EXPONENTIAL INTEGRALS

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CONTENTS

1. INTRODUCTION

2. FRACTIONAL DERIVATIVES AND BOREL TRANSFORM

Definition 2.1. Let $\alpha \in (0,1)$ and $n \in \mathbb{N}$, then the $n+\alpha$-Caputo's derivative of a smooth fucntion f is defined as

$$\partial_x^{n+\alpha} f(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f^{(n+1)}(s) ds \tag{2.1}$$

In particular, this definition is well suited for the differential calculus in the convolutive model $(\mathbb{C}[[\zeta]], *)$. Let $\varphi(z) := \sum_{k\geq 0} a_k z^{-k-1} \in \mathbb{C}[[z^{-1}]]$ be Gevrey 1, then assuming $a_k = 0$ for every $k < n$, the Borel transform of $z^{n+\alpha}\varphi(z)$ can be computed in two different ways:

$$\begin{aligned} \mathcal{B}\big(z^{n+\alpha}\varphi(z)\big)(\zeta) = \mathcal{B}(z^{\alpha+n}) * \hat{\varphi}(\zeta) &= \int_0^\zeta \frac{(\zeta-s)^{-1-n-\alpha}}{(-1-n-\alpha)!} \sum_{k\geq 0} \frac{a_k}{k!} s^k ds \\ &= \frac{1}{(-\alpha)!} \int_0^\zeta (\zeta-s)^{-\alpha} \sum_{k\geq 0} \frac{a_k}{(k-n-1)!} s^{k-n-1} ds = \partial_\zeta^{n+\alpha} \hat{\varphi}(\zeta) \end{aligned} \tag{2.2}$$

$$\mathcal{B}\big(z^{n+\alpha}\varphi(z)\big)(\zeta) = \mathcal{B}\left(\sum_{k\geq 0} a_k z^{-k-1+n+\alpha}\right)(\zeta) = \sum_{k>n} \frac{a_k}{(k-n-\alpha)!} \zeta^{k-n-\alpha} \tag{2.3}$$

and computitng the integral which defines the $n+\alpha$-derivative in (2.2) we get exaclty the same result as (2.3).

3. Resurgence of exponential integrals

Let X be a $N-$dim manifold, $f: X \to \mathbb{C}$ be a holomorphic Morse function with only simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$I(z) := \int_{\mathcal{C}} e^{-zf}\, \nu \tag{3.1}$$

where $\mathcal{C}$ is a suitable countur such that the integral is well defined. For any Morse cirtial points x_α of f, the saddle point approximation gives the following formal series

$$I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\, \nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)}(2\pi)^{N/2} z^{-N/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n} \qquad \text{as } z \to \infty \tag{3.2}$$

where $\mathcal{C}_\alpha$ is a steepest descendet path through the critical point x_α.

Theorem 3.1. *Let $N=1$. Let $\tilde{\varphi}_\alpha(z) := e^{-zf(x_\alpha)}(2\pi)^{1/2}\sum_{n\geq 0} a_{\alpha,n} z^{-n}$ and assume $f''(x_\alpha) \neq 0$ for every critical point x_α. Then*

(1) *$\tilde{\varphi}_\alpha$ is Gevrey-1;*
(2) *$\hat{\varphi}(\zeta) := \mathcal{B}(\tilde{\varphi})$ is a germ of analytic function at $\zeta = \zeta_\alpha = f(x_\alpha)$;*
(3) *the following formual holds true*

$$\hat{\varphi}_\alpha(\zeta) = \partial^{3/2}_{\zeta, \text{based at } \zeta_\alpha}\left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu\right) = \Gamma(-1/2)\int_{\zeta_\alpha}^{\zeta} (\zeta-\zeta')^{-1/2}\partial_{\zeta'}\left(\int_{f^{-1}(\zeta')} \frac{\nu}{df}\right) d\zeta' \tag{3.3}$$

Proof. Part (1): Let us assume that locally $\nu = \sum_{j\geq 0} b^\alpha_j (t-t_\alpha)^j\, dt$ for $|t-t_\alpha| < \varepsilon$. By the steepest descendet method, $I_\alpha(z)$ can be aproximated as $z \to \infty$ as

$$\begin{aligned}
I_\alpha(\zeta) &\sim e^{-zf(x_\alpha)} \int_{-\varepsilon}^{\varepsilon} \sum_{n\geq 0} t^{2n} b^\alpha_{2n} e^{-zf''(x_\alpha)\frac{t^2}{2}}\, dt \\
&= e^{-zf(x_\alpha)} \sum_{n\geq 0} b^\alpha_{2n} \int_{-\varepsilon}^{\varepsilon} t^{2n} e^{-zf''(x_\alpha)\frac{t^2}{2}}\, dt \\
&= \sqrt{2\pi} e^{-zf(x_\alpha)} z^{-1/2} \sum_{n\geq 0} \frac{b^\alpha_{2n}}{(f''(x_\alpha))^{n+1/2}} (2n-1)!! z^{-n}\, \mathrm{Erf}\left(\frac{\sqrt{zf''(x_\alpha)}}{\sqrt{2}}\varepsilon\right) + \\
&\quad - 2e^{-zf(x_\alpha)} \sum_{n\geq 1} b^\alpha_{2n} (2n-1)!! e^{-zf''(x_\alpha)\frac{\varepsilon^2}{2}} \sum_{j=1}^{n} \frac{\varepsilon^{2j-1}}{(2j-1)!!} \left(f''(x_\alpha) z\right)^{n-j+1} \\
&\sim_{\varepsilon << 1} 2\varepsilon\sqrt{2\pi} e^{-zf(x_\alpha)} z^{-1/2} \sum_{n\geq 0} \frac{b^\alpha_{2n}}{(f''(x_\alpha))^{n+1/2}} (2n-1)!! z^{-n}
\end{aligned}$$

therefore

$$a_{\alpha,n} := 2\varepsilon \frac{b^\alpha_{2n}}{(f''(x_\alpha))^{n+1/2}} (2n-1)!!$$

Since, $\frac{1}{\varepsilon} = \limsup_n \sqrt[n]{b^\alpha_n}$

$$|a_{\alpha,n}| \le CA^n n!$$

where we use $(2n-1)!! \sim \frac{2^n}{\sqrt{\pi n}} n!$ as $n \to \infty$.

Part (2):

$$\hat{\varphi}_\alpha(\zeta) = \mathcal{B}\left(e^{-zf(x_\alpha)}(2\pi)^{1/2}\sum_{n\ge0} a_{\alpha,n} z^{-n}\right)(\zeta) = T_{f(x_\alpha)}(2\pi)^{1/2}\left(\delta a_0 + \sum_{n\ge0} a_{n+1}\frac{\zeta^n}{n!}\right)$$

$$(2\pi)^{1/2}\left(\delta(f_{x_\alpha})a_0 + \sum_{n\ge0} a_{n+1}\frac{(\zeta - f(x_\alpha))^n}{n!}\right)$$

Since $a_n \le CA^n n!$, the series $\sum_{n\ge0} a_{n+1}\frac{(\zeta-f(x_\alpha))^n}{n!}$ has a finite radius of convergence.

Part (3): thanks to properties of Caputo's fractional derivatives, we have that the Borel transform of $\tilde{I}_\alpha(z) = z^{-1/2}\tilde{\varphi}_\alpha(z)$ is

$$\partial^{1/2}_{\zeta,\text{based at }\zeta_\alpha}\hat{I}_\alpha(\zeta) = \hat{\varphi}_\alpha(\zeta). \tag{3.4}$$

In addition, we notice

$$\begin{aligned} I_\alpha(z) &= \int_{\mathcal{C}_\alpha} e^{-zf}\,\nu & & f = \zeta \\ &= \int_{\mathcal{H}_\alpha} e^{-z\zeta}\partial_\zeta\left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)}\nu\right)d\zeta & & \mathcal{H}_\alpha \text{ is Hankel countour} \\ &=: \int_{\mathcal{H}_\alpha} e^{-\zeta z}\hat{I}_\alpha(\zeta)d\zeta \end{aligned}$$

hence

$$\hat{\varphi}_\alpha(\zeta) = \partial^{1/2}_{\zeta,\text{based at }\zeta_\alpha}\left(\partial_\zeta\left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)}\nu\right)\right) = \partial^{3/2}_{\zeta,\text{based at }\zeta_\alpha}\left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)}\nu\right) \tag{3.5}$$

□

Example 3.2 (Airy)**.** Let $f(t) = \frac{t^3}{3} - t$ and

$$I(z) := \int_\gamma e^{-zf(t)}\mathrm{dt}$$

where γ is a countour where the integral is well defined.

By the change of coordinates $z = x^{3/2}$, $I(z) = -2\pi i z^{-1/3}Ai(x)$ where

$$Ai(x) := \frac{1}{2\pi i}\int_{-\infty e^{-i\frac{\pi}{3}}}^{\infty e^{i\frac{\pi}{3}}} e^{\frac{t^3}{3} - zt}\mathrm{dt}$$

hence $I(z)$ solves the following ODE[1]

$$I''(z) - \frac{4}{9} I(z) + \frac{I'(z)}{z} - \frac{1}{9}\frac{I(z)}{z^2} = 0 \tag{3.6}$$

A formal solution of (3.6) can be computed by making the following ansatz

$$\tilde{I}(z) = \sum_{k\in\mathbb{N}^2} U^k e^{-\lambda\cdot k z} z^{-\tau\cdot k} w_k(z) \tag{3.7}$$

with $U^{(k_1,k_2)} = U_1^{k_1} U_2^{k_2}$ and $U_1, U_2 \in \mathbb{C}$ are constant parameter, $\lambda = (\frac{2}{3}, -\frac{2}{3})$, $\tau = (\frac{1}{2}, \frac{1}{2})$, and $\tilde{w}_k(z) \in \mathbb{C}[[z^{-1}]]$. In addition, we can check that the only non zero $\tilde{w}_k(z)$ occurs at $k = (1,0)$ and $k = (0,1)$, therefore

$$\tilde{I}(z) = U_1 e^{-2/3z} z^{-1/2} \tilde{w}_+(z) + U_2 e^{2/3z} z^{-1/2} \tilde{w}_-(z) \tag{3.8}$$

where from now on we denote $\tilde{w}_+(z) = \tilde{w}_{(1,0)}(z)$ and $\tilde{w}_-(z) = \tilde{w}_{(0,1)}(z)$. In particular, $\tilde{w}_+(z)$ and $\tilde{w}_-(z)$ are formal solution of

$$\tilde{w}_+'' - \frac{4}{3}\tilde{w}_+' + \frac{5}{36}\frac{\tilde{w}_+}{z^2} = 0 \tag{3.9}$$

$$\tilde{w}_-'' + \frac{4}{3}\tilde{w}_-' + \frac{5}{36}\frac{\tilde{w}_-}{z^2} = 0 \tag{3.10}$$

Taking the Borel transform of (3.9), (3.10) we get

$$\zeta^2 \hat{w}_+(\zeta) + \frac{4}{3}\zeta \hat{w}_+ + \frac{5}{36}\zeta * \hat{w}_+ = 0$$
$$\zeta^2 \hat{w}_+(\zeta) + \frac{4}{3}\zeta \hat{w}_+ + \frac{5}{36}\int_0^\zeta (\zeta - \zeta')\hat{w}_+(\zeta') d\zeta' = 0$$

$$\zeta^2 \hat{w}_-(\zeta) - \frac{4}{3}\zeta \hat{w}_- + \frac{5}{36}\zeta * \hat{w}_- = 0$$
$$\zeta^2 \hat{w}_-(\zeta) - \frac{4}{3}\zeta \hat{w}_- + \frac{5}{36}\int_0^\zeta (\zeta - \zeta')\hat{w}_-(\zeta') d\zeta' = 0$$

and taking derivatives we get

$$\zeta(\frac{4}{3} + \zeta)\hat{w}_+'' + (\frac{8}{3} + 4\zeta)\hat{w}_+' + \frac{77}{36}\hat{w}_+ = 0$$
$$\frac{4}{3}\zeta(1 + \frac{3}{4}\zeta)\hat{w}_+'' + (\frac{8}{3} + 4\zeta)\hat{w}_+' + \frac{77}{36}\hat{w}_+ = 0$$
$$u(1-u)\hat{w}_+''(u) + (2 - 4u)\hat{w}_+'(u) - \frac{77}{36}\hat{w}_+(u) = 0 \qquad u = -\frac{3}{4}\zeta$$

[1] $Ai(x)$ solves the Airy equation $y'' = xy$.

$$\zeta(-\frac{4}{3}+\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0$$
$$\frac{4}{3}\zeta(-1+\frac{3}{4}\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0$$
$$u(1-u)\hat{w}_-''(u)+(2-4u)\hat{w}_-'(u)-\frac{77}{36}\hat{w}(u)=0 \qquad u=\frac{3}{4}\zeta$$

Notice that the latter equations are hypergeometric, hence a solution is given by

$$\hat{w}_+(\zeta)=c_1\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right) \tag{3.11}$$

$$\hat{w}_-(\zeta)=c_2\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right) \tag{3.12}$$

for some constants $c_1,c_2\in\mathbb{C}$ (see DLMF 15.10.2). In addition $\hat{w}_\pm(\zeta)$ have a log singularity respectively at $\zeta=\mp\frac{4}{3}$, therefore they are $\{\mp\frac{4}{3}\}$-resurgent functions.[2]

Remark 3.3. $\hat{w}_+(\zeta)$ is Laplace summable along the positive real axis, and it can be analyticaly continued on $\mathbb{C}\setminus-\frac{4}{3}\mathbb{R}_{\leq 0}$ with (see 15.2.3 DLMF)

$$\begin{aligned}
\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0) &= -\frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\frac{3}{4}\zeta)^n \qquad \zeta<-\frac{4}{3}\\
&= \frac{36}{5}i\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\frac{3}{4}\zeta)^{n-1}\\
&= -\frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\left(\frac{5}{144}(4+3\zeta)\left(1+{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)\right)\right)\\
&= \mathbf{i}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)\\
&= \mathbf{i}\hat{w}_-(\zeta+\frac{4}{3})
\end{aligned}$$

Anolougusly, $\hat{w}_-(\zeta)$ is Laplace summable along the negative real axis, and it jumps across the branch cut $\frac{4}{3}\mathbb{R}_{\geq 0}$ as

$$\begin{aligned}
\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0) &= \frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1-\frac{3}{4}\zeta)^n \qquad \zeta>\frac{4}{3}\\
&= \frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\left(-\frac{5}{144}(-4+3\zeta){}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)\right)\\
&= -\mathbf{i}\pi\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)\\
&= -\mathbf{i}\hat{w}_+(\zeta-\frac{4}{3})
\end{aligned}$$

[2] The solution we find are equal to the ones in DLMF §9.7. They also agree with slide 10 of Maxim's talk for ERC and with the series (6.121) in Sauzin's book. However they do not agree with (2.16) in Mariño's Diablerets.

These relations manifest the resurgence property of $\tilde{I}$, indeed near the singularities in the Borel plane of either $\hat{w}_+$ or $\hat{w}_-$, $\hat{w}_-$ and $\hat{w}_+$ respectively contribute to the jump of the former solution.

Our next goal is to prove that the Borel transform of $\tilde{I}(z)$ can be written in terms of $1/f'(f^{-1}(\zeta))$, namely formula (3.3). It is convenient to consider the two asymptotic formal solutions separately, namely we define

$$\tilde{I}_{-1}(z) := e^{-2/3z} z^{-1/2} \tilde{w}_+(z) =: z^{-1/2} \tilde{u}_+(z) \tag{3.13}$$

$$\tilde{I}_{1}(z) := e^{2/3z} z^{-1/2} \tilde{w}_-(z) =: z^{-1/2} \tilde{u}_-(z) \tag{3.14}$$

In particular, $\tilde{u}_\pm(z)$ are solutions of

$$\tilde{u}''(z) - \frac{4}{9}\tilde{u}(z) + \frac{5}{36}\frac{\tilde{u}(z)}{z^2} = 0 \tag{3.15}$$

with asymptotic behaviour $\tilde{u}_\pm(z) \sim O(e^{\pm 2/3z})$ as $z \to \infty$.

The Borel transforms $\hat{u}_\pm(\zeta)$ solve the same equation

$$\zeta^2 \hat{u} - \frac{4}{9}\hat{u} + \frac{5}{36}\zeta * \hat{u}$$

$$\zeta^2 \hat{u} - \frac{4}{9}\hat{u} + \frac{5}{36}\int_0^\zeta (\zeta - \zeta')\hat{u}(\zeta')d\zeta'$$

taking derivatives is equivalent to

$$(\zeta^2 - \frac{4}{9})\hat{u}''(\zeta) + 4\zeta\hat{u}'(\zeta) + \frac{77}{36}\hat{u}(\zeta) = 0$$

and Mathematica gives the following solutions

$$\hat{u}(\zeta) = c_1\, {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{9}{4}\zeta^2\right) + \frac{3i}{2}\zeta c_2\, {}_1F_2\left(\frac{13}{12}, \frac{17}{12}, \frac{3}{2}, \frac{9}{4}\zeta^2\right) =$$

$$= c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}}\left({}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} + \frac{3}{4}\zeta\right)\right) \quad \text{see DLMF 15.8.27}$$

$$+ \frac{3i}{2}\zeta c_2\left(\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{3\zeta\Gamma(-\frac{1}{2})\Gamma(2)}\right)\left({}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right)\right) \quad \text{see DLMF 15.8.28}$$

$$= \left(c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} - c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) +$$

$$+ \left(c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} + c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right)$$

Since $\hat{u}_+$ has a simple singularity at $\zeta=-2/3$ and $\hat{u}_-$ has a simple singularity at $\zeta=2/3$, we have

$$\hat{u}_+(\zeta)=C_1\,T_{-2/3}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right)=C_1\,T_{-2/3}\,\hat{w}_+(\zeta)$$

$$\hat{u}_-(\zeta)=C_2\,T_{2/3}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right)=C_2\,T_{2/3}\,\hat{w}_-(\zeta)$$

Lemma 3.4. *The following identity holds true*

$$(3.16)\qquad {}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right)=\frac{1}{1-u^2}\qquad \zeta=\frac{u^3}{3}-u$$

Proof. From the special case of hypergeometric function (see 15.4.14 DLMF) we have the following identity:

$$\begin{aligned}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right)&=\frac{\cos(y)}{\cos(3y)} & 3y=\arcsin\left(\frac{3}{2}\zeta\right)\\ &=\frac{\cos(y)}{\cos(2y)\cos(y)-\sin(2y)\sin(y)}\\ &=\frac{1}{\cos(2y)-2\sin^2(y)}\\ &=\frac{1}{1-4\sin^2(y)} & \zeta=2\sin(y)-\frac{8}{3}\sin^3(y)\end{aligned}$$

Therefore, if $u:=-2\sin(y)$, we have $\zeta=\frac{u^3}{3}-u=f(u)$ and

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right)=\frac{1}{1-u^2}=-\frac{1}{f'(u)}$$

□

Then equations (3.3) is equivalent to

Claim 3.5.

$$(3.17)\quad \hat{w}_+(\zeta-2/3)=-\frac{1}{\sqrt{\pi}}\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}\partial_s\left[{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}s^2\right)\right]ds\qquad \zeta\in(2/3,+\infty)$$

Let us study the RHS of claim (3.5)

$$\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}\partial_s\left[{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}s^2\right)\right]ds=2\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}s\,{}_2F_1\left(\frac{4}{3},\frac{5}{3};\frac{3}{2};\frac{9}{4}s^2\right)ds$$

$$=-\frac{2}{9}\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};\frac{1}{2}-\frac{3s}{4}\right)ds+\frac{2}{9}\int_{2/3}^{\zeta}(\zeta-s)^{-1/2}{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};\frac{1}{2}+\frac{3s}{4}\right)ds \qquad \text{15.8.28 DLMF}$$

$$=\frac{4}{9\sqrt{3}}\int_0^x(x-y)^{-1/2}\left[{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};1+y\right)-{}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2};y\right)\right]dy \qquad x\in(-\infty,0)$$

$$=-\frac{\sqrt{3}\pi}{4}\Gamma(1/6)e^{-\frac{3}{2}\pi i}\int_0^x(x-y)^{-1/2}y^{-5/3}{}_2F_1\left(\frac{5}{3},\frac{1}{6};\frac{1}{3};\frac{1}{y}\right)dy$$

$$=-\frac{\sqrt{3}\pi}{4}\frac{\Gamma(1/6)\Gamma(7/6)}{\Gamma(5/3)}e^{-\frac{3}{2}\pi i}|x|^{-7/6}{}_2F_1\left(\frac{7}{6},\frac{1}{6};\frac{1}{3};\frac{1}{x}\right) \qquad (4.3)$$

$$=\frac{5\pi^2}{24\sqrt{3}}\frac{\Gamma(1/3)}{\Gamma(5/3)}\left({}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\frac{1}{2}-\frac{\zeta}{4}\right)-e^{-\frac{11}{6}\pi i}{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\frac{1}{2}+\frac{\zeta}{4}\right)\right) \qquad \zeta\in(2/3,+\infty)$$

however, ${}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\frac{1}{2}+\frac{\zeta}{4}\right)$ has a branch cut at $\zeta\in(2/3,+\infty)$, thus the claim holds true.

Analougusly, it can be verified for $\hat{w}_{-}(\zeta+2/3)$ for $\zeta\in(-\infty,-2/3)$.

Example 3.6 (Bessel)**.** Let $X=\mathbb{C}^*$, $f(x)=x+\frac{1}{x}$ and $\nu=\frac{dx}{x}$, then the ciritcal points of f are $x=\pm1$ and

(3.18)
$$I(z):=\int_0^{\infty}e^{-zf(x)}\frac{dx}{x}.$$

By change of cooridnates $t=zx$

$$I(z)=\int_0^{\infty}e^{-z\left(\frac{t}{z}+\frac{z}{t}\right)}\frac{dt}{t}=\int_0^{\infty}e^{-\left(t+\frac{z^2}{t}\right)}\frac{dt}{t}=2K_0(2z) \qquad |\arg z|<\frac{\pi}{4}$$

where $K_0(z)$ is the modified Bessel function (see definition 10.32.10 DLMF). In particular, since $K_0(z)$ solves

(3.19)
$$\frac{d^2}{dz^2}w(z)+\frac{1}{z}\frac{d}{dz}w(z)-w(z)=0$$

and $K_0(z)\sim\left(\frac{\pi}{2}\right)^{1/2}e^{-z}z^{-1/2}\sum_{k\geq0}\frac{(1/2)_k(1/2)_k}{(-2)^k k!}z^{-k}$ as $z\to\infty$ (see DLMF 10.40.2), then $I(z)$ is a solution of

(3.20)
$$\frac{d^2}{dz^2}I(z)+\frac{1}{z}\frac{d}{dz}I(z)-4I(z)=0.$$

The formal integral of (3.20) is given by a two parameter formal solution $\tilde{I}_1(z)$

(3.21)
$$\tilde{I}(z)=\sum_{\mathbf{k}\in\mathbb{N}^2}U^k e^{-\mathbf{k}\cdot\lambda z}z^{-\tau\cdot\mathbf{k}}\tilde{w}_{\mathbf{k}}(z)$$

where $\lambda = (2,-2)$, $\tau = (-\frac{1}{2},-\frac{1}{2})$, $U^k := U_1^{k_1} U_2^{k_2}$ with $k = (k_1, k_2)$ and $U_1, U_2 \in \mathbb{C}$, and $\tilde{w}_{\mathbf{k}}(z) \in \mathbb{C}[[z^{-1}]]$ is a formal solution of

$$\tilde{w}''_{\mathbf{k}}(z) - 4(k_1-k_2)\tilde{w}'_{\mathbf{k}}(z) + 4(1-(k_1-k_2)^2)\tilde{w}_{\mathbf{k}}(z) + \frac{(k_1+k_2-1)}{z}\tilde{w}'_{\mathbf{k}}(z) + \\ -2(k_1-k_2)\frac{(k_1+k_2-1)}{z}\tilde{w}_{\mathbf{k}}(z) + \frac{(k_1+k_2)^2}{4z^2}\tilde{w}_{\mathbf{k}}(z) = 0 \tag{3.22}$$

The only non zero $\tilde{w}_{\mathbf{k}}(z)$ occurs for $\mathbf{k} = (1,0)$ and $\mathbf{k} = (0,1)$, hence

$$\tilde{I}(z) = U_1 e^{-2z} z^{-1/2} \tilde{w}_{(1,0)}(z) + U_2 e^{2z} z^{-1/2} \tilde{w}_{(0,1)}(z) \tag{3.23}$$

and according to our convention, we define

$$\tilde{I}_1(z) := e^{-2z} z^{-1/2} \tilde{w}_{(1,0)}(z) \tag{3.24}$$

$$\tilde{I}_{-1}(z) := e^{2z} z^{-1/2} \tilde{w}_{(0,1)}(z). \tag{3.25}$$

We set $\tilde{w}_{(1,0)} = \tilde{w}_+$ and $\tilde{w}_{(0,1)} = \tilde{w}_-$, then their Borel transforms are solutions respectively of the following equations

$$(+) \quad \zeta^2 \hat{w}_+(\zeta) + 4\zeta \hat{w}_+(\zeta) + \frac{1}{4}\zeta * \hat{w}_+(\zeta) = 0$$

$$(-) \quad \zeta^2 \hat{w}_+(\zeta) - 4\zeta \hat{w}_+(\zeta) + \frac{1}{4}\zeta * \hat{w}_+(\zeta) = 0$$

taking twice derivative in ζ we get

$$(+) \quad (\zeta^2 + 4\zeta)\frac{d^2}{d\zeta^2}\hat{w}_+ + 4(\zeta-1)\frac{d}{d\zeta}\hat{w}_+ + \frac{9}{4}\hat{w}_+ = 0$$

$$(-) \quad (\zeta^2 - 4\zeta)\frac{d^2}{d\zeta^2}\hat{w}_- + 4(\zeta+1)\frac{d}{d\zeta}\hat{w}_+ + \frac{9}{4}\hat{w}_- = 0$$

$$(+) \quad \xi(1-\xi)\frac{d^2}{d\xi^2}\hat{w}_+ + (1-4\xi)\frac{d}{d\xi}\hat{w}_+ - \frac{9}{4}\hat{w}_+ = 0 \qquad \xi = -\frac{\zeta}{4}$$

$$(-) \quad \xi(1-\xi)\frac{d^2}{d\xi^2}\hat{w}_- + (1-4\xi)\frac{d}{d\xi}\hat{w}_- - \frac{9}{4}\hat{w}_- = 0 \qquad \xi = \frac{\zeta}{4}$$

therefore, since equation $(+), (-)$ are hypergeometric the fundamental solution is (see DLMF 15.10.2)

$$\hat{w}_+(\zeta) = {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; -\frac{\zeta}{4}\right) \tag{3.26}$$

$$\hat{w}_-(\zeta) = {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{\zeta}{4}\right) \tag{3.27}$$

In particular, we notice that taking the series expansion of $\hat{w}_+$ and $\hat{w}_-$ we get numerically that

$$\hat{w}_+(\zeta-4) = \frac{1}{\pi}\log(z)\hat{w}_-(z)+\phi_{\text{reg}}$$
$$\hat{w}_-(\zeta+4) = \frac{1}{\pi}\log(z)\hat{w}_+(z)+\psi_{\text{reg}}$$

and analytically (thanks to 15.2.3 DLMF)

$$\begin{aligned}
\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0) &= {}_2F_1\left(\frac{3}{2},\frac{3}{2};2;-\frac{\zeta}{4}+i0\right)-{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;-\frac{\zeta}{4}-i0\right) \qquad \zeta<-4\\
&= -8i\left(-\frac{\zeta}{4}-1\right)^{-1}\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\frac{\zeta}{4}+1\right)^n\\
&= 8i\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\frac{\zeta}{4}+1\right)^{n-1}\\
&= 8i\sum_{n\geq 0}\frac{(1/2)_{n+1}(1/2)_{n+1}}{(n+1)!\Gamma(n+1)}\left(\frac{\zeta}{4}+1\right)^{n}\\
&= \mathbf{2i}\sum_{n\geq 0}\frac{(3/2)_n(3/2)_n}{(n)!\Gamma(n+2)}\left(\frac{\zeta}{4}+1\right)^{n}\\
&= \mathbf{2i}{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{\zeta}{4}+1\right)
\end{aligned}$$

$$\begin{aligned}
\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0) &= {}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{\zeta}{4}+i0\right)-{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{\zeta}{4}-i0\right) \qquad \zeta>4\\
&= 8i\left(\frac{\zeta}{4}-1\right)^{-1}\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\frac{\zeta}{4}-1\right)^n\\
&= -8i\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(1-\frac{\zeta}{4}\right)^{n-1}\\
&= -8i\sum_{n\geq 0}(-1)^n\frac{(1/2)_{n+1}(1/2)_{n+1}}{(n+1)!\Gamma(n+1)}\left(1-\frac{\zeta}{4}\right)^{n}\\
&= -\mathbf{2i}\sum_{n\geq 0}(-1)^n\frac{(3/2)_n(3/2)_n}{(n)!\Gamma(n+2)}\left(1-\frac{\zeta}{4}\right)^{n}\\
&= -\mathbf{2i}{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;1-\frac{\zeta}{4}\right)
\end{aligned}$$

These are evidence of the resurgent properties of $\tilde{I}_{\pm 1}(z)$.

Lemma 3.7. *The following identity holds true*

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{1}{2};\frac{\zeta^2}{4}\right) = 2i\frac{u}{u^2-1} \qquad \zeta = u+\frac{1}{u} \tag{3.28}$$

Proof. From 15.4.13 DLMF, we have

$$
\begin{aligned}
{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{1}{2};\frac{\zeta^2}{4}\right) &= \frac{2}{\sqrt{4-\zeta^2}} && y=\operatorname{arccsc}(\zeta/2)\\
&= \frac{1}{\sqrt{1-\csc^2(y)}}\\
&= -i\tan(y) && \zeta=\frac{2}{\sin(y)}
\end{aligned}
$$

therefore if $u=\tan\left(\frac{y}{2}\right)$, we have $\zeta=\frac{1+u^2}{u}=f(u)$ and

$$
{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{1}{2};\frac{\zeta^2}{4}\right)=2i\frac{u}{u^2-1}=\frac{2i}{f'(u)u}
$$

□

Claim 3.8.

$$
\hat{w}_+(\zeta-2)=i\pi\int_2^{\zeta}(\zeta-\zeta')^{-1/2}2\zeta'\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};\frac{3}{2};\frac{\zeta'^2}{4}\right)d\zeta' \qquad \zeta\in(2,+\infty) \tag{3.29}
$$

Proof. Let us first consider the RHS of (3.8)

$$
\begin{aligned}
&2\pi\int_2^{\zeta}(\zeta-\zeta')^{-1/2}\zeta'\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};\frac{3}{2};\frac{\zeta'^2}{4}\right)d\zeta'=\\
&=\frac{4}{3}\int_2^{\zeta}(\zeta-\zeta')^{-1/2}\left[{}_2F_1\left(2,2;\frac{5}{2};\frac{1}{2}+\frac{\zeta'}{4}\right)-{}_2F_1\left(2,2;\frac{5}{2};\frac{1}{2}-\frac{\zeta'}{4}\right)\right]d\zeta'\\
&=\frac{8}{3}\int_0^{x}(y-x)^{-1/2}\left[{}_2F_1\left(2,2,;\frac{5}{2};y\right)-{}_2F_1\left(2,2;\frac{5}{2};1-y\right)\right]dy && x\in(-\infty,0)\\
&=2\pi\int_0^{x}(x-y)^{-1/2}y^{-2}F\left(2,\frac{1}{2};1;\frac{1}{y}\right)dy && (4.3)\\
&=\pi^2|x|^{-3/2}{}_2F_1\left(\frac{3}{2},\frac{1}{2};1;\frac{1}{x}\right) && x\in(-\infty,0)\\
&=\frac{\pi^2}{2}\left({}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{1}{2}-\frac{\zeta}{4}\right)-i\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{1}{2}+\frac{\zeta}{4}\right)\right) && \zeta\in(2,+\infty)
\end{aligned}
$$

however ${}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{1}{2}+\frac{\zeta}{4}\right)$ has a branch cut at $\zeta\in(2,+\infty)$, thus the claim holds true. □

Analougusly, it can be verified for $\hat{w}_-(\zeta+2)$ for $\zeta\in(-\infty,-2)$.

4. USEFUL IDENTITIES FOR GAUSS HYPERGEOMTRIC FUNCTIONS

$$
\begin{aligned}
(4.1) \quad {}_2F_1(a,b;c;z) = e^{b\pi i}\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(a+b-c+1)\Gamma(c-b)}{}_2F_1(a,b;c;1-z)+\\
-e^{(a+b-c)\pi i}\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(b)\Gamma(a-b+1)}|z|^{-a}{}_2F_1\left(a,a-c+1;a-b+1;\frac{1}{z}\right)
\end{aligned}
$$

(4.2)

$$
\int_0^x |y|^{a-\mu-1}{}_2F_1\left(a,b;c,y\right)(x-y)^{\mu-1}\,dy = \frac{\Gamma(\mu)\Gamma(a-\mu)}{\Gamma(a)}|x|^{a-1}{}_2F_1\left(a-\mu,b;c;x\right)
$$

$$
x \in (-\infty,0)\cup(0,1),\ \Re a > \Re\mu > 0
$$

which can be rewritten as

(4.3)

$$
\int_{y>x} |y|^{-a}|x-y|^{\mu-1}{}_2F_1\left(a,b;c;y^{-1}\right)dy = \frac{\Gamma(\mu)\Gamma(a-\mu)}{\Gamma(a)}|x|^{-a+\mu}{}_2F_1\left(a-\mu,b;c;x^{-1}\right)
$$

$$
x \in (-\infty,0)\cup(1,\infty),\ \Re a > \Re\mu > 0
$$

5. RESURGENCE FOR DEGREE 3 POLYNOMIALS

Let f be a degree 3 polynomial, and t_1, t_2 its critical points (not necessarily distinguished):

(1) if $t_1 \neq t_2$, then

$$
I(z) = \int_{\mathcal{C}_j} e^{-zf}\,dt
$$

is a solution of

$$
I'' + aI' + bI + c\frac{I'}{z} + \frac{d}{z}I + \frac{e}{z^2}I = 0 \tag{5.1}
$$

where a, b, c, d, e are determined in terms of f.

(2) if $t_1 = t_2$, then

$$
I(z) = \int_{\mathcal{C}_1} e^{-zf}\,dt
$$

is a solution of a first order ODE

$$
I' + \left(a_4 - \frac{a_2^3}{27a_1^2} + \frac{1}{3z}\right)I = 0 \tag{5.2}
$$

Proof. Let $f(t) = a_1 t^3 + a_2 t^2 + a_3 t + a_4$ with $a_1 \neq 0$,

$$\int_{\mathcal{C}_j} e^{-fz}\,dt = \int_{\mathcal{C}_j + \frac{a_2}{3a_1}} e^{-(a_1 t^3 + pt + q)z}\,dt \qquad\qquad t \to t - \frac{a_2}{3a_1}$$

where $p = a_3 - \frac{a_2^2}{3a_1}$ and $q = a_4 - \frac{a_2 a_3}{3a_1} + \frac{2a_2^3}{27a_1^2}$.

Case (1): if $p \neq 0$,

$$\begin{aligned} I(z) = \int t(3a_1 t^2 + p) z e^{-fz} = \int (3a_1 t^3 + pt) z e^{-ft} = \\ \int 2a_1 t^3 z e^{-fz} + \int (a_1 t^3 + pt + q) z e^{-fz} - qzI \\ 2z \int a_1 t^3 e^{-fz} - zI' - qzI \end{aligned}$$

$$\begin{aligned} 2z \int a_1 t^3 e^{-fz} = 2z^2 \int \frac{t^4}{4} a_1 (3a_1 t^2 + p) e^{-fz} = \frac{z^2}{2} \int (3a_1^2 t^6 + pa_1 t^4) e^{-fz} = \\ \frac{z^2}{2} \int (3a_1^2 t^6 + 6pa_1 t^4 + 3q^2 + 3p^2 t^2 + 6pqt + 6a_1 q t^3) e^{-fz} + \\ + \frac{z^2}{2} \int (pa_1 t^4 - 6pa_1 t^4) e^{-fz} - \frac{z^2}{2} \int (3q^2 + 3p^2 t^2 + 6pqt + 6a_1 q t^3) e^{-fz} \\ = \frac{3z^2}{2} I'' + \frac{3z^2}{2} q^2 I + 3qz^2 I' - \frac{z^2}{2} p \int (3a_1 t^4 + pt^2) e^{-fz} - z^2 p \int (a_1 t^4 + pt^2) e^{-fz} \\ = \frac{3z^2}{2} I'' + \frac{3z^2}{2} q^2 I + 3qz^2 I' - \frac{5}{3} zp \int t e^{-fz} - \frac{2}{3} z^2 p^2 \int t^2 e^{-fz} \end{aligned}$$

hence

(5.3)
$$I = -zI' - qzI + \frac{3z^2}{2} I'' + \frac{3z^2}{2} q^2 I + 3qz^2 I' - \frac{5}{3} zp \int t e^{-fz} - \frac{2}{3} z^2 p^2 \int t^2 e^{-fz}$$

(5.4)
$$\frac{3z^2}{2} \left(I'' + q^2 I + 2qI' - \frac{2}{3z} I' - \frac{2q}{3z} I - \frac{2}{3z^2} I - \frac{10}{9z} p \int t e^{-fz}\,dt - \frac{4}{9} p^2 \int t^2 e^{-fz} \right) = 0$$

Notice that

$$\frac{4}{9} p^2 \int t^2 e^{-fz} = \frac{4}{27a_1} p^2 \int (3a_1 t^2 + p) e^{-fz} - \frac{4}{27a_1} p^3 I = -\frac{4}{27a_1} p^3 I$$

$$-\frac{10}{9z}p\int te^{-fz}dt = \frac{5}{9z}\int pte^{-fz} - \frac{5}{3z}\int (pt+q)e^{-fz} + \frac{5}{3z}qI =$$

$$\frac{5}{9z}\int pte^{-fz} - \frac{5}{3z}\int (pt+q+a_1t^3)e^{-fz} + \frac{5}{3z}\int a_1t^3e^{-fz} + \frac{5}{3z}qI =$$

$$\frac{5}{9z}\int t(3a_1t^2+p)e^{-fz} + \frac{5}{3z}I' + \frac{5}{3z}qI$$

$$= \frac{5}{9z^2}I + \frac{5}{3z}I' + \frac{5}{3z}qI$$

thefore, collecting all the contributions together we find

$$I'' + q^2I + 2qI' - \frac{2}{3z}I' - \frac{2q}{3z}I - \frac{2}{3z^2}I + \frac{5}{9z^2}I + \frac{5}{3z}I' + \frac{5}{3z}qI + \frac{4}{27a_1}p^3I = 0$$

$$I'' + 2qI' + \left(\frac{4p^3}{27a_1} + q^2\right)I + \frac{1}{z}I' + \frac{q}{z}I - \frac{1}{9z^2}I = 0$$

Case (2): if $p = 0$, then integrating by part we have

$$\begin{aligned} I(z) &= \int_{\mathcal{C}_1+\frac{a_2}{3a_1}} e^{-(a_1t^3+q)z}dt \\ &= \left[te^{-(a_1t^3+q)z}\right]_{\mathcal{C}_1+\frac{a_2}{3a_1}} + \int_{\mathcal{C}_1+\frac{a_2}{3a_1}} 3a_1t^3ze^{-(a_1t^3+q)z}dt \\ &= 3z\int_{\mathcal{C}_1+\frac{a_2}{3a_1}} (a_1t^3+q)e^{-(a_1t^3+q)z}dt - 3qz\int_{\mathcal{C}_1+\frac{a_2}{3a_1}} e^{-(a_1t^3+q)z}dt \\ &= -3zI'(z) - 3qzI(z) \end{aligned}$$

□

We would like to verify that for every cubic function f, the Borel transform of the exponential integral can be expressed by an hypergeometric function and hence deduce its resurgent properties in full generality. If $p \neq 0$, $I(z)$ is a solution of

$$I'' + 2qI' + \left(\frac{4p^3}{27a_1} + q^2\right)I + \frac{1}{z}I' + \frac{q}{z}I - \frac{1}{9z^2}I = 0 \tag{5.5}$$

hence a formal solution as $z \to \infty$ is given (up to constants $U_1, U_2 \in \mathbb{C}$) by

$$\tilde{I}_+(z) := U_1e^{-(q+\sqrt{\frac{4p^3}{27a_1}})z}z^{1/2}\tilde{w}_+(z) \tag{5.6}$$

$$\tilde{I}_-(z) := U_2e^{-(q-\sqrt{\frac{4p^3}{27a_1}})z}z^{1/2}\tilde{w}_-(z) \tag{5.7}$$

where $\tilde{w}_\pm(z) \in \mathbb{C}[\![z^{-1}]\!]$ is the formal solution of

$$\tilde{w}_\pm'' \mp 2\sqrt{\frac{4p^3}{27a_1}}\tilde{w}_\pm' + \frac{5}{36}\frac{\tilde{w}_\pm}{z^2} = 0 \tag{5.8}$$

with $\tilde{w}_\pm(z) = 1 + \sum_{k\geq 1} a_{\pm,k} z^{-k}$.

We can now compute the Borel transform of (5.8): for $\tilde{w}_+(z)$

$$\zeta^2 \hat{w} - 2\sqrt{\frac{4p^3}{27a_1}}\zeta\hat{w} + \frac{5}{36}\int_0^\zeta (\zeta-\zeta')\hat{w}(\zeta')d\zeta' = 0$$

$$\zeta^2\hat{w}' + 2\zeta\hat{w} + 2\sqrt{\frac{4p^3}{27a_1}}\hat{w} + 2\sqrt{\frac{4p^3}{27a_1}}\zeta\hat{w}' + \frac{5}{36}\int_0^\zeta \hat{w}(\zeta') = 0$$

$$\left(\zeta^2 + 2\sqrt{\frac{4p^3}{27a_1}}\zeta\right)\hat{w}'' + 4\left(\zeta + \sqrt{\frac{4p^3}{27a_1}}\right)\hat{w}' + \frac{77}{36}\hat{w} = 0$$

$$t(1-t)\hat{w}'' + (2-4t)\hat{w}' - \frac{77}{36}\hat{w} = 0 \qquad \zeta = -2t\sqrt{\frac{4p^3}{27a_1}}$$

hence

$$\hat{w}_+(\zeta) = {}_2F_1\left(\frac{7}{6},\frac{11}{6};2;-\frac{3}{4p}\sqrt{\frac{3a_1}{p}}\zeta\right)$$

and analougusly,

$$\hat{w}_-(\zeta) = {}_2F_1\left(\frac{7}{6},\frac{11}{6};2;\frac{3}{4p}\sqrt{\frac{3a_1}{p}}\zeta\right).$$

Notice that $\hat{w}_\pm(\zeta)$ has a branch cut singularity respectively at $\zeta = \zeta_\pm := \pm\sqrt{\frac{16p^3}{27a_1}}$, and thanks to the well known formulas for the analytic continuation of hypergeometric functions (see 15.2.3 DLMF), if we assume the branch cut is from $\zeta_\pm$ to $+\infty$

$$\begin{aligned}
\hat{w}_-(\zeta+i0) - \hat{w}_-(\zeta-i0) &= \frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\left(\frac{\zeta}{\zeta_+}-1\right)^{-1}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_+}\right)^k \qquad \zeta\in(\zeta_+,+\infty)\\
&= -\frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_+}\right)^{k-1}\\
&= -\mathbf{i}\frac{1}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{\Gamma\left(\frac{7}{6}+k\right)\Gamma\left(\frac{11}{6}+k\right)}{\Gamma(2+k)\Gamma(k+1)}\left(1-\frac{\zeta}{\zeta_+}\right)^k\\
&= -\mathbf{i}\,{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;1-\frac{\zeta}{\zeta_+}\right)\\
&= -\mathbf{i}\hat{w}_+(\zeta-\zeta_+)
\end{aligned}$$

Similarly, if we assume that the branch cut is from $\zeta_\pm$ to $-\infty$ then

$$\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0) = -\frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\left(\frac{\zeta}{\zeta_-}-1\right)^{-1}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_-}\right)^k \qquad \zeta\in(-\infty,\zeta_-)$$
$$= \frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_-}\right)^{k-1}$$
$$= \mathbf{i}\frac{1}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{\Gamma\left(\frac{7}{6}+k\right)\Gamma\left(\frac{11}{6}+k\right)}{\Gamma(2+k)\Gamma(k+1)}\left(1-\frac{\zeta}{\zeta_-}\right)^k$$
$$= \mathbf{i}\,{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;1-\frac{\zeta}{\zeta_-}\right)$$
$$= \mathbf{i}\,\hat{w}_-(\zeta-\zeta_-)$$

therefore we see that the Stokes factors are given by $\pm i$ (as for Airy).

I think it will be nice to add the geometric interpretation of Maxim in term of Lefschetz thimbles

Let us first compute the Borel transform of (5.1) (indeed as in the proof of Theorem 3.1 we know that (5.1) admits a formal solution which is Gevrey-1)

$$\zeta^2\hat{I}-a\zeta\hat{I}+b\hat{I}-\int_0^\zeta \zeta'\hat{I}(\zeta')+d\int_0^\zeta \hat{I}(\zeta')-\frac{1}{9}\int_0^\zeta(\zeta-\zeta')\hat{I}(\zeta')=0$$
$$2\zeta\hat{I}+\zeta^2\hat{I}'-a\hat{I}-a\zeta\hat{I}'+b\hat{I}'-\zeta\hat{I}+d\hat{I}-\frac{1}{9}\int\hat{I}(\zeta')=0$$
$$(\zeta^2-a\zeta+b)\hat{I}''+(3\zeta-2a+d)\hat{I}'+\frac{8}{9}\hat{I}=0$$

Now we denote by λ_1,λ_2 the distinguished (we assume that $p\neq 0$) roots of $\zeta^2-a\zeta+b$, then

$$(\zeta-\lambda_1)(\zeta-\lambda_2)\hat{I}''+(3\zeta-2a+d)\hat{I}'+\frac{8}{9}\hat{I}=0 \tag{5.9}$$
$$(t+\lambda_2-\lambda_1)t\hat{I}''+(3t+3\lambda_2-2a+d)\hat{I}'+\frac{8}{9}=0 \qquad t=\zeta-\lambda_2 \tag{5.10}$$
$$s(1-s)\hat{I}''-\left(3s+\frac{3\lambda_2-2a+d}{\lambda_1-\lambda_2}\right)\hat{I}'-\frac{8}{9}\hat{I}=0 \qquad t=(\lambda_1-\lambda_2)s \tag{5.11}$$

where (5.11) is an hypergeometric equation[3] and a solution is given by

[3]Notice that $\lambda_{1,2}=q\pm\frac{2i}{3}p\sqrt{\frac{p}{3a_1}}$, $a=2q$ and $d=q$. Hence

$$\frac{2a-d-3\lambda_2}{\lambda_1-\lambda_2}=\frac{4q-q-3q-2ip\sqrt{\frac{p}{3a_1}}}{-\frac{4i}{3}p\sqrt{\frac{p}{3a_1}}}=\frac{3}{2}$$

(5.12)
$$\hat{I}_{\lambda_1}(\zeta;U_1,U_2)=U_1\,{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};\frac{\zeta-\lambda_2}{\lambda_1-\lambda_2}\right)+U_2\left(\frac{\zeta-\lambda_2}{\lambda_1-\lambda_2}\right)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{\zeta-\lambda_2}{\lambda_1-\lambda_2}\right)$$

which has a branch cut at $\zeta=\lambda_1$, where U_1,U_2 are constants. Of course, reversing the role of λ_1 and λ_2 we find

(5.13)
$$\hat{I}_{\lambda_2}(\zeta,;U_1,U_2)=U_1\,{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)+U_2\left(\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)$$

is the Borel transform of $\tilde{I}_{\lambda_2}(z)$ and it has a branch cut singularity at $\zeta=\lambda_2$. It is remarkable that the dependece on the function f is only on the location of the singularities, but it is always an hypergeometric function with the same parameters. In addition, we can compute the Stokes constants thanks to the well known formula for analyitic continuation of hypergeometric (see 15.2.3 in DLMF)

$$\begin{aligned}\hat{I}_{\lambda_1}(\zeta+i0;U_1,0)-\hat{I}_{\lambda_1}(\zeta-i0;U_1,0)&=-U_1\frac{2\pi i}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}\left(\frac{\lambda_1-\zeta}{\lambda_2-\lambda_1}\right)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)\\&=-\mathbf{i}U_1\frac{2\pi i}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}\left(\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)\\&=-\mathbf{i}\hat{I}_{\lambda_2}\left(\zeta;0,\frac{U_1}{\Gamma(2/3)\Gamma(4/3)}\right)\end{aligned}$$