Resurgence of the Airy function and other exponential integrals

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1 Introduction

1.1 The unreasonable effectiveness of Borel summation

You can often find a formal power series [double-check that this matches the τ from the position domain]

$$\tilde{\Phi} = \frac{c_0}{z^\tau} + \frac{c_1}{z^{\tau+1}} + \frac{c_2}{z^{\tau+2}} + \frac{c_3}{z^{\tau+3}} + \dots,$$

with $\tau \in (0, 1]$, that looks or acts like a solution to a problem whose actual solutions are holomorphic functions of z. For example, if you want to understand how the solutions of the holomorphic ordinary differential equation (ODE)

$$\textbf{[one-third Bessel equation, rescaled to match integral example]} \tag{1}$$

behave near $z = \infty$, you might start by looking for formal *transmonomial* solutions $e^{-\alpha z}\,\tilde{\Phi}$, where $\tilde{\Phi}$ is a formal power series of the kind above. Setting $\alpha = -\frac{1}{12}$ and $\tau = \frac{1}{2}$ gives a well-behaved recurrence relation for $\tilde{\Phi}$, which produces the solution [check]

$$e^{z/12}\left[\frac{(-1)!!}{z^{1/2}} + \frac{5}{6}\cdot\frac{1!!}{z^{3/2}} + \frac{385}{216}\cdot\frac{3!!}{z^{5/2}} + \frac{17017}{3888}\cdot\frac{5!!}{z^{7/2}} + \dots\right] \tag{2}$$

and its constant multiples. As another example, you might rewrite the integral

$$\Phi(z) = \int_\Lambda \exp\left[-\tfrac{1}{12}z\left(4u^3 - 3u\right)\right]\,du$$

$$\Phi(z) = \int_\Lambda \exp\left[-z\left(\tfrac{1}{3}u^3 - \tfrac{1}{4}u\right)\right]\,du$$

as

$$e^{z/12}\int_{-\infty}^{\infty} e^{-z\tau^2/2}\left[1 - \frac{2}{3}\tau + \frac{5}{6}\tau^2 - \frac{32}{27}\tau^3 + \frac{385}{216}\tau^4 - \frac{224}{81}\tau^5 + \frac{17017}{3888}\tau^6 - \dots\right]d\tau$$

using the substitution $\frac{1}{2}\tau^2 = \frac{1}{3}u^3 - \frac{1}{4}u + \frac{1}{12}$. Naïvely integrating term by term, you again get the transmonomial (2).

$$
\begin{aligned}
&e^{z/12} z^{-1/2} \left[(-1)!! + \frac{5}{6}\, 1!!\, z^{-1} + \frac{385}{216}\, 3!!\, z^{-2} + \ldots \right] \\
&= e^{z/12} z^{1/2} \left[\frac{(-1)!!}{z} + \frac{5}{6} \cdot \frac{1!!}{z^2} + \frac{385}{216} \cdot \frac{3!!}{z^3} + \frac{17017}{3888} \cdot \frac{5!!}{z^4} + \ldots \right] \\
&= e^{z/12} \left[\frac{(-1)!!}{z^{1/2}} + \frac{5}{6} \cdot \frac{1!!}{z^{3/2}} + \frac{385}{216} \cdot \frac{3!!}{z^{5/2}} + \frac{17017}{3888} \cdot \frac{5!!}{z^{7/2}} + \ldots \right]
\end{aligned}
$$

Once you have the formal solution $\tilde{\Phi}$, you might try to get an actual solution by applying *Borel summation*, which turns a formal power series into a function asymptotic to it. Borel summation works in three steps.

1. Thinking of z as a "frequency variable," we take the formal inverse Laplace transform of $\tilde{\Phi}$, producing a formal power series $\tilde{\phi}$ in a new "position variable" ζ.

2. With luck, $\tilde{\phi}$ has a positive radius of convergence. In this case, we say $\tilde{\Phi}$ is 1-*Gevrey*. We sum $\tilde{\phi}$ to get a holomorphic function $\hat{\phi}$ on a neighborhood of $\zeta = 0$. Then, by analytic continuation, we expand the domain of $\hat{\phi}$ to a Riemann surface B with a distinguished 1-form λ—the continuation of $d\zeta$. [Nikita has a complementary picture where the Borel plane is the cotangent fiber? Ask more about this.]

3. With more luck, $\hat{\phi}$ grows slowly enough along an infinite ray $b + e^{i\theta}[0, \infty)$ [change, explain, or link to notation] for its Laplace transform $\mathcal{L}^\theta_b \hat{\phi}$ [link to definition] to be a holomorphic function of z, well-defined on some sector of the frequency plane. In this case, we say $\tilde{\Phi}$ is *Borel-summable*, and we call $\mathcal{L}^\theta_b \hat{\phi}$ its *Borel sum* at b.

The Borel summation process is summarized in the following diagram.

$$
\begin{tikzcd}
 & \text{problem} \arrow[dl, dotted, mapsto] \arrow[dr, mapsto, "\text{formally solves}"] & \\
\hat{\Phi} & & \tilde{\Phi} \arrow[d, mapsto, "\mathcal{B}"] \\
\hat{\phi} \arrow[u, mapsto, "\mathcal{L}^\theta_b"] & & \tilde{\phi}(\zeta) \arrow[ll, mapsto, "\text{sum}"]
\end{tikzcd}
$$

- You can often find a formal power series $\tilde{\Phi} = \ldots$ that looks or acts like a solution to a problem whose actual solutions are holomorphic functions of z. For example...
  - Exponential integral: naïve saddle point expansion.
  - ODE: formal solution. Existence theorem in [**?**]?
  - Feynman diagram series?

- Once you have $\tilde{\Phi}$, you might try applying *Borel summation*, which turns a formal power series into a function asymptotic to it.
  - Borel summation work in three steps. First we turn the formal power series $\tilde{\Phi}$ [$\tilde{\Phi}(z)$] in the "frequency variable" z into a formal power series $\tilde{\phi}$ in a new "position variable" ζ.
  - The series $\tilde{\phi}$ turns out to have a [finite] positive radius of convergence, so it defines a holomorphic function $\hat{\phi}$. By analytic continuation, we can expand the domain of $\hat{\phi}$ to a Riemann surface B with a distinguished 1-form λ—the continuation of $d\zeta$.
  - If $\hat{\phi}$ grows slowly enough along an infinite ray $\Gamma_b^\theta := b + e^{i\theta}[0, \infty)$ [decide notation], its Laplace transform $\mathcal{L}_b^\theta \hat{\phi} := \ldots$ is a holomorphic function of z, well-defined on some sector of the frequency plane. In this case, we say $\tilde{\Phi}$ is *Borel-summable*, and we call $\mathcal{L}_b^\theta \hat{\phi}$ its *Borel sum* at b.
  - [Draw the square!]
- Different functions can be asymptotic to the same power series, and Borel summation picks one of them.
- In many cases, it picks correctly, producing an actual solution to your problem.
  - The question of how that happens is the starting point for this paper.
  - (In both of the cases that we study, the Borel sum of $\tilde{\Phi}$ is always taken at a zero of λ, rather than an arbitrary point in B. For ODEs, our treatment explains why the zeroes of λ play a special role.)

1.2 A new perspective: Borel regularity

- The central goal of this paper is to present a new perspective on Borel summation which helps explain why it works for at least two kinds of problems.
  - The first problem is evaluating a certain kind of exponential integral: a one-dimensional *thimble integral*.
  - The second problem is solving a certain kind of ODE.
  - These two problems are closely linked. By playing with derivatives of an exponential integral, you can often find a linear ODE that the integral satisfies. Conversely, for many classical ODEs, there are useful bases of exponential integral solutions.
- Suppose you have a holomorphic function Φ which is asymptotic to a formal power series $\tilde{\Phi}$ as z goes to ∞ along a given ray.
- If $\tilde{\Phi}$ is Borel-summable, as described in Section 1.1, its Borel sum is a new holomorphic function.
- Since different functions can be asymptotic to the same power series, taking the Borel sum of the asymptotic series of Φ must smooth away some details. We'll therefore call this process *Borel regularization*.

- We'll say Φ is *Borel-regularizable* if it's asymptotic to a Borel-summable power series.
- We'll say Φ is *Borel regular* if it's Borel-regularizable and Borel regularization leaves it unchanged.
- We have the following vector space inclusions:

  Borel regular functions $\subset$ Borel-regularizable functions $\subset$ holomorphic functions.

- Borel regular functions have been characterized in the literature before; Watson showed a century ago [Watson] that a function Φ is Borel regular whenever there's a constant $c \in (0, \infty)$ with
  $$|\Phi(z) - \sum_{k=0}^{N} c_k z^{-k}| \leq c^{N+1} N! |z|^{-N}$$
  over all orders N and all z in a wide enough wedge around infinity.

  Nevanlinna's improvement of Watson's theorem (Sokal, "An improvement of Watson's theorem on Borel summability") tells us that for a function Φ with a well-defined inverse Laplace transform ϕ, the following conditions are equivalent [double check]:

  - Φ is Borel regular.
  - Φ is approximated well, in a certain sense (a weak version of being "Gevrey-asymptotic"), by its asymptotic series.
  - ϕ grows at most exponentially along the Laplace transform ray.
- When its domain is restricted to the space of Gevrey power series, Borel summation is invertible. [Need to find a good reference for this, because a small change in the statement would mess up our conclusion. Does `arXiv:2112.08792`, §A.2 have the statement we want?] Thus, Borel regularization acts as a projection operator on the space of Borel-regularizable functions.[Veronica: Do you mean that if $\Phi \sim \tilde{\Phi}$ and $\tilde{\Phi}$ is Gevrey and Borel summable, then $\Phi = \mathcal{L} \circ \mathcal{B}\tilde{\Phi}$ (i.e. it is Borel regular)? P.Ramis proved that if Φ is Gevrey asymptotic to $\tilde{\Phi}$, then $\tilde{\Phi}$ is of Gevrey type. In addition, every Gevrey type series $\tilde{\Phi}$ is the Gevrey asymptotics of a holomorphic function. When it is defined, the Borel sum of $\tilde{\Phi}$ is the natural holomorphic function asymptotic to $\tilde{\Phi}$.]
- Borel regularity can help explain why Borel summation is so effective at solving certain kinds of problems, like the ones in Section 1.1.
- The formal solutions of a problem are typically supposed to generalize the actual solutions, in the sense that they include the asymptotic series of the actual solutions.
- In this case, every Borel regular solution can be found by taking the Borel sum of a formal solution.
- ...

1.3 Goals and Results

- The central goal of this paper is to lay out two kinds of problems where we can prove that the Borel sum of a formal power series solution is always an actual solution.
  - The first problem is evaluating a certain kind of exponential integral: a one-dimensional *thimble integral*.
  - The second problem is solving a certain kind of ODE.
  - These two problems are closely linked. By playing with derivatives of an exponential integral, you can often find a linear ODE that the integral satisfies. Conversely, for many classical ODEs, there are useful bases of exponential integral solutions.
  - (Does this touch the Picard-Lefshetz perspective? Betti / de Rham relationship: ODE is a connection, and exponential integrals give flat sections?)

- Clearly separate the parts of the theory that deal with holomorphic functions and formal power series.

- (Super-motivation: why do the zeroes of λ play a special role?) As part of the treatment, we've made use of some new perspectives on the Laplace transform.
  - **Geometric picture.** The spatial domain B is a translation surface. If $b \in B$ is non-singular, the frequency domain for $\mathcal{L}_b^\theta$ is T^*B_b. If b is a conical singularity, the frequency domain is more interesting, as we'll see in our main example.
  - **A new dictionary for ODEs.** The Laplace transform is often used to solve ODEs on the frequency domain by relating them to ODEs on the spatial domain. We find, however, that it's much easier and more natural to relate ODEs on the frequency domain to integral equations on the spatial domain. This clarifies why we take the Borel sums at zeroes of λ when we're trying to solve an ODE. [Veronica—I dont't see why]

- Our picture helps explain why it's useful to work on the Borel plane (the position domain).
  - Integral equations are more regular than differential equations.
  - A thimble integral in the frequency domain can be recast as the Laplace transform of a function in the position domain.

- Illustrate with detailed treatments of several examples.
  - Some have been discussed many times, using different approaches and conventions. We'll try to give an idea of how all these different treatments fit together. • The Airy function (Marino, Sauzin). • The anharmonic oscillator (Bender–Wu, Schiappa).
  - Others haven't been discussed much.

- Recently, resurgence theory (first developed by Écalle in the '80) has attracted interests in math and physics as a powerful alternative to Borel summability. Resurgence of lienar ODEs have been studied (see Costin slides for ReNewQuantum). Many results are also known for non linear ODEs (see Schiappa PI, Costin PI). For algebraic exponential integrals of the type we studied in this paper, resurgence of their asymptotic expansion can be understood geometrically (see Maxim's slides ReNewQuantum), however for more general exponential integrals (see examples in Maxim's talk) resurgence remains a conjecture. Despite their simplicity, our examples of linear ODEs and of exponential integrals show some features of resurgence and they are toy model to get a feeling on Écalle formalism.
- The examples give a place to compare more complicated formalisms like the Picard-Lefshetz (Morse theory) or Ecalle formalisms? [How do we work this into the introduction?]

1.4 Results

- what does it mean being Borel regular?
- when does it happen?
  - State new Borel regularity results
    * Linear, homogeneous ODE with regular singularity at 0 and irregular singularity at infinity [big idea in `airy-resurgence`]
      - Contextualize with previous work of Braaksma ("Multisummability and Stokes multipliers of linear meromorphic differential equations")
      - Also contextualize with Balser, Braaksma, Sibuya, and Ramis ("Multisummability of formal power series solutions of linear ordinary differential equations")
      - Also contextualize with Loday-Richaud
    * *Borel regularity* for **thimbles integrals** can be stated a the commutativity of the following diagram:

$$\begin{tikzcd} I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\nu \arrow[r, "\sim"] & \tilde{I}_\alpha(z) \arrow[d, "\mathcal{B}"] \\ \hat{\imath}_\alpha(\zeta) \arrow[u, "\mathcal{L}^\theta"'] \arrow[r, equal, "\text{sum}"'] & \tilde{\imath}_\alpha(\zeta) \end{tikzcd} \tag{3}$$

    * A priori, the Laplace transform of $\hat{\imath}_\alpha(\zeta)$ and $I_\alpha(z)$ have the same asymptotic behaviour in a given sector (indeed taking the asymptotic of $I_\alpha(z)$ we *loose* information); however Borel regularity guarantees that $I_\alpha(z) = \mathcal{L}^\theta \hat{\imath}_\alpha$ in a given sector.
    * fractional derivative formula
    * Conjecturally, we expect $\hat{\varphi}_\alpha(\zeta)$ to have simple singularities.

- * in the examples, $\hat{\varphi}_\alpha(\zeta)$ turn out be an hypergeometric function of type ${}_pF_{p-1}$ where p is the number of critical values.
  * We expect that hypergeometric functions play a special role in resurgence theory as they may always appear when there are only finitely many singularities.
  * maybe we can say more about algebraic hypergeomtric functions
- Recall Watson condition (old): Let R_N be the difference between a function and the partial sum

$$\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \ldots + \frac{\varphi_{N-2}}{z^{N-1}}$$

of its asymptotic series. Watson showed a century ago that the function is Borel regular whenever there's a constant $c \in (0, \infty)$ with

$$|R_N| \leq \frac{c^{N+1} N!}{|z|^N}$$

over all orders N and all z in a wide enough wedge around infinity.

1.5 Why does Borel resummation work?

Borel resummation is a way of turning a formal power series

$$\tilde{\varphi} = z^\sigma \left(\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \frac{\varphi_3}{z^4} + \ldots \right),$$

with $\sigma \in [0, 1)$, into a function which is asymptotic to $\tilde{\varphi}$ as $z \to \infty$. Different functions can be asymptotic to the same power series, and Borel resummation picks one of them, performing an implicit regularization [**arXiv:1705.03071, or maybe arXiv:1412.6614**]. When a function matches the Borel sum of its asymptotic series, we'll say it's *Borel regular*. Several familiar kinds of regularity imply Borel regularity, and shed light on why it occurs.

- **Having a good asymptotic approximation**

  Let R_N be the difference between a function and the partial sum

$$\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \ldots + \frac{\varphi_{N-2}}{z^{N-1}}$$

  of its asymptotic series. Watson showed a century ago that the function is Borel regular whenever there's a constant $c \in (0, \infty)$ with

$$|R_N| \leq \frac{c^{N+1} N!}{|z|^N}$$

  over all orders N and all z in a wide enough wedge around infinity (Sokal, "An improvement of Watson's theorem on Borel summability"; Hardy, *Divergent Series*, Theorem 136; Watson, "A theory of asymptotic series," §8?).

- **Satisfying a singular differential equation**

- This is the setup. We restrict to ODEs with irregular singularity at ∞ and of Poincar rank 1:
$$\big[P(\partial_z) + \frac{1}{z}Q(\partial_z) + R(z^{-1})\big]\Psi = 0 \tag{4}$$
where $P(\lambda)$ is a (monic) degree d polynomial, $Q(\lambda)$ is a degree $d-1$ polynomial and $R(z^{-1}) = O(z^{-2})$ [Looking at the existence theorem in `airy.resurgence 2.1.3` we could apply this reasoning on the analytic side for more general equations, but this particular case makes it easier to talk about the formal side as well]. Furthermore we assume P has simple zeros $P(\alpha_j) = 0$, $j = 1, ..., d$ and $Q(\alpha_j) \neq 0$.
- (backgrounds) Under these assumptions,
  * (4) admits a basis of formal solutions: let $\tau_j := Q(-\alpha_j)/P'(-\alpha_j) \in \mathbb{Q}^*$, then the formal solutions $\tilde{\Psi}_1, ..., \tilde{\Psi}_d$ are of the form
  $$\tilde{\Psi}_j(z) = e^{-\alpha_j z} z^{-\tau_j} \tilde{F}_j(z) \in e^{-\alpha_j z}\mathbb{C}[\![z^{-\tau_j}]\!] \tag{5}$$
  Notice that this basis is distinguished only up to scaling, so we have a distinguished frame in the space of formal solutions.
  * (5) is Borel-Laplace summable and its Borel-Laplace sum Ψ_j satisfies the original equation (4)
  * as a consequence, the distinguished frame of formal solutions become a distinguished frame of analytic solutions $\Psi_1, ..., \Psi_d$.
- Can we see there exists a distinguished basis in a purely analytic way? YES. The reason is the existence theorem gives for every α_j a unique solution in the ζ-plane which blows-up in a certain way at α_j: $\phi_j(\zeta_j) = \zeta_j^{-\tau_j} + \tilde{f}_j$, $\zeta_j = \zeta - \alpha_j$.
- why Borel summations of $\tilde{\Psi}_1, ..., \tilde{\Psi}_d$ finds this solutions? because they are an analytic frame of Borel regular functions.
- Maybe the correct place is the setting of Ecalle's formal integral. See §5.2.2.1 of Delabaere's *Divergent Series, Summability and Resurgence III*.
- Say there's a unique solution (up to scaling) that shrinks as you go right; everything else blows up exponentially. Then this is the only solution that can be expressed as a Laplace transform. [Follows from Aaron's argument in Airy resurgence, even if Aaron works with more general ODEs]
- If the Borel-transformed equation has a subexponential solution $\hat{f}$ which is "shifted holomorphic" (we called this having a "fractional power singularity" in `airy-resurgence`), then $\mathcal{L}\hat{f}$ satisfies the original equation, because there are no boundary terms.
- Draw diagram showing formal vs. holomorphic solutions in time vs. frequency domains.

$$
\begin{tikzcd}
 & & ODE \arrow[dd, "\mathcal{L}^{-1}"', "\mathcal{B}"] & \\
\Phi \arrow[urr, dashed] & \hat{\Phi} \arrow[ur, red] & & \tilde{\Phi} \arrow[ul, "formal"'] \arrow[dd, "\mathcal{B}"] \\
 & & IE & \\
\phi \arrow[uu, "\mathcal{L}"] \arrow[urr] & \hat{\phi} \arrow[uu, "\mathcal{L}"] \arrow[ur] & & \tilde{\phi} \arrow[ul] \arrow[ll, mapsto, red, "sum"']
\end{tikzcd}
$$

where the arrow in red are a consequence of multisummability. In addition, we can distinguish on the right hand side of the diagram the formal solutions and on the left hand side the holomorphic ones. On the upper part of the diagram the functions in the z-plane while on the lower part the functions in the Borel plane ζ-plane.

- here is the way there are many ways to see this porblem have a distinguished base of solutions, Poincaré see it formally in the z-plane, Ecalle figured it out how to see it formally in the Borel plane, our results shows how to see it analytically in the Borel plane.
- M.A.E.T. says you can start in formal z-plane but it is not really constructive (see Balser chap 14); going from formal ζ to analytic is constructive and it's essentially Borel-Laplace summation. Our method uses just Laplace transform.
- How do we know we are picking the same frame? from properties of Laplace transform we get solution asymptotics to Poincaré frame. From uniqueness result we get a frame equivalent to Ecalle's frame.
- multi-summability is a regularity result starting form the formal solution in the z-plane. Borel regularity is instead based on the analytic solution in the z-plane. The argument we gave about getting the same frame is what proves Borel regularity of Ψ_j.

• **Being a thimble integral**

Let X be a translation surface—a Riemann surface carrying a holomorphic 1-form ν. Suppose X is of *meromorphic type*, meaning that we got it by puncturing a compact Riemann surface $\overline{X}$ at finitely many points, and ν has a pole at each puncture. A *translation coordinate* on X is a local coordinate whose derivative is ν.

Take another meromorphic-type translation surface B and a holomorphic Morse[1] map $f\colon \overline{X} \to \overline{B}$ that sends punctures to punctures [actually, don't require this; the Orr–Sommerfeld integrals, for example, don't satisfy it]. Suppose every singularity of B is a critical value of f. **[Typical usage of "Borel plane" seems ambiguous, so maybe we can use "Borel plane" for B and "Borel cover" for the Riemann surface of the Borel-transformed series. How to handle the OrrSommerfeld**

[1]This condition means that the critical points of f are isolated (the compactness of $\overline{X}$ guarantees this) and the 2-jet of f is non-zero at every critical point.

functions (DLMF §9.13)? We know $f = 4u^3 - 3u$ is the pullback of a translation coordinate, but we also need a puncture at $f(0)\ldots$] For each critical point p, let Γ_p be the ray going rightward from $f(p)$, and let ζ_p be the translation coordinate around Γ_p which vanishes at $f(p)$. These are well-defined as long as Γ_p misses the critical values of f. The preimage $f^{-1}(\Gamma_p)$ is a bunch of disjoint curves, as long as Γ_p misses the other critical values of f. The *Lefschetz thimble* Λ_p is the component of $f^{-1}(\Gamma_p)$ that goes through p, oriented so that shifting it to its left would make its projection run clockwise around Γ_p. The *thimble integral*

$$I_p = \int_{\Lambda_p} e^{-zf^*\zeta_p}\nu$$

is a holomorphic function on the right half-plane parametrized by z, and it turns out [**we hope**] to be Borel regular.

[**Talk about exponential integrals and their decomposition into thimble integrals.**]

In higher-dimensional complex manifolds, integrals over Lefschetz thimbles are still Borel regular [**"Exponential integrals, Lefschetz thimbles and linear resurgence"**][**"Exponential Integral" lectures?**]. This fact plays an important technical role in quantum mechanics, where infinite-dimensional exponential integrals are supposed to give the expectation values of observable quantities. Physicists often use Borel summation and related techniques to assign values to these integrals [**Costin & Kruskal, "On optimal truncation..."**].

Choose a path $\gamma\colon \mathbb{R} \to X$ whose projection $f \circ \gamma$ starts out going leftward out of a puncture, ends up going rightward into a puncture, and never touches a critical value of f. Choose a translation coordinate ζ on B and continue it along $f \circ \gamma$, noting that it may become multi-valued if $f \circ \gamma$ intersects itself. This data defines the *exponential integral*

$$I = \int_\gamma e^{-zf^*\zeta}\nu,$$

a holomorphic function on the right half-plane parametrized by z. It turns out [**we hope**] that we can get I by summing $e^{-\alpha_p z} I_p$ over various critical points—as long as none of the Γ_p run into each other. [**We get jumps at phases where the Γ_p do hit each other.**] The constants α_p are values of ζ, continued to the critical points along certain paths.

- Each resummation method for asymptotic series makes some implicit assumption that allows us to reconstruct a holomorphic function from its asymptotic behaviour.
- The resummation method works correctly for functions which satisfy that assumption.
- For the modified Bessel function $K_{1/3}$, Borel resummation works because the asymptotic series encodes a second-order differential equation.
  - Different aspects of this example appear in various places (Mariño, Kawai–Takei, Sauzin). We give a detailed, unified treatment.

- We can generalize this argument to all K_ν with $\nu \in \mathbb{Q}$.
- We can also generalize to all third-order exponential integrals.
  - Most of them are equivalent to the $K_{1/3}$ integral, but there's also an interesting degeneration.

1.6 Fractional derivative formula

- Theorem **??** says that for a certain class of exponential integrals

  $$I(z) = \int_\Gamma e^{-zf}\ \nu,$$

  the inverse Laplace [**better to say Borel?**] transform is the $\frac{3}{2}$ derivative of $d\zeta/df$, where $f^* d\zeta = \nu$ [**check**].
- the asymptotic expansion of $I(z)$ is a resurgent function.
- Is it always a *simple* resurgent function?
  - **Maxim belies it is in general, and indeed in our examples we get simple resurgent functions. But how to prove it in general?**

1.7 Stokes phenomenon

- For Bessel functions, we can see explicitly how solutions jump when the Laplace transform angle crosses a critical value.
- The jump comes from the branch cut difference identity for hypergeometric functions.
- Possible interpretation of the Stokes factors as intersections numbers in Morse–Novikov theory [**ask Maxim**]

2 The Laplace and Borel transforms

2.1 The Laplace transform

- Action on differential equations.
  - Can we find a way to prove this when the differential operator spits out a function that's not integrable around zero?
- Global picture?

2.2 The Borel transform

- Action on differential equations.
  - No inhomogeneous terms! How is this consistent with the Laplace transform's action? Is there always an inhomogeneous solution with subexponential asymptotics?

3 Third-order exponential integrals

- Reduce to
$$I(z) = \int \exp\left[-z\big(u^3 + pu + q\big)\right]\, du$$
using change of coordinate.

- When $p \neq 0$, can reduce further to
$$I(z) = p^{1/2} e^{-qz} K_{1/3}(p^{3/2} z).$$

- As p goes to zero, $I(z)$ degenerates to
$$\left(\tfrac{1}{2}\right)^{2/3} e^{-qz}\Gamma(\tfrac{1}{3}) z^{-1/3} = \left(\tfrac{1}{2}\right)^{2/3} e^{-qz}\mathcal{L}_{\zeta,0}(\zeta^{-2/3}) = \left(\tfrac{1}{2}\right)^{2/3} \mathcal{L}_{\zeta_{-q},q}(\zeta^{-2/3}).$$

Outline

Title: Borel regularity and Resurgence of Exponential Integrals

1. introduction
   - Exponential integrals
     - they are function of z and they are defined from the data of (X, f) and $[\mathcal{C}]$, $[\nu]$
     - the choice of the path $\mathcal{C}$:
       * $\mathcal{C} \in H_n^{B,z}(X, f)$
       * Witten's formalism, $\mathcal{C}$ is a Lefschetz thimbles (or steepest descendent path)
     - they define a paring between the relative homology (rapid decaying homology) $H_\bullet^{B,z}(X, f)$ and the twisted de Rham cohomology $H^\bullet_{dR,z}(X, f)$
       * there is a comparison isomorphism (Maxim)
     - varying z we have the Stokes phenomena
     - as $z \to \infty$, the asymptotic expansion of I is a divergent series $\tilde{I}$, usually of Gevrey-class
       * *exact resurgence relation* (Berry–Howls): divergence encodes contributions from other critical values
       * it is an example of resurgent series (Écalle)
       * $\tilde{I}$ is resurgent in $\mathbb{C} \setminus \{\text{poles of } \nu\,, \text{cirtical values of } f\}$
       * it is a toy example of resurgent series because there are only finitely many singularities in the Borel plane
       * we have to compute the **Stokes constants** relative to the singular points in B to fully understand B. There are two methods to compute Stokes constants: • geometric: using intersection theory of thimbles (Picard–Lefschetz, Witten, Maxim), • analytic: using Écalle formalism

- what are exponential integrals? has to be done
  - motivation
    - In the classical theory of special functions, exponential integrals are often used to express solutions of linear differential and difference equations.
    - In physics ??
    - Geometrically they represent a Poincaré pairing (as explained by Kontsevich in **IHES lectures**).
- What is the class of ODEs that we study? has to be done
- State results about resurgence of exponential integrals and Stokes phenomena
  - Thimbles integrals [Kontsevich]: geometric computation of Stokes constants has to be done
  - ODE and fractional derivative formula [`draft2`]
  - if hypergeometric functions appear in a large class of examples: integral formulas for hypergeometric functions has to be done

2. Formalism for Laplace transform [`draft2`, "The geometry of the Laplace transform"]

   (a) Analytic

      i. Introduction

      ii. Brief revew of translation surfaces (we can refer to this from the introduction if we need to)

      iii. The Laplace transform of a holomorphic function

         A. Over an ordinary point

         B. Over a branch point

         C. Differential equation

      iv. Relating differential equations in the frequency domain to integral equations in the position domain

   (b) Formal

      i. Laplace transform of a formal series

      ii. Borel transform

      iii. Relating differential equations in the frequency variable to integral equations in the position variable

3. Review of integral equations

   - Existence of solutions
   - Fractional integrals and derivatives
   - Going between integral and differential equations (slight functions)

4. General cases

   (a) Borel regularity

- General ODE of the form

$$\left[P\left(\tfrac{\partial}{\partial z}\right) + z^{-1}Q\left(\tfrac{\partial}{\partial z}\right) + z^{-2}R(z^{-1})\right]\Phi = 0,$$

  where P is a polynomial, Q is a polynomial of one degree lower, and R is an entire function [see `airy-resurgence` and written notes]

  - More generally, for P of degree n, we should be able to handle

$$\left[P\left(\tfrac{\partial}{\partial z}\right) + z^{-1}Q_1\left(\tfrac{\partial}{\partial z}\right) + z^{-2}Q_2\left(\tfrac{\partial}{\partial z}\right) + \ldots + z^{-(n-1)}Q_{n-1}\left(\tfrac{\partial}{\partial z}\right) + z^{-n}R(z^{-1})\right]\Phi = 0,$$

    where Q_k has degree $n-k$. has to be done

    * We want the most general ODE with a regular singularity at $z = 0$ and its only other singularity, typically irregular, at $z = \infty$. has to be done
    * The singularity at ∞ should only be regular for an Euler equation. has to be done
  - Show that we can find a slight solution at each critical value.
  - Show that $\hat{\iota} = \tilde{\iota}$, where:
    * $I = \mathcal{L}\iota$
    * $\hat{\iota}$ is the Taylor expansion of ι
    * $\tilde{I}$ is the asymptotic series of I
    * $\tilde{\iota} = \mathcal{B}\tilde{I}$
    * Idea: Show that $\hat{\iota}$ and $\tilde{\iota}$ have matching asymptotics at $\zeta = 0$. Since they both satisfy the position-domain integral equation, they must coincide.
- General thimble integral (conditions?)
  - Proof of Borel regularity
  - 3/2-derivative formula
  - Contour argument

(b) Resurgence

- Explain how Borel regularity relates resurgence of formal series to resurgence of holomorphic functions in the position domain. think more about what we're trying to say here
- Relate to Ecalle's formalism and the alien derivative
- Stokes factors
  - For ODEs
  - For thimble integrals

5. Examples make sure each example contains a computation of the Borel transform, so we can see it matches

(a) The Airy example

- $I(z)$ is a solution of a linear ODE. We explicitly find its Borel transform, knowing the nature of singularities and the asymptotic behaviour of a basis of solution for the ODE [`airy-resurgence`]

- Compute Stokes constants
  - Using fractional derivative formula and Borel transform computation [`draft2`]
  - Using Picard-Lefschetz theory (Pham, Kontsevich, etc.)
- Comparison with the literature has to be done
  - Mariño
  - Sauzin
  - Kontsevich slides
  - Kawai–Takei? [might take too long to understand well enough]

(b) The Airy–Lucas examples

- Compute Borel transform [`airy-resurgence`]
- Compute Stokes constants has to be done

(c) Bessel 0 (it is different because we have infinite cover)

- Compute Stokes constants [`draft2`]

(d) Bessel μ (follows from Bessel 0)

- Compute Stokes constants [`modified Bessel`]

(e) The generalized Airy example

(f) The vibrating beam example

- In addition to the simple example, maybe we can do an example where the equation on the spatial domain includes fractional integrals, since Andy is interested in that sort of thing