

# EXPONENTIAL INTEGRALS

VERONICA FANTINI

## CONTENTS

1. Introduction	1
2. Resurgence of exponential integrals	1
2.1. Second tentative formula	5

## 1. INTRODUCTION

### 2. RESURGENCE OF EXPONENTIAL INTEGRALS

Let  $X$  be a  $N$  – dim manifold,  $f: X \rightarrow \mathbb{C}$  be a holomorphic Morse function with only simple critical points, and  $\nu \in \Gamma(X, \Omega^N)$ , and set

$$(2.1) \quad I(z) := \int_{\mathcal{C}} e^{-zf} \nu$$

where  $\mathcal{C}$  is a suitable contour such that the integral is well defined. For any Morse critical points  $x_\alpha$  of  $f$ , the saddle point approximation gives the following formal series

$$(2.2) \quad I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf} \nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)} (2\pi)^{N/2} z^{-N/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n}.$$

**Theorem 2.1.** Let  $\tilde{\varphi}_\alpha(z) := e^{-zf(x_\alpha)} (2\pi)^{N/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n}$

- (1)  $\tilde{\varphi}_\alpha$  is Gevrey-1;
- (2)  $\hat{\varphi}(\zeta) := \mathcal{B}(\tilde{\varphi})$  is a germ of analytic function at  $\zeta = \zeta_\alpha = f(x_\alpha)$ ;
- (3) the following formula holds true

$$(2.3) \quad \hat{\varphi}_\alpha(\zeta) = \partial_{\zeta, \text{based at } \zeta_\alpha}^{N/2} \left( \int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu \right) = \Gamma(-N/2) \int_{\zeta_\alpha}^{\zeta} (\zeta - \zeta')^{-N/2} \int_{f^{-1}(\zeta')} \frac{\nu}{df} d\zeta'$$

**Definition 2.2.** Let  $\alpha \in (0, 1)$ , then the  $\alpha$ -Caputo's derivative of a smooth function  $f$  is defined as

$$(2.4) \quad \partial_x^\alpha f(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f'(s) ds$$

---

**Example 2.3 (Airy).** Let  $f(t) = \frac{t^3}{3} - t$  and

$$I(z) := \int_{\gamma} e^{-zf(t)} dt$$

where  $\gamma$  is a contour where the integral is well defined.

By the change of coordinates  $z = x^{3/2}$ ,  $I(z) = -2\pi i z^{-1/3} Ai(x)$  where

$$Ai(x) := \frac{1}{2\pi i} \int_{-\infty e^{-i\frac{\pi}{3}}}^{\infty e^{i\frac{\pi}{3}}} e^{\frac{t^3}{3} - zt} dt$$

hence  $I(z)$  solves the following ODE<sup>1</sup>

$$(2.5) \quad I''(z) - \frac{4}{9}I(z) + \frac{I'(z)}{z} - \frac{1}{9} \frac{I(z)}{z^2} = 0$$

A formal solution of (2.5) can be computed by making the following ansatz

$$(2.6) \quad \tilde{I}(z) = \sum_{k \in \mathbb{N}^2} U^k e^{-\lambda \cdot k z} z^{-\tau \cdot k} w_k(z)$$

with  $U^{(k_1, k_2)} = U_1^{k_1} U_2^{k_2}$  and  $U_1, U_2 \in \mathbb{C}$  are constant parameter,  $\lambda = (\frac{2}{3}, -\frac{2}{3})$ ,  $\tau = (\frac{1}{2}, \frac{1}{2})$ , and  $\tilde{w}_k(z) \in \mathbb{C}[[z^{-1}]]$ . In addition, we can check that the only non zero  $\tilde{w}_k(z)$  occurs at  $k = (1, 0)$  and  $k = (0, 1)$ , therefore

$$(2.7) \quad \tilde{I}(z) = U_1 e^{-2/3 z} z^{-1/2} \tilde{w}_+(z) + U_2 e^{2/3 z} z^{-1/2} \tilde{w}_-(z)$$

where from now on we denote  $\tilde{w}_+(z) = \tilde{w}_{(1,0)}(z)$  and  $\tilde{w}_-(z) = \tilde{w}_{(0,1)}(z)$ . In particular,  $\tilde{w}_+(z)$  and  $\tilde{w}_-(z)$  are formal solution of

$$(2.8) \quad \tilde{w}_+'' - \frac{4}{3} \tilde{w}_+' + \frac{5}{36} \frac{\tilde{w}_+}{z^2} = 0$$

$$(2.9) \quad \tilde{w}_-'' + \frac{4}{3} \tilde{w}_-'' + \frac{5}{36} \frac{\tilde{w}_-}{z^2} = 0$$

Taking the Borel transform of (2.8), (2.9) we get

$$\begin{aligned} \zeta^2 \hat{w}_+(\zeta) + \frac{4}{3} \zeta \hat{w}_+ + \frac{5}{36} \zeta * \hat{w}_+ &= 0 \\ \zeta^2 \hat{w}_+(\zeta) + \frac{4}{3} \zeta \hat{w}_+ + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}_+(\zeta') d\zeta' &= 0 \\ \zeta^2 \hat{w}_-(\zeta) - \frac{4}{3} \zeta \hat{w}_- + \frac{5}{36} \zeta * \hat{w}_- &= 0 \\ \zeta^2 \hat{w}_-(\zeta) - \frac{4}{3} \zeta \hat{w}_- + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}_-(\zeta') d\zeta' &= 0 \end{aligned}$$

---

<sup>1</sup> $Ai(x)$  solves the Airy equation  $y'' = xy$ .

and taking derivatives we get

$$\begin{aligned}\zeta\left(\frac{4}{3} + \zeta\right)\hat{w}_+'' + \left(\frac{8}{3} + 4\zeta\right)\hat{w}_+' + \frac{77}{36}\hat{w}_+ &= 0 \\ \frac{4}{3}\zeta\left(1 + \frac{3}{4}\zeta\right)\hat{w}_+'' + \left(\frac{8}{3} + 4\zeta\right)\hat{w}_+' + \frac{77}{36}\hat{w}_+ &= 0 \\ u(1-u)\hat{w}_+''(u) + (2-4u)\hat{w}_+'(u) - \frac{77}{36}\hat{w}_+(u) &= 0 \quad u = -\frac{3}{4}\zeta\end{aligned}$$

$$\begin{aligned}\zeta\left(-\frac{4}{3} + \zeta\right)\hat{w}_-'' + \left(-\frac{8}{3} + 4\zeta\right)\hat{w}_-' + \frac{77}{36}\hat{w}_- &= 0 \\ \frac{4}{3}\zeta\left(-1 + \frac{3}{4}\zeta\right)\hat{w}_-'' + \left(-\frac{8}{3} + 4\zeta\right)\hat{w}_-' + \frac{77}{36}\hat{w}_- &= 0 \\ u(1-u)\hat{w}_-''(u) + (2-4u)\hat{w}_-'(u) - \frac{77}{36}\hat{w}_-(u) &= 0 \quad u = \frac{3}{4}\zeta\end{aligned}$$

Notice that the latter equations are hypergeometric, hence a solution is given by

$$(2.10) \quad \hat{w}_+(\zeta) = c_1 {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}\zeta\right)$$

$$(2.11) \quad \hat{w}_-(\zeta) = c_2 {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{3}{4}\zeta\right)$$

for some constants  $c_1, c_2 \in \mathbb{C}$  (see DMLF 15.10.2). In addition  $\hat{w}_\pm(\zeta)$  have a log singularity respectively at  $\zeta = \mp \frac{4}{3}$ , therefore they are  $\{\mp \frac{4}{3}\}$ -resurgent functions.<sup>2</sup>

**Remark 2.4.**  $\hat{w}_+(\zeta)$  is Laplace summable along the positive real axis, and it can be analytically continued on  $\mathbb{C} \setminus -\frac{4}{3}\mathbb{R}_{\geq 0}$  with

$$\begin{aligned}\hat{w}_+(\zeta + i0) - \hat{w}_+(\zeta - i0) &= \frac{36}{5}i\left(-\frac{3}{4}\zeta - 1\right)^{-1} \sum_{k \geq 0} \frac{(5/6)_n (1/6)_n}{\Gamma(n)n!} \left(1 + \frac{3}{4}\zeta\right)^n \quad \zeta < -\frac{4}{3} \\ &= \frac{36}{5}i\left(-\frac{3}{4}\zeta - 1\right)^{-1} \left(\frac{5}{144}(4 + 3\zeta) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 + \frac{3}{4}\zeta\right)\right) \\ &= -i {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 + \frac{3}{4}\zeta\right) \\ &= -i \hat{w}_-\left(\zeta + \frac{4}{3}\right)\end{aligned}$$

<sup>2</sup>The solution we find are equal to the ones in DMLF §9.7. They also agree with slide 10 of Maxim's talk for ERC and with the series (6.121) in Sauzin's book. However they do not agree with (2.16) in Mariño's Diablerets.

Anolougsly,  $\hat{w}_-(\zeta)$  is Laplace summable along the negative real axis, and it jumps across the branch cut  $\frac{4}{3}\mathbb{R}_{\geq 0}$  as

$$\begin{aligned}\hat{w}_-(\zeta + i0) - \hat{w}_-(\zeta - i0) &= -\frac{36}{5}i\left(\frac{3}{4}\zeta - 1\right)^{-1} \sum_{k \geq 0} \frac{(5/6)_n (1/6)_n}{\Gamma(n)n!} \left(1 - \frac{3}{4}\zeta\right)^n & \zeta > \frac{4}{3} \\ &= -\frac{36}{5}i\left(\frac{3}{4}\zeta - 1\right)^{-1} \left(-\frac{5}{144}(-4 + 3\zeta)_1 F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 - \frac{3}{4}\zeta\right)\right) \\ &= i {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 - \frac{3}{4}\zeta\right) \\ &= i\hat{w}_+(\zeta - \frac{4}{3})\end{aligned}$$

These relations manifest the resurgence property of  $\tilde{I}$ , indeed near the singularities in the Borel plane of either  $\hat{w}_+$  or  $\hat{w}_-$ ,  $\hat{w}_-$  and  $\hat{w}_+$  respectively contribute to the jump of the former solution.

Our next goal is to prove that the Borel transform of  $\tilde{I}(z)$  can be written in terms of  $1/f'(f^{-1}(\zeta))$ , namely formula (2.3). First of all we define  $\tilde{u}(z) := z^{1/2}\tilde{I}(z)$  which is a solution of  $\hat{\varphi}_1(\zeta)$  and  $\hat{\varphi}_2(\zeta)$ :

$$(2.12) \quad \tilde{u}''(z) - \frac{4}{9}\tilde{u}(z) + \frac{5}{36} \frac{\tilde{u}(z)}{z^2} = 0$$

As a consequence, the Borel transform  $\hat{u}(\zeta)$  solves

$$\begin{aligned}\zeta^2 \hat{u} - \frac{4}{9} \hat{u} + \frac{5}{36} \zeta * \hat{u} \\ \zeta^2 \hat{u} - \frac{4}{9} \hat{u} + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{u}(\zeta') d\zeta' \\ \text{taking derivatives is equivalent to} \\ (\zeta^2 - \frac{4}{9}) \hat{u}''(\zeta) + 4\zeta \hat{u}'(\zeta) + \frac{77}{36} \hat{u}(\zeta) = 0\end{aligned}$$

and Mathematica gives the following solutions

$$\begin{aligned}\hat{u}(\zeta) &= c_1 {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{9}{4}\zeta^2\right) + \frac{3i}{2}\zeta c_2 {}_1F_2\left(\frac{13}{12}, \frac{17}{12}, \frac{3}{2}, \frac{9}{4}\zeta^2\right) = \\ &= c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} \left({}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} + \frac{3}{4}\zeta\right)\right) & \text{see DMLF 15.8.27} \\ &\quad + \frac{3i}{2}\zeta c_2 \left(\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{3\zeta\Gamma(-\frac{1}{2})\Gamma(2)}\right) \left({}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right)\right) & \text{see DMLF 15.8.28} \\ &= \left(c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} - c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) + \\ &\quad + \left(c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} + c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right)\end{aligned}$$

---


$$\begin{aligned}\hat{u}(\zeta) &= C_1 T_{-2/31} F_2\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}\zeta\right) + C_2 T_{2/31} F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{3}{4}\zeta\right) \\ &= C_1 T_{-2/3} \hat{w}_+(\zeta) + C_2 T_{2/3} \hat{w}_-(\zeta)\end{aligned}$$

Using properties of Caputo's  $1/2$ -derivative we would like to express  $\hat{w}_\pm(\zeta)$  as the  $1/2$ -derivatives of other hypergeometric series. From the integral representation of hypergeometric functions (see DMLF 15.6.1)

$$\begin{aligned}{}_2F_1\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}\zeta\right) &= \frac{3}{5\pi} \int_0^1 t^{5/6} (1-t)^{-5/6} \left(1 + \frac{3}{4}\zeta t\right)^{-7/6} dt & |u| < \frac{4}{3} \\ &= \frac{3}{5\pi} \zeta^{-1} \int_0^\zeta u^{5/6} (\zeta - u)^{-5/6} \left(1 + \frac{3}{4}u\right)^{-7/6} du & u = \zeta t \\ &= \frac{3}{5\pi} \int_0^\zeta (\zeta - u)^{-1/2} \left(\zeta^{-1/2} \left(1 - \frac{u}{\zeta}\right)^{-1/3} \left(\frac{u}{\zeta}\right)^{5/6} \left(1 + \frac{3}{4}u\right)^{-7/6}\right) du \\ &= \frac{3}{5\sqrt{\pi}} \frac{1}{\Gamma(1/2)} \int_0^\zeta (\zeta - u)^{-1/2} \left(\zeta^{-1/2} \left(1 - \frac{u}{\zeta}\right)^{-1/3} \left(\frac{u}{\zeta}\right)^{5/6} \left(1 + \frac{3}{4}u\right)^{-7/6}\right) du\end{aligned}$$

For formula (2.3) to hold true we must have

$$(2.13) \quad \frac{3}{5\sqrt{\pi}} \left( \zeta^{-1/2} \left(1 - \frac{u}{\zeta}\right)^{-1/3} \left(\frac{u}{\zeta}\right)^{5/6} \left(1 + \frac{3}{4}u\right)^{-7/6} \right) = \frac{1}{f'(f^{-1}(u))}$$

however,  $\frac{1}{f'(f^{-1}(u))}$  does not depend on  $\zeta$ , thus (2.13) is false, and the Airy exponential integral is a counter example of the formula (2.3).

**2.1. Second tentative formula.** Let us assume that the correct formula that replaces (2.3) in Theorem 2.1 is

$$(2.14) \quad \hat{I}_\alpha(\zeta) = \partial_{\zeta, \text{based at } \zeta_\alpha}^{N/2} \left( \int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} v \right) = \Gamma(-N/2) \int_{\zeta_\alpha}^\zeta (\zeta - \zeta')^{-N/2} \int_{f^{-1}(\zeta')} \frac{v}{df} d\zeta'$$

then we compute  $\hat{I}$  using properties of the Borel transform of a product:

$$\begin{aligned}\hat{I}(\zeta) &= \mathcal{B}(z^{-1/2}) * \hat{u}(\zeta) \\ &= \frac{1}{\Gamma(1/2)} \int_0^\zeta (\zeta - s)^{-1/2} \hat{u}(s) ds \\ &= \frac{1}{\Gamma(1/2)} \int_0^\zeta (\zeta - s)^{-1/2} (T_{-2/3} \hat{w}_+(s) + T_{2/3} \hat{w}_-(s)) ds\end{aligned}$$

---


$$\begin{aligned}
&= \frac{1}{\Gamma(1/2)} \int_0^\zeta (\zeta - s)^{-1/2} \hat{w}_+(s - 2/3) + \frac{1}{\Gamma(1/2)} \int_0^\zeta (\zeta - s)^{-1/2} \hat{w}_-(s + 2/3) ds \\
&= \frac{1}{\Gamma(1/2)} \int_{-2/3}^{\zeta-2/3} (\zeta - \frac{2}{3} - s)^{-1/2} \hat{w}_+(s) + \frac{1}{\Gamma(1/2)} \int_{2/3}^{\zeta+2/3} (\zeta + \frac{2}{3} - s)^{-1/2} \hat{w}_-(s) ds \\
&= \frac{C_1}{\Gamma(1/2)} \int_{-2/3}^{\zeta-2/3} (\zeta - \frac{2}{3} - s)^{-1/2} {}_2F_1\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}s\right) ds + \frac{C_2}{\Gamma(1/2)} \int_{2/3}^{\zeta+2/3} (\zeta + \frac{2}{3} - s)^{-1/2} {}_2F_1\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{3}{4}s\right) ds \\
&= \frac{C_3}{\Gamma(1/2)} \int_{-2/3}^{\zeta-2/3} (\zeta - \frac{2}{3} - s)^{-\frac{1}{2}} \frac{\partial}{\partial s} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, -\frac{3}{4}s\right) ds + \frac{C_4}{\Gamma(1/2)} \int_{2/3}^{\zeta+2/3} (\zeta + \frac{2}{3} - s)^{-\frac{1}{2}} \frac{\partial}{\partial s} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, \frac{3}{4}s\right) ds \\
&= C_3 \partial_{\zeta, \text{based at } -2/3}^{1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, -\frac{3}{4}\zeta\right) + C_4 \partial_{\zeta, \text{based at } 2/3}^{1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, \frac{3}{4}\zeta\right)
\end{aligned}$$

However,  ${}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, \mp \frac{3}{4}\zeta\right)$  are not algebraic, hence they can not be equal to  $\frac{1}{f'(f^{-1}(\zeta))}$  as it is in in Theorem 2.1 part II (2.14).