EXPONENTIAL INTEGRALS

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CONTENTS

1. INTRODUCTION

2. FRACTIONAL DERIVATIVES AND BOREL TRANSFORM

Definition 2.1. Let $\alpha \in (0,1)$ and $n \in \mathbb{N}$, then the $n+\alpha$-Caputo's derivative of a smooth fucntion f is defined as

$$\partial_x^{n+\alpha} f(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f^{(n+1)}(s) ds \tag{2.1}$$

In particular, this definition is well suited for the differential calculus in the convolutive model $(\mathbb{C}[[\zeta]], *)$. Let $\varphi(z) := \sum_{k\geq 0} a_k z^{-k-1} \in \mathbb{C}[[z^{-1}]]$ be Gevrey 1, then assuming $a_k = 0$ for every $k < n$, the Borel transform of $z^{n+\alpha}\varphi(z)$ can be computed in two different ways:

$$\begin{aligned} \mathcal{B}\big(z^{n+\alpha}\varphi(z)\big)(\zeta) = \mathcal{B}(z^{\alpha+n}) * \hat{\varphi}(\zeta) &= \int_0^\zeta \frac{(\zeta-s)^{-1-n-\alpha}}{(-1-n-\alpha)!} \sum_{k\geq 0} \frac{a_k}{k!} s^k ds \\ &= \frac{1}{(-\alpha)!} \int_0^\zeta (\zeta-s)^{-\alpha} \sum_{k\geq 0} \frac{a_k}{(k-n-1)!} s^{k-n-1} ds = \partial_\zeta^{n+\alpha} \hat{\varphi}(\zeta) \end{aligned} \tag{2.2}$$

$$\mathcal{B}\big(z^{n+\alpha}\varphi(z)\big)(\zeta) = \mathcal{B}\left(\sum_{k\geq 0} a_k z^{-k-1+n+\alpha}\right)(\zeta) = \sum_{k>n} \frac{a_k}{(k-n-\alpha)!} \zeta^{k-n-\alpha} \tag{2.3}$$

and computitng the integral which defines the $n+\alpha$-derivative in (2.2) we get exaclty the same result as (2.3).

3. Resurgence of exponential integrals

Let X be a $N-$dim manifold, $f : X \to \mathbb{C}$ be a holomorphic Morse function with only simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$I(z) := \int_{\mathcal{C}} e^{-zf}\, \nu \tag{3.1}$$

where $\mathcal{C}$ is a suitable countur such that the integral is well defined. For any Morse cirtial points x_α of f, the saddle point approximation gives the following formal series

$$I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\, \nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)}(2\pi)^{N/2} z^{-N/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n} \qquad \text{as } z \to \infty \tag{3.2}$$

where $\mathcal{C}_\alpha$ is a steepest descendet oath through the critical point x_α.

Theorem 3.1. *Let $N = 1$. Let $\tilde{\varphi}_\alpha(z) := e^{-zf(x_\alpha)}(2\pi)^{1/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n}$*

(1) *$\tilde{\varphi}_\alpha$ is Gevrey-1;*
(2) *$\hat{\varphi}(\zeta) := \mathcal{B}(\tilde{\varphi})$ is a germ of analytic function at $\zeta = \zeta_\alpha = f(x_\alpha)$;*
(3) *the following formual holds true*

$$\hat{\varphi}_\alpha(\zeta) = \partial^{3/2}_{\zeta,\text{based at }\zeta_\alpha}\left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu\right) = \Gamma(-1/2)\int_{\zeta_\alpha}^{\zeta} (\zeta-\zeta')^{-1/2}\partial_{\zeta'}\left(\int_{f^{-1}(\zeta')} \frac{\nu}{df}\right) d\zeta' \tag{3.3}$$

Proof. Part (1): Let us assume that locally $\nu = \sum_{j\geq 0} b_j^\alpha (t-t_\alpha)^j\, dt$ for $|t-t_\alpha| < \varepsilon$. By the steepest descendet method, $I_\alpha(z)$ can be aproximated as $z \to \infty$ as

$$\begin{aligned}
I_\alpha(\zeta) &\sim e^{-zf(x_\alpha)} \int_{-\varepsilon}^{\varepsilon} \sum_{n\geq 0} t^{2n} b_{2n}^\alpha e^{-zf''(x_\alpha)\frac{t^2}{2}}\, dt \\
&= e^{-zf(x_\alpha)} \sum_{n\geq 0} b_{2n}^\alpha \int_{-\varepsilon}^{\varepsilon} t^{2n} e^{-zf''(x_\alpha)\frac{t^2}{2}}\, dt \\
&= \sqrt{2\pi} e^{-zf(x_\alpha)} z^{-1/2} \sum_{n\geq 0} \frac{b_{2n}^\alpha}{(f''(x_\alpha))^{n+1/2}} (2n-1)!! z^{-n} \operatorname{Erf}\left(\frac{\sqrt{zf''(x_\alpha)}}{\sqrt{2}}\varepsilon\right) + \\
&\quad - 2e^{-zf(x_\alpha)} \sum_{n\geq 1} b_{2n}^\alpha (2n-1)!! e^{-zf''(x_\alpha)\frac{\varepsilon^2}{2}} \sum_{j=1}^{n} \frac{\varepsilon^{2j-1}}{(2j-1)!!} \left(f''(x_\alpha)z\right)^{n-j+1} \\
&\sim_{\varepsilon<<1} 2\varepsilon\sqrt{2\pi} e^{-zf(x_\alpha)} z^{-1/2} \sum_{n\geq 0} \frac{b_{2n}^\alpha}{(f''(x_\alpha))^{n+1/2}} (2n-1)!! z^{-n}
\end{aligned}$$

therefore

$$a_{\alpha,n} := 2\varepsilon \frac{b_{2n}^\alpha}{(f''(x_\alpha))^{n+1/2}} (2n-1)!!$$

Since, $\frac{1}{\varepsilon} = \limsup_n \sqrt[n]{b_n^\alpha}$

$$|a_{\alpha,n}| \leq C A^n n!$$

where we use $(2n-1)!! \sim \frac{2^n}{\sqrt{\pi n}} n!$ as $n \to \infty$.

Part (2):

$$\hat{\varphi}_\alpha(\zeta) = \mathcal{B}\left(e^{-zf(x_\alpha)}(2\pi)^{1/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n}\right)(\zeta) = T_{f(x_\alpha)}(2\pi)^{1/2}\left(\delta a_0 + \sum_{n\geq 0} a_{n+1} \frac{\zeta^n}{n!}\right)$$
$$(2\pi)^{1/2}\left(\delta(f_{x_\alpha}) a_0 + \sum_{n\geq 0} a_{n+1} \frac{(\zeta - f(x_\alpha))^n}{n!}\right)$$

Since $a_n \leq C A^n n!$, the series $\sum_{n\geq 0} a_{n+1} \frac{(\zeta - f(x_\alpha))^n}{n!}$ has a finite radius of convergence.

Part (3): thanks to properties of Caputo's fractional derivatives, we have that the Borel transform of $\tilde{I}_\alpha(z) = z^{-1/2} \tilde{\varphi}_\alpha(z)$ is

$$\partial^{1/2}_{\zeta,\text{based at } \zeta_\alpha} \hat{I}_\alpha(\zeta) = \hat{\varphi}_\alpha(\zeta). \tag{3.4}$$

In addition, we notice

$$\begin{aligned} I_\alpha(z) &= \int_{\mathcal{C}_\alpha} e^{-zf}\, \nu & f = \zeta \\ &= \int_{\mathcal{H}_\alpha} e^{-z\zeta} \partial_\zeta \left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu\right) d\zeta & \mathcal{H}_\alpha \text{ is Heckel countour} \\ &=: \int_{\mathcal{H}_\alpha} e^{-\zeta z} \hat{I}_\alpha(\zeta) d\zeta \end{aligned}$$

hence

$$\hat{\varphi}_\alpha(\zeta) = \partial^{1/2}_{\zeta,\text{based at } \zeta_\alpha}\left(\partial_\zeta\left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu\right)\right) = \partial^{3/2}_{\zeta,\text{based at } \zeta_\alpha}\left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu\right) \tag{3.5}$$

□

Example 3.2 (Airy)**.** Let $f(t) = \frac{t^3}{3} - t$ and

$$I(z) := \int_\gamma e^{-zf(t)} \text{dt}$$

where γ is a countour where the integral is well defined.

By the change of coordinates $z = x^{3/2}$, $I(z) = -2\pi i z^{-1/3} Ai(x)$ where

$$Ai(x) := \frac{1}{2\pi i} \int_{-\infty e^{-i\frac{\pi}{3}}}^{\infty e^{i\frac{\pi}{3}}} e^{\frac{t^3}{3} - zt} \text{dt}$$

hence $I(z)$ solves the following ODE[1]

$$I''(z)-\frac{4}{9}I(z)+\frac{I'(z)}{z}-\frac{1}{9}\frac{I(z)}{z^2}=0 \tag{3.6}$$

A formal solution of (3.6) can be computed by making the following ansatz

$$\tilde{I}(z)=\sum_{k\in\mathbb{N}^2}U^k e^{-\lambda\cdot kz}z^{-\tau\cdot k}w_k(z) \tag{3.7}$$

with $U^{(k_1,k_2)}=U_1^{k_1}U_2^{k_2}$ and $U_1,U_2\in\mathbb{C}$ are constant parameter, $\lambda=(\frac{2}{3},-\frac{2}{3})$, $\tau=(\frac{1}{2},\frac{1}{2})$, and $\tilde{w}_k(z)\in\mathbb{C}[[z^{-1}]]$. In addition, we can check that the only non zero $\tilde{w}_k(z)$ occurs at $k=(1,0)$ and $k=(0,1)$, therefore

$$\tilde{I}(z)=U_1e^{-2/3z}z^{-1/2}\tilde{w}_+(z)+U_2e^{2/3z}z^{-1/2}\tilde{w}_-(z) \tag{3.8}$$

where from now on we denote $\tilde{w}_+(z)=\tilde{w}_{(1,0)}(z)$ and $\tilde{w}_-(z)=\tilde{w}_{(0,1)}(z)$. In particular, $\tilde{w}_+(z)$ and $\tilde{w}_-(z)$ are formal solution of

$$\tilde{w}_+''-\frac{4}{3}\tilde{w}_+'+\frac{5}{36}\frac{\tilde{w}_+}{z^2}=0 \tag{3.9}$$

$$\tilde{w}_-''+\frac{4}{3}\tilde{w}_-'+\frac{5}{36}\frac{\tilde{w}_-}{z^2}=0 \tag{3.10}$$

Taking the Borel transform of (3.9), (3.10) we get

$$\zeta^2\hat{w}_+(\zeta)+\frac{4}{3}\zeta\hat{w}_++\frac{5}{36}\zeta*\hat{w}_+=0$$

$$\zeta^2\hat{w}_+(\zeta)+\frac{4}{3}\zeta\hat{w}_++\frac{5}{36}\int_0^\zeta(\zeta-\zeta')\hat{w}_+(\zeta')d\zeta'=0$$

$$\zeta^2\hat{w}_-(\zeta)-\frac{4}{3}\zeta\hat{w}_-+\frac{5}{36}\zeta*\hat{w}_-=0$$

$$\zeta^2\hat{w}_-(\zeta)-\frac{4}{3}\zeta\hat{w}_-+\frac{5}{36}\int_0^\zeta(\zeta-\zeta')\hat{w}_-(\zeta')d\zeta'=0$$

and taking derivatives we get

$$\zeta(\frac{4}{3}+\zeta)\hat{w}_+''+(\frac{8}{3}+4\zeta)\hat{w}_+'+\frac{77}{36}\hat{w}_+=0$$

$$\frac{4}{3}\zeta(1+\frac{3}{4}\zeta)\hat{w}_+''+(\frac{8}{3}+4\zeta)\hat{w}_+'+\frac{77}{36}\hat{w}_+=0$$

$$u(1-u)\hat{w}_+''(u)+(2-4u)\hat{w}_+'(u)-\frac{77}{36}\hat{w}_+(u)=0 \qquad u=-\frac{3}{4}\zeta$$

[1] $Ai(x)$ solves the Airy equation $y''=xy$.

$$
\begin{aligned}
&\zeta(-\frac{4}{3}+\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0\\
&\frac{4}{3}\zeta(-1+\frac{3}{4}\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0\\
&\quad u(1-u)\hat{w}_-''(u)+(2-4u)\hat{w}_-'(u)-\frac{77}{36}\hat{w}(u) \qquad\qquad u=\frac{3}{4}\zeta
\end{aligned}
$$

Notice that the latter equations are hypergeometric, hence a solution is given by

$$\hat{w}_+(\zeta)=c_1\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right) \tag{3.11}$$

$$\hat{w}_-(\zeta)=c_2\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right) \tag{3.12}$$

for some constants $c_1, c_2 \in \mathbb{C}$ (see DLMF 15.10.2). In addition $\hat{w}_\pm(\zeta)$ have a log singularity respectively at $\zeta=\mp\frac{4}{3}$, therefore they are $\{\mp\frac{4}{3}\}$-resurgent functions.[2]

Remark 3.3. $\hat{w}_+(\zeta)$ is Laplace summable along the positive real axis, and it can be analyticaly continued on $\mathbb{C}\setminus -\frac{4}{3}\mathbb{R}_{\leq 0}$ with (see 15.2.3 DLMF)

$$
\begin{aligned}
\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0)&=-\frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\frac{3}{4}\zeta)^n \qquad\qquad \zeta<-\frac{4}{3}\\
&=\frac{36}{5}i\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\frac{3}{4}\zeta)^{n-1}\\
&=-\frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\left(\frac{5}{144}(4+3\zeta)\left(1+{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)\right)\right)\\
&=\mathbf{i}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)\\
&=\mathbf{i}\,\hat{w}_-(\zeta+\frac{4}{3})
\end{aligned}
$$

Anolougusly, $\hat{w}_-(\zeta)$ is Laplace summable along the negative real axis, and it jumps across the branch cut $\frac{4}{3}\mathbb{R}_{\geq 0}$ as

$$
\begin{aligned}
\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0)&=\frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1-\frac{3}{4}\zeta)^n \qquad\qquad \zeta>\frac{4}{3}\\
&=\frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\left(-\frac{5}{144}(-4+3\zeta){}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)\right)\\
&=-\mathbf{i}\pi\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)\\
&=-\mathbf{i}\,\hat{w}_+(\zeta-\frac{4}{3})
\end{aligned}
$$

[2]The solution we find are equal to the ones in DLMF §9.7. They also agree with slide 10 of Maxim's talk for ERC and with the series (6.121) in Sauzin's book. However they do not agree with (2.16) in Mariño's Diablerets.

These relations manifest the resurgence property of $\tilde{I}$, indeed near the singularities in the Borel plane of either $\hat{w}_+$ or $\hat{w}_-$, $\hat{w}_-$ and $\hat{w}_+$ respectively contribute to the jump of the former solution.

Our next goal is to prove that the Borel transform of $\tilde{I}(z)$ can be written in terms of $1/f'(f^{-1}(\zeta))$, namely formula (3.3). It is convenient to consider the two asymptotic formal solutions separately, namely we define

$$\tilde{I}_{-1}(z) := e^{-2/3z} z^{-1/2} \tilde{w}_+(z) =: z^{-1/2} \tilde{u}_+(z) \tag{3.13}$$

$$\tilde{I}_1(z) := e^{2/3z} z^{-1/2} \tilde{w}_-(z) =: z^{-1/2} \tilde{u}_-(z) \tag{3.14}$$

In particular, $\tilde{u}_\pm(z)$ are solutions of

$$\tilde{u}''(z) - \frac{4}{9}\tilde{u}(z) + \frac{5}{36}\frac{\tilde{u}(z)}{z^2} = 0 \tag{3.15}$$

with asymptotic behaviour $\tilde{u}_\pm(z) \sim O(e^{\pm 2/3z})$ as $z \to \infty$.

The Borel transforms $\hat{u}_\pm(\zeta)$ solve the same equation

$$\zeta^2 \hat{u} - \frac{4}{9}\hat{u} + \frac{5}{36}\zeta * \hat{u}$$

$$\zeta^2 \hat{u} - \frac{4}{9}\hat{u} + \frac{5}{36}\int_0^\zeta (\zeta - \zeta')\hat{u}(\zeta')d\zeta'$$

taking derivatives is equivalent to

$$(\zeta^2 - \frac{4}{9})\hat{u}''(\zeta) + 4\zeta\hat{u}'(\zeta) + \frac{77}{36}\hat{u}(\zeta) = 0$$

and Mathematica gives the following solutions

$$\hat{u}(\zeta) = c_1\, {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{9}{4}\zeta^2\right) + \frac{3i}{2}\zeta c_2\, {}_1F_2\left(\frac{13}{12}, \frac{17}{12}, \frac{3}{2}, \frac{9}{4}\zeta^2\right) =$$

$$= c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}}\left({}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} + \frac{3}{4}\zeta\right)\right) \quad \text{see DLMF 15.8.27}$$

$$+ \frac{3i}{2}\zeta c_2\left(\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{3\zeta\Gamma(-\frac{1}{2})\Gamma(2)}\right)\left({}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right)\right) \quad \text{see DLMF 15.8.28}$$

$$= \left(c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} - c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) +$$

$$+ \left(c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} + c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right)$$

Since $\hat{u}_+$ has a simple singularity at $\zeta = -2/3$ and $\hat{u}_-$ has a simple singularity at $\zeta = 2/3$, we have

$$\hat{u}_+(\zeta) = C_1 T_{-2/3}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right) = C_1 T_{-2/3}\,\hat{w}_+(\zeta)$$
$$\hat{u}_-(\zeta) = C_2 T_{2/3}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right) = C_2 T_{2/3}\,\hat{w}_-(\zeta)$$

Lemma 3.4. *The following identity holds true*

$$ {}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right) = \frac{1}{1-u^2} \qquad \zeta = \frac{u^3}{3} - u \tag{3.16}$$

Proof.

$$\begin{aligned}
{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right) &= 2\cos\left(\frac{1}{3}\arcsin\left(\frac{3}{2}\zeta\right)\right)(4-9\zeta^2)^{-1/2} && \text{Mathematica [Fullsimplify]}\\
&= \frac{\cos(y)}{\cos(3y)} && 3y = \arcsin\left(\frac{3}{2}\zeta\right)\\
&= \frac{\cos(y)}{\cos(2y)\cos(y)-\sin(2y)\sin(y)}\\
&= \frac{1}{\cos(2y)-2\sin^2(y)}\\
&= \frac{1}{1-4\sin^2(y)} && \zeta = 2\sin(y)-\frac{8}{3}\sin^3(y)
\end{aligned}$$

Therefore, if $u := -2\sin(y)$, we have $\zeta = \frac{u^3}{3} - u = f(u)$ and

$$ {}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right) = \frac{1}{1-u^2} = -\frac{1}{f'(u)}$$

□

Then equations (3.3) is equivalent to

Claim 3.5.

$$\hat{w}_+(\zeta-2/3) = -\frac{1}{\sqrt{\pi}}\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}\partial_s\left[{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}s^2\right)\right]ds \qquad \zeta\in(2/3,+\infty) \tag{3.17}$$

Let us study the RHS of claim (3.5)

$$\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}\partial_s\left[{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}s^2\right)\right]ds=2\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}s\,{}_2F_1\left(\frac{4}{3},\frac{5}{3};\frac{3}{2};\frac{9}{4}s^2\right)ds$$

$$=-\frac{2}{9}\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};\frac{1}{2}-\frac{3s}{4}\right)ds+\frac{2}{9}\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};\frac{1}{2}+\frac{3s}{4}\right)ds \quad \text{15.8.28 DLMF}$$

$$=\frac{4}{9\sqrt{3}}\int_0^x(x-y)^{-1/2}\left[{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};1+y\right)-{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};y\right)\right]dy \qquad x\in(-\infty,0)$$

$$=-\frac{\sqrt{3}\pi}{4}\Gamma(1/6)e^{-\frac{3}{2}\pi i}\int_0^x(x-y)^{-1/2}y^{-5/3}{}_2F_1\left(\frac{5}{3},\frac{1}{6};\frac{1}{3};\frac{1}{y}\right)dy$$

$$=-\frac{\sqrt{3}\pi}{4}\frac{\Gamma(1/6)\Gamma(7/6)}{\Gamma(5/3)}e^{-\frac{3}{2}\pi i}|x|^{-7/6}{}_2F_1\left(\frac{7}{6},\frac{1}{6};\frac{1}{3};\frac{1}{x}\right) \qquad (4.3)$$

$$=\frac{5\pi^2}{24\sqrt{3}}\frac{\Gamma(1/3)}{\Gamma(5/3)}\left({}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\frac{1}{2}-\frac{\zeta}{4}\right)-e^{-\frac{11}{6}\pi i}{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\frac{1}{2}+\frac{\zeta}{4}\right)\right) \qquad \zeta\in(2/3,+\infty)$$

however, ${}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\frac{1}{2}+\frac{\zeta}{4}\right)$ has a branch cut at $\zeta\in(2/3,+\infty)$, thus the claim holds true.

Analougusly, it can be verified for $\hat{w}_-(\zeta+2/3)$ for $\zeta\in(-\infty,-2/3)$.

Example 3.6 (Bessel)**.** Let $X=\mathbb{C}^*$, $f(x)=x+\frac{1}{x}$ and $\nu=\frac{dx}{x}$, then the ciritcal points of f are $x=\pm1$ and

(3.18) $$I(z):=\int_0^{\infty}e^{-zf(x)}\frac{dx}{x}.$$

By change of cooridnates $t=zx$

$$I(z)=\int_0^{\infty}e^{-z\left(\frac{t}{z}+\frac{z}{t}\right)}\frac{dt}{t}=\int_0^{\infty}e^{-\left(t+\frac{z^2}{t}\right)}\frac{dt}{t}=2K_0(2z) \qquad |\arg z|<\frac{\pi}{4}$$

where $K_0(z)$ is the modified Bessel function (see definition 10.32.10 DLMF). In particular, since $K_0(z)$ solves

(3.19) $$\frac{d^2}{dz^2}w(z)+\frac{1}{z}\frac{d}{dz}w(z)-w(z)=0$$

and $K_0(z)\sim\left(\frac{\pi}{2}\right)^{1/2}e^{-z}z^{-1/2}\sum_{k\geq0}\frac{(1/2)_k(1/2)_k}{(-2)^k k!}z^{-k}$ as $z\to\infty$ (see DLMF 10.40.2), then $I(z)$ is a solution of

(3.20) $$\frac{d^2}{dz^2}I(z)+\frac{1}{z}\frac{d}{dz}I(z)-4I(z)=0.$$

The formal integral of (3.20) is given by a two parameter formal solution $\tilde{I}_1(z)$

(3.21) $$\tilde{I}(z)=\sum_{\mathbf{k}\in\mathbb{N}^2}U^k e^{-\mathbf{k}\cdot\lambda z}z^{-\tau\cdot\mathbf{k}}\tilde{w}_{\mathbf{k}}(z)$$

where $\lambda=(2,-2)$, $\tau=(-\frac{1}{2},-\frac{1}{2})$, $U^k:=U_1^{k_1}U_2^{k_2}$ with $k=(k_1,k_2)$ and $U_1,U_2\in\mathbb{C}$, and $\tilde{w}_{\mathbf{k}}(z)\in\mathbb{C}[[z^{-1}]]$ is a formal solution of

$$\begin{aligned}\tilde{w}''_{\mathbf{k}}(z)-4(k_1-k_2)\tilde{w}'_{\mathbf{k}}(z)+4(1-(k_1-k_2)^2)\tilde{w}_{\mathbf{k}}(z)+\frac{(k_1+k_2-1)}{z}\tilde{w}'_{\mathbf{k}}(z)+\\ -2(k_1-k_2)\frac{(k_1+k_2-1)}{z}\tilde{w}_{\mathbf{k}}(z)+\frac{(k_1+k_2)^2}{4z^2}\tilde{w}_{\mathbf{k}}(z)=0\end{aligned}\tag{3.22}$$

The only non zero $\tilde{w}_{\mathbf{k}}(z)$ occurs for $\mathbf{k}=(1,0)$ and $\mathbf{k}=(0,1)$, hence

$$\tilde{I}(z)=U_1e^{-2z}z^{-1/2}\tilde{w}_{(1,0)}(z)+U_2e^{2z}z^{-1/2}\tilde{w}_{(0,1)}(z)\tag{3.23}$$

and according to our convention, we define

$$\tilde{I}_1(z):=e^{-2z}z^{-1/2}\tilde{w}_{(1,0)}(z)\tag{3.24}$$

$$\tilde{I}_{-1}(z):=e^{2z}z^{-1/2}\tilde{w}_{(0,1)}(z).\tag{3.25}$$

We set $\tilde{w}_{(1,0)}=\tilde{w}_+$ and $\tilde{w}_{(0,1)}=\tilde{w}_-$, then their Borel transforms are solutions respectively of the following equations

$$\begin{aligned}&(+)\quad \zeta^2\hat{w}_+(\zeta)+4\zeta\hat{w}_+(\zeta)+\frac{1}{4}\zeta*\hat{w}_+(\zeta)=0\\ &(-)\quad \zeta^2\hat{w}_+(\zeta)-4\zeta\hat{w}_+(\zeta)+\frac{1}{4}\zeta*\hat{w}_+(\zeta)=0\end{aligned}$$

taking twice derivative in ζ we get

$$\begin{aligned}&(+)\quad (\zeta^2+4\zeta)\frac{d^2}{d\zeta^2}\hat{w}_++4(\zeta-1)\frac{d}{d\zeta}\hat{w}_++\frac{9}{4}\hat{w}_+=0\\ &(-)\quad (\zeta^2-4\zeta)\frac{d^2}{d\zeta^2}\hat{w}_-+4(\zeta+1)\frac{d}{d\zeta}\hat{w}_++\frac{9}{4}\hat{w}_-=0\\ &(+)\quad \xi(1-\xi)\frac{d^2}{d\xi^2}\hat{w}_++(1-4\xi)\frac{d}{d\xi}\hat{w}_+-\frac{9}{4}\hat{w}_+=0 \qquad \xi=-\frac{\zeta}{4}\\ &(-)\quad \xi(1-\xi)\frac{d^2}{d\xi^2}\hat{w}_-+(1-4\xi)\frac{d}{d\xi}\hat{w}_--\frac{9}{4}\hat{w}_-=0 \qquad \xi=\frac{\zeta}{4}\end{aligned}$$

therefore, since equation $(+),(-)$ are hypergeometric the fundamental solution is (see DLMF 15.10.2)

$$\hat{w}_+(\zeta)={}_2F_1\left(\frac{3}{2},\frac{3}{2};2;-\frac{\zeta}{4}\right)\tag{3.26}$$

$$\hat{w}_-(\zeta)={}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{\zeta}{4}\right)\tag{3.27}$$

In particular, we notice that taking the series expansion of $\hat{w}_+$ and $\hat{w}_-$ we get numerically that

$$\hat{w}_+(\zeta-4) = \frac{1}{\pi}\log(z)\hat{w}_-(z)+\phi_{\text{reg}}$$
$$\hat{w}_-(\zeta+4) = \frac{1}{\pi}\log(z)\hat{w}_+(z)+\psi_{\text{reg}}$$

and analytically (thanks to 15.2.3 DLMF)

$$\begin{aligned}
\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0) &= {}_2F_1\left(\frac{3}{2},\frac{3}{2};2;-\frac{\zeta}{4}+i0\right)-{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;-\frac{\zeta}{4}-i0\right) \qquad \zeta<-4\\
&= -8i\left(-\frac{\zeta}{4}-1\right)^{-1}\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\frac{\zeta}{4}+1\right)^n\\
&= 8i\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\frac{\zeta}{4}+1\right)^{n-1}\\
&= 8i\sum_{n\geq 0}\frac{(1/2)_{n+1}(1/2)_{n+1}}{(n+1)!\Gamma(n+1)}\left(\frac{\zeta}{4}+1\right)^n\\
&= \mathbf{2i}\sum_{n\geq 0}\frac{(3/2)_n(3/2)_n}{(n)!\Gamma(n+2)}\left(\frac{\zeta}{4}+1\right)^n\\
&= \mathbf{2i}\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{\zeta}{4}+1\right)
\end{aligned}$$

$$\begin{aligned}
\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0) &= {}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{\zeta}{4}+i0\right)-{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{\zeta}{4}-i0\right) \qquad \zeta>4\\
&= 8i\left(\frac{\zeta}{4}-1\right)^{-1}\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\frac{\zeta}{4}-1\right)^n\\
&= -8i\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(1-\frac{\zeta}{4}\right)^{n-1}\\
&= -8i\sum_{n\geq 0}(-1)^n\frac{(1/2)_{n+1}(1/2)_{n+1}}{(n+1)!\Gamma(n+1)}\left(1-\frac{\zeta}{4}\right)^n\\
&= -\mathbf{2i}\sum_{n\geq 0}(-1)^n\frac{(3/2)_n(3/2)_n}{(n)!\Gamma(n+2)}\left(1-\frac{\zeta}{4}\right)^n\\
&= -\mathbf{2i}\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;1-\frac{\zeta}{4}\right)
\end{aligned}$$

These are evidence of the resurgent properties of $\tilde{I}_{\pm 1}(z)$.

Lemma 3.7. *The following identity holds true*

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{1}{2};\frac{\zeta^2}{4}\right) = 2i\frac{u}{u^2-1} \qquad \zeta = u+\frac{1}{u} \tag{3.28}$$

Proof. From 15.4.13 DLMF, we have

$$
\begin{aligned}
{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{1}{2};\frac{\zeta^2}{4}\right) &= \frac{2}{\sqrt{4-\zeta^2}} && y = \operatorname{arccsc}(\zeta/2) \\
&= \frac{1}{\sqrt{1-\csc^2(y)}} \\
&= -i\tan(y) && \zeta = \frac{2}{\sin(y)}
\end{aligned}
$$

therefore if $u = \tan\left(\frac{y}{2}\right)$, we have $\zeta = \frac{1+u^2}{u} = f(u)$ and

$$
{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{1}{2};\frac{\zeta^2}{4}\right) = 2i\frac{u}{u^2-1} = \frac{2i}{f'(u)u}
$$

□

Claim 3.8.

$$
\hat{w}_+(\zeta-2) = i\pi\int_2^{\zeta}(\zeta-\zeta')^{-1/2}2\zeta'\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};\frac{3}{2};\frac{\zeta'^2}{4}\right)d\zeta' \qquad \zeta\in(2,+\infty) \tag{3.29}
$$

Proof. Let us first consider the RHS of (3.8)

$$
\begin{aligned}
&2\pi\int_2^{\zeta}(\zeta-\zeta')^{-1/2}\zeta'\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};\frac{3}{2};\frac{\zeta'^2}{4}\right)d\zeta' = \\
&= \frac{4}{3}\int_2^{\zeta}(\zeta-\zeta')^{-1/2}\left[{}_2F_1\left(2,2;\frac{5}{2};\frac{1}{2}+\frac{\zeta'}{4}\right) - {}_2F_1\left(2,2;\frac{5}{2};\frac{1}{2}-\frac{\zeta'}{4}\right)\right]d\zeta' \\
&= \frac{8}{3}\int_0^{x}(y-x)^{-1/2}\left[{}_2F_1\left(2,2,;\frac{5}{2};y\right) - {}_2F_1\left(2,2;\frac{5}{2};1-y\right)\right]dy && x\in(-\infty,0) \\
&= 2\pi\int_0^{x}(x-y)^{-1/2}y^{-2}F\left(2,\frac{1}{2};1;\frac{1}{y}\right)dy && (4.3) \\
&= \pi^2|x|^{-3/2}{}_2F_1\left(\frac{3}{2},\frac{1}{2};1;\frac{1}{x}\right) && x\in(-\infty,0) \\
&= \frac{\pi^2}{2}\left({}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{1}{2}-\frac{\zeta}{4}\right) - i\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{1}{2}+\frac{\zeta}{4}\right)\right) && \zeta\in(2,+\infty)
\end{aligned}
$$

however ${}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{1}{2}+\frac{\zeta}{4}\right)$ has a branch cut at $\zeta\in(2,+\infty)$, thus the claim holds true. □

Analougusly, it can be verified for $\hat{w}_-(\zeta+2)$ for $\zeta\in(-\infty,-2)$.

4. USEFUL IDENTITIES FOR GAUSS HYPERGEOMTRIC FUNCTIONS

$$ {}_2F_1(a,b;c;z) = e^{b\pi i}\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(a+b-c+1)\Gamma(c-b)}{}_2F_1(a,b;c;1-z)+ \\ - e^{(a+b-c)\pi i}\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(b)\Gamma(a-b+1)}|z|^{-a}{}_2F_1\left(a,a-c+1;a-b+1;\frac{1}{z}\right) \tag{4.1} $$

$$ \int_0^x |y|^{a-\mu-1}{}_2F_1\big(a,b;c,y\big)(x-y)^{\mu-1}dy = \frac{\Gamma(\mu)\Gamma(a-\mu)}{\Gamma(a)}|x|^{a-1}{}_2F_1\big(a-\mu,b;c;x\big) \\ x\in(-\infty,0)\cup(0,1),\ \Re a > \Re\mu > 0 \tag{4.2} $$

which can be rewritten as

$$ \int_{y>x} |y|^{-a}|x-y|^{\mu-1}{}_2F_1\big(a,b;c;y^{-1}\big)dy = \frac{\Gamma(\mu)\Gamma(a-\mu)}{\Gamma(a)}|x|^{-a+\mu}{}_2F_1\big(a-\mu,b;c;x^{-1}\big) \\ x\in(-\infty,0)\cup(1,\infty),\ \Re a > \Re\mu > 0 \tag{4.3} $$