RESURGENCE AND BOREL REGULARITY FOR ODES
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1. What is resurgence

Theory of resurgence was introduced by Ecalle in the '80 and it deals with divergent power series

$$\tilde{\Phi}(z) = \sum_{n\geq 0} a_n z^{-n-1} \in \mathbb{C}[[z^{-1}]], \quad \text{with } a_n \sim n! \tag{1.1}$$

they have zero radius of convergence.

1.1. Paradigma of Borel–Laplace sum.

$$\mathbb{C}[[z^{-1}]] \ni \tilde{\Phi} := \sum_{n\geq 0} a_n z^{-n-1} \xrightarrow{\mathcal{B}} \tilde{\phi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!} \zeta^n \in \mathbb{C}\{\zeta\}$$

[make drawings of the Borel plane]

Study the analytic continuation of $\tilde{\phi}(\zeta)$ and if $\hat{\phi}(\zeta)$ **behaves well** you can go back to the z-plane via Laplace transform

$$\begin{tikzcd}
\mathbb{C}[[z^{-1}]] \ni \tilde{\Phi} := \sum_{n\geq 0} a_n z^{-n-1} \arrow[rr, "\mathcal{B}"] & & \tilde{\phi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!} \zeta^n \in \mathbb{C}\{\zeta\} \arrow[dl, "\mathcal{L}^\theta"] \\
& \Phi \in \mathcal{O}(H_\theta) &
\end{tikzcd}$$

[make drawings of the z-plane]

The relation with $\tilde{F}$ is via asymptotic expansion as $\Re z e^{i\theta} \to \infty$

$$\begin{tikzcd}
\mathbb{C}[[z^{-1}]] \ni \tilde{\Phi} := \sum_{n\geq 0} a_n z^{-n-1} \arrow[rr, "\mathcal{B}"] & & \tilde{\phi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!} \zeta^n \in \mathbb{C}\{\zeta\} \arrow[dl, "\mathcal{L}^\theta"] \\
& \Phi \in \mathcal{O}(H_\theta) \arrow[ul, dashed, "\text{asymptotics}"] &
\end{tikzcd}$$

The Laplace transform is defined

$$F(z) \cong \mathcal{L}^\theta f(z) = \int_0^{+\infty e^{i\theta}} e^{-z\zeta} f(\zeta) d\zeta \tag{1.2}$$

for $\theta \in [0, 2\pi)$ and $F \in \mathcal{O}(H_\theta)$ if and only if $|f| \leq Ae^{-c|z|}$ for every z in a tubular neighbourhood. In general one has to check

- $\hat{f}$ can be defined through infinity;
- $\hat{f}$ has the right decaying at ∞. HARDER

For the first one, we have to study the singularities of $\hat{f}$.

1.1.1. *Ecalle's resurgent function.* $f \in \mathbb{C}\{\zeta\}$ is resurgent if it endlessly analytically continuable, i.e.

[draw picture]

He defined the **theory of singularities** and **relaxed the definition of Laplace transform** to deal with log- singularities, square root, poles, etc.

Examples of (*minor* of) singularities

$$\overset{\vee}{I}_c(\zeta) := \frac{\zeta^{c-1}}{(1 - e^{-2\pi i c})\Gamma(c)} \quad c \in \mathbb{C} \setminus \mathbb{Z}_{>0}$$

$$\overset{\vee}{I}_c(\zeta) := \frac{\zeta^{c-1}}{2\pi i \Gamma(c)} \log \zeta \quad c \in \mathbb{Z}_{>0}$$

draw singularity in a row (like Painlevé)

1.1.2. *The role of θ.* Varying θ, as long as $|\theta - \theta'| < \pi$, $\mathcal{L}^\theta f = \mathcal{L}^{\theta'}$ on $H_\theta \cap H_{\theta'}$. More generally, they disagree

$$\mathcal{L}^\theta - \mathcal{L}^{\theta'} = S_{\theta\theta'} \mathcal{L}^{\theta'} \quad S_{\theta\theta'} \in \mathbb{C}$$

These phenomena is called the Stokes phenomena. Computing the Stokes constant is crucial to understand the structure of F, and Ecalle developed the *Alien calculus* to study the Stokes phenomena.

Summarizing: the core of resurgence theory is

(1) study the Borel plane;
(2) compute the Stokes constants.

stress that resurgence is about Borel plane, but knows about the z-plane.

1.2. Motivation: why divergent series?

- **thimbles integrals**:
  COOL: if f is algebraic, $I(z)$ is a period, i.e. it is a geometric object studied (Deligne, Malgrange, Pham, Kontsevich–Soibelman $\sim$ generalization when f is local coordinate for 1-form $f = \int \alpha$) and the resurgence of $\tilde{I}$ has a geometric nature
  - critical values of f $\sim$ singularity in the Borel plane

- Picard–Lefschetz theory ~ Stokes phenomena
- gradient lines ~ Stokes indexes

- **ODEs** with irregular singularity at ∞: they admits formal solutions, and if the ODE is regular enough M.A.E.T. assures the existence of a holomorphic solution asymptotic to the formal one.
  - non linear ODEs have interesting behaviours (maybe be resonant)
- **q–difference equation**: like ODEs,

$$n! \rightsquigarrow q^{n(n+1)/2} \; |q|>1 \tag{1.3}$$

there is a dictionary between ODEs and q-difference equation.

2. ODEs

2.1. Which class of ODEs we consider.

$$[P(\partial/\partial_z)+\frac{1}{z}Q(\partial/\partial_z)+\sum_{j=2}^{d} z^{-j}R_j(\partial/\partial_z)]\Phi(z)=0 \tag{2.1}$$

with

- $P(\lambda)$ a degree d polynomial
- $Q(\lambda)$ a degree $d-1$ polynomial
- $R_j(\lambda)$ are degree $d-j$ polynomials

they are defined by Poincaré as *series normal de Ier ordre.* As a general fact, if $P(-\lambda)$ has simple roots $\alpha_1,\dots,\alpha_d$ then (2.1) admits d formal solution of the form

$$\tilde{\Phi}_j(z)=e^{-\alpha_j z}z^{-\tau_j}\tilde{\phi}_j(z)\in e^{-\alpha_j z}z^{-\tau_j}\mathbb{C}[[z^{-1}]][\log(z)] \tag{2.2}$$

where $\tau_j=-Q(\alpha_j)/P'(\alpha_j)$.

The M.A.E.T theorem, guarantees under suitable assumptions on the ODE, the existence of an holomorphic solution $\Phi_j(z)$ asymptotic to $\tilde{\Phi}_j$ in a suitable sector.

Our goal is to prove that $\Phi_j(z)\propto \mathcal{L}^{\theta}_{\zeta_j}\mathcal{B}\tilde{\Phi}_j$ for some angle θ.

Equivalently said, Borel–Laplace summability picks an actual solution of the ODE.

2.2. **Borel regularity for ODEs.**

$$
\begin{tikzcd}
& P\left(\frac{\partial}{\partial_z}\right)+\frac{1}{z}Q\left(\frac{\partial}{\partial_z}\right)+\sum_{j=2}^{d} z^{-j}R_j\left(\frac{\partial}{\partial_z}\right) & \\
\Phi_j \arrow[ur, red] & & \tilde{\Phi}_j \arrow[ul] \arrow[d, "\mathcal{B}"] \\
\hat{\phi}_j \arrow[u, "\mathcal{L}^{\theta}_{\zeta,\alpha_j}"] & & \tilde{\phi}_j(\zeta) \arrow[ll, "\text{sum}"']
\end{tikzcd}
$$

a priori it is not guarantee that $\mathsf{L}^{\theta}_{\zeta,\alpha_j}\mathcal{B}\tilde{\Phi}_j$ is a solution of (2.1), because different functions may have the same asymptotic.

Idea of the proof is based on the following diagram:

$$
\begin{tikzcd}
& P\left(\frac{\partial}{\partial_z}\right)+\frac{1}{z}Q\left(\frac{\partial}{\partial_z}\right)+\sum_{j=2}^{d} z^{-j}R_j\left(\frac{\partial}{\partial_z}\right) \arrow[dd, "\mathcal{B}"] & \\
\Phi_j \arrow[ur, red] & & \tilde{\Phi}_j \arrow[ul] \arrow[dd, "\mathcal{B}"] \\
& P(-\zeta)+\partial_{\zeta}^{-1}\circ Q(-\zeta)+\sum_{j=2}^{d}\partial_{\zeta}^{-j}\circ R_j(-\zeta) & \\
\hat{\phi}_j \arrow[uu, "\mathcal{L}^{\theta}_{\zeta,\alpha_j}"] & & \tilde{\phi}_j(\zeta) \arrow[ll, "\text{sum}"'] \arrow[ul]
\end{tikzcd}
$$

there exists a solution $\tilde{\phi}_j(\zeta)$ which is also slight, and in suitable coordinates it is a convergent series ~~in~~ $\overset{\vee}{I}_{n+\tau_j}(\zeta_j)$.

$$
\begin{tikzcd}
& P\left(\frac{\partial}{\partial_z}\right)+\frac{1}{z}Q\left(\frac{\partial}{\partial_z}\right)+\sum_{j=2}^{d} z^{-j}R_j\left(\frac{\partial}{\partial_z}\right) \arrow[dd, shift left, "\mathcal{B}"] & \\
\Phi_j \arrow[ur, red] & & \tilde{\Phi}_j \arrow[ul] \arrow[dd, "\mathcal{B}"] \\
& P(-\zeta)+\partial_{\zeta}^{-1}\circ Q(-\zeta)+\sum_{j=2}^{d}\partial_{\zeta}^{-j}\circ R_j(-\zeta) \arrow[uu, shift left, "\mathcal{L}"] & \\
\hat{\phi}_j \arrow[uu, "\mathcal{L}^{\theta}_{\zeta,\alpha_j}"] & & \tilde{\phi}_j(\zeta) \arrow[ll, "\text{sum}"'] \arrow[ul]
\end{tikzcd}
$$

the Laplace transform along a Hankel contour gives an inverse for $\mathcal{B}$.

[repeat again the idea: a solution of (2.1) is Borel regular, because its Borel–Laplace sum gives an actual solution.]

Remark 2.1. **[Extend the definition of $\mathcal{B}$ and $\mathcal{L}$]**

- As far as $\Gamma(\tau)$ is well defined, we can allow $\tau \in \mathbb{R}$ and extend the definition of $\mathcal{B}$.
- Ecalle's theory of singularity introduces a generalized Laplace transform which take as contour a Hankel contour rather than a straight line. This is not the only generalization of Laplace transform Ecalle introduced.

2.3. **Proof of Borel regularity.**

- Borel transform the ODE$_z$ and we get IE$_\zeta$.
  - IE are not easy to be solved so usually in the application we differentiate them to get an ODE$_\zeta$. If $\hat{\phi}$ is slight we don't loose informations by differentiating.
- by Prop 1, there exists a solution $\hat{\phi}(\zeta_j)$

$$\begin{aligned}\hat{\phi}(\zeta_j) &= \zeta_j^{\tau_j-1} + \tilde{g}_j && \tilde{g}_j \in \mathcal{HL}^{\infty,1-\tau_j-\epsilon}\\ &= \sum_{k\geq 0} a_k \zeta_j^{\tau_j-1+k} + \text{h.f.} \\ &= (1-e^{-2\pi i\tau_j})\sum_{n\geq 0}\tilde{a}_n \overset{\vee}{I}_{\tau_j+n}(\zeta_j) && \tilde{a}_n = a_n\Gamma(\tau_j+n),\\ & && \overset{\vee}{I}_c(\xi) = \frac{\xi^{c-1}}{(1-e^{-2\pi i c})\Gamma(c)}\end{aligned}$$

- $\hat{\phi}(\zeta_j)$ is a germ of meromorphic function

$$\begin{aligned}\limsup_{n\to\infty}\sqrt[n]{\frac{|\tilde{a}_n|}{\Gamma(\tau_j+n)(1-e^{-2\pi i(\tau_j+n)})}} &= \limsup_{n\to\infty}\sqrt[n]{\frac{a_n\Gamma(\tau_j+n)}{\Gamma(\tau_j+n)(1-e^{-2\pi i(\tau_j+n)})}} = \\ &= \limsup_{n\to\infty}\sqrt[n]{\frac{a_n}{(1-e^{-2\pi i(\tau_j+n)})}} < +\infty\end{aligned}$$

where in the last step we use

$$\infty + > \|\tilde{g}_j\zeta_j^{\tau_j+\epsilon-1}\|_\infty = \|\sum_{n\geq 1}\zeta_j^{\tau_j-1+n}\zeta_j^{-(\tau_j+\epsilon)+1}\|_\infty = \|\sum_{n\geq 1}a_n\zeta_j^{n-\epsilon}\|_\infty \tag{2.3}$$

- by Ecalle definition of Laplace transform [see Sauzin], $\hat{\phi}(\zeta_j)$ has a well defined Laplace transform which is asymptotic to $\sum_{n\geq 0}\tilde{a}_n z^{-\tau_j-n}(1-e^{-2\pi i\tau_j}) = (1-e^{-2\pi i\tau_j})\sum_{n\geq 0}a_n\Gamma(\tau_j+n)z^{-\tau_j-n}$

$$(1-e^{-2\pi i\tau_j})\sum_{n\geq 0}a_n\Gamma(\tau_j+n)z^{-\tau_j-n} \sim \mathcal{L}^\theta_{\zeta_j}\hat{\phi}_j = e^{\alpha_j z}\mathcal{L}^\theta_{\zeta,\alpha_j}\hat{\phi}$$

hence

$$\mathcal{L}^{\theta}_{\zeta,\alpha_j}\hat{\phi} \sim e^{-\alpha_j z} z^{-\tau_j} \sum_{n\geq 0} a_n \Gamma(\tau_j + n) z^{-n} \propto \tilde{\Phi}_j(z) \tag{2.4}$$

2.4. **Corollary: construct an explicit holomorphic solution predicted by M.A.E.T..**

Corollary 2.2. The holomorphic solution which exists by M.A.E.T. can be characterized as the Borel–Laplace sum of $\tilde{\Phi}_j$.

3. Q-DIFFERENCE EQUATIONS

3.1. **Formal solutions vs actual solutions: paradigma of q-Borel Laplace summability.**

3.1.1. *From ODEs to q - difference equations.*

ODE	q-difference, $\lvert q\rvert > 1$
$\sum_{n\geq 0}^{d} a_n(z)\frac{\partial^n}{\partial z^n}\Phi = a \in \mathbb{C}(z)$ and $a_n \in \mathbb{C}(z)$	$\sum_{n\geq 0} a_{q,n}\sigma_q^n f = a_q \in \mathbb{C}(x)$ and $a_{q,n} \in \mathbb{C}(x)$
$z^2\Phi'(z) + \Phi(z) = z$ $\tilde{\Phi}(z) = \sum_{n\geq 0}(-1)^{n+1} n! z^{-n-1}$	$x\sigma_q f(x) + f(x) = x$ $\hat{f}(x) = \sum_{n\geq 0}(-1)^n q^{n(n+1)/2} x^{n+1}$
General notation	
$z\frac{d}{dz}$	$\sigma_q f(x) = f(qx)$
$\frac{d}{dz}$	$\delta_q f(x) = \frac{f(qx)-f(x)}{(q-1)x}$
$n!$	$q^{n(n+1)/2}$
e^z	$e_q(x) = \log(1/q)\Theta_q(x)$

Remark 3.1. Why $n!$ corresponds to $q^{-n(n+1)/2}$

$$\Gamma(n) = \int_0^{+\infty} e^{-t} t^n \frac{dt}{t} \tag{3.1}$$

$$q^{-n(n-1)/2} = \int_0^{+\infty} \frac{t^n}{e_q(t)} \frac{dt}{t} \tag{3.2}$$

There is also another interesting relation between q- factorial and $n!$

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n} \to_{q\to 1} n! \tag{3.3}$$

$$
\begin{tikzcd}
\hat{f}(x)=\sum_{n\geq 0} a_n x^n \in \mathbb{C}[\![x]\!]_{(q;1)} \arrow[rr, "\mathcal{B}_{(q;1)}"] & & \varphi(\xi)=\sum_{n\geq 0} a_n q^{-n(n-1)/2}\xi^n \arrow[dl, "\mathsf{L}^{[\lambda]}_{q;1}"] \\
& f(x) \arrow[ul, dashed, "\text{asymptotics}"] &
\end{tikzcd}
$$

3.2. **Borel regularity: Dreyfus's theorem.** Let $\mathcal{B}_\mu$ be the q- Borel transform for q-Gevrey μ series

$$
\mathcal{B}_\mu\colon \sum_{n\geq 0} a_n x^n \to \sum_{n\geq 0} a_n q^{-n(n-1)/(2\mu)}\xi^n \tag{3.4}
$$

and let $\mathsf{L}^{[\lambda]}_{\mu,\kappa}$ be the q- Laplace transform with parameters $\mu\in\mathbb{Q}_{>0}$, $\kappa\in\mathbb{N}^*$ ($\mathsf{L}^{[\lambda]}_{q;1} := \mathsf{L}^{[\lambda]}_{1,1}$)

$$
\mathsf{L}^{[\lambda]}_{\mu,\kappa}\varphi(x)=\frac{\mu}{\kappa}\sum_{l\in\kappa^{-1}\mathbb{Z}}\frac{\varphi(q^l\lambda)}{\Theta_{q^{1/\mu}}(\frac{q^{1/\mu+l}\lambda}{x})}\quad \lambda\in\mathbb{C}^*/q^{\kappa^{-1}\mathbb{Z}} \tag{3.5}
$$

Let $\mathbb{H}^{[\lambda]}_{\mu,\kappa}$ be the space of functions $\varphi\in\mathcal{M}(\mathbb{C}^*)$, such that there exists $\varepsilon>0$, $\Omega\subset\mathbb{C}$ connected

- $\bigcup_{l\in\kappa^{-1}\mathbb{Z}}\{x\in\mathbb{C}^* | |x-\lambda q^l|<\varepsilon|\lambda q^l|\}\subset\Omega$
- φ can be continued analytically in Ω with $q^{1/\mu}$ exponential growth

$$
|\varphi(\xi)| < C|\Theta_{|q|^{1/\mu}}(A|\xi|)|
$$

For every $\varphi(\xi)\in\mathbb{H}^{[\lambda]}_{\mu,\kappa}$, $\mathsf{L}^{[}_{\mu,\kappa}\lambda]\varphi(x)\in$.

Under suitable assumptions on the q-difference equation, the following result holds true

Theorem 3.2. Let $\hat{h}$ be a formal power series solution of a linear q-difference equation with coefficients in $\mathbb{C}(x)$. There exist $\kappa_1,\ldots,\kappa_r\in\mathbb{Q}_{>0}$, $n,K\in\mathbb{N}^*$ and a finite set $\Sigma\subset\mathbb{C}^*/q^{n^{-1}\mathbb{Z}}$, we may compute from the q-difference equation, such that for all $\lambda\in(\mathbb{C}^*/q^{n^{-1}\mathbb{Z}})\setminus\mathbb{Z}$,

$$
\mathsf{S}^{[\lambda]}(\hat{h}) := \mathsf{L}^{[\lambda]}_{\kappa_r,n}\circ\mathsf{L}^{[\lambda]}_{\kappa_{r-1},K}\circ\ldots\circ\mathsf{L}^{[\lambda]}_{\kappa_1,K}\circ\mathcal{B}_{\kappa_1}\circ\ldots\circ\mathcal{B}_{\kappa_r}\hat{h}
$$

is meromorphic on $\mathbb{C}^*$, and is solution of the same equation as $\hat{h}$. Moreover, $\mathsf{S}^{[\lambda]}(\hat{h})$ is asymptotic to $\hat{h}$ and, for $|x|$ close to 0 it has poles of order at most 1 that are contained in $\lambda q^{n-1}, n\in\mathbb{Z}$.

Idea of the proof: let $P(\hat{f})=\left[\sum_{n=0}^m a_n\sigma_q^n\right]\hat{f}$

$$
\begin{tikzcd}
 & P\hat{f} = a \arrow[d] & \\
f(x) \arrow[ur, red] & q-\text{difference equation} & \hat{f}(x) \arrow[ul] \arrow[d, "\mathcal{B}_{\kappa_1}\circ\ldots\circ\mathcal{B}_{\kappa_r}"] \\
\varphi(\xi)\in\mathbb{H}^{[\lambda]}_{\kappa_1,K},\ \text{for some } \lambda \arrow[u, "\mathsf{L}^{[\lambda]}_{\kappa_r,n}\circ\ldots\circ\mathsf{L}^{[\lambda]}_{\kappa_1,K}"] \arrow[rr, Leftrightarrow] & & \varphi(\xi)\in\mathbb{C}\{\xi\},\ \text{of slope } 0 \arrow[ul]
\end{tikzcd}
$$

3.3. **Summary.**

Formal series	
$\tilde{\Phi}(z) = \sum_{n\geq 0} a_n z^{-n-1} \in \mathbb{C}[\![z^{-1}]\!]_1$	$\hat{f}(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{C}[\![x]\!]_{(q;1)}$
Borel transform	
$\tilde{\phi}(\zeta) = \sum_{n\geq 0} a_n \frac{\zeta^n}{n!} \in \mathbb{C}\{\zeta\}$	$\varphi(\xi) = \sum_{n\geq 0} a_n q^{n(n-1)/2} \xi^n \in \mathbb{C}(\!(\xi)\!)$
Laplace transform	
$\mathcal{L}^{\theta}_{\zeta} \hat{\phi}(z) = \int_0^{+e^{i\theta}\infty} e^{-z\zeta} \hat{\phi}(\zeta) d\zeta \in \mathcal{O}(H_\theta)$	$\mathcal{L}^{\theta}_{q;1} \varphi(x) = \int_0^{e^{i\theta}\infty} \frac{\varphi(\xi)}{e_q(\frac{\xi}{x})} d\xi$ Θ function: $\Theta_q(x) = \sum_{n\in\mathbb{Z}} q^{-n(n+1)/2} x^n$ $\mathsf{L}^{[\lambda]}_{q;1} \varphi(x) = \sum_{n\geq 0} \frac{\varphi(\lambda q^n)}{\Theta_q(\frac{\lambda q^n}{x})} \in \mathcal{O}(H^\lambda)$, $\lambda \in \mathbb{C}^*$ $H^\lambda = D^*_r \setminus \{-\lambda q^n, n \in \mathbb{Z}\}$
Domain of definition	
$\hat{\phi}(\zeta)$ s.t. $\exists C, a > 0$ $\lvert\hat{\phi}(\zeta)\rvert < C e^{a\lvert\zeta\rvert}$, $\zeta \in S_\delta$ S_δ is an half-strip	$\varphi(\xi)$ s.t. $\exists C, a > 0$ $\lvert\varphi(\xi)\rvert < C\lvert\xi\rvert^a q^{\frac{1}{2}\left(\frac{\log\lvert\xi\rvert}{\log q}\right)^2}$, $\xi \in S_\delta \cap \mathbb{C}^*$ $\varphi(\xi) \in \mathbb{H}^{[\lambda]}_{\mu,\kappa}$
Gevrey asymptotics	
$\mathcal{L}^{\theta}_{\zeta} \hat{\phi}(z) \sim_1 \tilde{\Phi}(z)$	$\mathsf{L}^{[\lambda]}_{q;1} \varphi(x) \sim_{q;1} \hat{f}(x)$
Stokes phenomena	
varying θ, $\mathcal{L}^{\theta}$ jumps	varying $\mu \not\equiv_q \lambda$, $\mathsf{L}^{[\mu]}_{q;1}$ jumps
Borel regularity	
M.A.E.T: existence of hol. solutions F.,Feynes : BL sum gives an actual solution *slight functions*	[Praagman, 86]: existence of merom. solutions [Dreyfus, 14]: q–BL gives an actual solution *slope* 0 operators