu) &= \int_{\mathcal{C}_0(\zeta)} \cosh(\mu u) du \\ &= \frac{1}{\mu} \left[\sinh(\mu u) \right]_{\text{start } \mathcal{C}_0(\zeta)}^{\text{end } \mathcal{C}_0(\zeta)} \\ &= \frac{1}{\mu} (\sinh(\mu \text{acosh}(\zeta)) - \sinh(-\mu \text{acosh}(\zeta))) \\ &= \frac{2}{\mu} \sinh(\mu \text{acosh}(\zeta))\end{aligned}$$

The we set $\xi = \frac{1}{2}(\zeta - 1)$, thanks to identity 15.4.16 **DLMF**

$$\begin{aligned}
\sinh(\tau) {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; -\sinh^2(\tau)\right) &= \frac{1}{2\mu} \sinh(2\mu\tau) \\
\xi^{1/2} {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; -\xi\right) &= \frac{1}{2\mu} \sinh(2\mu\tau) & \sinh^2(\tau) &= \xi \\
\xi^{1/2} {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; -\xi\right) &= \frac{1}{2\mu} \sinh(\mu \operatorname{acosh}(\zeta)) & \cosh(2\tau) &= \zeta \\
&= \frac{1}{4} \int_{C_0(\zeta)} \pi^*(\nu)
\end{aligned}$$

Thus we take 3/2-derivative based at $\zeta = 1$

$$\begin{aligned}
\partial_\zeta^{3/2} \left(\int_{C_0(\zeta)} \pi^*(\nu) \right) &= \partial_\zeta^2 \left(\frac{1}{\Gamma(\frac{1}{2})} \int_1^\zeta (\zeta - \zeta')^{-1/2} \left(\int_{C_0(\zeta')} \pi^*(\nu) \right) d\zeta' \right) \\
&= 4\partial_\zeta^2 \left[\frac{1}{\Gamma(\frac{1}{2})} \int_0^\xi \frac{1}{\sqrt{2}} (\xi - \xi')^{-1/2} (\xi')^{1/2} {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; -\xi'\right) 2 d\xi' \right] \\
&= \frac{8}{\sqrt{2}} \partial_\zeta^2 \left[\Gamma\left(\frac{3}{2}\right) \xi {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; 2; -\xi\right) \right] \\
&= \frac{8}{\sqrt{2}} \frac{\sqrt{\pi}}{2} \frac{1}{4} \partial_\xi^2 \left[\xi {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; 2; -\xi\right) \right] \\
&= -\frac{\sqrt{\pi}}{\sqrt{2}} \partial_\xi {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; 1; -\xi\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{2}} \Gamma\left(\frac{1}{2} - \mu\right) \Gamma\left(\frac{1}{2} + \mu\right) {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; -\xi\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{2}} \frac{\pi}{\cos(\mu\pi)} {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; \frac{1}{2} - \frac{\zeta}{2}\right)
\end{aligned}$$

Let us now consider the integral whose asymptotic behavior is given in terms of $\hat{w}_{\mu,-}(z)$:

I have to check the correct form of the integral I which correspond to the path C_π . I suspect a scaling factor of $\cos(\pi\mu)$ that will adjust the Stokes factor computations.

set $\zeta = -\cosh(u)$,

$$\begin{aligned}
\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu) &= \int_{\mathcal{C}_\pi(\zeta)} \cosh(\mu u) du \\
&= \frac{1}{\mu} \left[\sinh(\mu u) \right]_{\text{start}\mathcal{C}_\pi(\zeta)}^{\text{end}\mathcal{C}_\pi(\zeta)} \\
&= \frac{1}{\mu} (\sinh(\mu \operatorname{acosh}(-\zeta)) - \sinh(-\mu \operatorname{acosh}(-\zeta))) \\
&= \frac{2}{\mu} \sinh(\mu \operatorname{acosh}(-\zeta))
\end{aligned}$$

The we set $\xi = \frac{1}{2}(\zeta + 1)$, thanks to identity 15.4.16 **DLMF**

$$\begin{aligned}
\sinh(\tau) {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; -\sinh^2(\tau)\right) &= \frac{1}{2\mu} \sinh(2\mu\tau) \\
(-\xi)^{1/2} {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; \xi\right) &= \frac{1}{2\mu} \sinh(2\mu\tau) & \sinh^2(\tau) &= -\xi \\
(-\xi)^{1/2} {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; \xi\right) &= \frac{1}{2\mu} \sinh(\mu \operatorname{acosh}(-\zeta)) & \cosh(2\tau) &= -\zeta \\
&= \frac{1}{4} \int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu)
\end{aligned}$$

Thus we take 3/2-derivative based at $\zeta = -1$

$$\begin{aligned}
\partial_\zeta^{3/2} \left(\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu) \right) &= \partial_\zeta^2 \left(\frac{1}{\Gamma(\frac{1}{2})} \int_{-1}^\zeta (\zeta - \zeta')^{-1/2} \left(\int_{\mathcal{C}_\pi(\zeta')} \pi^*(\nu) \right) d\zeta' \right) \\
&= 4\partial_\zeta^2 \left(\frac{1}{\Gamma(\frac{1}{2})} \int_0^\xi \frac{1}{\sqrt{2}} (\xi - \xi')^{-1/2} (-\xi')^{1/2} {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; \xi'\right) 2 d\xi' \right) \\
&= -i \frac{8}{\sqrt{2}} \partial_\zeta^2 \left(\Gamma\left(\frac{3}{2}\right) (-\xi) {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; 2; \xi\right) \right) \\
&= i \frac{8}{\sqrt{2}} \frac{\sqrt{\pi}}{2} \frac{1}{4} \partial_\xi^2 \left(\xi {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; 2; \xi\right) \right) \\
&= i \frac{\sqrt{\pi}}{\sqrt{2}} \partial_\xi {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; 1; \xi\right) \\
&= i \frac{\sqrt{\pi}}{\sqrt{2}} \Gamma\left(\frac{1}{2} - \mu\right) \Gamma\left(\frac{1}{2} + \mu\right) {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; \xi\right) \\
&= i \frac{\sqrt{\pi}}{\sqrt{2}} \frac{\pi}{\cos(\mu\pi)} {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; \frac{1}{2} - \frac{\zeta}{2}\right)
\end{aligned}$$

1.3 Stokes factors

Thanks to explicit formula for the analytic continuation of hypergeometric functions (see [?] 15.2.3) and using the constants prescribed by the fractional derivative formula we are able to compute the Stokes constants: set $\hat{w}_{+,\mu}(\zeta) = \frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} {}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}-\mu; 2, 1-\frac{\zeta}{2}\right)$ and $\hat{w}_{-,\mu}(\zeta) := i \frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} {}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}-\mu; 2, 1+\frac{\zeta}{2}\right)$

$$\begin{aligned}
\hat{w}_{\mu,+}(\zeta+i0) - \hat{w}_{\mu,+}(\zeta-i0) &= \frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \left(-\frac{\zeta}{2}-1\right)^{-1} {}_2F_1\left(\frac{1}{2}+\mu, \frac{1}{2}-\mu; 0; 1+\frac{\zeta}{2}\right) \quad \zeta > -2 \\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \sum_{k \geq 0} \frac{(\frac{1}{2}-\mu)_k (\frac{1}{2}+\mu)_k}{\Gamma(k)k!} \left(1+\frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \sum_{k \geq 1} \frac{(\frac{1}{2}-\mu)_k (\frac{1}{2}+\mu)_k}{\Gamma(k)k!} \left(1+\frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \frac{1}{\Gamma(\frac{1}{2}-\mu)\Gamma(\frac{1}{2}+\mu)} \\
&\quad \cdot \sum_{k \geq 1} \frac{\Gamma(\frac{1}{2}-\mu+k)\Gamma(\frac{1}{2}+\mu+k)}{\Gamma(k)k!} \left(1+\frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \frac{1}{\Gamma(\frac{1}{2}-\mu)\Gamma(\frac{1}{2}+\mu)} \\
&\quad \cdot \sum_{k \geq 0} \frac{\Gamma(\frac{3}{2}-\mu+k)\Gamma(\frac{3}{2}+\mu+k)}{\Gamma(k+1)(k+1)!} \left(1+\frac{\zeta}{2}\right)^k \\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{1}{2}-\mu)\Gamma(\frac{1}{2}+\mu)} {}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; 1+\frac{\zeta}{2}\right) \\
&= -2\pi i \frac{\sqrt{\pi}}{2} {}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; 1+\frac{\zeta}{2}\right) \\
&= -2\cos(\pi\mu)\hat{w}_{-,\mu}(\zeta+2)
\end{aligned}$$

and for $\hat{w}_{\mu,-}(\zeta)$

$$\begin{aligned}
\hat{w}_{\mu,-}(\zeta+i0) - \hat{w}_{\mu,-}(\zeta-i0) &= i \frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \left(\frac{\zeta}{2}-1\right)^{-1} {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; 0; 1-\frac{\zeta}{2}\right) \quad \zeta < 2 \\
&= -i \frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \sum_{k \geq 0} \frac{(\frac{1}{2}-\mu)_k (\frac{1}{2}+\mu)_k}{\Gamma(k)k!} \left(1-\frac{\zeta}{2}\right)^{k-1} \\
&= -2i\cos(\mu\pi) \frac{i\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} {}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; 1-\frac{\zeta}{2}\right) \\
&= +2\cos(\mu\pi)\hat{w}_{\mu,+}(\zeta-2)
\end{aligned}$$

Therefore we have shown that Stokes constants are independent on μ and equal to ± 2 .