

Airy function: Kawai+Takei vs. Mariño

Aaron Fenyes, Veronica Fantini

December 2, 2022

Kawai and Takei want to solve

$$\left[\left(\frac{d}{dx} \right)^2 - \eta^2 x \right] \psi(x, \eta) = 0.$$

They define $\psi_B(x, y)$ as the inverse Laplace transform of $\psi(x, \eta)$ with respect to η .

With $w = x\eta^{2/3}$, the equation above is equivalent to

$$\left[\left(\frac{d}{dw} \right)^2 - w \right] \psi(w\eta^{-2/3}, \eta) = 0.$$

Proof: substitute back to get

$$\begin{aligned} \left[\eta^{-4/3} \left(\frac{d}{dx} \right)^2 - \eta^{2/3} x \right] \psi(x, \eta) &= 0 \\ \left[\eta^{-4/3} \left(\frac{d}{dx} \right)^2 - \eta^{-4/3} \eta^2 x \right] \psi(x, \eta) &= 0 \\ \eta^{-4/3} \left[\left(\frac{d}{dx} \right)^2 - \eta^2 x \right] \psi(x, \eta) &= 0. \end{aligned}$$

Hence, $\psi(w\eta^{-2/3}, \eta) = k(\eta)\text{Ai}(w)$ is a solution for any holomorphic function k .

1 Veronica's change of coordinates

Kawai and Takei study the WKB analysis of the equation

$$\left[\left(\frac{d}{dx} \right)^2 - \eta^2 x \right] \psi(x, \eta) = 0 \tag{1}$$

as $\eta \rightarrow \infty$. They define $\psi_B(x, y)$ as the inverse Laplace transform of $\psi(x, \eta)$ with respect to η . In the coordinates $t = yx^{-3/2}$ they find an explicit formula for $\psi_B(x, y)$ in terms of

Gauss hypergeometric functions:

$$\begin{aligned}\psi_{+,B}(x,y) &= \frac{1}{x}\phi_+(t) = \frac{\sqrt{3}}{2\sqrt{\pi}} \frac{1}{x} s^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; s\right) \\ \psi_{-,B}(x,y) &= \frac{1}{x}\phi_-(t) = \frac{\sqrt{3}}{2\sqrt{\pi}} \frac{1}{x} (1-s)^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; 1-s\right)\end{aligned}$$

where $s = 3t/4 + 1/2$. The same hypergeometric functions have been computed in Section ?? as the Borel transform of the formal solutions of the Airy equation

$$\left[\left(\frac{d}{dw}\right)^2 - w\right]f(w) = 0. \quad (2)$$

Although the two equations look closely related (they are equivalent by the change of coordinates $w = x\eta^{2/3}$), the Borel transform of ψ is computed with respect to $\eta x^{3/2}$ (which is the conjugate variable of t) while the Borel transform of $f(w)$ is computed with respect to w . So we need to find a different change of coordinates to explain why the Borel transforms of $\psi(x, \eta)$ and $f(w)$ are given by the same hypergeometric function.

First of all notice that if η and y are conjugate variables under Borel transform, meaning

$$\sum_{n \geq 0} a_n \eta^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n \geq 0} \frac{a_n}{n!} y^n$$

then $t = yx^{-3/2}$ is the conjugate variable of $q = \eta x^{3/2}$ up to correction by a factor of $x^{-3/2}$

$$\sum_{n \geq 0} a_n q^{-n-1} = \sum_{n \geq 0} a_n x^{-3/2(n+1)} \eta^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n \geq 0} \frac{a_n x^{-3/2(n+1)}}{n!} y^n = x^{-3/2} \sum_{n \geq 0} \frac{a_n}{n!} t^n.$$

In addition, $\psi_{B,\pm}(x,y) = \frac{1}{x}\phi_{\pm}(t)$, therefore we expect that $\psi(x, \eta) = x^{1/2}\Phi(q)$. Assume that $\psi(x, y)$ is a solution of (1), then $\Phi(q)$ solves

$$\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi(q) = 0 \quad (3)$$

Proof.

$$\begin{aligned}
& \left[\left(\frac{d}{dx} \right)^2 - \eta^2 x \right] \psi(x, \eta) = 0 \\
& \left[\left(\frac{d}{dx} \right)^2 - \eta^2 x \right] x^{1/2} \Phi(q) = 0 \\
& \frac{d}{dx} \left[\frac{1}{2} x^{-1/2} \Phi + x^{1/2} \frac{d}{dx} \Phi \right] - \eta^2 x^{3/2} \Phi = 0 \\
& -\frac{1}{4} x^{-3/2} \Phi + \frac{1}{2} x^{-1/2} \frac{d}{dx} \Phi + \frac{1}{2} x^{-1/2} \frac{d}{dx} \Phi + x^{1/2} \left(\frac{d}{dx} \right)^2 \Phi - \eta^2 x^{3/2} \Phi = 0 \\
& \left[x^{1/2} \left(\frac{d}{dx} \right)^2 + x^{-1/2} \frac{d}{dx} - \frac{1}{4} x^{-3/2} - \eta^2 x^{3/2} \right] \Phi = 0 \\
& \left[\left(\frac{d}{dx} \right)^2 + x^{-1} \frac{d}{dx} - \frac{1}{4} x^{-2} - \eta^2 x \right] \Phi = 0
\end{aligned}$$

□

Now rewrite (3) in the coordinates $q = \eta x^{3/2}$:

$$\begin{aligned}
& \left[\left(\frac{d}{dx} \right)^2 + x^{-1} \frac{d}{dx} - \frac{1}{4} x^{-2} - \eta^2 x \right] \Phi = 0 \\
& \left[\frac{9}{4} \eta^2 x \left(\frac{d}{dq} \right)^2 + \frac{3}{4} \eta x^{-1/2} \frac{d}{dq} + x^{-1} \cdot \frac{3}{2} \eta x^{1/2} \frac{d}{dq} - \frac{1}{4} x^{-2} - \eta^2 x \right] \Phi = 0 \\
& \left[\eta^2 x \left(\frac{d}{dq} \right)^2 + \frac{1}{3} \eta x^{-1/2} \frac{d}{dq} + \frac{2}{3} \eta x^{-1/2} \frac{d}{dq} - \frac{1}{9} x^{-2} - \frac{4}{9} \eta^2 x \right] \Phi = 0 \\
& \left[\eta^2 \left(\frac{d}{dq} \right)^2 + \eta x^{-3/2} \frac{d}{dq} - \frac{1}{9} x^{-3} - \frac{4}{9} \eta^2 \right] \Phi = 0 \\
& \left[\left(\frac{d}{dq} \right)^2 + \eta^{-1} x^{-3/2} \frac{d}{dq} - \frac{1}{9} \eta^{-2} x^{-3} - \frac{4}{9} \right] \Phi = 0 \\
& \left[\left(\frac{d}{dq} \right)^2 + q^{-1} \frac{d}{dq} - \frac{1}{9} q^{-2} - \frac{4}{9} \right] \Phi = 0
\end{aligned}$$

therefore $\Phi(q)$ is a solution of the transform Airy equation (see draft2).

2 Weber equation: WKB vs modifield Bessel ODE

In [Takei] the author studied the Borel summation of WKB solutions of the harmonic oscillator

$$\left[\frac{d^2}{dx^2} - \eta^2 x^2 - \lambda \eta \right] \psi = 0 \quad (4)$$

with a parameter λ . He proved that the WKB solutions

$$\psi_{\pm}(x, \eta) = e^{\pm \eta x^2/2} \sum_{n=0}^{\infty} \frac{\psi_{\pm, n}}{x^{2n+(1\pm\lambda)/2}} \eta^{-(\frac{1}{2}\pm\frac{\lambda}{4}+n)} \quad (5)$$

where $\psi_{\pm, n}$ are constants independent of x and η have Borel transform (in the variable y conjugate to η)

$$\psi_{+, B}(x, y) = \frac{x^{-3/2}}{\Gamma(\frac{1}{2} + \frac{\lambda}{4})} s^{-1/2+\lambda/4} F\left(\frac{1}{4} + \frac{\lambda}{4}, \frac{3}{4} + \frac{\lambda}{4}; \frac{1}{2} + \frac{\lambda}{4}; s\right) \quad (6)$$

$$\psi_{-, B}(x, y) = \frac{x^{-3/2}}{\Gamma(\frac{1}{2} + \frac{\lambda}{4})} (s-1)^{-1/2+\lambda/4} F\left(\frac{1}{4} + \frac{\lambda}{4}, \frac{3}{4} + \frac{\lambda}{4}; \frac{1}{2} + \frac{\lambda}{4}; 1-s\right) \quad (7)$$

where $s = y/x^2 + 1/2$.

Lemma 1. *Set $\lambda = 0$. Let $\eta = 2zx^{-2}$ then $\psi_{-}(x, \eta) = \frac{1}{2\sqrt{\pi}} x^{1/2} \tilde{K}_{1/4}(z)$ is a solution of (4) if and only if $\tilde{K}_{1/4}(z)$ is a formal solution of the modified Bessel equation*

$$\left[\frac{d^2}{dz^2} - 1 + \frac{1}{z} \frac{d}{dz} - \frac{1}{16z^2} \right] \tilde{K}_{1/4}(z) = 0 \quad (8)$$

Proof. We start with the Weber equation (4)

$$\begin{aligned} & \left[\frac{d^2}{dx^2} - \eta^2 x^2 \right] \psi_{-} = 0 \\ & \left[\frac{d^2}{dx^2} - \eta^2 x^2 \right] x^{1/2} \tilde{K}_{1/4}(z) = 0 \\ & \frac{d}{dx} \left(\frac{1}{2} x^{-1/2} \tilde{K}_{1/4}(z) + x^{1/2} \frac{d}{dx} \tilde{K}_{1/4}(z) \right) - \eta^2 x^{5/2} \tilde{K}_{1/4}(z) = 0 \\ & -\frac{1}{4} x^{-3/2} \tilde{K}_{1/4}(z) + x^{-1/2} \frac{d}{dx} \tilde{K}_{1/4}(z) + x^{1/2} \frac{d^2}{dx^2} \tilde{K}_{1/4}(z) - \eta^2 x^{5/2} \tilde{K}_{1/4}(z) = 0 \\ & -\frac{1}{4} x^{-2} \tilde{K}_{1/4}(z) + x^{-1} \frac{d}{dx} \tilde{K}_{1/4}(z) + \frac{d^2}{dx^2} \tilde{K}_{1/4}(z) - \eta^2 x^2 \tilde{K}_{1/4}(z) = 0 \end{aligned}$$

Under the change of coordinates $\eta = 2zx^{-2}$ we have

$$\frac{d}{dx} = \eta x \frac{d}{dz} \quad \frac{d^2}{dx^2} = \eta \frac{d}{dz} + \eta^2 x^2 \frac{d^2}{dz^2}$$

therefore

$$\begin{aligned}
& -\frac{1}{4}x^{-2}\tilde{K}_{1/4}(z) + x^{-1}\frac{d}{dx}\tilde{K}_{1/4}(z) + \frac{d^2}{dx^2}\tilde{K}_{1/4}(z) - \eta^2x^2\tilde{K}_{1/4}(z) = 0 \\
& -\frac{1}{4}x^{-2}\tilde{K}_{1/4}(z) + \eta\frac{d}{dz}K_{1/4}(z) + \left(\eta\frac{d}{dz} + \eta^2x^2\frac{d^2}{dz^2}\right)\tilde{K}_{1/4}(z) - \eta^2x^2\tilde{K}_{1/4}(z) = 0 \\
& -\frac{1}{4}\frac{\eta}{2z}\tilde{K}_{1/4}(z) + \eta\frac{d}{dz}\tilde{K}_{1/4}(z) + \eta\frac{d}{dz}\tilde{K}_{1/4}(z) + 2z\eta\frac{d^2}{dz^2}\tilde{K}_{1/4}(z) - 2z\eta\tilde{K}_{1/4}(z) = 0 \\
& -\frac{1}{4}\frac{1}{4z^2}K_{1/4}(z) + \frac{1}{z}\frac{d}{dz}\tilde{K}_{1/4}(z) + \frac{d^2}{dz^2}K_{1/4}(z) - \tilde{K}_{1/4}(z) = 0 \\
& \left[\frac{d^2}{dz^2} - 1 + \frac{1}{z}\frac{d}{dz} - \frac{1}{16z^2}\right]\tilde{K}_{1/4}(z) = 0
\end{aligned}$$

□

Lemma 2. Set $\lambda = 0$. Let $\eta = 2zx^{-2}$ then

$$\psi_{-,B}(x, y) = x^{-3/2}\hat{\kappa}_{1/4}(\zeta)$$

where $y = \frac{\zeta}{2}x^2$ and $\hat{\kappa}_{1/4}(\zeta) = \mathcal{B}(\tilde{K}_{1/4})(\zeta)$.

Proof. We move to the Borel plane: let ζ be the variable conjugate to z and let y the conjugate to η , i.e.

$$\begin{aligned}
\mathcal{B}: \sum_{n \geq 0} a_n z^{-n-1} &\rightarrow \sum_{n \geq 0} a_n \frac{\zeta^n}{n!} \\
\mathcal{B}: \sum_{n \geq 0} a_n \eta^{-n-1} &\rightarrow \sum_{n \geq 0} a_n \frac{y^n}{n!}
\end{aligned}$$

if $\eta = 2zx^{-2}$ then $y = \frac{\zeta}{2}x^2$, indeed

$$\begin{aligned}
& \sum_{n \geq 0} a_n \eta^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n \geq 0} a_n \frac{y^n}{n!} \\
& \parallel \\
& \sum_{n \geq 0} a_n \left(\frac{2}{x^2}\right)^{-n-1} z^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n \geq 0} a_n \left(\frac{2}{x^2}\right)^{-n} z^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n \geq 0} a_n \left(\frac{x^2}{2}\right)^n \frac{\zeta^n}{n!}
\end{aligned}$$

In particular, $\tilde{K}_{1/4}(z)$ is known as the asymptotics of the modified Bessel function $K_{1/4}(z)$ which is equal to

$$\tilde{K}_{1/4}(z) = \sqrt{\pi}e^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!} (2z)^{-n-1/2}$$

Then we can compute its Borel transform:

$$\begin{aligned}
\hat{\kappa}_{1/4} &= \sqrt{\pi} \sum_{n \geq 0} \frac{(-1)^n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{\Gamma(n+1/2)n!} \left(\frac{\zeta-1}{2}\right)^{n-1/2} \\
&= \sqrt{\pi} \left(\frac{\zeta-1}{2}\right)^{-1/2} \sum_{n \geq 0} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{\Gamma(n+1/2)n!} \left(\frac{1-\zeta}{2}\right)^n \\
&= \sqrt{2}(\zeta-1)^{-1/2} F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \frac{1-\zeta}{2}\right)
\end{aligned}$$

hence

$$\begin{array}{ccc}
\psi_-(x, \eta) & \xrightarrow{B} & \psi_{-,B}(x, y) = \frac{x^{-3/2}}{\sqrt{\pi}} (s-1)^{-1/2+\lambda/4} F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}+; 1-s\right) \\
\parallel & & \parallel \cdot \frac{x^2}{2} \\
\frac{1}{2\sqrt{\pi}} x^{1/2} \tilde{K}_{1/4}(z) & \xrightarrow{B} & \frac{1}{2\sqrt{\pi}} x^{1/2} \sqrt{2}(\zeta-1)^{-1/2} F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \frac{1-\zeta}{2}\right)
\end{array}$$

□

Furthermore, the modified Bessel equation (8) has another holomorphic solution $I_{1/4}(z)$ (which together with $K_{1/4}$ forms a basis) whose asymptotic behaviour is

$$\tilde{I}_{1/4}(z) = \frac{1}{\sqrt{\pi}} e^z \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!} (2z)^{-n-1/2} \quad (9)$$