BOREL REGULARITY FOR EXPONENTIAL INTEGRALS AND ODES

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CONTENTS

1. INTRODUCTION

1.1. **Motivation.**

1.2. **What are exponential integrals.** Let X be an n-dimensional algebraic variety, we consider exponential integrals of the form

$$I(z) = \int_{\mathcal{C}} e^{-zf}\, \nu \tag{1.1}$$

where $f : X \to \mathbb{C}$ is an algebraic function, $\nu \in \Gamma(X, \Omega^n)$, $z \in \mathbb{C}$ and $\mathcal{C}$ is an n-cycle of integration such that the integral is well defined. Assuming $\theta := \arg(z)$ fixed, we define for every $c > 0$

$$S_c^+ := \{\zeta \in \mathbb{C} | \operatorname{Re}(\zeta e^{i\theta}) \geq c\}$$
$$S_c^- := \{\zeta \in \mathbb{C} | \operatorname{Re}(\zeta e^{i\theta}) \leq c\}$$

and we define $H_\bullet(X,zf)$ as the relative homology groups, namely

$$H_\bullet(X,zf) := H_\bullet\left(X, f^{-1}(S_c^+); \mathbb{Z}\right)$$

for c large enough so that $f^{-1}(S_c^+)$ does not contain critical values of f (i.e. points $\zeta_\alpha := f(x_\alpha)$ where $df(x_\alpha)=0$ [1]). Then we assume $[\mathcal{C}] \in H_n(X,zf)$.

Notice that $I(z)$ only depends on $[\mathcal{C}] \in H_n(X,zf)$ and on $[\nu] \in H^n_{dR}(X,zf) := \mathbb{H}^n\left(X, (\Omega_X^\bullet, d_W := e^{zf}\circ d \circ e^{-zf})\right)$: indeed let $\mathcal{C}' = \mathcal{C} + \partial\gamma \in [\mathcal{C}]$,

$$\int_{\mathcal{C}'} e^{-zf}\,\nu = \int_{\mathcal{C}} e^{-zf}\,\nu + \int_{\partial\gamma} e^{-zf}\,\nu = \int_{\mathcal{C}} e^{-zf}\,\nu$$

because e^{-zf} is rapidly decaying at $\partial\gamma$. Similarly, if $\nu' = \nu + d_W\eta \in [\nu]$

$$\int_{\mathcal{C}} e^{-zf}\,\nu' = \int_{\mathcal{C}} e^{-zf}\,\nu + \int_{\mathcal{C}} e^{-zf}\,d_W\eta = \int_{\mathcal{C}} e^{-zf}\,\nu + \int_{\mathcal{C}} d(e^{-zf}\,\eta) =$$
$$= \int_{\mathcal{C}} e^{-zf}\,\nu + \int_{\partial\mathcal{C}} e^{-zf}\,\eta = \int_{\mathcal{C}} e^{-zf}\,\nu.$$

Moreover, for θ generic, i.e. when $\theta \neq \arg(\zeta_\alpha - \zeta_\beta)$ and $\alpha \neq \beta$, there is a decomposition of $H_\bullet(X,zf)$ as direct sum over the critical values of f of the homology relative to these critical values (see [12][8]):

$$H_\bullet(X,zf) = \bigoplus_\alpha H_\bullet\left(f^{-1}(B_\alpha(\varepsilon)) \cap f^{-1}(S_\alpha^+(\theta,\varepsilon))\right) \tag{1.2}$$

where $B_\alpha(\varepsilon) := \{\zeta \in \mathbb{C} \big| \, |\zeta - \zeta_\alpha| < \varepsilon\}$ and $S_\alpha^+(\theta,\varepsilon) := B_\alpha(\varepsilon) \cap \{\mathrm{Re}(\zeta e^{i\theta}) \geq \frac{\varepsilon}{2}\}$ (see Figure 1).

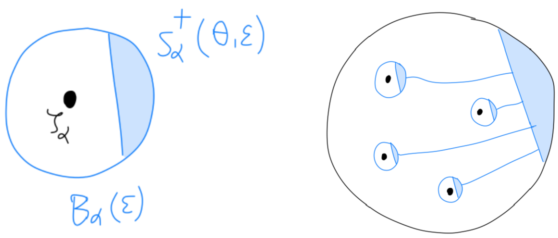

FIGURE 1. Relative homology

Let $\theta = 0$ (if $\theta \neq 0$ we consider $x e^{i\theta/2}$), locally in a neighbourhood of a critcal point x_α we can choose a coordinate system $x_1, \ldots, x_n$ such that $f(x) - f(x_\alpha) = x_1^2 + \ldots + x_n^2$. Then $\mathrm{Re}(f)$ is a Morse function with simple, isolated and non degenerate critical points. We choose a Riemannian metric g on X and we define the Lefschetz thimbles

[1] In dimension $n > 1$ there are other critical values due to the fact that f is not proper. However under suitable assumptions it is possible to overcome this issue (see [12] part 1, §2).

$\mathcal{C}_{\alpha,z}$ as the steepstet descendet path issuing from a critcal point x_α (see [13]):

$$\mathcal{C}_{\alpha,z} = \{x \in X \,\big|\, \lim_{t \to -\infty} x(t) = x_\alpha\} \tag{1.3}$$

where $x(t)$ is the solution of the gradient flow equation

$$\frac{\mathrm{d}\,x_i(t)}{\mathrm{d}\,t} = -g_{ij} x_j(t)$$

and $x_i(0) = x_i$ [2].

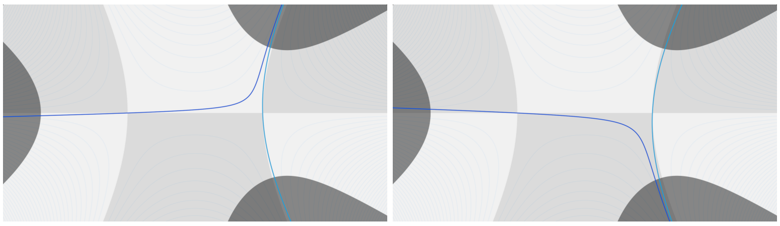

FIGURE 2. Lefschetz thimbles for the Airy integral $f(t) = t^3/3 - t$ at $\theta = ?$ (left) and $\theta = ??$ (right).

Notice that $[\mathcal{C}_{\alpha,z}] \in H_n(X, zf)$ and they generate the relative homology for every critical values $H_n\left(f^{-1}(B_\alpha(\varepsilon)) \cap f^{-1}(S_\alpha^+(\theta,\varepsilon))\right)$. In fact,

- $H_k\left(f^{-1}(B_\alpha(\varepsilon)) \cap f^{-1}(S_\alpha^+(\theta,\varepsilon))\right)$ is isomorphic to $H_{k-1}(B_\alpha^M(\varepsilon) \cap f^{-1}(\zeta - \zeta_\alpha))$, where $B_\alpha^M(\varepsilon)$ is an open ball centred at x_α of radius ε such that for every $\varepsilon' < \varepsilon$ $B_\alpha(\varepsilon')$ is transverse to $f^{-1}(\zeta_\alpha)$.
- $H_{k-1}(B_\alpha^M(\varepsilon) \cap f^{-1}(\zeta - \zeta_\alpha)) = 0$ if $k \neq n$ and $H_{n-1}(B_\alpha^M(\varepsilon) \cap f^{-1}(\zeta - \zeta_\alpha)) = \mathbb{Z}^\mu$, where μ is the Milnor number;

hence $H_n\left(f^{-1}(B_\alpha(\varepsilon)) \cap f^{-1}(S_\alpha^+(\theta,\varepsilon))\right)$ has rank μ. In addition from singularity theory we know that the set of vanishing cycles $\partial\mathcal{C}_{\alpha,z}$ forms a basis of $H_{n-1}(B_\alpha^M(\varepsilon) \cap f^{-1}(\zeta - \zeta_\alpha))$ (see Theorem 2.1 [2]).

We furthermore assume the following:

Assumption 1.1. *The map $f : X \to \mathbb{C}$ has isolated, simple and non degenerate critical values ζ_α.*

Since for θ generic the $\mathcal{C}_{\alpha,z}$ are a basis for the relative homology $H_n(X, zf)$, the integral $I(z)$ can be decomposed as sum over Lefschetz thimbles: we denote

$$I_\alpha(z) := \int_{\mathcal{C}_{\alpha,z}} e^{-zf}\, \nu \tag{1.4}$$

and we refer to $I_\alpha(z)$ as a thimble integral. Then for every $[\mathcal{C}] \in H_n(X, zf)$

$$I(z) = \sum_\alpha n_\alpha I_\alpha(z).$$

[2] An equivalent definition is given by F. Pham in [12].

Moreover the coeficients n_α can be computed via intersection theory (see §5 [12], §3.1.5 [13]). When X is not compact the intersection paring is defined introducing dual cycles: let

$$\mathcal{K}_{\alpha,z} := \{x \in X \,\big|\, \lim_{t \to -\infty} x(t) = x_\alpha\}$$

where $x(t)$ solve the upward flow equation

$$\frac{\mathrm{d}\, x_i(t)}{\mathrm{d}\, t} = g_{ij} x_j(t) \qquad \text{with } x_i(0) = x_i$$

then $\mathcal{K}_{\alpha,z} \in H_n(X, f^{-1}(S_c^-); \mathbb{Z})$ for c large enough and they define a basis (for θ generic). Furthermore, $\mathcal{K}_{\alpha,z}$ and $\mathcal{C}_{\alpha,z}$ intersect transversely at the critical point x_α, hence they define a non degenerate pairing

$$\begin{array}{cccc} \langle -,-\rangle : H_n(X, f^{-1}(S_c^+); \mathbb{Z}) \otimes H_n(X, f^{-1}(S_c^-); \mathbb{Z}) \to \mathbb{Z} & & & \\ \mathcal{C} \qquad , \qquad \mathcal{K} & \to & \langle \mathcal{C}, \mathcal{K} \rangle. & \end{array}$$

Now looking at the intersection of $\mathcal{C} \in H_n(X, zf)$ with the basis $\mathcal{K}_{\alpha,z}$ we can compute the constants n_α in terms of intersection pairing:

$$\langle \mathcal{C}, \mathcal{K}_{\alpha,z} \rangle = \sum_\beta n_\beta \langle \mathcal{C}_{\beta,z}, \mathcal{K}_{\alpha,z} \rangle = n_\alpha$$

ORIENTATION?

$n_\alpha = (-1)^{n(n-1)/2} \langle \mathcal{C}_{\alpha,\theta+\pi}, \mathcal{C} \rangle$

1.3. **Borel regularity.** Borel resummation is a way of turning a formal power series

$$\varphi_\bullet = z^\sigma \left(\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \frac{\varphi_3}{z^4} + \dots \right),$$

with $\sigma \in [0,1)$, into a function which is asymptotic to $\varphi_\bullet$ as $z \to \infty$. Different functions can be asymptotic to the same power series, and Borel resummation picks one of them, performing an implicit regularization [**arXiv:1705.03071, or maybe arXiv:1412.6614**]. When a function matches the Borel sum of its asymptotic series, we'll say it's *Borel regular*. Several familiar kinds of regularity imply Borel regularity, and shed light on why it occurs.

- **Having a good asymptotic approximation**

  Let R_N be the difference between a function and the partial sum

$$\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \dots + \frac{\varphi_{N-2}}{z^{N-1}}$$

of its asymptotic series. Watson showed a century ago that the function is Borel regular whenever there's a constant $c \in (0, \infty)$ with

$$|R_N| \le \frac{c^{N+1} N!}{|z|^N}$$

over all orders N and all z in a wide enough wedge around infinity.

- **Satisfying a singular differential equation**
  - Think about conditions where this works.
  - Maybe the correct place is the setting of Ecalle's formal integral. See §5.2.2.1 of Delabaere's *Divergent Series, Summability and Resurgence III*.
  - Say there's a unique solution (up to scaling) that shrinks as you go right; everything else blows up exponentially. Then this is the only solution that can be expressed as a Laplace transform.
  - If the Borel-transformed equation has a subexponential solution $\hat{f}$ which is "shifted holomorphic" (we called this having a "fractional power singularity" in `airy-resurgence`), then $\mathcal{L}\hat{f}$ satisfies the original equation, because there are no boundary terms.
  - Draw diagram showing formal vs. holomorphic solutions in time vs. frequency domains.
- **Being a thimble integral**

  Under the previous assumptions, we can now state a Borel regularity result for thimble integrals:

Theorem 1.2. Borel regularity for thimbles integrals can be stated a the commutativity of the following diagram:

$$\begin{tikzcd} I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\,\nu \arrow[r, "\sim"] \arrow[d, "\mathcal{L}^\theta"'] & \tilde{I}_\alpha(z) \arrow[d, "\mathcal{B}"] \\ \hat{\iota}_\alpha(\zeta) \arrow[r, equal, "\text{sum}"'] & \tilde{\iota}_\alpha(\zeta) \end{tikzcd} \tag{1.5}$$

A priori, the Laplace transform of $\hat{\iota}_\alpha(\zeta)$ and $I_\alpha(z)$ have the same asymptotic behaviour in a given sector (indeed taking the asymptotic of $I_\alpha(z)$ we *loose* information); however Borel regularity guarantees that $I_\alpha(z) = \mathcal{L}^\theta \hat{\iota}_\alpha$ in a given sector. We give a proof of Theorem 1.2 in Section ?? 3.1 when $N = 1$, and the proof is based on the fact that $I_\alpha(z)$ can be rewritten as the Laplace transform–type integral. In higher dimension the same result was stated in [**?**] (but the proof was not written).

Generically, $\tilde{I}_\alpha(z)$ is a divergent series whose coefficients factorially grow and, as proved by Berry and Howls [3][**?**] the divergence of $\tilde{I}_\alpha(z)$ encodes contributions from the other critical points of f in the form of *exact resurgence relation* (see equation 19 [3]). Indeed, the Borel transform $\hat{\imath}_\alpha(\zeta)$ yet contains information about the Borel transform $\hat{\imath}_\beta(\zeta)$ at other critical values ζ_β. This is the idea of resurgence of $\tilde{I}_\alpha(z)$ [5][6][7]. However, since f has finitely many critical points and ν is meromorphic, Borel–Laplace summability properties and resurgence of exponential integrals are straightforward to be proven. However, in order to completely describe the Borel plane one needs to compute the Stokes data which provide the analytic continuation across the branch cut. In particular for a certain class of exponential integrals the analytic theory developed by J. Ecalle to compute the Stokes data (see [**?**][11][4][1]) has a geometric interpretation relying on intersection theory of dual relative homology classes in the sense of F. Pham [12] (see also [10][9]).

1.4. **Stokes phenomena for exponential integrals.** So far we have considered $\theta = \arg(z)$ fixed and generic (i.e. $\theta \neq \arg(\zeta_\alpha - \zeta_\beta)$), but as we let θ varying $I_\alpha(z)$ becomes a multivalued function. Indeed when θ crosses a Stokes ray $\ell_{\alpha\beta} \coloneqq \arg(\zeta_\alpha - \zeta_\beta)\mathbb{R}$, $I_\alpha(z)$ jumps. Computing the jumps of $I_\alpha(z)$ is equivalent to determine the analytic continuation of $I_\alpha(z)$ for $z \in \mathbb{C}$. There are different ways to compute the Stokes jumps across the rays $\ell_{\alpha\beta}$: one is purely geometric, based on the way the decomposition (1.2) varies as θ varies. Another one is based on the resurgence analysis of the asymptotic expansion of $I_\alpha(z)$ as $\mathrm{Re}(ze^{i\theta}) \to +\infty$. In Section **??** we prove that the geometric and the resurgence approach to compute the Stokes jumps are equivalent:

Theorem 1.3 (Theorem **??**)**.**

1.4.1. *Geometric computation of Stokes constants.* As we discuss in §1.2, at θ generic, the relative homology groups $H_n(X, zf)$ admits a decomposition in terms of the homology relative ot each critical values $H_n(f^{-1}(B_\alpha(\varepsilon) \cap f^{-1}(S^+_\alpha(\theta, \varepsilon)))$.

1.4.2. *ODE and fractional derivative formula [draft2].*

1.4.3. *If hypergeometric functions appear in a large class of examples: integral formulas for hypergeometric functions.*

1.5. **Plan of the paper.**

2. Formalism of the Laplace transform

3. Borel regularity

3.1. **ODEs.**

3.2. **Thimble integrals.** We are going to prove Theorem 1.2. Let X be a n-dimensional algebraic variety, $f : X \to \mathbb{C}$ be an algebraic function with simple, isolated, non-degenerate critical points, and $\nu \in \Gamma(X, \Omega^n)$, and we consider

$$I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\, \nu \tag{3.1}$$

where $\mathcal{C}_{\alpha,z}$ is a Lefschetz thimble $[\mathcal{C}_{\alpha,z}] \in H_n(X, zf)$ for $\theta = \arg(z)$ generic.

Let us restrict to one dimensional X. In particular, $\mathcal{C}_\alpha$ is a steepest descent path through the critical point x_α and generic θ are such that $f(x_\beta) \notin f(x_\alpha) + [0, e^{i\theta}\infty)$ for $\beta \neq \alpha$. For any critical points x_α (satisfying the previous assumptions), the saddle point approximation allows to compute the asymptotic expansion of $I_\alpha(z)$

$$I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\, \nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)} \sqrt{2\pi} z^{-1/2} \sum_{k\geq 0} a_{\alpha,k} z^{-k} \qquad \text{as } \operatorname{Re}(ze^{i\theta}) \to \infty. \tag{3.2}$$

Notice that $f \circ \mathcal{C}_\alpha$ lies in the ray $\zeta_\alpha + [0, e^{i\theta}\infty)$, where $\zeta_\alpha := f(x_\alpha)$.

Theorem 3.1 (Theorem ??)**.** Let $n = 1$. Let $I_\alpha(z)$ defined as in (3.2) for every critical point x_α. Then $\tilde{I}_\alpha$ is Borel regular for $\operatorname{Re}(ze^{i\theta}) > 0$:

(1) The series $\tilde{I}_\alpha(z) = e^{-zf(x_\alpha)} \sqrt{2\pi} z^{-1/2} \sum_{k\geq 0} a_{\alpha,k} z^{-k}$ is Gevrey-1.
(2) The series $\tilde{\iota}_\alpha(\zeta) := \mathcal{B}(\tilde{I}_\alpha)$ converges near $\zeta = \zeta_\alpha$.
(3) If you continue the sum of $\tilde{\iota}_\alpha$ along the ray going rightward from ζ_α in the direction θ, and take its Laplace transform along that ray, you'll recover I_α.

Remark 3.2. (1) We may drop the assumption of non degenerate critical points for f, however the asymptotic expansion of $I_\alpha(z)$ will depend on the order m such that $f^{(m)}(x_\alpha) \neq 0$ and $f^{(j)}(x_\alpha) = 0$ for every $j = 1, \dots, m-1$ (see [Zorich] Theorem 1 Section 19.2.5).
(2) in [Malgrange74] (see also Chapter 5 of [Mistergard Phd thesis] for a general review), the author computes the asymptotic expansion of exponential integrals for $n > 1$ which get logarithmic terms like

$$\tilde{I}(z) = \sum_{j\in A} \sum_{k\geq 0} \sum_{q=0}^{n-1} a_{k,q,j} z^{-k-j} (\log z)^q,$$

for $A \subset \mathbb{Q}_{\geq 0}$ finite. Due to the presence of logarithmic terms, the definition of Borel transform has to be further extended (see [Mistergard phd] Definition pag 5) and the study of Borel regularity becomes more involved.
(3) in the proof of Theorem 3.1 we will derive formula (3.2) using Watson's lemma. However, the same result can be computed from geometric arguments as in Theorem 5.3.3 [Mistergard phd].

Proof. Part (1): Since f is Morse, we can find a holomorphic chart τ around x_α with $\frac{1}{2}\tau^2 = f - \zeta_\alpha$. Let $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$ be the parts of $\mathcal{C}_\alpha$ that go from the past to x_α and from x_α to the future, respectively. We can arrange for τ to be valued in $(-\infty e^{i\theta}, 0]$ and $[0, e^{i\theta}\infty)$ on $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$, respectively. [**We should explicitly spell out and check the conditions that make this possible. I think we're implicitly orienting $\mathcal{C}_\alpha$ so that τ in the upper half-plane.**] Since ν is holomorphic, we can express it as a Taylor series

$$\nu = \sum_{k\geq 0} b_k^\alpha \tau^k \, d\tau$$

that converges in some disk $|\tau| < \varepsilon$.

In coordinates τ the integral $I_\alpha(z)$ can be approximated as

$$I_\alpha(z) \sim e^{-z\zeta_\alpha} \int_{\tau\in[-\varepsilon,\varepsilon]} e^{-z\tau^2/2}\, \nu$$

as $\mathrm{Re}(z e^{i\theta}) \to \infty$ (see Lemma 1 in Section 19.2.2 Zorich). [**I need to learn how this works! Do we get asymptoticity at all orders? —Aaron**] Plugging in the Taylor series above, we get

$$\begin{aligned}
I_\alpha(z) &\sim e^{-z\zeta_\alpha} \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{k\geq 0} b_k^\alpha \tau^k \, d\tau \\
&= e^{-z\zeta_\alpha} \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{k\geq 0} b_{2k}^\alpha \tau^{2k} \, d\tau \\
&= 2e^{-z\zeta_\alpha} \int_{0}^{\varepsilon} e^{-z\tau^2/2} \sum_{k\geq 0} b_{2k}^\alpha \tau^{2k} \, d\tau.
\end{aligned}$$

By Watson's Lemma (see Lemma 4 Section 19.2.2 Zorich)

$$\begin{aligned}
I_\alpha(z) &\sim e^{-z\zeta_\alpha} \sum_{k\geq 0} b_{2k}^\alpha \Gamma\left(k + \tfrac{1}{2}\right) 2^{k+1/2} z^{-k-1/2} \\
&= e^{-z\zeta_\alpha} \sqrt{2\pi} \sum_{k\geq 0} b_{2k}^\alpha (2k-1)!! z^{-k-1/2}
\end{aligned}$$

Call the right-hand side $\tilde{I}_\alpha$. We now see that $a_{\alpha,k} = (2k-1)!!\, b_{2k}^\alpha$ in the statement of the theorem. We know from the definition of ε that $\left|b_k^\alpha\right| \varepsilon^k \lesssim 1$. Recalling that $(2k-1)!! \sim (\pi k)^{-1/2} 4^k\, k!$ as $k \to \infty$, we deduce that $|a_{\alpha,k}| \lesssim \left(\frac{4}{\varepsilon^2}\right)^k k!$, showing that $\tilde{I}_\alpha$ is Gevrey-1.

Part (2): note that [**explain formally what it means to center at ζ_α**]

$$\tilde{\iota}_\alpha := \mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha = \sqrt{2\pi} \sum_{k\geq 0} (2k-1)!!\, b_{2k}^\alpha \frac{(\zeta - \zeta_\alpha)^{k-1/2}}{\Gamma\left(k + \frac{1}{2}\right)}$$

FIGURE 3. The contour $\mathcal{C}_\alpha$, its image under f which is the Hankel contour $\mathcal{H}_\alpha = f(\mathcal{C}_\alpha)$ and the ray $[\zeta_\alpha, +\infty]$.

Since $(2k-1)!! = \pi^{-1/2} 2^k \Gamma\left(k+\frac{1}{2}\right)$ and $|b_k^\alpha|\epsilon^n \lesssim 1$, then $\tilde{\iota}_\alpha(\zeta)$ has a finite radius of convergence.

Part (3): Let's recast the integral I_α into the f plane. As ζ goes rightward from ζ_α, the start and end points of $\mathcal{C}_\alpha(\zeta)$ sweep backward along $\mathcal{C}_\alpha^-(\zeta)$ and forward along $\mathcal{C}_\alpha^+(\zeta)$, respectively. Hence, we have

$$\begin{aligned}
I_\alpha(z) &= \int_{\mathcal{C}_\alpha} e^{-zf}\,\nu \\
&= \int_{\mathcal{H}_\alpha} e^{-z\zeta}\left(\int_{f^{-1}(\zeta)} \frac{\nu}{df}\right) d\zeta \\
&= \int_{\zeta_\alpha}^{e^{i\theta}\infty} e^{-z\zeta}\left[\frac{\nu}{df}\right]_{\mathrm{start}\,\mathcal{C}_\alpha(\zeta)}^{\mathrm{end}\,\mathcal{C}_\alpha(\zeta)} d\zeta.
\end{aligned}$$

where $\mathcal{H}_\alpha$ is the Hanckel contour through the point ζ_α (see Figure [**?**]) with ends in the θ direction. Noticing that the last integral is a Laplace transform for the initial choice of θ, we learn that

$$\hat{\iota}_\alpha(\zeta) = \left[\frac{\nu}{df}\right]_{\mathrm{start}\,\mathcal{C}_\alpha(\zeta)}^{\mathrm{end}\,\mathcal{C}_\alpha(\zeta)}. \tag{3.3}$$

In Ecalle's formalism, $\overset{\triangledown}{\iota}_\alpha := \int_{f^{-1}(\zeta)} \frac{\nu}{df}$ and $\hat{\iota}_\alpha$ are respectively a major and a minor of the singularity and they differ by an holomorphic function (we will see this in the examples Section Airy, Bessel).

We can rewrite our Taylor series for ν as

$$\begin{aligned}
\nu &= \sum_{k\geq 0} b_n^\alpha [2(f-\zeta_\alpha)]^{k/2} \frac{df}{[2(f-\zeta_\alpha)]^{1/2}} \\
&= \sum_{k\geq 0} b_n^\alpha [2(f-\zeta_\alpha)]^{(k-1)/2}\, df,
\end{aligned}$$

taking the positive branch of the square root on $\mathcal{C}_\alpha^+$ and the negative branch on $\mathcal{C}_\alpha^-$. Plugging this into our expression for $\hat{\iota}_\alpha$, we learn that

$$
\begin{aligned}
\hat{\iota}_\alpha(\zeta) &= \left[\sum_{k\geq 0} b_k^\alpha [2(f-\zeta_\alpha)]^{(k-1)/2}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)} \\
&= \sum_{k\geq 0} b_n^\alpha \Big([2(\zeta-\zeta_\alpha)]^{(k-1)/2} - (-1)^{k-1}[2(\zeta-\zeta_\alpha)]^{(k-1)/2}\Big) \\
&= \sum_{k\geq 0} 2b_{2k}^\alpha [2(\zeta-\zeta_\alpha)]^{k-1/2} \\
&= \sum_{k\geq 0} 2^{k+1/2} b_{2k}^\alpha (\zeta-\zeta_\alpha)^{k-1/2} \\
&= \mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha.
\end{aligned}
$$

We have now shown that the sum of $\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha$ is actually equal to $\hat{\iota}_\alpha$ as $\zeta \in \zeta_\alpha + [0, e^{i\theta}\infty)$. □

Remark 3.3. Different choices of admissible θ correspond to different choices of thimbles $[\mathcal{C}_\alpha] \in H_n^B(X, zf)$, but the Borel transform of $\tilde{I}_\alpha$ does not depend on θ. However, if $\theta_* := \arg(\zeta_\alpha - \zeta_\beta)$ and $\theta_\pm := \theta_* \pm \delta$ for small δ, then $I_\alpha(z)$ jumps on the intersection between $\mathrm{Re}(e^{i\theta_+}z) > 0$ and $\mathrm{Re}(e^{i\theta_-}z) > 0$. This is known as the Stokes phenomenon (see Section resurgence thimbles integrals).

3.2.1. 3/2 *derivative formula.* In Theorem 3.1 we have seen that the asymptotic behaviour of $I_\alpha(z)$ has a fractional power contribution, namely

$$\tilde{I}_\alpha(z) = e^{-z\zeta_\alpha} z^{-1/2}\sqrt{2\pi}\sum_{k\geq 0} a_{\alpha,k} z^{-k},$$

hence we have used the extended notion of Borel transform to deal with fractional powers. Now we will focus on the formal series $\tilde{\Phi}_\alpha(z) := e^{-z\zeta_\alpha}\sqrt{2\pi}\sum_{k\geq 0} a_{\alpha,k} z^{-k} = z^{1/2}\tilde{I}_\alpha(z)$ which does not contain any fractional power and we prove a fractional derivative formula which relates the Borel transforms $\hat{\varphi}_\alpha(\zeta)$ and $\hat{\iota}_\alpha(\zeta)$. Moreover we show that the $\hat{\varphi}_\alpha(\zeta)$ depends on ν and df as well as $\hat{\iota}_\alpha(\zeta)$ does.

Corollary 3.4. Under the same assumptions of Theorem 3.1, for any ζ on the ray going rightward from ζ_α in the direction of θ, we have

$$\hat{\varphi}_\alpha(\zeta) = \partial^{3/2}_{\zeta \text{ from } \zeta_\alpha}\left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right) = \left(\frac{\partial}{\partial\zeta}\right)^2 \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{\zeta_\alpha}^{\zeta} (\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_\alpha(\zeta')} \nu\right) d\zeta', \tag{3.4}$$

where $\mathcal{C}_\alpha(\zeta)$ is the part of $\mathcal{C}_\alpha$ that goes through $e^{-i\theta} f^{-1}([\zeta_\alpha, \zeta])$. Notice that $\mathcal{C}_\alpha(\zeta)$ starts and ends in $e^{-i\theta} f^{-1}(\zeta)$. **[Be careful about the orientation of $\mathcal{C}_\alpha$.]**

Proof. Theorem **??** tells us that

$$\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha = \mathcal{B}_{\zeta_\alpha} z^{-1/2}\tilde{\varphi}_\alpha = \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha}\mathcal{B}\tilde{\varphi}_\alpha = \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha.$$

It follows, from the proof of part 3 of Theorem 3.1, that

$$\hat{\iota}_\alpha(\zeta) = \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha. \tag{3.5}$$

Since fractional integrals form a semigroup, equation (3.5) implies that

$$\partial^{-1}_{\zeta \text{ from } \zeta_\alpha}\hat{\iota}_\alpha(\zeta) = \partial^{-3/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha.$$

Rewriting equation (3.3) as

$$\hat{\iota}_\alpha(\zeta) = \partial_\zeta\left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right),$$

we can see that

$$\partial^{-1}_{\zeta \text{ from } \zeta_\alpha}\hat{\iota}_\alpha(\zeta) = \int_{\mathcal{C}_\alpha(\zeta)} \nu - \int_{\mathcal{C}_\alpha(0)} \nu.$$

The initial value term vanishes, because the path $\mathcal{C}_\alpha(0)$ is a point. Hence,

$$\int_{\mathcal{C}_\alpha(\zeta)} \nu = \partial^{-3/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha(\zeta).$$

Recalling that the Riemann-Liouville fractional derivative is a left inverse of the fractional integral, we conclude that

$$\partial^{3/2}_{\zeta \text{ from } \zeta_\alpha}\left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right) = \hat{\varphi}_\alpha(\zeta).$$

□

3.2.2. *Singularities.* From equation (3.5) we see that singularities of $\hat{\iota}_\alpha(\zeta)$ in the Borel plane comes from either poles of ν or zeros of df. Instead, the fractional derivatives formula tells that singularities of $\hat{\varphi}_\alpha$ are given by convolutions of $\zeta^{-1/2}/\Gamma(1/2)$ with $\hat{\iota}_\alpha$. Since $\zeta^{-1/2}/\Gamma(1/2)$ is singular at $\zeta = 0$ the set of singularities of $\hat{\varphi}_\alpha(\zeta)$ is exactly the same as the one of $\hat{\iota}_\alpha(\zeta)$. However, the type of singularities will change and we expect $\hat{\varphi}_\alpha(\zeta)$ to have only simple singularities.

In the examples we noticed that $\hat{\varphi}_\alpha(\zeta)$ is always an hypergeometric function. In particular when there are only two critical values (see Airy, Bessel) the $\hat{\varphi}_\alpha(\zeta)$ is a Gaussian hypergeometric function ${}_2F_1\left(a, b; c; \frac{\zeta}{\zeta_\alpha}\right)$ with $c = 2$ and $a + b = c + 1$. Whereas, in the generalized Airy example (see Section ??) we get generalized hypergeometric functions ${}_3F_2\left(\mathbf{a}; \mathbf{b}; (\frac{\zeta}{\zeta_\alpha} - 1)^2\right)$ and ${}_3F_2\left(\mathbf{a}_0; \mathbf{b}_0; (\frac{\zeta}{\zeta_\alpha})^2\right)$ with $|\mathbf{a}| = |\mathbf{b}| + 1$. This behaviour reflects the resurgence properties of $\hat{\varphi}_\alpha$ (as well as the one of $\hat{\iota}_\alpha$), meaning the analytic continuation of $\hat{\varphi}_\alpha(\zeta)$ at ζ_α is given in terms of $\hat{\varphi}_\beta(\zeta)$, $\zeta_\beta \neq \zeta_\alpha$ when $\hat{\varphi}_\alpha(\zeta), \hat{\varphi}_\beta(\zeta)$ are hypergeometric functions of the previous type.

Lemma 3.5. Let us assume f has only two critical values $\zeta_\alpha = -\zeta_\beta$ and let $\hat{\varphi}_\alpha(\zeta) = {}_2F_1(a,b;2;\frac{\zeta}{\zeta_\alpha})$ with $a+b=c+1$, then across the branch cut

$$\hat{\varphi}_\alpha(\zeta+i0) - \hat{\varphi}_\alpha(\zeta-i0) = C\ {}_2F_1\left(a,b;2;1+\frac{\zeta}{\zeta_\beta}\right) \tag{3.6}$$

$$\hat{\varphi}_\beta(\zeta+i0) - \hat{\varphi}_\beta(\zeta-i0) = -C\ {}_2F_1\left(a,b;2;1+\frac{\zeta}{\zeta_\alpha}\right) \tag{3.7}$$

Proof. It follows from DLMF eq. 15.2.2. □

It would be interesting to further investigate the relationship between the properties of resurgent functions (with finitely many singularities in the Borel plane) and hypergeometric functions.

3.2.3. *Contour argument.* As noticed in proof of Theorem 3.1, the integral $I_\alpha(z)$ can be written as

(i) the Laplace transform of $\hat{\imath}_\alpha(\zeta)$
(ii) the Hankel contour integral of the major $\overset{\triangledown}{\imath}_\alpha(\zeta)$

and $\overset{\triangledown}{\imath}_\alpha(\zeta) = \hat{\imath}_\alpha(\zeta) + \text{hol.fct.}$. In the applications we have evidence that $\overset{\triangledown}{\imath}_\alpha(\zeta)$ is an algebraic hypergeometric function and when there are only two critical values, it decomposes as a sum of two germs of holomorphic functions at each critical values respectively (see `airy-resurgence` Section 6.1, 6.3).

4. Stokes phenomenon for thimble integrals

5. Examples

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