Exercise 3.20

Aaron Fenyes

June 2, 2022

Let $\mathcal{D}$ be the noncommutative $\mathbb{C}$-algebra generated by the number operator N and the lowering operator A, with the relation $NA = A(N-1)$. Let N act as $t\frac{\partial}{\partial t}$ on $\mathbb{C}(\!(t)\!)$ and $\zeta\frac{\partial}{\partial \zeta}$ on $\mathbb{C}(\!(\zeta)\!)$. Let A act as t^{-1} on $\mathbb{C}(\!(t)\!)$ and

$$\begin{aligned} \frac{\zeta^n}{n!} &\mapsto \frac{\zeta^{n-1}}{(n-1)!} && n > 0 \\ \zeta^n &\mapsto \zeta^{n-1} && n \leq 0 \end{aligned}$$

on $\mathbb{C}(\!(\zeta)\!)$. These actions commute with the physicist's Borel transform

$$\begin{aligned} \mathcal{B}_{\text{ph}} \colon \mathbb{C}[\![t]\!] &\to \mathbb{C}[\![\zeta]\!] \\ t^n &\mapsto \frac{\zeta^n}{n!} \end{aligned}$$

whenever both paths through the commutative square are well-defined.

Exercise. Suppose $f \in \mathbb{C}[\![t]\!]$ is annihilated by $D \in \mathcal{D}$. Prove that $\mathcal{B}_{\text{ph}} f \in \mathbb{C}[\![\zeta]\!]$ is annihilated by the operator $\mathcal{F}(D)$ given by the alternate representation

$$\begin{aligned} \mathcal{F} \colon \mathcal{D} &\to \left\{\mathbb{C}[\![\zeta]\!] \to \mathbb{C}(\!(\zeta)\!)\right\} \\ A &\mapsto \tfrac{\partial}{\partial \zeta} \\ N &\mapsto \zeta\tfrac{\partial}{\partial \zeta}. \end{aligned}$$

Solution. Moving A to the right, we can write

$$\begin{aligned} D &= P_0(N)\, A^n + P_1(N)\, A^{n-1} + \ldots + P_{n-1}(N)\, A + P_n(N) \\ \mathcal{F}(D) &= P_0(N) \left(\tfrac{\partial}{\partial \zeta}\right)^n + P_1(N) \left(\tfrac{\partial}{\partial \zeta}\right)^{n-1} + \ldots + P_{n-1}(N)\, \tfrac{\partial}{\partial \zeta} + P_n(N) \end{aligned}$$

in terms of polynomials $P_0, \ldots, P_n$. Let Π be the projection $\mathbb{C}(\!(\zeta)\!) \to \mathbb{C}[\![\zeta]\!]$ in the monomial basis. Notice that $\left(\frac{\partial}{\partial \zeta}\right)^k$ and ΠA^k act the same on elements of $\mathbb{C}[\![\zeta]\!]$. Since Π commutes with N, it follows that $\mathcal{F}(D) = \Pi D$. Since the action of $\mathcal{D}$ commutes with $\mathcal{B}_{\text{ph}}$, the assumption that $Df = 0$ implies that

$$\begin{aligned} \mathcal{F}(D)\, \mathcal{B}_{\text{ph}} f &= \Pi D \mathcal{B}_{\text{ph}} f \\ &= \Pi \mathcal{B}_{\text{ph}} D f \\ &= 0. \end{aligned}$$

□

Note from Alex: For holomorphic functions, the correct statement might be that $Df = 0$ *implies* $\frac{\partial}{\partial\zeta}\mathcal{F}(D)\,\mathcal{B}_{ph}f = 0$. *The idea is that* $\mathcal{F}(A)$ *only differs from* A *because it annihilates constants, so a "translation error" in the differential equation can only leave constants left over.*