EXPONENTIAL INTEGRALS

VERONICA FANTINI

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2. Fractional derivatives and Borel transform

Definition 2.1. For $\alpha \in (0,1)$, the fractional integral $\partial^{\alpha-1}_{x \text{ from } 0}$ is defined by

$$\partial^{\alpha-1}_{x \text{ from } 0} f(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-x')^{-\alpha} f(x')\, dx'.$$

Fractional derivatives $\partial^{n+\alpha}_{x \text{ from } 0}$ are defined for integers $n \geq -1$ by composing the fractional integral with powers of $\frac{\partial}{\partial x}$. However, $\partial^{\alpha-1}_{x \text{ from } 0}$ and $\frac{\partial}{\partial x}$ don't commute: their commutator is an initial value operator **[check, clarify]**. Various ordering conventions give various definitions of $\partial^{n+\alpha}_{x \text{ from } 0} f(x)$, which differ by operators that depend on the germ of f at zero **[Lazarević, §1.3—original source Podlubny]**. We'll use the *Riemann-Liouville* convention.

Definition 2.2. For $\alpha \in (0,1)$ and $n \geq -1$ integer, the *Riemann-Liouville fractional derivative* $\partial^{n+\alpha}_{x \text{ from } 0}$ is defined by

$$\partial^{n+\alpha}_{x \text{ from } 0} := \left(\tfrac{\partial}{\partial x}\right)^{n+1} \partial^{\alpha-1}_{x \text{ from } 0}.$$

Each convention brings its own annoyances to interactions with the Borel transform.[1] The Riemann-Liouville convention will be the least annoying for our purposes.

Theorem 2.3. *Take a Gevrey-*1 *formal series* $\varphi(z) = \sum_{k \geq 0} a_k z^{-(k+1)}$*. For* $\alpha \in (0,1)$ *and* $n \geq -1$*, we have*

$$\partial^{n+\alpha}_{\zeta \text{ from } 0} [\mathcal{B}\varphi](\zeta) = [\mathcal{B} z^{n+\alpha} \varphi](\zeta)$$

[1] See Remark **??** for more.

Lemma 2.4. *For $\alpha \in (0,1)$ and $n \geq -1$,*

$$\partial^{n+\alpha}_{\zeta \text{ from } 0}\left[\mathcal{B}z^{-(k+1)}\right](\zeta) = \left[\mathcal{B}z^{n+\alpha}z^{-(k+1)}\right](\zeta)$$

Proof. First, evaluate

$$\begin{aligned}
\partial^{\alpha-1}_{\zeta \text{ from } 0}\left[\mathcal{B}z^{-(k+1)}\right](\zeta) &= \frac{1}{\Gamma(1-\alpha)}\int_0^\zeta (\zeta-\zeta')^{-\alpha}\frac{\zeta'^k}{\Gamma(k+1)}\,d\zeta' \\
&= \frac{1}{\Gamma(1-\alpha)\Gamma(k+1)}\int_0^1 (\zeta-\zeta t)^{-\alpha}(\zeta t)^k\,\zeta\,dt \\
&= \frac{\zeta^{k-(\alpha-1)}}{\Gamma(1-\alpha)\Gamma(k+1)}\int_0^1 (1-t)^{-\alpha}t^k\,dt \\
&= \frac{\zeta^{k-(\alpha-1)}}{\Gamma(k-\alpha)}
\end{aligned}$$

by reducing the integral to Euler's beta function **[DLMF 5.12.1]**. This establishes that

$$\left(\frac{\partial}{\partial\zeta}\right)^{n+1}\partial^{\alpha-1}_{\zeta \text{ from } 0}\left[\mathcal{B}z^{-(k+1)}\right](\zeta) = \frac{\zeta^{k-(n+\alpha)}}{\Gamma\big(k-(n+\alpha)+1\big)} \tag{2.1}$$

for $n=-1$. If (2.1) holds for $n=m$, it also holds for $n=m+1$, because

$$\begin{aligned}
\frac{\partial}{\partial\zeta}\left(\frac{\partial}{\partial\zeta}\right)^{m+1}\partial^{\alpha-1}_{\zeta \text{ from } 0}\left[\mathcal{B}z^{-(k+1)}\right](\zeta) &= \frac{\partial}{\partial\zeta}\left(\frac{\zeta^{k-(m+\alpha)}}{\big(k-(m+\alpha)\big)\Gamma\big(k-(m+\alpha)\big)}\right) \\
&= \frac{\zeta^{k-(m+1+\alpha)}}{\Gamma\big(k-(m+\alpha)\big)}
\end{aligned}$$

Hence, (2.1) holds for all $n \geq -1$, and the desired result quickly follows. The condition $\alpha \in (0,1)$ saves us from the trouble we'd run into if $k-(m+\alpha)$ were in $\mathbb{Z}_{\leq 0}$. This is how we avoid the initial value corrections that appear in integer-order derivatives of Borel transforms. □

Definition 2.5. Let $\alpha \in (0,1)$ and $n \in \mathbb{N}$, then the $n+\alpha$-Caputo's derivative of a smooth fucntion f is defined as

$$\partial^{n+\alpha}_x f(x) := \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-s)^{-\alpha}f^{(n+1)}(s)ds \tag{2.2}$$

In particular, this definition is well suited for the differential calculus in the convolutive model $(\mathbb{C}[[\zeta]],*)$. Let $\varphi(z) := \sum_{k\geq 0} a_k z^{-k-1} \in \mathbb{C}[[z^{-1}]]$ be Gevrey 1, then assuming $a_k = 0$ for every $k < n$, the Borel transform of $z^{n+\alpha}\varphi(z)$ can be computed in two different ways: **[Can we do this when φ isn't in $o(z^n)$?]**

$$\mathcal{B}\big(z^{n+\alpha}\varphi(z)\big)(\zeta) = \mathcal{B}(z^{\alpha+n}) * \hat{\varphi}(\zeta) = \int_0^\zeta \frac{(\zeta-s)^{-1-n-\alpha}}{(-1-n-\alpha)!}\sum_{k\geq 0}\frac{a_k}{k!}s^k\,ds \tag{2.3}$$

$$= \frac{1}{(-\alpha)!}\int_0^\zeta (\zeta-s)^{-\alpha}\sum_{k\geq 0}\frac{a_k}{(k-n-1)!}s^{k-n-1}\,ds = \partial_\zeta^{n+\alpha}\hat{\varphi}(\zeta)$$

$$\mathcal{B}\big(z^{n+\alpha}\varphi(z)\big)(\zeta) = \mathcal{B}\left(\sum_{k\geq 0} a_k z^{-k-1+n+\alpha}\right)(\zeta) = \sum_{k>n}\frac{a_k}{(k-n-\alpha)!}\zeta^{k-n-\alpha} \tag{2.4}$$

and computitng the integral which defines the $n+\alpha$-derivative in (2.3) we get exaclty the same result as (2.4).

3. Resurgence of exponential integrals

Let X be a $N-$dim manifold, $f: X \to \mathbb{C}$ be a holomorphic Morse function with only simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$I(z) := \int_{\mathcal{C}} e^{-zf}\,\nu \tag{3.1}$$

where $\mathcal{C}$ is a suitable countur such that the integral is well defined. For any Morse cirtial points x_α of f, the saddle point approximation gives the following formal series

$$I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\,\nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)}(2\pi)^{N/2}z^{-N/2}\sum_{n\geq 0}a_{\alpha,n}z^{-n} \qquad \text{as } z\to\infty \tag{3.2}$$

where $\mathcal{C}_\alpha$ is a steepest descent path through the critical point x_α. Notice that $f\circ\mathcal{C}_\alpha$ lies in the ray $\zeta_\alpha + [0,\infty)$, where $\zeta_\alpha := f(x_\alpha)$.

Theorem 3.1. *Let $N=1$. Let $\tilde{\varphi}_\alpha(z) := e^{-zf(x_\alpha)}(2\pi)^{1/2}\sum_{n\geq 0}a_{\alpha,n}z^{-n}$ and assume $f''(x_\alpha)\neq 0$ for every critical point x_α. Then:*

1. *The series $\tilde{\varphi}_\alpha$ is Gevrey-1.*
2. *The series $\hat{\varphi}(\zeta) := \mathcal{B}(\tilde{\varphi})$ converges near $\zeta = \zeta_\alpha$.*
3. *For any ζ on the ray going rightward from ζ_α, we have*

$$\hat{\varphi}_\alpha(\zeta) = \partial^{3/2}_{\zeta \text{ from } \zeta_\alpha}\left(\int_{\mathcal{C}_\alpha(\zeta)}\nu\right)$$

$$= \left(\frac{\partial}{\partial\zeta}\right)^2\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{\zeta_\alpha}^{\zeta}(\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_\alpha(\zeta)}\nu\right)d\zeta',$$

where $\mathcal{C}_\alpha(\zeta)$ is the part of $\mathcal{C}_\alpha$ that goes through $f^{-1}([\zeta_\alpha, \zeta])$. Notice that $\mathcal{C}_\alpha(\zeta)$ starts and ends in $f^{-1}(\zeta)$. ***[The integral over $\mathcal{C}_\alpha(\zeta)$ needs to satisfy some condition near $\zeta = 0$ for this to hold. Figure out what that is! Also, be careful about the orientation of $\mathcal{C}_\alpha$.]***

(4)

Proof. Part (1): Let's write $\approx$ when two functions are asymptotic (at all orders around the base point **[is this the right condition?]**), and $\sim$ when a function is asymptotic to a formal power series (at the truncation order of each partial sum).

Since f is Morse, we can find a holomorphic chart τ around x_α with $\frac{1}{2}\tau^2 = f - \zeta_\alpha$. Let $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$ be the parts of $\mathcal{C}_\alpha$ that go from the past to x_α and from x_α to the future, respectively. We can arrange for τ to be valued in $(-\infty, 0]$ and $[0, \infty)$ on $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$, respectively. **[We should explicitly spell out and check the conditions that make this possible. I think we're implicitly orienting $\mathcal{C}_\alpha$ so that τ in the upper half-plane.]** Since ν is holomorphic, we can express it as a Taylor series

$$\nu = \sum_{n\geq 0} b_n^\alpha \tau^n \, d\tau$$

that converges in some disk $|\tau| < \varepsilon$.

By the steepest descent method,

$$e^{-z\zeta_\alpha} I_\alpha(z) \approx \int_{\tau\in[-\varepsilon,\varepsilon]} e^{-z\tau^2/2}\,\nu$$

as $z \to \infty$. **[I need to learn how this works! Do we get asymptoticity at all orders? —Aaron]** Plugging in the Taylor series above, we get

$$\begin{aligned} e^{-z\zeta_\alpha} I_\alpha(z) &\approx \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n\geq 0} b_n^\alpha \tau^n \, d\tau \\ &= \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n\geq 0} b_{2n}^\alpha \tau^{2n} \, d\tau. \end{aligned}$$

By the dominated convergence theorem,[2]

$$\begin{aligned} e^{-z\zeta_\alpha} I_\alpha(z) &\approx \sum_{n\geq 0} b_{2n}^\alpha \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \tau^{2n} \, d\tau \\ &= \sum_{n\geq 0} (2n-1)!!\, b_{2n}^\alpha \left[\sqrt{2\pi}\, z^{-(n+1/2)} \operatorname{erf}\!\big(\varepsilon\sqrt{z/2}\big) - 2e^{-z\varepsilon^2/2} \sum_{\substack{k\in\mathbb{N}_+ \\ k\leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{n-k+1} \right]. \end{aligned}$$

[2] Notice that the sum over k is empty when $n = 0$. Following convention, we extend the double factorial to all odd integers by its recurrence relation, giving $(-1)!! = 1$.

The annoying $e^{-z\varepsilon^2/2}$ correction terms are dwarfed by their $z^{-(n+1/2)}$ counterparts when z is large. These terms are crucial, however, for the convergence of the sum. To see why, consider their absolute sum C_{exp}. When $z \in [0, \infty)$,

$$\begin{aligned}
C_{\text{exp}} &= 2e^{-\operatorname{Re}(z)\varepsilon^2/2} \sum_{n\geq 1} (2n-1)!! \left| b_{2n}^{\alpha} \sum_{\substack{k\in\mathbb{N}_+ \\ k\leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{n-k+1} \right| \\
&= 2e^{-z\varepsilon^2/2} \sum_{n\geq 1} (2n-1)!! \left| b_{2n}^{\alpha} \right| \sum_{\substack{k\in\mathbb{N}_+ \\ k\leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{n-k+1} \\
&\geq -2\varepsilon e^{-z\varepsilon^2/2} \sum_{n\geq 1} (2n-1)!! \left| b_{2n}^{\alpha} \right| z^n,
\end{aligned}$$

which diverges for typical f and ν.

This argument suggests that no matter how tiny the correction terms get, we can't expect to swat them all aside. We can, however, set aside any finite set of them. For each cutoff N, the tail

$$\sum_{n\geq N} b_{2n}^{\alpha} \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \tau^{2n}\, d\tau$$

is in $o_{z\to\infty}(z^{-N})$ **[check]**, and the absolute sum

$$\begin{aligned}
C_{\text{exp}}^{N} &= 2e^{-\operatorname{Re}(z)\varepsilon^2/2} \sum_{n=1}^{N-1} (2n-1)!! \left| b_{2n}^{\alpha} \sum_{\substack{k\in\mathbb{N}_+ \\ k\leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{n-k+1} \right| \\
&\leq 2e^{-\operatorname{Re}(z)\varepsilon^2/2} \sum_{n=1}^{N-1} (2n-1)!! \left| b_{2n}^{\alpha} \right| \sum_{\substack{k\in\mathbb{N}_+ \\ k\leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} |z|^{n-k+1} \\
&\geq -2\varepsilon e^{-z\varepsilon^2/2} \sum_{n\geq 1} (2n-1)!! \left| b_{2n}^{\alpha} \right| z^n,
\end{aligned}$$

is in $o_{z\to\infty}(z^{-m})$ for every m **[check]**. Hence,

$$e^{-z\zeta_\alpha} I_\alpha(z) \sim \sqrt{2\pi} \sum_{n\geq 0} (2n-1)!!\, b_{2n}^{\alpha}\, z^{-(n+1/2)} \operatorname{erf}\!\left(\varepsilon\sqrt{z/2}\right).$$

The differences $1-\operatorname{erf}\!\left(\varepsilon\sqrt{z/2}\right)$ shrink exponentially as z grows, allowing the simpler estimate

$$e^{-z\zeta_\alpha} I_\alpha(z) \sim \sqrt{2\pi} \sum_{n\geq 0} (2n-1)!!\, b_{2n}^{\alpha}\, z^{-(n+1/2)}.$$

Call the right-hand side $\tilde{I}_\alpha$. We now see that $a_{\alpha,n} = (2n-1)!!\, b_{2n}^{\alpha}$ in the statement of the theorem. **[Resolve discrepancy with previous calculation.]** Note that **[explain**

formally what it means to center at ζ_α]

$$\mathcal{B}_{\zeta_\alpha}(\tilde{I}_\alpha)=\sqrt{2\pi}\sum_{n\geq 0}\frac{2^n}{\sqrt{\pi}}\Gamma\left(n+\tfrac{1}{2}\right)b^\alpha_{2n}\frac{(\zeta-\zeta_\alpha)^{n-1/2}}{\Gamma\left(n+\frac{1}{2}\right)}$$
$$=\sum_{n\geq 0}2^{n+1/2}\,b^\alpha_{2n}(\zeta-\zeta_\alpha)^{n-1/2}.$$

We know from the definition of ε that $\left|b^\alpha_n\right|\varepsilon^n\lesssim 1$. Recalling that $(2n-1)!!\approx(\pi n)^{-1/2}2^n\,n!$ as $n\to\infty$, we deduce that $|a_{\alpha,n}|\lesssim\left(\frac{2}{\varepsilon^2}\right)^n n!$, showing that $\tilde{\varphi}_\alpha$ is Gevrey-1.

Part (1): Let us assume that locally $\nu=\sum_{j\geq 0}b^\alpha_j(t-t_\alpha)^j\,dt$ for $|t-t_\alpha|<\varepsilon$. By the steepest descent method, $I_\alpha(z)$ can be aproximated as $z\to\infty$ as

$$I_\alpha(\zeta)\sim e^{-zf(x_\alpha)}\int_{-\varepsilon}^{\varepsilon}\sum_{n\geq 0}t^{2n}b^\alpha_{2n}e^{-zf''(x_\alpha)\frac{t^2}{2}}\,dt$$
$$=e^{-zf(x_\alpha)}\sum_{n\geq 0}b^\alpha_{2n}\int_{-\varepsilon}^{\varepsilon}t^{2n}e^{-zf''(x_\alpha)\frac{t^2}{2}}\,dt$$
$$=\sqrt{2\pi}e^{-zf(x_\alpha)}z^{-1/2}\sum_{n\geq 0}\frac{b^\alpha_{2n}}{(f''(x_\alpha))^{n+1/2}}(2n-1)!!z^{-n}\operatorname{Erf}\left(\frac{\sqrt{zf''(x_\alpha)}}{\sqrt{2}}\varepsilon\right)+$$
$$-2e^{-zf(x_\alpha)}\sum_{n\geq 1}b^\alpha_{2n}(2n-1)!!e^{-zf''(x_\alpha)\frac{\varepsilon^2}{2}}\sum_{j=1}^{n}\frac{\varepsilon^{2j-1}}{(2j-1)!!}\left(f''(x_\alpha)z\right)^{n-j+1}$$
$$\sim_{\varepsilon\ll 1}2\varepsilon\sqrt{2\pi}e^{-zf(x_\alpha)}z^{-1/2}\sum_{n\geq 0}\frac{b^\alpha_{2n}}{(f''(x_\alpha))^{n+1/2}}(2n-1)!!z^{-n}$$

therefore

$$a_{\alpha,n}:=2\varepsilon\frac{b^\alpha_{2n}}{(f''(x_\alpha))^{n+1/2}}(2n-1)!!$$

[The value of ε shouldn't affect the asymptotic series! I think we want the limit $z\to\infty$ rather than $\varepsilon\ll 1$.]

Since, $\frac{1}{\varepsilon}=\limsup_n\sqrt[n]{b^\alpha_n}$

$$|a_{\alpha,n}|\leq CA^n n!$$

where we use $(2n-1)!!\sim\frac{2^n}{\sqrt{\pi n}}n!$ as $n\to\infty$.

Part (2):

$$\hat{\varphi}_\alpha(\zeta)=\mathcal{B}\left(e^{-zf(x_\alpha)}(2\pi)^{1/2}\sum_{n\geq 0}a_{\alpha,n}z^{-n}\right)(\zeta)=T_{f(x_\alpha)}(2\pi)^{1/2}\left(\delta a_0+\sum_{n\geq 0}a_{n+1}\frac{\zeta^n}{n!}\right)$$
$$(2\pi)^{1/2}\left(\delta(f_{x_\alpha})a_0+\sum_{n\geq 0}a_{n+1}\frac{(\zeta-f(x_\alpha))^n}{n!}\right)$$

Since $a_n\leq CA^n n!$, the series $\sum_{n\geq 0}a_{n+1}\frac{(\zeta-f(x_\alpha))^n}{n!}$ has a finite radius of convergence.

Part (3): thanks to properties of Caputo's fractional derivatives, we have that the Borel transform of $\tilde{I}_\alpha(z) = z^{-1/2}\tilde{\varphi}_\alpha(z)$ is

$$\partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \mathcal{B}(\tilde{I}_\alpha)(\zeta) = \hat{\varphi}_\alpha(\zeta). \tag{3.3}$$

Let's recast the integral I_α into the f plane. As ζ goes rightward from ζ_α, the start and end points of $\mathcal{C}_\alpha(\zeta)$ sweep backward along $\mathcal{C}^-_\alpha(\zeta)$ and forward along $\mathcal{C}^+_\alpha(\zeta)$, respectively. Hence, we have

$$\begin{aligned} I_\alpha(z) &= \int_{\mathcal{C}_\alpha} e^{-zf}\,\nu \\ &= \int_{\zeta_\alpha}^{\infty} e^{-z\zeta}\left[\frac{\nu}{df}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)} d\zeta. \end{aligned}$$

Noticing that the right-hand side is a Laplace transform, we learn that

$$\hat{I}_\alpha(\zeta) = \left[\frac{\nu}{df}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)}.$$

Observing that

$$\partial_\zeta\left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right) = \left[\frac{\nu}{df}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)},$$

we conclude that

$$\hat{I}_\alpha(\zeta) = \partial_\zeta\left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right) d\zeta.$$

We can rewrite our Taylor series for ν as

$$\begin{aligned} \nu &= \sum_{n\geq 0} b^\alpha_n [2(f-\zeta_\alpha)]^{n/2} \frac{df}{[2(f-\zeta_\alpha)]^{1/2}} \\ &= \sum_{n\geq 0} b^\alpha_n [2(f-\zeta_\alpha)]^{(n-1)/2}\, df, \end{aligned}$$

taking the positive branch of the square root on $\mathcal{C}^+_\alpha$ and the negative branch on $\mathcal{C}^-_\alpha$. Plugging this into our expression for $\hat{I}_\alpha$, we learn that

$$\begin{aligned} \hat{I}_\alpha(\zeta) &= \left[\sum_{n\geq 0} b^\alpha_n [2(f-\zeta_\alpha)]^{(n-1)/2}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)} \\ &= \sum_{n\geq 0} b^\alpha_n \Big([2(\zeta-\zeta_\alpha)]^{(n-1)/2} - (-1)^{n-1}[2(\zeta-\zeta_\alpha)]^{(n-1)/2}\Big) \\ &= \sum_{n\geq 0} 2 b^\alpha_{2n} [2(\zeta-\zeta_\alpha)]^{n-1/2} \\ &= \sum_{n\geq 0} 2^{n+1/2} b^\alpha_{2n} (\zeta-\zeta_\alpha)^{n-1/2} \\ &= \mathcal{B}_{\zeta_\alpha}(\tilde{I}_\alpha). \end{aligned}$$

We already knew, from the general theory of the Borel transform, that the sum of $\mathcal{B}_{\zeta_\alpha}(\tilde{I}_\alpha)$ would be asymptotic to $\hat{I}_\alpha$. We've now shown that the sum of $\mathcal{B}_{\zeta_\alpha}(\tilde{I}_\alpha)$ is actually equal to $\hat{I}_\alpha$, strengthening the formal series identity 3.3 to the analytic identity

$$\begin{aligned}\hat{\varphi}_\alpha(\zeta) &= \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \hat{I}_\alpha \\ &= \partial^{3/2}_{\zeta \text{ from } \zeta_\alpha} \left(\int_{\mathcal{C}_\alpha(\zeta)} \nu \right) d\zeta.\end{aligned}$$

□

Example 3.2 (Airy)**.** Let $f(t) = \frac{t^3}{3} - t$ and

$$I(z) := \int_\gamma e^{-zf(t)} \mathrm{dt}$$

where γ is a countour where the integral is well defined.

By the change of coordinates $z = x^{3/2}$, $I(z) = -2\pi i z^{-1/3} Ai(x)$ where

$$Ai(x) := \frac{1}{2\pi i} \int_{-\infty e^{-i\frac{\pi}{3}}}^{\infty e^{i\frac{\pi}{3}}} e^{\frac{t^3}{3} - zt} \mathrm{dt}$$

hence $I(z)$ solves the following ODE[3]

$$I''(z) - \frac{4}{9} I(z) + \frac{I'(z)}{z} - \frac{1}{9}\frac{I(z)}{z^2} = 0 \tag{3.4}$$

A formal solution of (3.4) can be computed by making the following ansatz

$$\tilde{I}(z) = \sum_{k \in \mathbb{N}^2} U^k e^{-\lambda \cdot k z} z^{-\tau \cdot k} w_k(z) \tag{3.5}$$

with $U^{(k_1,k_2)} = U_1^{k_1} U_2^{k_2}$ and $U_1, U_2 \in \mathbb{C}$ are constant parameter, $\lambda = (\frac{2}{3}, -\frac{2}{3})$, $\tau = (\frac{1}{2}, \frac{1}{2})$, and $\tilde{w}_k(z) \in \mathbb{C}[[z^{-1}]]$. In addition, we can check that the only non zero $\tilde{w}_k(z)$ occurs at $k = (1,0)$ and $k = (0,1)$, therefore

$$\tilde{I}(z) = U_1 e^{-2/3z} z^{-1/2} \tilde{w}_+(z) + U_2 e^{2/3z} z^{-1/2} \tilde{w}_-(z) \tag{3.6}$$

where from now on we denote $\tilde{w}_+(z) = \tilde{w}_{(1,0)}(z)$ and $\tilde{w}_-(z) = \tilde{w}_{(0,1)}(z)$. In particular, $\tilde{w}_+(z)$ and $\tilde{w}_-(z)$ are formal solution of

$$\tilde{w}''_+ - \frac{4}{3}\tilde{w}'_+ + \frac{5}{36}\frac{\tilde{w}_+}{z^2} = 0 \tag{3.7}$$

$$\tilde{w}''_- + \frac{4}{3}\tilde{w}'_- + \frac{5}{36}\frac{\tilde{w}_-}{z^2} = 0 \tag{3.8}$$

[3] $Ai(x)$ solves the Airy equation $y'' = xy$.

Taking the Borel transform of (3.7), (3.8) we get

$$\zeta^2\,\hat{w}_+(\zeta)+\frac{4}{3}\zeta\,\hat{w}_+ +\frac{5}{36}\zeta*\hat{w}_+=0$$
$$\zeta^2\,\hat{w}_+(\zeta)+\frac{4}{3}\zeta\,\hat{w}_+ +\frac{5}{36}\int_0^{\zeta}(\zeta-\zeta')\hat{w}_+(\zeta')d\zeta'=0$$

$$\zeta^2\,\hat{w}_-(\zeta)-\frac{4}{3}\zeta\,\hat{w}_- +\frac{5}{36}\zeta*\hat{w}_-=0$$
$$\zeta^2\,\hat{w}_-(\zeta)-\frac{4}{3}\zeta\,\hat{w}_- +\frac{5}{36}\int_0^{\zeta}(\zeta-\zeta')\hat{w}_-(\zeta')d\zeta'=0$$

and taking derivatives we get

$$\zeta(\frac{4}{3}+\zeta)\hat{w}_+''+(\frac{8}{3}+4\zeta)\hat{w}_+'+\frac{77}{36}\hat{w}_+=0$$
$$\frac{4}{3}\zeta(1+\frac{3}{4}\zeta)\hat{w}_+''+(\frac{8}{3}+4\zeta)\hat{w}_+'+\frac{77}{36}\hat{w}_+=0$$
$$u(1-u)\hat{w}_+''(u)+(2-4u)\hat{w}_+'(u)-\frac{77}{36}\hat{w}_+(u)=0 \qquad u=-\frac{3}{4}\zeta$$

$$\zeta(-\frac{4}{3}+\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0$$
$$\frac{4}{3}\zeta(-1+\frac{3}{4}\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0$$
$$u(1-u)\hat{w}_-''(u)+(2-4u)\hat{w}_-'(u)-\frac{77}{36}\hat{w}(u)=0 \qquad u=\frac{3}{4}\zeta$$

Notice that the latter equations are hypergeometric, hence a solution is given by

$$\hat{w}_+(\zeta)=c_1\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right) \tag{3.9}$$

$$\hat{w}_-(\zeta)=c_2\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right) \tag{3.10}$$

for some constants $c_1,c_2\in\mathbb{C}$ (see DLMF 15.10.2). In addition $\hat{w}_\pm(\zeta)$ have a log singularity respectively at $\zeta=\mp\frac{4}{3}$, therefore they are $\{\mp\frac{4}{3}\}$-resurgent functions.[4]

[4]The solution we find are equal to the ones in DLMF §9.7. They also agree with slide 10 of Maxim's talk for ERC and with the series (6.121) in Sauzin's book. However they do not agree with (2.16) in Mariño's Diablerets.

Remark 3.3. $\hat{w}_+(\zeta)$ is Laplace summable along the positive real axis, and it can be analyticaly continued on $\mathbb{C}\setminus -\frac{4}{3}\mathbb{R}_{\leq 0}$ with (see 15.2.3 DLMF)

$$\begin{aligned}
\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0) &= -\frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\frac{3}{4}\zeta)^n \qquad \zeta<-\frac{4}{3}\\
&= \frac{36}{5}i\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\frac{3}{4}\zeta)^{n-1}\\
&= -\frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\left(\frac{5}{144}(4+3\zeta)\left(1+{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)\right)\right)\\
&= \mathbf{i}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)\\
&= \mathbf{i}\hat{w}_-(\zeta+\frac{4}{3})
\end{aligned}$$

Anolougusly, $\hat{w}_-(\zeta)$ is Laplace summable along the negative real axis, and it jumps across the branch cut $\frac{4}{3}\mathbb{R}_{\geq 0}$ as

$$\begin{aligned}
\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0) &= \frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1-\frac{3}{4}\zeta)^n \qquad \zeta>\frac{4}{3}\\
&= \frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\left(-\frac{5}{144}(-4+3\zeta)_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)\right)\\
&= -\mathbf{i}\pi\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)\\
&= -\mathbf{i}\hat{w}_+(\zeta-\frac{4}{3})
\end{aligned}$$

These relations manifest the resurgence property of $\tilde{I}$, indeed near the singularities in the Borel plane of either $\hat{w}_+$ or $\hat{w}_-$, $\hat{w}_-$ and $\hat{w}_+$ respectively contribute to the jump of the former solution.

Our next goal is to prove that the Borel transform of $\tilde{I}(z)$ can be written in terms of $1/f'(f^{-1}(\zeta))$, namely formula (**??**). It is convenient to consider the two asymptotic formal solutions separately, namely we define

$$\tilde{I}_{-1}(z) := e^{-2/3z}z^{-1/2}\tilde{w}_+(z) =: z^{-1/2}\tilde{u}_+(z) \tag{3.11}$$

$$\tilde{I}_1(z) := e^{2/3z}z^{-1/2}\tilde{w}_-(z) =: z^{-1/2}\tilde{u}_-(z) \tag{3.12}$$

In particular, $\tilde{u}_\pm(z)$ are solutions of

$$\tilde{u}''(z)-\frac{4}{9}\tilde{u}(z)+\frac{5}{36}\frac{\tilde{u}(z)}{z^2}=0 \tag{3.13}$$

with asymptotic behaviour $\tilde{u}_\pm(z)\sim O(e^{\pm 2/3z})$ as $z\to\infty$.

The Borel transforms $\hat{u}_{\pm}(\zeta)$ solve the same equation

$$\zeta^2 \hat{u} - \frac{4}{9}\hat{u} + \frac{5}{36}\zeta * \hat{u}$$

$$\zeta^2 \hat{u} - \frac{4}{9}\hat{u} + \frac{5}{36}\int_0^{\zeta}(\zeta-\zeta')\hat{u}(\zeta')d\zeta'$$

taking derivatives is equivalent to

$$(\zeta^2 - \frac{4}{9})\hat{u}''(\zeta) + 4\zeta\hat{u}'(\zeta) + \frac{77}{36}\hat{u}(\zeta) = 0$$

and Mathematica gives the following solutions

$$\begin{aligned}
\hat{u}(\zeta) &= c_1\,{}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{9}{4}\zeta^2\right) + \frac{3i}{2}\zeta c_2\,{}_1F_2\left(\frac{13}{12},\frac{17}{12},\frac{3}{2},\frac{9}{4}\zeta^2\right) = \\
&= c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}}\left({}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{1}{2}-\frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{1}{2}+\frac{3}{4}\zeta\right)\right) && \text{see DLMF 15.8.27} \\
&\quad + \frac{3i}{2}\zeta c_2\left(\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{3\zeta\Gamma(-\frac{1}{2})\Gamma(2)}\right)\left({}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}-\frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}+\frac{3}{4}\zeta\right)\right) && \text{see DLMF 15.8.28} \\
&= \left(c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} - c_2 i\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right){}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}-\frac{3}{4}\zeta\right) + \\
&\quad + \left(c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} + c_2 i\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right){}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}+\frac{3}{4}\zeta\right)
\end{aligned}$$

Since $\hat{u}_+$ has a simple singularity at $\zeta = -2/3$ and $\hat{u}_-$ has a simple singularity at $\zeta = 2/3$, we have

$$\hat{u}_+(\zeta) = C_1 T_{-2/3}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right) = C_1 T_{-2/3}\hat{w}_+(\zeta)$$

$$\hat{u}_-(\zeta) = C_2 T_{2/3}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right) = C_2 T_{2/3}\hat{w}_-(\zeta)$$

Lemma 3.4. *The following identity holds true*

$$ {}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right) = \frac{1}{1-u^2} \qquad \zeta = \frac{u^3}{3} - u \tag{3.14}$$

Proof. From the special case of hypergeometric function (see 15.4.14 DLMF) we have the following identity:

$$
\begin{aligned}
{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right) &= \frac{\cos(y)}{\cos(3y)} & 3y = \arcsin\left(\frac{3}{2}\zeta\right)\\
&= \frac{\cos(y)}{\cos(2y)\cos(y)-\sin(2y)\sin(y)}\\
&= \frac{1}{\cos(2y)-2\sin^2(y)}\\
&= \frac{1}{1-4\sin^2(y)} & \zeta = 2\sin(y)-\frac{8}{3}\sin^3(y)
\end{aligned}
$$

Therefore, if $u := -2\sin(y)$, we have $\zeta = \frac{u^3}{3} - u = f(u)$ and

$$
{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right) = \frac{1}{1-u^2} = -\frac{1}{f'(u)}
$$

□

Then equations (**??**) is equivalent to

Claim 3.5. [There should be a way to predict the correct normalization of the LHS.]

$$
\hat{w}_+(\zeta-2/3) = -\frac{1}{\sqrt{\pi}}\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}\partial_s\left[{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}s^2\right)\right]ds \qquad \zeta\in(2/3,+\infty) \tag{3.15}
$$

Using $a=\frac{5}{3}$, $b=\frac{7}{3}$, $c=\frac{5}{2}$:

$$
\begin{aligned}
w_1 &= \frac{\Gamma(\frac{5}{2})\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{6})}w_5 + \frac{\Gamma(\frac{5}{2})\Gamma(-\frac{2}{3})}{\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}w_6\\
w_3 &= \frac{1}{e^{(5/3)\pi i}-e^{(7/3)\pi i}}\left[e^{(-1/6)\pi i}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{6})}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})}w_5 - e^{(-5/6)\pi i}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{5}{6})}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})}w_6\right]
\end{aligned}
$$

Common denominators

$$
\begin{aligned}
w_1 &= \frac{\Gamma(\frac{5}{2})\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{6})\Gamma(\frac{1}{3})}w_5 + \frac{\Gamma(\frac{5}{2})\Gamma(-\frac{2}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})\Gamma(\frac{5}{3})}w_6\\
w_3 &= \frac{i}{\sqrt{3}}\left[e^{(-1/6)\pi i}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}w_5 - e^{(-5/6)\pi i}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{5}{6})\Gamma(\frac{1}{6})}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}w_6\right]
\end{aligned}
$$

Extract

$$
\begin{aligned}
w_1 &= \frac{\Gamma(\frac{5}{2})\frac{2}{\sqrt{3}}\pi}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{6})\Gamma(\frac{1}{3})}w_5 - \frac{\Gamma(\frac{5}{2})\frac{2}{\sqrt{3}}\pi}{\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})\Gamma(\frac{5}{3})}w_6\\
w_3 &= \frac{2\pi i}{\sqrt{3}}\Gamma(\tfrac{5}{2})\left[e^{(-1/6)\pi i}\frac{1}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}w_5 - e^{(-5/6)\pi i}\frac{1}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}w_6\right]
\end{aligned}
$$

...

$$w_1 = \frac{2\pi}{\sqrt{3}}\Gamma(\tfrac{5}{2})\left[\frac{1}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{6})\Gamma(\frac{1}{3})}w_5 - \frac{1}{\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})\Gamma(\frac{5}{3})}w_6\right]$$

$$w_3 = \frac{2\pi}{\sqrt{3}}\Gamma(\tfrac{5}{2})\left[e^{(1/3)\pi i}\frac{1}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}w_5 - e^{-(1/3)\pi i}\frac{1}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}w_6\right]$$

Now subtract!

$$w_1 - w_3 = \frac{2\pi}{\sqrt{3}}\Gamma(\tfrac{5}{2})\left[e^{-(1/3)\pi i}\frac{1}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}w_5 + e^{-(2/3)\pi i}\frac{1}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}w_6\right]$$

Let us study the RHS of claim (3.5). **Note that $x = \frac{1}{2} - \frac{3}{4}\zeta$ and $y = \frac{1}{2} - \frac{3}{4}s$.**

$$\int_{2/3}^{\zeta}(\zeta - s)^{-1/2}\partial_s\left[{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}s^2\right)\right]ds$$

$$= 2\int_{2/3}^{\zeta}(\zeta - s)^{-1/2}s\,{}_2F_1\left(\frac{4}{3},\frac{5}{3};\frac{3}{2};\frac{9}{4}s^2\right)ds$$

$$= -\frac{2}{9}\int_{2/3}^{\zeta}(\zeta - s)^{-1/2}\left[{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};\frac{1}{2} - \frac{3s}{4}\right) - {}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};\frac{1}{2} + \frac{3s}{4}\right)\right]ds \qquad \text{15.8.28 } \textbf{DLMF}$$

$$= \frac{4}{9\sqrt{3}}\int_0^x (y - x)^{-1/2}\left[{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};y\right) - {}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};1-y\right)\right]dy \qquad x \in (-\infty, 0)$$

$$= \frac{4}{9\sqrt{3}}\frac{2\pi}{\sqrt{3}}\Gamma(\tfrac{5}{2})\int_0^x (y - x)^{-1/2}\left[e^{-(1/3)\pi i}\frac{e^{(5/3)\pi i}y^{-5/3}{}_2F_1\left(\frac{5}{3},\frac{1}{6};\frac{1}{3};\frac{1}{y}\right)}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}\right.$$

$$\left. + e^{-(2/3)\pi i}\frac{e^{(7/3)\pi i}y^{-7/3}{}_2F_1\left(\frac{7}{3},\frac{5}{6};\frac{5}{3};\frac{1}{y}\right)}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}\right]dy \qquad x \in (-\infty, 0)$$

Continue w_5 and w_6 counterclockwise around their singularities at zero. Notice that $|y-x| = y-x$ on the integration path.

$$= \frac{8\pi}{27}\Gamma(\tfrac{5}{2})\int_0^x |y-x|^{-1/2}\left[e^{-(1/3)\pi i}\frac{e^{(10/3)\pi i}|y|^{-5/3}{}_2F_1\left(\frac{5}{3},\frac{1}{6};\frac{1}{3};\frac{1}{y}\right)}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}\right.$$
$$\left.+e^{-(2/3)\pi i}\frac{e^{(14/3)\pi i}|y|^{-7/3}{}_2F_1\left(\frac{7}{3},\frac{5}{6};\frac{5}{3};\frac{1}{y}\right)}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}\right]dy \qquad x\in(-\infty,0)$$
$$= \frac{8\pi}{27}\Gamma(\tfrac{5}{2})\int_0^x |y-x|^{-1/2}\left[-\frac{|y|^{-5/3}{}_2F_1\left(\frac{5}{3},\frac{1}{6};\frac{1}{3};\frac{1}{y}\right)}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}\right.$$
$$\left.+\frac{|y|^{-7/3}{}_2F_1\left(\frac{7}{3},\frac{5}{6};\frac{5}{3};\frac{1}{y}\right)}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}\right]dy \qquad x\in(-\infty,0)$$
$$= \frac{8\pi}{27}\Gamma(\tfrac{5}{2})\left[-\frac{\Gamma(\frac{1}{2})\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{3})}\frac{|x|^{-7/6}{}_2F_1\left(\frac{7}{6},\frac{1}{6};\frac{1}{3};\frac{1}{x}\right)}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}\right.$$
$$\left.+\frac{\Gamma(\frac{1}{2})\Gamma(\frac{11}{6})}{\Gamma(\frac{7}{3})}\frac{|x|^{-11/6}{}_2F_1\left(\frac{11}{6},\frac{5}{6};\frac{5}{3};\frac{1}{x}\right)}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}\right] \qquad (4.3)$$
$$= \frac{8\pi}{27}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{3})}\left[-\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}e^{(7/6)\pi i}x^{-7/6}{}_2F_1\left(\frac{7}{6},\frac{1}{6};\frac{1}{3};\frac{1}{x}\right)\right.$$
$$\left.+\frac{\Gamma(\frac{11}{6})}{\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}e^{(11/6)\pi i}x^{-11/6}{}_2F_1\left(\frac{11}{6},\frac{5}{6};\frac{5}{3};\frac{1}{x}\right)\right]$$
$$= \frac{8\pi}{27}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{3})}\left[-\frac{\Gamma(2)\Gamma(\frac{2}{3})}{\Gamma(\frac{11}{6})\Gamma(\frac{5}{6})}\frac{\Gamma(\frac{11}{6})}{\Gamma(2)\Gamma(\frac{2}{3})}\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{1}{3})}w_5(\tfrac{7}{6},\tfrac{11}{6},2)\right.$$
$$\left.+\frac{\Gamma(2)\Gamma(-\frac{2}{3})}{\Gamma(\frac{7}{6})\Gamma(\frac{1}{6})}\frac{\Gamma(\frac{7}{6})}{\Gamma(2)\Gamma(-\frac{2}{3})}\frac{\Gamma(\frac{11}{6})}{\Gamma(\frac{5}{3})}w_6(\tfrac{7}{6},\tfrac{11}{6},2)\right]$$
$$= \frac{8\pi}{27}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})\Gamma(\frac{7}{6})\Gamma(\frac{11}{6})}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{3})}\left[-\frac{\Gamma(2)\Gamma(\frac{2}{3})}{\Gamma(\frac{11}{6})\Gamma(\frac{5}{6})}\frac{1}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})}w_5(\tfrac{7}{6},\tfrac{11}{6},2)\right.$$
$$\left.+\frac{\Gamma(2)\Gamma(-\frac{2}{3})}{\Gamma(\frac{7}{6})\Gamma(\frac{1}{6})}\frac{1}{\Gamma(-\frac{2}{3})\Gamma(\frac{5}{3})}w_6(\tfrac{7}{6},\tfrac{11}{6},2)\right]$$
$$= \frac{8\pi}{27}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})\Gamma(\frac{7}{6})\Gamma(\frac{11}{6})}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{3})}\left[-\frac{\Gamma(2)\Gamma(\frac{2}{3})}{\Gamma(\frac{11}{6})\Gamma(\frac{5}{6})}\frac{\sin(\frac{2}{3}\pi)}{\pi}w_5(\tfrac{7}{6},\tfrac{11}{6},2)\right.$$
$$\left.+\frac{\Gamma(2)\Gamma(-\frac{2}{3})}{\Gamma(\frac{7}{6})\Gamma(\frac{1}{6})}\frac{\sin(-\frac{2}{3}\pi)}{\pi}w_6(\tfrac{7}{6},\tfrac{11}{6},2)\right]$$
$$= \frac{8\pi}{27}\frac{\sqrt{3}}{2\pi}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})\Gamma(\frac{7}{6})\Gamma(\frac{11}{6})}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{3})}\left[-\frac{\Gamma(2)\Gamma(\frac{2}{3})}{\Gamma(\frac{11}{6})\Gamma(\frac{5}{6})}w_5(\tfrac{7}{6},\tfrac{11}{6},2)-\frac{\Gamma(2)\Gamma(-\frac{2}{3})}{\Gamma(\frac{7}{6})\Gamma(\frac{1}{6})}w_6(\tfrac{7}{6},\tfrac{11}{6},2)\right]$$

Using DLMF 15.10.25, we see this is

$$
\begin{aligned}
&= -\frac{8}{27}\frac{\sqrt{3}}{2}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})\Gamma(\frac{7}{6})\Gamma(\frac{11}{6})}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{3})} w_1(\tfrac{7}{6},\tfrac{11}{6},2) \\
&= -\frac{8}{27}\frac{\sqrt{3}}{2}\frac{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\frac{1}{6}\Gamma(\frac{1}{6})\frac{5}{6}\Gamma(\frac{5}{6})}{\frac{4}{3}\frac{1}{3}\Gamma(\frac{1}{3})\frac{2}{3}\Gamma(\frac{2}{3})} w_1(\tfrac{7}{6},\tfrac{11}{6},2) \\
&= -\frac{5\sqrt{3}}{6\cdot 2^4}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} w_1(\tfrac{7}{6},\tfrac{11}{6},2) \\
&= -\frac{5\sqrt{3}}{6\cdot 2^4}\frac{\pi\sin(\frac{1}{3}\pi)}{\sin(\frac{1}{6}\pi)} w_1(\tfrac{7}{6},\tfrac{11}{6},2) \\
&= -\frac{5\pi}{32} w_1(\tfrac{7}{6},\tfrac{11}{6},2) \\
&= -\frac{5\pi}{32}\, {}_2F_1\left(\frac{7}{6},\frac{11}{6};2;x\right) \\
&= -\frac{5\pi}{32}\, {}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\frac{1}{2}-\frac{3\zeta}{4}\right) \\
&= -\frac{5\pi}{32}\hat{w}_+(\zeta-2/3)
\end{aligned}
$$

This matches $-\sqrt{\pi}$ times the LHS of claim (3.5) **(up to scaling)**.

Alternate calculation with $\frac{1}{2}$ derivative:

$$
\begin{aligned}
&\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}s^2\right)ds \\
&= \frac{2}{3}\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}\left[{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};\frac{1}{2}-\frac{3s}{4}\right)+{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};\frac{1}{2}+\frac{3s}{4}\right)\right]ds && \text{15.8.27 DLMF} \\
&= \frac{4}{3\sqrt{3}}\int_0^x (x-y)^{-1/2}\left[{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};1-y\right)+{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};y\right)\right]dy && x\in(-\infty,0) \\
&= \frac{2i}{3\sqrt{3}}\int_0^x (x-y)^{-1/2}\left[(1+\tfrac{1}{\sqrt{3}})y^{-2/3}\,{}_2F_1\left(\frac{2}{3},\frac{1}{6},\frac{1}{3};\frac{1}{y}\right)+(1-\tfrac{1}{\sqrt{3}})y^{-4/3}\,{}_2F_1\left(\frac{4}{3},\frac{5}{6},\frac{5}{3};\frac{1}{y}\right)\right]ds \\
&= \frac{2i}{3\sqrt{3}}\left[\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})}(1+\tfrac{1}{\sqrt{3}})x^{-1/6}\,{}_2F_1\left(\frac{1}{6},\frac{1}{6},\frac{1}{3};\frac{1}{x}\right)+\frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{6})}{\Gamma(\frac{4}{3})}(1-\tfrac{1}{\sqrt{3}})x^{-5/6}\,{}_2F_1\left(\frac{5}{6},\frac{5}{6},\frac{5}{3};\frac{1}{x}\right)\right]ds && (4.3) \\
&= ??? && \zeta\in(2/3,+\infty)
\end{aligned}
$$

Analogously, it can be verified for $\hat{w}_-(\zeta+2/3)$ for $\zeta\in(-\infty,-2/3)$.

Example 3.6 (Bessel)**.** Let $X=\mathbb{C}^*$, $f(x)=x+\frac{1}{x}$ and $\nu=\frac{dx}{x}$, then the ciritcal points of f are $x=\pm 1$ and

$$I(z) := \int_0^{\infty} e^{-zf(x)}\frac{dx}{x}. \tag{3.16}$$

By change of cooridnates $t = zx$

$$I(z) = \int_0^\infty e^{-z\left(\frac{t}{z}+\frac{z}{t}\right)} \frac{dt}{t} = \int_0^\infty e^{-\left(t+\frac{z^2}{t}\right)} \frac{dt}{t} = 2K_0(2z) \qquad |\arg z| < \frac{\pi}{4}$$

where $K_0(z)$ is the modified Bessel function (see definition 10.32.10 DLMF). In particular, since $K_0(z)$ solves

$$\frac{d^2}{dz^2} w(z) + \frac{1}{z}\frac{d}{dz} w(z) - w(z) = 0 \tag{3.17}$$

and $K_0(z) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-z} z^{-1/2} \sum_{k\geq 0} \frac{(1/2)_k(1/2)_k}{(-2)^k k!} z^{-k}$ as $z \to \infty$ (see DLMF 10.40.2), then $I(z)$ is a solution of

$$\frac{d^2}{dz^2} I(z) + \frac{1}{z}\frac{d}{dz} I(z) - 4I(z) = 0. \tag{3.18}$$

The formal integral of (3.18) is given by a two parameter formal solution $\tilde{I}_1(z)$

$$\tilde{I}(z) = \sum_{\mathbf{k}\in\mathbb{N}^2} U^k e^{-\mathbf{k}\cdot\lambda z} z^{-\tau\cdot\mathbf{k}} \tilde{w}_{\mathbf{k}}(z) \tag{3.19}$$

where $\lambda = (2,-2)$, $\tau = (-\frac{1}{2},-\frac{1}{2})$, $U^k := U_1^{k_1} U_2^{k_2}$ with $k = (k_1,k_2)$ and $U_1, U_2 \in \mathbb{C}$, and $\tilde{w}_{\mathbf{k}}(z) \in \mathbb{C}[[z^{-1}]]$ is a formal solution of

$$\begin{aligned} \tilde{w}''_{\mathbf{k}}(z) - 4(k_1-k_2)\tilde{w}'_{\mathbf{k}}(z) + 4(1-(k_1-k_2)^2)\tilde{w}_{\mathbf{k}}(z) + \frac{(k_1+k_2-1)}{z}\tilde{w}'_{\mathbf{k}}(z) + \\ -2(k_1-k_2)\frac{(k_1+k_2-1)}{z}\tilde{w}_{\mathbf{k}}(z) + \frac{(k_1+k_2)^2}{4z^2}\tilde{w}_{\mathbf{k}}(z) = 0 \end{aligned} \tag{3.20}$$

The only non zero $\tilde{w}_{\mathbf{k}}(z)$ occurs for $\mathbf{k} = (1,0)$ and $\mathbf{k} = (0,1)$, hence

$$\tilde{I}(z) = U_1 e^{-2z} z^{-1/2} \tilde{w}_{(1,0)}(z) + U_2 e^{2z} z^{-1/2} \tilde{w}_{(0,1)}(z) \tag{3.21}$$

and according to our convention, we define

$$\tilde{I}_1(z) := e^{-2z} z^{-1/2} \tilde{w}_{(1,0)}(z) \tag{3.22}$$

$$\tilde{I}_{-1}(z) := e^{2z} z^{-1/2} \tilde{w}_{(0,1)}(z). \tag{3.23}$$

We set $\tilde{w}_{(1,0)} = \tilde{w}_+$ and $\tilde{w}_{(0,1)} = \tilde{w}_-$, then their Borel transforms are solutions respectively of the following equations

$$(+) \quad \zeta^2 \hat{w}_+(\zeta) + 4\zeta \hat{w}_+(\zeta) + \frac{1}{4}\zeta * \hat{w}_+(\zeta) = 0$$

$$(-) \quad \zeta^2 \hat{w}_+(\zeta) - 4\zeta \hat{w}_+(\zeta) + \frac{1}{4}\zeta * \hat{w}_+(\zeta) = 0$$

taking twice derivative in ζ we get

$$\begin{aligned}
&(+) \quad (\zeta^2+4\zeta)\frac{d^2}{d\zeta^2}\hat{w}_+ + 4(\zeta-1)\frac{d}{d\zeta}\hat{w}_+ + \frac{9}{4}\hat{w}_+ = 0 \\
&(-) \quad (\zeta^2-4\zeta)\frac{d^2}{d\zeta^2}\hat{w}_- + 4(\zeta+1)\frac{d}{d\zeta}\hat{w}_+ + \frac{9}{4}\hat{w}_- = 0 \\
&(+) \quad \xi(1-\xi)\frac{d^2}{d\xi^2}\hat{w}_+ + (1-4\xi)\frac{d}{d\xi}\hat{w}_+ - \frac{9}{4}\hat{w}_+ = 0 \qquad \xi = -\frac{\zeta}{4} \\
&(-) \quad \xi(1-\xi)\frac{d^2}{d\xi^2}\hat{w}_- + (1-4\xi)\frac{d}{d\xi}\hat{w}_- - \frac{9}{4}\hat{w}_- = 0 \qquad \xi = \frac{\zeta}{4}
\end{aligned}$$

therefore, since equation $(+), (-)$ are hypergeometric the fundamental solution is (see DLMF 15.10.2)

$$\hat{w}_+(\zeta) = {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; -\frac{\zeta}{4}\right) \tag{3.24}$$

$$\hat{w}_-(\zeta) = {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{\zeta}{4}\right) \tag{3.25}$$

In particular, we notice that taking the series expansion of $\hat{w}_+$ and $\hat{w}_-$ we get numerically that

$$\begin{aligned}
\hat{w}_+(\zeta-4) &= \frac{1}{\pi}\log(z)\hat{w}_-(z) + \phi_{\text{reg}} \\
\hat{w}_-(\zeta+4) &= \frac{1}{\pi}\log(z)\hat{w}_+(z) + \psi_{\text{reg}}
\end{aligned}$$

and analytically (thanks to 15.2.3 DLMF)

$$\begin{aligned}
\hat{w}_+(\zeta+i0) - \hat{w}_+(\zeta-i0) &= {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; -\frac{\zeta}{4}+i0\right) - {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; -\frac{\zeta}{4}-i0\right) \qquad \zeta < -4 \\
&= -8i\left(-\frac{\zeta}{4}-1\right)^{-1}\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\frac{\zeta}{4}+1\right)^n \\
&= 8i\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\frac{\zeta}{4}+1\right)^{n-1} \\
&= 8i\sum_{n\geq 0}\frac{(1/2)_{n+1}(1/2)_{n+1}}{(n+1)!\Gamma(n+1)}\left(\frac{\zeta}{4}+1\right)^n \\
&= \mathbf{2i}\sum_{n\geq 0}\frac{(3/2)_n(3/2)_n}{(n)!\Gamma(n+2)}\left(\frac{\zeta}{4}+1\right)^n \\
&= \mathbf{2i}\,{}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{\zeta}{4}+1\right)
\end{aligned}$$

$$\begin{aligned}
\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0) &= {}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{\zeta}{4}+i0\right)-{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{\zeta}{4}-i0\right) \qquad \zeta>4\\
&= 8i\left(\frac{\zeta}{4}-1\right)^{-1}\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\frac{\zeta}{4}-1\right)^n\\
&= -8i\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(1-\frac{\zeta}{4}\right)^{n-1}\\
&= -8i\sum_{n\geq 0}(-1)^n\frac{(1/2)_{n+1}(1/2)_{n+1}}{(n+1)!\Gamma(n+1)}\left(1-\frac{\zeta}{4}\right)^n\\
&= \mathbf{-2i}\sum_{n\geq 0}(-1)^n\frac{(3/2)_n(3/2)_n}{(n)!\Gamma(n+2)}\left(1-\frac{\zeta}{4}\right)^n\\
&= \mathbf{-2i}\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;1-\frac{\zeta}{4}\right)
\end{aligned}$$

These are evidence of the resurgent properties of $\tilde{I}_{\pm 1}(z)$.

Lemma 3.7. *The following identity holds true*

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{1}{2};\frac{\zeta^2}{4}\right)=2i\frac{u}{u^2-1} \qquad \zeta=u+\frac{1}{u} \tag{3.26}$$

Proof. From 15.4.13 DLMF, we have

$$\begin{aligned}
{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{1}{2};\frac{\zeta^2}{4}\right) &= \frac{2}{\sqrt{4-\zeta^2}} \qquad y=\operatorname{arccsc}(\zeta/2)\\
&= \frac{1}{\sqrt{1-\csc^2(y)}}\\
&= -i\tan(y) \qquad \zeta=\frac{2}{\sin(y)}
\end{aligned}$$

therefore if $u=\tan\left(\frac{y}{2}\right)$, we have $\zeta=\frac{1+u^2}{u}=f(u)$ and

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{1}{2};\frac{\zeta^2}{4}\right)=2i\frac{u}{u^2-1}=\frac{2i}{f'(u)u}$$

□

Claim 3.8.

$$\hat{w}_+(\zeta-2)=i\pi\int_2^\zeta(\zeta-\zeta')^{-1/2}2\zeta'\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};\frac{3}{2};\frac{\zeta^2}{4}\right)d\zeta' \qquad \zeta\in(2,+\infty) \tag{3.27}$$

Proof. Let us first consider the RHS of (3.8)

$$
\begin{aligned}
&2\pi\int_2^\zeta(\zeta-\zeta')^{-1/2}\zeta'\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};\frac{3}{2};\frac{\zeta^2}{4}\right)d\zeta'=\\
&=\frac{4}{3}\int_2^\zeta(\zeta-\zeta')^{-1/2}\left[{}_2F_1\left(2,2;\frac{5}{2};\frac{1}{2}+\frac{\zeta'}{4}\right)-{}_2F_1\left(2,2;\frac{5}{2};\frac{1}{2}-\frac{\zeta'}{4}\right)\right]d\zeta'\\
&=\frac{8}{3}\int_0^x(y-x)^{-1/2}\left[{}_2F_1\left(2,2,;\frac{5}{2};y\right)-{}_2F_1\left(2,2;\frac{5}{2};1-y\right)\right]dy \qquad x\in(-\infty,0)\\
&=2\pi\int_0^x(x-y)^{-1/2}y^{-2}F\left(2,\frac{1}{2};1;\frac{1}{y}\right)dy \qquad (4.3)\\
&=\pi^2|x|^{-3/2}{}_2F_1\left(\frac{3}{2},\frac{1}{2};1;\frac{1}{x}\right) \qquad x\in(-\infty,0)\\
&=\frac{\pi^2}{2}\left({}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{1}{2}-\frac{\zeta}{4}\right)-i\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{1}{2}+\frac{\zeta}{4}\right)\right) \qquad \zeta\in(2,+\infty)
\end{aligned}
$$

however ${}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{1}{2}+\frac{\zeta}{4}\right)$ has a branch cut at $\zeta\in(2,+\infty)$, thus the claim holds true. □

Analougusly, it can be verified for $\hat{w}_-(\zeta+2)$ for $\zeta\in(-\infty,-2)$.

4. Useful identities for Gauss hypergeomtric functions

(4.1)
$$
\begin{aligned}
{}_2F_1(a,b;c;z)=e^{b\pi i}&\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(a+b-c+1)\Gamma(c-b)}{}_2F_1(a,b;c;1-z)+\\
&-e^{(a+b-c)\pi i}\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(b)\Gamma(a-b+1)}|z|^{-a}{}_2F_1\left(a,a-c+1;a-b+1;\frac{1}{z}\right)
\end{aligned}
$$

(4.2)
$$
\int_0^x|y|^{a-\mu-1}{}_2F_1(a,b;c,y)(x-y)^{\mu-1}dy=\frac{\Gamma(\mu)\Gamma(a-\mu)}{\Gamma(a)}|x|^{a-1}{}_2F_1(a-\mu,b;c;x)
$$
$$
x\in(-\infty,0)\cup(0,1),\ \Re a>\Re\mu>0
$$

which can be rewritten as (`arXiv:1504.08144`**, formula 4.8)**

(4.3)
$$
\int_{y>x}|y|^{-a}|x-y|^{\mu-1}{}_2F_1\left(a,b;c;y^{-1}\right)dy=\frac{\Gamma(\mu)\Gamma(a-\mu)}{\Gamma(a)}|x|^{-a+\mu}{}_2F_1\left(a-\mu,b;c;x^{-1}\right)
$$
$$
x\in(-\infty,0)\cup(1,\infty),\ \Re a>\Re\mu>0
$$

5. RESURGENCE FOR DEGREE 3 POLYNOMIALS

Let f be a degree 3 polynomial, and t_1, t_2 its critical points (not necessarily distinguished):

(1) if $t_1 \neq t_2$, then

$$I(z) = \int_{\mathcal{C}_j} e^{-zf} \, dt$$

is a solution of

$$I'' + aI' + bI + c\frac{I'}{z} + \frac{d}{z}I + \frac{e}{z^2}I = 0 \tag{5.1}$$

where a, b, c, d, e are determined in terms of f.

(2) if $t_1 = t_2$, then

$$I(z) = \int_{\mathcal{C}_1} e^{-zf} \, dt$$

is a solution of a first order ODE

$$I' + \left(a_4 - \frac{a_2^3}{27a_1^2} + \frac{1}{3z}\right) I = 0 \tag{5.2}$$

Proof. Let $f(t) = a_1 t^3 + a_2 t^2 + a_3 t + a_4$ with $a_1 \neq 0$,

$$\int_{\mathcal{C}_j} e^{-fz} \, dt = \int_{\mathcal{C}_j + \frac{a_2}{3a_1}} e^{-(a_1 t^3 + pt + q)z} \, dt \qquad\qquad t \to t - \frac{a_2}{3a_1}$$

where $p = a_3 - \frac{a_2^2}{3a_1}$ and $q = a_4 - \frac{a_2 a_3}{3a_1} + \frac{2a_2^3}{27a_1^2}$.

Case (1): if $p \neq 0$,

$$I(z) = \int t(3a_1 t^2 + p) z e^{-fz} = \int (3a_1 t^3 + pt) z e^{-ft} =$$

$$\int 2a_1 t^3 z e^{-fz} + \int (a_1 t^3 + pt + q) z e^{-fz} - qzI$$

$$2z \int a_1 t^3 e^{-fz} - zI' - qzI$$

$$2z\int a_1t^3e^{-fz} = 2z^2\int \frac{t^4}{4}a_1(3a_1t^2+p)e^{-fz} = \frac{z^2}{2}\int(3a_1^2t^6+pa_1t^4)e^{-fz} =$$

$$\frac{z^2}{2}\int(3a_1^2t^6+6pa_1t^4+3q^2+3p^2t^2+6pqt+6a_1qt^3)e^{-fz}+$$

$$+\frac{z^2}{2}\int(pa_1t^4-6pa_1t^4)e^{-fz}-\frac{z^2}{2}\int(3q^2+3p^2t^2+6pqt+6a_1qt^3)e^{-fz}$$

$$=\frac{3z^2}{2}I''+\frac{3z^2}{2}q^2I+3qz^2I'-\frac{z^2}{2}p\int(3a_1t^4+pt^2)e^{-fz}-z^2p\int(a_1t^4+pt^2)e^{-fz}$$

$$=\frac{3z^2}{2}I''+\frac{3z^2}{2}q^2I+3qz^2I'-\frac{5}{3}zp\int te^{-fz}-\frac{2}{3}z^2p^2\int t^2e^{-fz}$$

hence

(5.3)

$$I=-zI'-qzI+\frac{3z^2}{2}I''+\frac{3z^2}{2}q^2I+3qz^2I'-\frac{5}{3}zp\int te^{-fz}-\frac{2}{3}z^2p^2\int t^2e^{-fz}$$

(5.4)

$$\frac{3z^2}{2}\left(I''+q^2I+2qI'-\frac{2}{3z}I'-\frac{2q}{3z}I-\frac{2}{3z^2}I-\frac{10}{9z}p\int te^{-fz}dt-\frac{4}{9}p^2\int t^2e^{-fz}\right)=0$$

Notice that

$$\frac{4}{9}p^2\int t^2e^{-fz}=\frac{4}{27a_1}p^2\int(3a_1t^2+p)e^{-fz}-\frac{4}{27a_1}p^3I=-\frac{4}{27a_1}p^3I$$

$$-\frac{10}{9z}p\int te^{-fz}dt=\frac{5}{9z}\int pte^{-fz}-\frac{5}{3z}\int(pt+q)e^{-fz}+\frac{5}{3z}qI=$$

$$\frac{5}{9z}\int pte^{-fz}-\frac{5}{3z}\int(pt+q+a_1t^3)e^{-fz}+\frac{5}{3z}\int a_1t^3e^{-fz}+\frac{5}{3z}qI=$$

$$\frac{5}{9z}\int t(3a_1t^2+p)e^{-fz}+\frac{5}{3z}I'+\frac{5}{3z}qI$$

$$=\frac{5}{9z^2}I+\frac{5}{3z}I'+\frac{5}{3z}qI$$

thefore, collecting all the contributions together we find

$$I''+q^2I+2qI'-\frac{2}{3z}I'-\frac{2q}{3z}I-\frac{2}{3z^2}I+\frac{5}{9z^2}I+\frac{5}{3z}I'+\frac{5}{3z}qI+\frac{4}{27a_1}p^3I=0$$

$$I''+2qI'+\left(\frac{4p^3}{27a_1}+q^2\right)I+\frac{1}{z}I'+\frac{q}{z}I-\frac{1}{9z^2}I=0$$

Case (2): if $p=0$, then integrating by part we have

$$
\begin{aligned}
I(z) &= \int_{\mathcal{C}_1+\frac{a_2}{3a_1}} e^{-(a_1t^3+q)z}\,dt \\
&= \Big[t e^{-(a_1t^3+q)z}\Big]_{\mathcal{C}_1+\frac{a_2}{3a_1}} + \int_{\mathcal{C}_1+\frac{a_2}{3a_1}} 3a_1t^3 z e^{-(a_1t^3+q)z}\,dt \\
&= 3z\int_{\mathcal{C}_1+\frac{a_2}{3a_1}} (a_1t^3+q)e^{-(a_1t^3+q)z}\,dt - 3qz\int_{\mathcal{C}_1+\frac{a_2}{3a_1}} e^{-(a_1t^3+q)z}\,dt \\
&= -3zI'(z) - 3qzI(z)
\end{aligned}
$$

$\square$

We would like to verify that for every cubic function f, the Borel transform of the exponential integral can be expressed by an hypergeometric function and hence deduce its resurgent properties in full generality. If $p\neq 0$, $I(z)$ is a solution of

(5.5) $$I''+2qI'+\left(\frac{4p^3}{27a_1}+q^2\right)I+\frac{1}{z}I'+\frac{q}{z}I-\frac{1}{9z^2}I=0$$

hence a formal solution as $z\to\infty$ is given (up to constants $U_1,U_2\in\mathbb{C}$) by

(5.6) $$\tilde{I}_+(z) := U_1 e^{-(q+\sqrt{\frac{4p^3}{27a_1}})z} z^{1/2}\tilde{w}_+(z)$$

(5.7) $$\tilde{I}_-(z) := U_2 e^{-(q-\sqrt{\frac{4p^3}{27a_1}})z} z^{1/2}\tilde{w}_-(z)$$

where $\tilde{w}_\pm(z)\in\mathbb{C}[[z^{-1}]]$ is the formal solution of

(5.8) $$\tilde{w}''_\pm \mp 2\sqrt{\frac{4p^3}{27a_1}}\,\tilde{w}'_\pm + \frac{5}{36}\frac{\tilde{w}_\pm}{z^2}=0$$

with $\tilde{w}_\pm(z)=1+\sum_{k\geq 1} a_{\pm,k}z^{-k}$.

We can now compute the Borel transform of (5.8): for $\tilde{w}_+(z)$

$$\zeta^2\hat{w}-2\sqrt{\frac{4p^3}{27a_1}}\zeta\hat{w}+\frac{5}{36}\int_0^\zeta(\zeta-\zeta')\hat{w}(\zeta')d\zeta'=0$$

$$\zeta^2\hat{w}'+2\zeta\hat{w}+2\sqrt{\frac{4p^3}{27a_1}}\hat{w}+2\sqrt{\frac{4p^3}{27a_1}}\zeta\hat{w}'+\frac{5}{36}\int_0^\zeta\hat{w}(\zeta')=0$$

$$\left(\zeta^2+2\sqrt{\frac{4p^3}{27a_1}}\zeta\right)\hat{w}''+4\left(\zeta+\sqrt{\frac{4p^3}{27a_1}}\right)\hat{w}'+\frac{77}{36}\hat{w}=0$$

$$t(1-t)\hat{w}''+(2-4t)\hat{w}'-\frac{77}{36}\hat{w}=0 \qquad \zeta=-2t\sqrt{\frac{4p^3}{27a_1}}$$

hence

$$\hat{w}_+(\zeta) = {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; -\frac{3}{4p}\sqrt{\frac{3a_1}{p}}\zeta\right)$$

and analougusly,

$$\hat{w}_-(\zeta) = {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; \frac{3}{4p}\sqrt{\frac{3a_1}{p}}\zeta\right).$$

Notice that $\hat{w}_\pm(\zeta)$ has a branch cut singularity respectively at $\zeta = \zeta_\pm := \pm\sqrt{\frac{16p^3}{27a_1}}$, and thanks to the well known formulas for the analytic continuation of hypergeometric functions (see 15.2.3 DLMF), if we assume the branch cut is from $\zeta_\pm$ to $+\infty$

$$\begin{aligned}
\hat{w}_-(\zeta+i0) - \hat{w}_-(\zeta-i0) &= \frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\left(\frac{\zeta}{\zeta_+}-1\right)^{-1}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_+}\right)^k \quad \zeta\in(\zeta_+,+\infty)\\
&= -\frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_+}\right)^{k-1}\\
&= -\mathbf{i}\frac{1}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{\Gamma\left(\frac{7}{6}+k\right)\Gamma\left(\frac{11}{6}+k\right)}{\Gamma(2+k)\Gamma(k+1)}\left(1-\frac{\zeta}{\zeta_+}\right)^k\\
&= -\mathbf{i}\,{}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; 1-\frac{\zeta}{\zeta_+}\right)\\
&= -\mathbf{i}\,\hat{w}_+(\zeta-\zeta_+)
\end{aligned}$$

Similarly, if we assume that the branch cut is from $\zeta_\pm$ to $-\infty$ then

$$\begin{aligned}
\hat{w}_+(\zeta+i0) - \hat{w}_+(\zeta-i0) &= -\frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\left(\frac{\zeta}{\zeta_-}-1\right)^{-1}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_-}\right)^k \quad \zeta\in(-\infty,\zeta_-)\\
&= \frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_-}\right)^{k-1}\\
&= \mathbf{i}\frac{1}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{\Gamma\left(\frac{7}{6}+k\right)\Gamma\left(\frac{11}{6}+k\right)}{\Gamma(2+k)\Gamma(k+1)}\left(1-\frac{\zeta}{\zeta_-}\right)^k\\
&= \mathbf{i}\,{}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; 1-\frac{\zeta}{\zeta_-}\right)\\
&= \mathbf{i}\,\hat{w}_-(\zeta-\zeta_-)
\end{aligned}$$

therefore we see that the Stokes factors are given by $\pm i$ (as for Airy).

I think it will be nice to add the geometric interpretation of Maxim in term of Lefschetz thimbles

The situation is quite different if we consider the degenerate case, where we have only a singular point: indeed there is a one parameter family of solutions of

$$I'(z)+\left(\frac{1}{3z}+q\right)I(z)=0 \tag{5.9}$$

namely for $U \in \mathbb{C}$

$$\tilde{I}(z)=U e^{-qz} z^{1/3}\tilde{w}(z) \qquad \text{where } \tilde{w}(z)\in\mathbb{C}[[z^{-1}]] \tag{5.10}$$

The Borel transform of $\tilde{w}$ is a solution of

$$\zeta\,\hat{w}'+\frac{\hat{w}}{3}=0 \tag{5.11}$$

hence, up to rescaling by a constant,

$$\hat{w}(\zeta)\propto\zeta^{-1/3}={}_2F_1\left(a,\frac{1}{3};a;1-\zeta\right)$$

for every $a\in\mathbb{C}$. In the degenerate case we get an hypergeometric function as well, but the resurgent structure is trivial, i.e. $\hat{w}(\zeta)$ is holomorphic on the Riemann surface of $\zeta^{1/3}$.

5.0.1. *Alternative computation of the Borel transform of I.* Let us first compute the Borel transform of (5.1) (indeed as in the proof of Theorem 3.1 we know that (5.1) admits a formal solution which is Gevrey-1)

$$\zeta^2\hat{I}-a\zeta\hat{I}+b\hat{I}-\int_0^{\zeta}\zeta'\hat{I}(\zeta')+d\int_0^{\zeta}\hat{I}(\zeta')-\frac{1}{9}\int_0^{\zeta}(\zeta-\zeta')\hat{I}(\zeta')=0$$

$$2\zeta\hat{I}+\zeta^2\hat{I}'-a\hat{I}-a\zeta\hat{I}'+b\hat{I}'-\zeta\hat{I}+d\hat{I}-\frac{1}{9}\int\hat{I}(\zeta')=0$$

$$(\zeta^2-a\zeta+b)\hat{I}''+(3\zeta-2a+d)\hat{I}'+\frac{8}{9}\hat{I}=0$$

Now we denote by λ_1,λ_2 the distinguished (we assume that $p\neq 0$) roots of $\zeta^2-a\zeta+b$, then

$$(\zeta-\lambda_1)(\zeta-\lambda_2)\hat{I}''+(3\zeta-2a+d)\hat{I}'+\frac{8}{9}\hat{I}=0 \tag{5.12}$$

$$(t+\lambda_2-\lambda_1)t\hat{I}''+(3t+3\lambda_2-2a+d)\hat{I}'+\frac{8}{9}=0 \qquad t=\zeta-\lambda_2 \tag{5.13}$$

$$s(1-s)\hat{I}''-\left(3s+\frac{3\lambda_2-2a+d}{\lambda_1-\lambda_2}\right)\hat{I}'-\frac{8}{9}\hat{I}=0 \qquad t=(\lambda_1-\lambda_2)s \tag{5.14}$$

where (5.14) is an hypergeometric equation[5] and a solution is given by

(5.15)
$$\hat{I}_{\lambda_1}(\zeta; U_1, U_2) = U_1\, {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; \frac{\zeta-\lambda_2}{\lambda_1-\lambda_2}\right) + U_2\left(\frac{\zeta-\lambda_2}{\lambda_1-\lambda_2}\right)^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta-\lambda_2}{\lambda_1-\lambda_2}\right)$$

which has a branch cut at $\zeta = \lambda_1$, where U_1, U_2 are constants. Of course, reversing the role of λ_1 and λ_2 we find

(5.16)
$$\hat{I}_{\lambda_2}(\zeta,; U_1, U_2) = U_1 {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; \frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right) + U_2\left(\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)$$

is the Borel transform of $\tilde{I}_{\lambda_2}(z)$ and it has a branch cut singularity at $\zeta = \lambda_2$. It is remarkable that the dependece on the function f is only on the location of the singularities, but it is always an hypergeometric function with the same parameters. In addition, we can compute the Stokes constants thanks to the well known formula for analyitic continuation of hypergeometric (see 15.2.3 in DLMF)

$$\begin{aligned}
\hat{I}_{\lambda_1}(\zeta+i0; U_1, 0) - \hat{I}_{\lambda_1}(\zeta-i0; U_1, 0) &= -U_1 \frac{2\pi i}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}\left(\frac{\lambda_1-\zeta}{\lambda_2-\lambda_1}\right)^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right) \\
&= -\mathbf{i} U_1 \frac{2\pi i}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}\left(\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right) \\
&= -\mathbf{i}\hat{I}_{\lambda_2}\left(\zeta; 0, \frac{U_1}{\Gamma(2/3)\Gamma(4/3)}\right)
\end{aligned}$$

[5]Notice that $\lambda_{1,2} = q \pm \frac{2i}{3} p\sqrt{\frac{p}{3a_1}}$, $a = 2q$ and $d = q$. Hence

$$\frac{2a-d-3\lambda_2}{\lambda_1-\lambda_2} = \frac{4q-q-3q-2ip\sqrt{\frac{p}{3a_1}}}{-\frac{4i}{3}p\sqrt{\frac{p}{3a_1}}} = \frac{3}{2}$$