EXPONENTIAL INTEGRALS

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Let X be a $N-$dim manifold, $f : X \to \mathbb{C}$ be a holomorphic Morse function with only simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$I(z) := \int_{\mathcal{C}} e^{-zf}\, \nu \tag{2.1}$$

where $\mathcal{C}$ is a suitable countur such that the integralis well defined. For any Morse cirtial points x_α of f, the saddle point approximation gives the following formal series

$$I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\, \nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)}(2\pi)^{N/2} z^{-N/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n}. \tag{2.2}$$

Theorem 2.1. *Let* $\tilde{\varphi}_\alpha(z) := e^{-zf(x_\alpha)}(2\pi)^{N/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n}$

(1) $\tilde{\varphi}_\alpha$ *is Gevrey-1;*
(2) $\hat{\varphi}(\zeta) := \mathcal{B}(\tilde{\varphi})$ *is a germ of analytic function at* $\zeta = \zeta_\alpha = f(x_\alpha)$*;*
(3) *the following formual holds true*

$$\hat{\varphi}_\alpha(\zeta) \;=\; \partial^{N/2}_{\zeta, based\, at\, \zeta_\alpha}\left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu\right) \;=\; \Gamma(-N/2)\int_{\zeta_\alpha}^{\zeta} (\zeta - \zeta')^{-N/2} \int_{f^{-1}(\zeta')} \frac{\nu}{df}\, d\zeta' \tag{2.3}$$

Definition 2.2. Let $\alpha \in (0,1)$, then the α-Caputo's derivative of a smooth fucntion f is defined as

$$\partial_x^\alpha f(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f'(s) ds \tag{2.4}$$

Example 2.3 (Airy)**.** Let $f(t) = \frac{t^3}{3} - t$ and

$$I(z) := \int_\gamma e^{-zf(t)} \mathrm{dt}$$

where γ is a countour where the integral is well defined.

By the change of coordinates $z = x^{3/2}$, $I(z) = -2\pi i z^{-1/3} Ai(x)$ where

$$Ai(x) := \frac{1}{2\pi i} \int_{-\infty e^{-i\frac{\pi}{3}}}^{\infty e^{i\frac{\pi}{3}}} e^{\frac{t^3}{3} - zt} \mathrm{dt}$$

hence $I(z)$ solves the following ODE[1]

$$I''(z) - \frac{4}{9} I(z) + \frac{I'(z)}{z} - \frac{1}{9}\frac{I(z)}{z^2} = 0 \tag{2.5}$$

A formal solution of (2.5) can be computed by making the following ansatz

$$\tilde{I}(z) = \sum_{k \in \mathbb{N}^2} U^k e^{-\lambda \cdot kz} z^{-\tau \cdot k} w_k(z) \tag{2.6}$$

with $U^{(k_1,k_2)} = U_1^{k_1} U_2^{k_2}$ and $U_1, U_2 \in \mathbb{C}$ are constant parameter, $\lambda = (\frac{2}{3}, -\frac{2}{3})$, $\tau = (\frac{1}{2}, \frac{1}{2})$, and $\tilde{w}_k(z) \in \mathbb{C}[[z^{-1}]]$. In addition, we can check that the only non zero $\tilde{w}_k(z)$ occurs at $k = (1,0)$ and $k = (0,1)$, therefore

$$\tilde{I}(z) = U_1 e^{-2/3z} z^{-1/2} \tilde{w}_+(z) + U_2 e^{2/3z} z^{-1/2} \tilde{w}_-(z) \tag{2.7}$$

where from now on we denote $\tilde{w}_+(z) = \tilde{w}_{(1,0)}(z)$ and $\tilde{w}_-(z) = \tilde{w}_{(0,1)}(z)$. In particular, $\tilde{w}_+(z)$ and $\tilde{w}_-(z)$ are formal solution of

$$\tilde{w}_+'' - \frac{4}{3}\tilde{w}_+' + \frac{5}{36}\frac{\tilde{w}_+}{z^2} = 0 \tag{2.8}$$

$$\tilde{w}_-'' + \frac{4}{3}\tilde{w}_-' + \frac{5}{36}\frac{\tilde{w}_-}{z^2} = 0 \tag{2.9}$$

Taking the Borel transform of (2.8), (2.9) we get

$$\zeta^2 \hat{w}_+(\zeta) + \frac{4}{3}\zeta \hat{w}_+ + \frac{5}{36}\zeta * \hat{w}_+ = 0$$
$$\zeta^2 \hat{w}_+(\zeta) + \frac{4}{3}\zeta \hat{w}_+ + \frac{5}{36}\int_0^\zeta (\zeta - \zeta')\hat{w}_+(\zeta') d\zeta' = 0$$

$$\zeta^2 \hat{w}_-(\zeta) - \frac{4}{3}\zeta \hat{w}_- + \frac{5}{36}\zeta * \hat{w}_- = 0$$
$$\zeta^2 \hat{w}_-(\zeta) - \frac{4}{3}\zeta \hat{w}_- + \frac{5}{36}\int_0^\zeta (\zeta - \zeta')\hat{w}_-(\zeta') d\zeta' = 0$$

[1] $Ai(x)$ solves the Airy equation $y'' = xy$.

and taking derivatives we get

$$\zeta(\frac{4}{3}+\zeta)\hat{w}_+''+(\frac{8}{3}+4\zeta)\hat{w}_+'+\frac{77}{36}\hat{w}_+=0$$
$$\frac{4}{3}\zeta(1+\frac{3}{4}\zeta)\hat{w}_+''+(\frac{8}{3}+4\zeta)\hat{w}_+'+\frac{77}{36}\hat{w}_+=0$$
$$u(1-u)\hat{w}_+''(u)+(2-4u)\hat{w}_+'(u)-\frac{77}{36}\hat{w}_+(u)=0 \qquad u=-\frac{3}{4}\zeta$$

$$\zeta(-\frac{4}{3}+\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0$$
$$\frac{4}{3}\zeta(-1+\frac{3}{4}\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0$$
$$u(1-u)\hat{w}_-''(u)+(2-4u)\hat{w}_-'(u)-\frac{77}{36}\hat{w}(u) \qquad u=\frac{3}{4}\zeta$$

Notice that the latter equations are hypergeometric, hence a solution is given by

$$\hat{w}_+(\zeta)=c_1\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right) \tag{2.10}$$

$$\hat{w}_-(\zeta)=c_2\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right) \tag{2.11}$$

for some constants $c_1, c_2 \in \mathbb{C}$ (see DLMF 15.10.2). In addition $\hat{w}_\pm(\zeta)$ have a log singularity respectively at $\zeta=\mp\frac{4}{3}$, therefore they are $\{\mp\frac{4}{3}\}$-resurgent functions.[2]

Remark 2.4. $\hat{w}_+(\zeta)$ is Laplace summable along the positive real axis, and it can be analyticaly continued on $\mathbb{C}\setminus -\frac{4}{3}\mathbb{R}_{\geq 0}$ with

$$\begin{aligned}\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0) &= \frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\sum_{k\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\frac{3}{4}\zeta)^n \qquad \zeta<-\frac{4}{3}\\ &= \frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\left(\frac{5}{144}(4+3\zeta)_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)\right)\\ &= -\mathbf{i}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)\\ &= -\mathbf{i}\,\hat{w}_-(\zeta+\frac{4}{3})\end{aligned}$$

[2] The solution we find are equal to the ones in DLMF §9.7. They also agree with slide 10 of Maxim's talk for ERC and with the series (6.121) in Sauzin's book. However they do not agree with (2.16) in Mariño's Diablerets.

Anolougusly, $\hat{w}_-(\zeta)$ is Laplace summable along the negative real axis, and it jums across the branch cut $\frac{4}{3}\mathbb{R}_{\geq 0}$ as

$$\begin{aligned}\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0) &= -\frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\sum_{k\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1-\frac{3}{4}\zeta)^n \qquad \zeta>\frac{4}{3}\\ &= -\frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\left(-\frac{5}{144}(-4+3\zeta)_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)\right)\\ &= \mathbf{i}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)\\ &= \mathbf{i}\,\hat{w}_+(\zeta-\frac{4}{3})\end{aligned}$$

These relations manifest the resurgence property of $\tilde{I}$, indeed near the singularities in the Borel plane of either $\hat{w}_+$ or $\hat{w}_-$, $\hat{w}_-$ and $\hat{w}_+$ respectively contribute to the jump of the former solution.

Our next goal is to prove that the Borel transform of $\tilde{I}(z)$ can be written in terms of $1/f'(f^{-1}(\zeta))$, namely formula (2.3). First of all we define $\tilde{u}(z) := z^{1/2}\tilde{I}(z)$ which is a solution of $\hat{\varphi}_1(\zeta)$ and $\hat{\varphi}_2(\zeta)$:

$$\tilde{u}''(z)-\frac{4}{9}\tilde{u}(z)+\frac{5}{36}\frac{\tilde{u}(z)}{z^2}=0 \tag{2.12}$$

As a consequence, the Borel transform $\hat{u}(\zeta)$ solves

$$\zeta^2\hat{u}-\frac{4}{9}\hat{u}+\frac{5}{36}\zeta*\hat{u}$$

$$\zeta^2\hat{u}-\frac{4}{9}\hat{u}+\frac{5}{36}\int_0^\zeta(\zeta-\zeta')\hat{u}(\zeta')d\zeta'$$

taking derivatives is equivalent to

$$(\zeta^2-\frac{4}{9})\hat{u}''(\zeta)+4\zeta\hat{u}'(\zeta)+\frac{77}{36}\hat{u}(\zeta)=0$$

and Mathematica gives the following solutions

$$\begin{aligned}\hat{u}(\zeta) &= c_1\,{}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{9}{4}\zeta^2\right)+\frac{3i}{2}\zeta c_2\,{}_1F_2\left(\frac{13}{12},\frac{17}{12},\frac{3}{2},\frac{9}{4}\zeta^2\right)=\\ &= c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}}\left({}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{1}{2}-\frac{3}{4}\zeta\right)-{}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{1}{2}+\frac{3}{4}\zeta\right)\right) \qquad \text{see DLMF 15.8.27}\\ &\quad+\frac{3i}{2}\zeta c_2\left(\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{3\zeta\Gamma(-\frac{1}{2})\Gamma(2)}\right)\left({}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}-\frac{3}{4}\zeta\right)-{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}+\frac{3}{4}\zeta\right)\right) \qquad \text{see DLMF 15.8.28}\\ &= \left(c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}}-c_2 i\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right){}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}-\frac{3}{4}\zeta\right)+\\ &\quad+\left(c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}}+c_2 i\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right){}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}+\frac{3}{4}\zeta\right)\end{aligned}$$

$$\begin{aligned}\hat{u}(\zeta) &= C_1 T_{-2/3}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right)+C_2 T_{2/3}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right)\\ &= C_1 T_{-2/3}\,\hat{w}_+(\zeta)+C_2 T_{2/3}\,\hat{w}_-(\zeta)\end{aligned}$$

Using properties of Caputo's 1/2-derivative we would like to express $\hat{w}_\pm(\zeta)$ as the 1/2-derivatives of other hypergeometric series. From the integral representation of hypergeomtric functions (see DLMF 15.6.1)

$$\begin{aligned}{}_2F_1\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right) &= \frac{3}{5\pi}\int_0^1 t^{5/6}(1-t)^{-5/6}(1+\frac{3}{4}\zeta t)^{-7/6}dt && |u|<\frac{4}{3}\\ &= \frac{3}{5\pi}\zeta^{-1}\int_0^\zeta u^{5/6}(\zeta-u)^{-5/6}(1+\frac{3}{4}u)^{-7/6}du && u=\zeta t\\ &= \frac{3}{5\pi}\int_0^\zeta(\zeta-u)^{-1/2}\left(\zeta^{-1/2}(1-\frac{u}{\zeta})^{-1/3}\left(\frac{u}{\zeta}\right)^{5/6}(1+\frac{3}{4}u)^{-7/6}\right)du\\ &= \frac{3}{5\sqrt{\pi}}\frac{1}{\Gamma(1/2)}\int_0^\zeta(\zeta-u)^{-1/2}\left(\zeta^{-1/2}(1-\frac{u}{\zeta})^{-1/3}\left(\frac{u}{\zeta}\right)^{5/6}(1+\frac{3}{4}u)^{-7/6}\right)du\end{aligned}$$

For formula (2.3) to hold true we must have

$$\frac{3}{5\sqrt{\pi}}\left(\zeta^{-1/2}(1-\frac{u}{\zeta})^{-1/3}\left(\frac{u}{\zeta}\right)^{5/6}(1+\frac{3}{4}u)^{-7/6}\right)=\frac{1}{f'(f^{-1}(u))} \tag{2.13}$$

however, $\frac{1}{f'(f^{-1}(u))}$ does not depend on ζ, thus (2.13) is false, and the Airy exponential integral is a counter example of the formula (2.3).

2.1. **Second temptative formula.**

Let us assume that the correct formula that replaces (2.3) in Theorem 2.1 is

$$\hat{I}_\alpha(\zeta) = \partial^{N/2}_{\zeta,\text{based at }\zeta_\alpha}\left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)}\nu\right) = \Gamma(-N/2)\int_{\zeta_\alpha}^{\zeta}(\zeta-\zeta')^{-N/2}\int_{f^{-1}(\zeta')}\frac{\nu}{df}d\zeta' \tag{2.14}$$

then we compute $\hat{I}$ using properties of the Borel transform of a product:

$$\begin{aligned}\hat{I}(\zeta) &= \mathcal{B}(z^{-1/2})*\hat{u}(\zeta)\\ &= \frac{1}{\Gamma(1/2)}\int_0^\zeta(\zeta-s)^{-1/2}\hat{u}(s)ds\\ &= \frac{1}{\Gamma(1/2)}\int_0^\zeta(\zeta-s)^{-1/2}\left(T_{-2/3}\hat{w}_+(s)+T_{2/3}\hat{w}_-(s)\right)ds\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1/2)}\int_0^{\zeta}(\zeta-s)^{-1/2}\hat{w}_+(s-2/3)+\frac{1}{\Gamma(1/2)}\int_0^{\zeta}(\zeta-s)^{-1/2}\hat{w}_-(s+2/3)ds\\
&= \frac{1}{\Gamma(1/2)}\int_{-2/3}^{\zeta-2/3}(\zeta-\frac{2}{3}-s)^{-1/2}\hat{w}_+(s)+\frac{1}{\Gamma(1/2)}\int_{2/3}^{\zeta+2/3}(\zeta+\frac{2}{3}-s)^{-1/2}\hat{w}_-(s)ds\\
&= \frac{C_1}{\Gamma(1/2)}\int_{-2/3}^{\zeta-2/3}(\zeta-\frac{2}{3}-s)^{-1/2}{}_2F_1\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}s\right)ds+\frac{C_2}{\Gamma(1/2)}\int_{2/3}^{\zeta+2/3}(\zeta+\frac{2}{3}-s)^{-1/2}{}_2F_1\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}s\right)ds\\
&= \frac{C_3}{\Gamma(1/2)}\int_{-2/3}^{\zeta-2/3}(\zeta-\frac{2}{3}-s)^{-\frac{1}{2}}\frac{\partial}{\partial s}{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,-\frac{3}{4}s\right)ds+\frac{C_4}{\Gamma(1/2)}\int_{2/3}^{\zeta+2/3}(\zeta+\frac{2}{3}-s)^{-\frac{1}{2}}\frac{\partial}{\partial s}{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,\frac{3}{4}s\right)ds\\
&= C_3\partial^{1/2}_{\zeta,\text{based at }-2/3}{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,-\frac{3}{4}\zeta\right)+C_4\partial^{1/2}_{\zeta,\text{based at }2/3}{}_2F_1\left(\frac{1}{6},\frac{5}{6},1,\frac{3}{4}\zeta\right)
\end{aligned}
$$

However, ${}_2F_1\left(\frac{1}{6},\frac{5}{6},1,\mp\frac{3}{4}\zeta\right)$ are not algebraic, hence they can not be equal to $\frac{1}{f'(f^{-1}(\zeta))}$ as it is in in Theorem 2.1 part II (2.14).

2.1.1. *Comparison with Aaron.* The Airy integral can be written in terms of the modified Bessel equation as

$$Ai(x)=\frac{1}{\pi\sqrt{3}}x^{1/2}K(\frac{2}{3}x^{3/2}). \tag{2.15}$$

On the other hand we have, for $z=x^{3/2}$

$$Ai(x)=-\frac{z^{1/3}}{2\pi i}I(z)=-\frac{1}{2\pi i}x^{1/2}I(x^{3/2}) \tag{2.16}$$

hence

$$
\begin{aligned}
-\frac{1}{2\pi i}I(x^{3/2})&=\frac{1}{\pi\sqrt{3}}K(\frac{2}{3}x^{3/2})\\
\frac{i}{2}I(z)&=\frac{1}{\sqrt{3}}K(\frac{2}{3}z)
\end{aligned}
$$

In particular, the Borel trasforms of LHS and RHS[3] must be equal, i.e.

$$\begin{aligned}
\frac{i}{2}\hat{I}(\zeta) &= \frac{\sqrt{3}}{2}\hat{K}(\frac{3}{2}\zeta) \\
&= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}(\frac{3}{2}\zeta-1)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{1}{2}-\frac{3}{4}\zeta\right) \\
&= \frac{\sqrt{3}}{2}(3\zeta-2)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{1}{2}-\frac{3}{4}\zeta\right) \\
&= \frac{\sqrt{3}}{2}T_{-2/3}\left((3\zeta)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)\right) \\
&= \frac{1}{2}T_{-2/3}\left(\frac{1}{\sqrt{\zeta}}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)\right) \\
\hat{I}(\zeta) &= -i\,T_{-2/3}\left(\frac{1}{\sqrt{\zeta}}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)\right)
\end{aligned}$$

In addition, since $\hat{I}(\zeta) = C_3\partial^{1/2}_{\zeta,\text{based at }-2/3}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)$ is a solution of the Borel transform of equation (2.5); thus

$$\begin{aligned}
C_3\partial^{1/2}_{\zeta,\text{based at }-2/3}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right) &= -i\,T_{-2/3}\left(\frac{1}{\sqrt{\zeta}}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)\right) \\
C_3\partial^{1/2}_{\zeta}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right) &= -i\frac{1}{\sqrt{\zeta}}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)
\end{aligned}$$

Equation 5 in Aaron is

$$\left[\frac{\partial^2}{\partial z^2}+2\frac{\partial}{\partial z}+\frac{1}{z}\frac{\partial}{\partial z}-\frac{1}{9z^2}\right]\kappa=0 \tag{2.17}$$

[3]The conjugate variable of z is ζ, hence $\hat{K}(\frac{2}{3}z)=\frac{3}{2}\hat{K}(\frac{3}{2}\zeta)$. Indeed assuming $K(z)=\sum_{n\geq 0}a_n z^{-n}$, the Borel transform of $K(\frac{2}{3}z)=\sum_{n\geq 0}a_n\left(\frac{3}{2}\right)^n z^{-n}$ is by definition

$$\hat{K}(\frac{2}{3}z)=\sum_{k\geq 0}a_{n+1}\left(\frac{3}{2}\right)^{n+1}\frac{\zeta^n}{n!}=\frac{3}{2}\sum_{k\geq 0}\frac{a_{n+1}}{n!}\left(\frac{3}{2}\zeta\right)^n=\frac{3}{2}\hat{K}(\frac{3}{2}\zeta).$$

Its Borel transform is

$$\zeta^2\hat{\kappa}-2\zeta\hat{\kappa}+1*(-\zeta\hat{\kappa})-\frac{1}{9}\zeta*\hat{\kappa}=0$$

$$\zeta^2\hat{\kappa}-2\zeta\hat{\kappa}-\int_0^{\zeta} s\hat{\kappa}(s)ds-\frac{1}{9}\int_0^{\zeta}(\zeta-s)\hat{\kappa}(s)ds=0$$

taking derivatives once

$$2\zeta\hat{\kappa}+\zeta^2\hat{\kappa}'-2\hat{\kappa}-2\zeta\hat{\kappa}'-\zeta\hat{\kappa}-\frac{1}{9}\int_0^{\zeta}\hat{k}(s)ds=0$$

$$\zeta\hat{\kappa}+\zeta^2\hat{\kappa}'-2\hat{\kappa}-2\zeta\hat{\kappa}'-\frac{1}{9}\int_0^{\zeta}\hat{k}(s)ds=0$$

taking derivatives once again

$$\hat{\kappa}+\zeta\hat{\kappa}'+2\zeta\hat{k}'+\zeta^2\hat{\kappa}''-2\hat{\kappa}'-2\hat{\kappa}'-2\zeta\hat{\kappa}''-\frac{1}{9}\hat{\kappa}=0$$

$$(\zeta^2-2\zeta)\hat{\kappa}''+(3\zeta-4)\hat{k}'+\frac{8}{9}\hat{\kappa}=0$$

The last equation is a hypergeomtric equation and a solution is given by

$$\hat{\kappa}(\zeta)={}_2F_1\left(\frac{2}{3},\frac{4}{3};2;\frac{\zeta}{2}\right) \tag{2.18}$$

However, at pag. 6 in Aaron there is a different solution, namely

$$\hat{\kappa}(\zeta-1)={}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)$$

$$\hat{\kappa}(\zeta)={}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};1+\frac{\zeta}{2}\right)$$