Regular singular Volterra equations on complex domains

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1   Introduction

1.1   Motivation

In its most basic form, the Laplace transform $\mathcal{L}$ turns L^1 functions of a real "position" variable ζ into holomorphic functions of a complex "frequency" variable z. Through identities like

$$\frac{\partial}{\partial z}\mathcal{L}\varphi = \mathcal{L}(z\varphi)$$
$$\mathcal{L}k\ \mathcal{L}f = k * \varphi$$
$$z^{-\lambda}\mathcal{L}\varphi = \mathcal{L}\,\partial_{,}^{-\lambda}\varphi,$$

where $\partial_{,}^{-\lambda}$ is the Riemann-Liouville fractional integral of order $\lambda \in (0, \infty)$, the Laplace transform pulls differential operators on the frequency domain back to Volterra integral operators on the position domain. The favorable regularity properties and comprehensive theory of Volterra equations can thus be brought to bear on differential equations.

If we complexify the position variable ζ, the functions on the position domain whose Laplace transforms satisfy a given differential equation will often extend holomorphically. [...]

1.2   Main result

Definition 1. *Consider a Volterra operator* $\mathcal{V}$ *of the form*

$$[\mathcal{V}\varphi](a) = m(a)\,\varphi(a) + \int_{\zeta=0}^{a} k(a, \cdot)\,\varphi\,d\zeta.$$

We'll say $\mathcal{V}$ *is* effectively finite-order *over a domain* Ω *if there are some* $c, c' \in (0, \infty)$ *for which*

- $|m(a)| \lesssim e^{c|\zeta(a)|}$ *over all* $a \in \Omega$

- $|k(a, a')| \lesssim e^{c|\zeta(a)|+c'|\zeta(a')|}$ *over all* $a, a' \in \Omega$

and at one of the following holds:

1. *m isn't the zero function.*

2. *m is the zero function, and there's some $\lambda \in (0, \infty)$ for which*

$$|k(a, a')| \in O_{\text{diag}}\left(|\zeta(a) - \zeta(a')|^{\lambda - 1}\right),$$

where "diag" means the diagonal in Ω^2.

We'll say $\mathcal{V}$ has order 0 in case 1, and order at least λ in case 2.

The result below might not be stated perfectly, but it gives at least a rough sense of what I expect. It's an improved version of the result stated in Section 3.1.3, which is copied from `airy-resurgence`. Note that `airy-resurgence` uses the opposite sign convention for the parameter σ in $\mathcal{H}L^{\infty,\sigma}$.

Theorem 1. *Let Ω be an acute-angled sector around 0 in the complex plane. I don't think it's really necessary for Ω to have a sharp point at 0, but the condition of being an acute-angled sector is sufficient and easy to state.*

Consider a Volterra operator

$$\mathcal{V} = p + \partial^{-1} \circ q + \mathcal{R}$$

over Ω which can be decomposed, as shown above, into the sum of a zeroth-order part p, a first-order part $\partial^{-1} \circ q$, and an effectively higher-order part $\mathcal{R}$. To be precise, $\mathcal{R}$ is supposed to be an integral operator of the form

$$[\mathcal{R}\varphi](a) = \int_{\zeta=0}^{a} k(a, \cdot)\, \varphi \, d\zeta$$

which is effectively finite-order over Ω in the sense of Definition 1, with order at least $1 + \epsilon$ for some $\epsilon \in (0, \infty)$.

Assume, furthermore, that:

- *There's some $c \in (0, \infty)$ for which $|p(a)| \lesssim e^{c|\zeta(a)|}$ and $|q(a)| \lesssim e^{c|\zeta(a)|}$ over all $a \in \Omega$.*

- *p and q extend holomorphically over $\zeta = 0$.*

- *p has a simple zero at $\zeta = 0$.*

Let p_1 and q_0 be the values of $\frac{\partial}{\partial \zeta}p$ and q, respectively, at $\zeta = 0$.

Under these conditions, let's try to solve the regular singular Volterra equation

$$\mathcal{V}f = 0.$$

The leading-order part of the equation,

$$\left[p_1\zeta + \partial^{-1} \circ q_0\right]f = 0,$$

has $\zeta^{\tau-1}$ as a solution, where $\tau = p_1/q_0$. If we guess that the function spaces defined in Section 2 are well suited for this problem, we might hope for the full equation to have a solution of the form

$$f = \zeta^{\tau-1} + f_*,$$

where $f_* \in \mathcal{H}L_Z^{\infty,1-\tau-\epsilon}(\Omega)$ for some $Z \in \mathbb{C}$. When $Z\Omega$ is in the right half-plane and $|Z|$ is large enough, there is in fact a unique solution of this form.

2 Function spaces for holomorphic Volterra operators

2.1 Weighted holomorphic L^∞ spaces

Throughout this paper, as described in Section 1.1, the "position" variable ζ will be the standard coordinate on $\mathbb{C}$. Take a simply connected open set $\Omega \subset \mathbb{C}$ that touches but doesn't contain $\zeta = 0$. Let $\mathcal{H}L^\infty(\Omega)$ be the space of bounded holomorphic functions on Ω with the supremum norm $\|\cdot\|_\infty$. For any $\sigma \in \mathbb{R}$ and $Z \in \mathbb{C}$, multiplying by $\zeta^\sigma e^{\mathrm{Re}(Z\zeta)}$ maps $\mathcal{H}L^\infty(\Omega)$ isomorphically onto another space of holomorphic functions on Ω. We'll call this space $\mathcal{H}L_Z^{\infty,\sigma}(\Omega)$ and give it the norm $\|f\|_{\infty,\sigma;Z} = \|\zeta^{-\sigma} e^{-\mathrm{Re}(Z\zeta)} f\|_\infty$, so that

$$\mathcal{H}L^\infty(\Omega) \to \mathcal{H}L_Z^{\infty,\sigma}(\Omega)$$
$$\varphi \mapsto \zeta^\sigma e^{\mathrm{Re}(Z\zeta)} \varphi$$

is an isometry. More generally,

$$\mathcal{H}L_Z^{\infty,\rho}(\Omega) \to \mathcal{H}L_Z^{\infty,\rho+\delta}(\Omega)$$
$$f \mapsto \zeta^\delta f$$

is an isometry for all $\rho, \delta \in \mathbb{R}$ and $Z \in \mathbb{C}$. This reduces to the previous statement when $\rho = 0$ and $Z = 0$.

2.2 Graded algebra structure

For each $\delta \in [0, \infty)$, the functions in $\mathcal{H}L^{\infty,\rho}(\Omega)$ belong to $\mathcal{H}L^{\infty,\rho-\delta}(\Omega)$ too, and the inclusion map $\mathcal{H}L^{\infty,\rho}(\Omega) \hookrightarrow \mathcal{H}L^{\infty,\rho-\delta}(\Omega)$ has norm $\|\zeta^\delta\|_\infty$.

$$\begin{aligned}
\|f\|_{\infty,\rho+\delta,Z} &= \|\zeta^{-(\rho-\delta)} e^{-\mathrm{Re}(Z\zeta)} f\|_\infty \\
&= \|\zeta^\delta \zeta^{-\rho} e^{-\mathrm{Re}(Z\zeta)} f\|_\infty \\
&= \|\zeta^\delta\|_\infty \|\zeta^{-\rho} e^{-\mathrm{Re}(Z\zeta)} f\|_\infty \qquad \text{(Banach algebra)} \\
&\le \|\zeta^\delta\|_\infty \|f\|_{\infty,\rho;Z}
\end{aligned}$$

Since $\mathcal{H}L^\infty(\Omega)$ is a Banach algebra, the function space $\mathcal{H}L_Z^{\infty,-\infty}(\Omega) := \bigcup_{\sigma \in \mathbb{R}} \mathcal{H}L_Z^{\infty,\sigma}(\Omega)$ is a graded algebra, with a different norm on each grade. For each $\rho, \delta \in \mathbb{R}$, multiplication by a function $m \in \mathcal{H}L^{\infty,\delta}(\Omega)$ gives a map $\mathcal{H}L^{\infty,\rho}(\Omega) \to \mathcal{H}L^{\infty,\rho+\delta}(\Omega)$ with norm $\|m\|_{\infty,\delta}$.

2.3 Norms of effectively finite-order operators

- Say $|m(a)| \lesssim e^{c|\zeta(a)|}$ over all $a \in \Omega$. For any $f \in \mathcal{H}L_Z^{\infty,\rho}$,

$$\|mf\|_{\infty,\sigma;Z} =$$

  which is effectively finite-order over Ω in the sense of Definition 1, with order at least $1 + \epsilon$ for some $\epsilon \in (0, \infty)$.

3   Integro-differential equations [from `airy-resurgence`]

Note that `airy-resurgence` uses the opposite sign convention for the parameter
σ in $\mathcal{H}L^{\infty,\sigma}$.

3.1   Existence of solutions

3.1.1   Algebraic integral operators

Take a simply connected open set $\Omega \subset \mathbb{C}$ that touches but doesn't contain $\zeta = 0$. Let
$\mathcal{H}L^{\infty}(\Omega)$ be the space of bounded holomorphic functions on Ω with the supremum norm
$\|\cdot\|_{\infty}$. For any $\sigma \in \mathbb{R}$, multiplying by $\zeta^{-\sigma}$ maps $\mathcal{H}L^{\infty}(\Omega)$ isomorphically onto another
space of holomorphic functions on Ω. We'll call this space $\mathcal{H}L^{\infty,\sigma}(\Omega)$ and give it the norm
$\|f\|_{\infty,\sigma} = \|\zeta^{\sigma} f\|_{\infty}$, so that

$$\mathcal{H}L^{\infty}(\Omega) \to \mathcal{H}L^{\infty,\sigma}(\Omega)$$

$$\phi \mapsto \zeta^{-\sigma}\phi$$

is an isometry. More generally,

$$\mathcal{H}L^{\infty,\rho}(\Omega) \to \mathcal{H}L^{\infty,\rho+\delta}(\Omega)$$

$$f \mapsto \zeta^{-\delta} f$$

is an isometry for all $\rho \in \mathbb{R}$ and $\delta \in [0,\infty)$. This reduces to the previous statement when
$\rho = 0$. For each $\delta \in [0,\infty)$, the functions in $\mathcal{H}L^{\infty,\rho}(\Omega)$ belong to $\mathcal{H}L^{\infty,\rho+\delta}(\Omega)$ too, and the
inclusion map $\mathcal{H}L^{\infty,\rho}(\Omega) \hookrightarrow \mathcal{H}L^{\infty,\rho+\delta}(\Omega)$ has norm $\|\zeta^{\delta}\|_{\infty}$. Conceptually, $\|\zeta^{\delta}\|_{\infty}$ measures
of the size of Ω, so let's write it as M^{δ} with $M = \|\zeta\|_{\infty}$.

Since $\mathcal{H}L^{\infty}(\Omega)$ is a Banach algebra, the function space $\mathcal{H}L^{\infty,\infty}(\Omega) := \bigcup_{\sigma \in \mathbb{R}} \mathcal{H}L^{\infty,\sigma}(\Omega)$
is a graded algebra, with a different norm on each grade. For each $\rho, \delta \in \mathbb{R}$, multiplication
by a function $m \in \mathcal{H}L^{\infty,\delta}(\Omega)$ gives a map $\mathcal{H}L^{\infty,\rho}(\Omega) \to \mathcal{H}L^{\infty,\rho+\delta}(\Omega)$ with norm $\|m\|_{\infty,\delta}$.

We'll study integral operators $\mathcal{G} \colon \mathcal{H}L^{\infty,\rho}(\Omega) \to \mathcal{H}L^{\infty,\sigma}(\Omega)$ of the form

$$[\mathcal{G}f](a) = \int_{\zeta=0}^{a} g(a, \cdot)\, f\, d\zeta,$$

where the kernel g is an algebraic function over $\mathbb{C}^2$ which can be singular on Δ, the diagonal.[1]
To avoid ambiguity, we fix a branch of g to use at the start of the integration path. The
domain of g is a covering of $\mathbb{C}^2$ which can be branched over Δ. Continuing g around Δ
changes its phase by a root of unity, leaving its absolute value the same [**check**]. That
makes $|g|$ a well-defined function on $\mathbb{C}^2 \setminus \Delta$, which we can use to bound $\|\mathcal{G}\|$.

For each $a \in \mathbb{C}$, the expression $|g(a, \cdot)\, d\zeta|$ defines a *density* on $\Omega \setminus \{a\}$—a norm on the
tangent bundle which is compatible with the conformal structure. The square of a density
is a Riemannian metric. Let $\ell_{g,\Omega}^{\sigma,\rho}(a)$ be the distance from $\zeta = 0$ to a with respect to the

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[1]Thanks to Alex Takeda for suggesting this.

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density $|\zeta(a)^\sigma\, g(a,\cdot)\, \zeta^{-\rho}\, d\zeta|$ on $\Omega \smallsetminus \{a\}$. The bound

$$\left|[\zeta^\sigma \mathcal{G} f](a)\right| \leq \left|\zeta(a)^\sigma \int_{\zeta=0}^{a} g(a,\cdot)\, f\, d\zeta\right|$$

$$\leq \int_{\zeta=0}^{a} |\zeta^\rho f|\, |\zeta(a)^\sigma\, g(a,\cdot)\, \zeta^{-\rho}\, d\zeta|$$

$$\leq \|f\|_{\infty,\rho} \int_{\zeta=0}^{a} |\zeta(a)^\sigma\, g(a,\cdot)\, \zeta^{-\rho}\, d\zeta|$$

holds for any integration path. Taking the infimum over all paths, we see that

$$\left|[\mathcal{G} f](a)\right| \leq \ell_{g,\Omega}^{\sigma,\rho}(a)\, \|f\|_{\infty,\rho}.$$

so $\|\mathcal{G}\| \leq \sup_{a\in\Omega} \ell_{g,\Omega}^{\sigma,\rho}(a)$. Crucially, we can always make $\|\mathcal{G}\|$ a contraction by restricting Ω.

3.1.2   The example of fractional integrals

Setting $g(a,a') = (\zeta(a) - \zeta(a'))^{-\lambda-1}$ with $\lambda \in (-\infty, 0)$, we get the fractional integral $\partial_{\zeta\ \text{from } 0}^{\lambda}$. The shortest path from $\zeta = 0$ to a with respect to $|\zeta(a)^{\rho+\lambda}\, g(a,\cdot)\, \zeta^{-\rho}\, d\zeta|$ is the same as the shortest path with respect to $|d\zeta|$ [check]. It follows that

$$\ell_{g,\Omega}^{\sigma,\rho}(a) = \int_0^{|\zeta(a)|} |\zeta(a)|^{\rho+\lambda}\, (|\zeta(a)| - r)^{-\lambda-1}\, r^{-\rho}\, dr$$

$$= |\zeta(a)|^{\rho+\lambda} \int_0^1 (|\zeta(a)| - |\zeta(a)|t)^{-\lambda-1}\, (|\zeta(a)|t)^{-\rho}\, |\zeta(a)|\, dt$$

$$= |\zeta(a)|^{\rho+\lambda-\lambda-1-\rho+1} \int_0^1 (1-t)^{-\lambda-1}\, t^{-\rho}\, dt$$

$$= \int_0^1 (1-t)^{-\lambda-1}\, t^{-\rho}\, dt$$

$$= B(-\lambda,\, 1-\rho).$$

The beta function B can be written more explicitly as

$$B(-\lambda,\, 1-\rho) = \frac{\Gamma(-\lambda)\,\Gamma(1-\rho)}{\Gamma(1-\lambda-\rho)}.$$

Now we can see that for each $\lambda \in (-\infty, 0)$ and $\rho \in \mathbb{R}$, the fractional integral $\partial_{\zeta\ \text{from } 0}^{\lambda}$ maps $\mathcal{H}L^{\infty,\rho}(\Omega)$ into $\mathcal{H}L^{\infty,\rho+\lambda}(\Omega)$, with norm $\|\partial_{\zeta\ \text{from } 0}^{\lambda}\| \leq B(-\lambda,\, 1-\rho)$.

3.1.3   Fractional integral equations near a regular singular point

Angeliki: Maybe Kato–Rellich perturbation theory can give existence immediately. It might not give uniqueness, though.

Consider an integral operator $\mathcal{J}$ of the form

$$p + \partial_{\zeta\ \text{from } 0}^{-1} \circ q + \sum_{\lambda\in\Lambda} \partial_{\zeta\ \text{from } 0}^{\lambda} \circ r_\lambda,$$

where:

- p is a function in $\mathcal{H}L^{\infty,-1}(\Omega)$ that extends holomorphically over $\zeta = 0$, and its derivative at $\zeta = 0$ is non-zero.

- q is a function in $\mathcal{H}L^{\infty}(\Omega)$ that extends holomorphically over $\zeta = 0$.

- r_λ are functions in $\mathcal{H}L^{\infty}(\Omega)$.

- Λ is a countable subset of $(-\infty, -1)$ whose supremum is less than -1.

Our demand that p and q have convergent power series at $\zeta = 0$ can probably be relaxed; having convergent Novikov series, for example, should be enough. We could also probably replace $\partial_{\zeta \text{ from } 0}^{-1}$ with $\partial_{\zeta \text{ from } 0}^{-1+\delta} \circ \zeta^\delta$ for some $\delta \in [0,1)$, or adjust the $\partial_{\zeta \text{ from } 0}^{\lambda}$ similarly.

We want to solve the equation $\mathcal{J}f = 0$. Let's look for a solution of the form $f = \zeta^{\tau-1} + \tilde{f}$ with $\tau \in (0, \infty)$ and $\tilde{f} \in \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega)$ for some $\epsilon \in (0,1]$. When ϵ is small enough that $\Lambda \subset (-\infty, -1-\epsilon]$, we'll see that we can always find such a solution, as long as we're willing to shrink Ω. In fact, there's exactly one such solution. [Add convergence conditions for ∂^λ terms.]

Let p_0' and q_0 be the values of $\frac{\partial}{\partial \zeta} p$ and q, respectively, at $\zeta = 0$. We're assuming that p and q extend holomorphically over $\zeta = 0$, and the additional assumption that $p \in \mathcal{H}L^{\infty,-1}(\Omega)$ implies that p has a first-order zero at $\zeta = 0$.

Since p and q extend holomorphically over $\zeta = 0$, and p vanishes at $\zeta = 0$, we can write

$$p = p_0'\zeta + \tilde{p}$$
$$q = q_0 \;\; + \tilde{q}$$

with $\tilde{p} \in \mathcal{H}L^{\infty,-2}(\Omega)$ and $\tilde{q} \in \mathcal{H}L^{\infty,-1}(\Omega)$. Then we have

$$\mathcal{J} = p_0'\zeta + q_0\, \partial_{\zeta \text{ from } 0}^{-1} + \tilde{\mathcal{J}}$$

with

$$\tilde{\mathcal{J}} = \tilde{p} + \partial_{\zeta \text{ from } 0}^{-1} \circ \tilde{q} + \sum_{\lambda \in \Lambda} \partial_{\zeta \text{ from } 0}^{\lambda} \circ r_\lambda$$

For any $\tau \in (0, \infty)$,

$$\mathcal{J}\zeta^{\tau-1} = (p_0' + q_0/\tau)\,\zeta^\tau + \tilde{\mathcal{J}}\zeta^{\tau-1}.$$

Setting $\tau = -q_0/p_0'$ makes the first term vanish, leaving

$$\mathcal{J}\zeta^{\tau-1} = \tilde{\mathcal{J}}\zeta^{\tau-1}.$$

Then the equation $\mathcal{J}f = 0$ becomes

$$0 = \tilde{\mathcal{J}}\zeta^{\tau-1} + \mathcal{J}\tilde{f}$$
$$0 = \tilde{\mathcal{J}}\zeta^{\tau-1} + \left[p_0'\zeta + q_0\,\partial_{\zeta \text{ from } 0}^{-1} + \tilde{\mathcal{J}}\right]\tilde{f}$$
$$-p_0'\zeta\tilde{f} = \tilde{\mathcal{J}}\zeta^{\tau-1} + \left[q_0\,\partial_{\zeta \text{ from } 0}^{-1} + \tilde{\mathcal{J}}\right]\tilde{f}$$
$$\tilde{f} = \left[-\tfrac{1}{p_0'}\,\zeta^{-1} \circ \tilde{\mathcal{J}}\right]\zeta^{\tau-1} + \left[\tau\,\zeta^{-1} \circ \partial_{\zeta \text{ from } 0}^{-1} - \tfrac{1}{p_0'}\,\zeta^{-1} \circ \tilde{\mathcal{J}}\right]\tilde{f}. \tag{1}$$

$$\mathcal{H}L^{\infty,\rho}(\Omega) \xrightarrow[\|\tilde{p}\|_{\infty,-2}]{\tilde{p}} \mathcal{H}L^{\infty,\rho-2}(\Omega) \begin{array}{c} \xrightarrow{\|\zeta\|_\infty} \mathcal{H}L^{\infty,\rho-1}(\Omega) \\ \xrightarrow{\|\zeta\|_\infty^{1-\epsilon}} \mathcal{H}L^{\infty,\rho-1-\epsilon}(\Omega) \end{array}$$

$$\mathcal{H}L^{\infty,\rho}(\Omega) \xrightarrow[\|\tilde{q}\|_{\infty,-1}]{\tilde{q}} \mathcal{H}L^{\infty,\rho-1}(\Omega) \xrightarrow[B(1,2-\rho)=\frac{1}{2-\rho}]{\partial^{-1}} \mathcal{H}L^{\infty,\rho-2}(\Omega) \begin{array}{c} \xrightarrow{\|\zeta\|_\infty} \mathcal{H}L^{\infty,\rho-1}(\Omega) \\ \xrightarrow{\|\zeta\|_\infty^{2}} \mathcal{H}L^{\infty,\rho}(\Omega) \end{array}$$

$$\mathcal{H}L^{\infty,\rho}(\Omega) \xrightarrow[\|r_\lambda\|_\infty]{r_\lambda} \mathcal{H}L^{\infty,\rho}(\Omega) \xrightarrow[B(-\lambda,1-\rho)]{\partial^\lambda} \mathcal{H}L^{\infty,\rho+\lambda}(\Omega) \begin{array}{c} \xrightarrow{\|\zeta\|_\infty^{-1-\lambda}} \mathcal{H}L^{\infty,\rho-1}(\Omega) \\ \xrightarrow{\|\zeta\|_\infty^{-1-\epsilon-\lambda}} \mathcal{H}L^{\infty,\rho-1-\epsilon}(\Omega) \end{array}$$

$$\mathcal{H}L^{\infty,\rho}(\Omega) \xrightarrow[B(1,1-\rho)=\frac{1}{1-\rho}]{\partial^{-1}} \mathcal{H}L^{\infty,\rho-1}(\Omega) \xrightarrow[\|\zeta^{-1}\|_{\infty,1}=1]{\zeta^{-1}} \mathcal{H}L^{\infty,\rho}(\Omega)$$

From [**our previous discussion**], we can work out that

$$\tilde{\mathcal{J}} \colon \mathcal{H}L^{\infty,1-\tau}(\Omega) \to \mathcal{H}L^{\infty,-\tau-\epsilon}(\Omega) \text{ with}$$
$$\|\tilde{\mathcal{J}}\| \le \left(\|\tilde{p}\|_{\infty,-2} + \tfrac{1}{1+\tau} \|\tilde{q}\|_{\infty,-1} \right) M^{1-\epsilon} + \sum_{\lambda\in\Lambda} B(-\lambda,\tau) \|r_\lambda\|_\infty M^{-1-\epsilon-\lambda}$$

and

$$\tilde{\mathcal{J}} \colon \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega) \to \mathcal{H}L^{\infty,-\tau-\epsilon}(\Omega) \text{ with}$$
$$\|\tilde{\mathcal{J}}\| \le \left(\|\tilde{p}\|_{\infty,-2} + \tfrac{1}{1+\tau+\epsilon} \|\tilde{q}\|_{\infty,-1} \right) M + \sum_{\lambda\in\Lambda} B(-\lambda,\tau+\epsilon) \|r_\lambda\|_\infty M^{-1-\lambda}.$$

Since $\|\zeta^{-1}\|_{\infty,1} = 1$, it follows that

$$\zeta^{-1} \circ \tilde{\mathcal{J}} \colon \mathcal{H}L^{\infty,1-\tau}(\Omega) \to \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega) \text{ with}$$
$$\|\zeta^{-1} \circ \tilde{\mathcal{J}}\| \le \left(\|\tilde{p}\|_{\infty,-2} + \tfrac{1}{1+\tau} \|\tilde{q}\|_{\infty,-1} \right) M^{1-\epsilon} + \sum_{\lambda\in\Lambda} B(-\lambda,\tau) \|r_\lambda\|_\infty M^{-1-\epsilon-\lambda} \quad (2)$$

and

$$\zeta^{-1} \circ \tilde{\mathcal{J}} \circlearrowright \mathcal{H}L^{\infty,1-\tau-\epsilon}(\Omega) \text{ with}$$
$$\|\zeta^{-1} \circ \tilde{\mathcal{J}}\| \le \left(\|\tilde{p}\|_{\infty,-2} + \tfrac{1}{1+\tau+\epsilon} \|\tilde{q}\|_{\infty,-1} \right) M + \sum_{\lambda\in\Lambda} B(-\lambda,\tau+\epsilon) \|r_\lambda\|_\infty M^{-1-\lambda}. \quad (3)$$

We can also see that

$$\tau \, \zeta^{-1} \circ \partial_{\zeta \text{ from } 0}^{-1} \circlearrowleft \mathcal{H}L^{\infty, 1-\tau-\epsilon}(\Omega) \text{ with}$$

$$\left\| \tau \, \zeta^{-1} \circ \partial_{\zeta \text{ from } 0}^{-1} \right\| = \tfrac{\tau}{\tau+\epsilon} < 1. \tag{4}$$

Now, let's return to equation 1, which tells us that $f = \zeta^{\tau-1} + \tilde{f}$ satisfies $\mathcal{J}f = 0$ when [and only when?] $\tilde{f}$ is a fixed point of the affine map $\mathcal{A}(\cdot) + b$, where

$$\mathcal{A} = \tau \, \zeta^{-1} \circ \partial_{\zeta \text{ from } 0}^{-1} - \tfrac{1}{p'_0} \, \zeta^{-1} \circ \tilde{\mathcal{J}}$$

$$b = \left[-\tfrac{1}{p'_0} \, \zeta^{-1} \circ \tilde{\mathcal{J}} \right] \zeta^{\tau-1}$$

Choosing $\epsilon \in (0, 1]$ so that $\Lambda \subset (-\infty, -1 - \epsilon]$ has given us the domain and codomain statements in bounds 2 and 3, which tell us that $\mathcal{A}(\cdot) + b$ sends $\mathcal{H}L^{\infty, 1-\tau-\epsilon}(\Omega)$ into itself. We'll show that when Ω is small enough, $\mathcal{A}(\cdot) + b$ contracts $\mathcal{H}L^{\infty, 1-\tau-\epsilon}(\Omega)$, and thus—by the contraction mapping theorem—has a unique fixed point.

An affine map is a contraction if and only if its linear part is a contraction. We know from bound 4 that $\tau \, \zeta^{-1} \circ \partial_{\zeta \text{ from } 0}^{-1}$ contracts $\mathcal{H}L^{\infty, 1-\tau-\epsilon}(\Omega)$. Since the supremum of Λ is less than -1, all the powers of $M = \|\zeta\|_\infty$ in bound 3 are positive. Thus, by shrinking Ω, we can make the norm of $\zeta^{-1} \circ \tilde{\mathcal{J}}$ on $\mathcal{H}L^{\infty, 1-\tau-\epsilon}(\Omega)$ as small as we want—small enough to make $\mathcal{A}$ a contraction.