Resurgence of the Airy function and other exponential integrals

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1 Introduction

1.1 Motivation

1.1.1 Setting

You can often find a formal power series

$$\tilde{\Phi} = z^{\sigma}\left(\frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \frac{c_3}{z^4} + \dots\right),$$

with $\sigma \in [0, 1)$, that looks or acts like a solution to a problem whose actual solutions are holomorphic functions of z. For example, you might rewrite the integral

$$\begin{aligned}
\zeta_\alpha &= \pm 1 \\
\tfrac{1}{2}\tau^2 &= f - \zeta_\alpha \\
&= 4u^3 - 3u \mp 1 \\
\tau &= 0 \text{ and } u_\tau = 0 \text{ at } u = \mp\tfrac{1}{2} \\
\tfrac{1}{2}\tau^2 &= \mp 6w^2 + 4w^3 \text{ with } u := \mp\tfrac{1}{2} + w \\
I_\alpha(z) &= e^{-z\zeta_\alpha} \int_{\tau\in[-\varepsilon,\varepsilon]} e^{-z\tau^2/2}\nu \\
&= e^{\mp z} \int_{\tau\in[-\varepsilon,\varepsilon]} e^{-z\tau^2/2}\nu
\end{aligned}$$

$$
\begin{aligned}
w &= \sum_{n=1}^{\infty} a_n \tau^n \\
w^2 &= \sum_{n=2}^{\infty} \left(\sum_{k+k'=n} a_k a_{k'} \right) \tau^n \\
&= a_1^2\tau^2 + 2a_1a_2\tau^3 + (2a_1a_3 + a_2^2)\tau^4 + (2a_1a_4 + 2a_2a_3)\tau^5 + (2a_1a_5 + 2a_2a_4 + a_3^2)\tau^6 + \dots \\
w^3 &= \sum_{n=3}^{\infty} \left(\sum_{k+k'+k''=n} a_k a_{k'} a_{k''} \right) \tau^n \\
&= a_1^3\tau^3 + 3a_1^2a_2\tau^4 + (3a_1a_2^2 + 3a_1^2a_3)\tau^5 + (3a_1^2a_4 + 6a_1a_2a_3 + a_2^3)\tau^6 + \dots
\end{aligned}
$$

$$
\begin{aligned}
\tau^2 &= \mp 12w^2 + 8w^3 \\
\tau^2 &= \mp 12a_1^2\tau^2 & 1 &= \mp 12a_1^2 \\
0 &= \mp 24a_1a_2\tau^3 + 8a_1^3\tau^3 & 0 &= \mp 36a_2 + 12a_1^2 \\
& & 0 &= \mp 36a_2 \mp 1 \\
& & \pm 1 &= \mp 36a_2 \\
& & -1 &= 36a_2 \\
0 &= \mp(24a_1a_3 + 12a_2^2)\tau^4 + 24a_1^2a_2\tau^4 & 0 &= \mp 24a_1a_3 \mp 12a_2^2 + 24a_1^2a_2 \\
& & &= \mp 24a_1a_3 \mp 12a_2^2 \mp 2a_2 \\
& & &= 24a_1a_3 + \frac{1}{3\cdot 36} - \frac{1}{18} \\
& & \frac{5}{108} &= 24a_1a_3 \\
& & \frac{5}{216}a_1 &= \mp a_3 \\
0 &= \mp(24a_1a_4 + 24a_2a_3)\tau^5 + (24a_1a_2^2 + 24a_1^2a_3)\tau^5 & 0 &= \mp 24a_1a_4 \mp 24a_2a_3 + 24a_1a_2^2 \mp 2a_3 \\
& & &= \mp 24a_1a_4 + \frac{5}{9}a_2a_1 + 24a_1a_2^2 + \frac{5}{108}a_1 \\
& & &= \mp 24a_4 + \frac{5}{9}a_2 + 24a_2^2 + \frac{5}{108} \\
& & \pm 24a_4 &= -\frac{5}{324} + \frac{1}{54} + \frac{5}{108} \\
& & &= \frac{4}{81} \\
& & a_4 &= \pm\frac{1}{486}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \mp 12(2a_1a_5 + 2a_2a_4 + a_3^2)\tau^6 + 8(3a_1^2a_4 + 6a_1a_2a_3 + a_2^3)\tau^6 \\
0 &= \mp 24a_1a_5 \mp 24a_2a_4 \mp 12a_3^2 + 24a_1^2a_4 + 48a_1a_2a_3 + 8a_2^3 \\
0 &= \mp 24a_1a_5 \mp 24a_2a_4 \mp \left(\mp\frac{5}{216}\right)^2 12a_1^2 \mp 2a_4 \mp \frac{5\cdot 48}{216}a_1^2a_2 + 8a_2^3 \\
0 &= \mp 24a_1a_5 \mp 24a_2a_4 + \left(\frac{5}{216}\right)^2 \mp 2a_4 + \frac{5}{54}a_2 + 8a_2^3 \\
0 &= \mp 24a_1a_5 - \frac{4}{81}a_2 + \left(\frac{5}{216}\right)^2 - \frac{1}{243} + \frac{5}{54}a_2 + 8a_2^3 \\
\pm 24a_1a_5 &= -\frac{77}{15552} \\
\pm 2(\mp 1)a_5 &= -\frac{77}{15552}a_1 \\
a_5 &= \frac{77}{31104}a_1
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= \{i,1\}\frac{1}{\sqrt{12}} \\
2a_2 &= -\frac{1}{18} \\
3a_3 &= \{-i,1\}\frac{5}{72\sqrt{12}} \\
4a_4 &= \pm\frac{2}{243} \\
5a_5 &= \{i,1\}\frac{385}{31104\sqrt{12}} \\
du &= dw = w_\tau\, d\tau \\
&= \left[\{i,1\}\frac{1}{\sqrt{12}} - \frac{1}{18}\tau + \{-i,1\}\frac{5}{72\sqrt{12}}\tau^2 \pm \frac{2}{243}\tau^3 + \{i,1\}\frac{385}{31104\sqrt{12}}\tau^4 + \dots\right] d\tau \\
&= \left[\{i,1\} - \frac{2}{3\sqrt{12}}\tau + \{-i,1\}\frac{5}{72}\tau^2 \pm \frac{8}{81\sqrt{12}}\tau^3 + \{i,1\}\frac{385}{31104}\tau^4 + \dots\right] \frac{d\tau}{\sqrt{12}} \\
&= \left[\{i,1\} - \frac{2}{3}\left(\frac{\tau}{\sqrt{12}}\right) + \{-i,1\}\frac{5}{6}\left(\frac{\tau}{\sqrt{12}}\right)^2 \pm \frac{32}{27}\left(\frac{\tau}{\sqrt{12}}\right)^3 + \{i,1\}\frac{385}{216}\left(\frac{\tau}{\sqrt{12}}\right)^4 + \dots\right] \frac{d\tau}{\sqrt{12}}
\end{aligned}
$$

$$
\Phi(z) = \int_\Lambda \exp\left[-z\left(4u^3 - 3u\right)\right]\, du
$$

as

$$
e^z \int_\Lambda e^{-z\tau^2/2}\left[1 - \frac{2}{3}\left(\frac{\tau}{\sqrt{12}}\right) + \frac{5}{6}\left(\frac{\tau}{\sqrt{12}}\right)^2 - \frac{32}{27}\left(\frac{\tau}{\sqrt{12}}\right)^3 + \frac{385}{216}\left(\frac{\tau}{\sqrt{12}}\right)^4 + \dots\right]\frac{d\tau}{\sqrt{12}}
$$

using the substitution $\frac{1}{2}\tau^2 = 4u^3 - 3u + 1$.

- You can often find a formal power series $\tilde{\Phi} = \ldots$ that looks or acts like a solution to a problem whose actual solutions are holomorphic functions of z. For example...
  - Exponential integral: naïve saddle point expansion.
  - ODE: formal solution. Existence theorem in [1]?
  - Feynman diagram series?
- Once you have $\tilde{\Phi}$, you might try applying *Borel summation*, which turns a formal power series into a function asymptotic to it.
  - Borel summation work in three steps. First we turn the formal power series $\tilde{\Phi}$ [$\tilde{\Phi}(z)$] in the "frequency variable" z into a formal power series $\tilde{\phi}$ in a new "position variable" ζ.
  - The series $\tilde{\phi}$ turns out to have a positive radius of convergence, so it defines a holomorphic function $\hat{\phi}$. By analytic continuation, we can expand the domain of $\hat{\phi}$ to a Riemann surface B with a distinguished 1-form λ—the continuation of $d\zeta$.
  - If $\hat{\phi}$ grows slowly enough along an infinite ray $\Gamma_b^\theta := b + e^{i\theta}[0, \infty)$ [decide notation], its Laplace transform $\mathcal{L}_b^\theta \hat{\phi} := \ldots$ is a holomorphic function of z, well-defined on some sector of the frequency plane. In this case, we say $\tilde{\Phi}$ is *Borel-summable*, and we call $\mathcal{L}_b^\theta \hat{\phi}$ its *Borel sum* at b.
  - [Draw the square!]
- Different functions can be asymptotic to the same power series, and Borel summation picks one of them.
- In many cases, it picks correctly, producing an actual solution to your problem.
  - The question of how that happens is the starting point for this paper.
  - (In both of the cases that we study, the Borel sum of $\tilde{\Phi}$ is always taken at a zero of λ, rather than an arbitrary point in B. For ODEs, our treatment explains why the zeroes of λ play a special role.)

1.1.2 Goals

- The central goal of this paper is to lay out two kinds of problems where we can prove that the Borel sum of a formal power series solution is always an actual solution.
  - The first problem is evaluating a certain kind of exponential integral: a one-dimensional *thimble integral.*
  - The second problem is solving a certain kind of ODE.
  - These two problems are closely linked. By playing with derivatives of an exponential integral, you can often find a linear ODE that the integral satisfies. Conversely, for many classical ODEs, there are useful bases of exponential integral solutions.
  - (Does this touch the Picard-Lefshetz perspective? Betti / de Rham relationship: ODE is a connection, and exponential integrals give flat sections?)

- Clearly separate the parts of the theory that deal with holomorphic functions and formal power series.
- (Super-motivation: why do the zeroes of λ play a special role?) As part of the treatment, we've made use of some new perspectives on the Laplace transform.
  - **Geometric picture.** The spatial domain B is a translation surface. If $b \in B$ is non-singular, the frequency domain for $\mathcal{L}_b^\theta$ is T^*B_b. If b is a conical singularity, the frequency domain is more interesting, as we'll see in our main example.
  - **A new dictionary for ODEs.** The Laplace transform is often usd to solve ODEs on the frequency domain by relating them to ODEs on the spatial domain. We find, however, that it's much easier and more natural to relate ODEs on the frequency domain to integral equations on the spatial domain. This clarifies why we take the Borel sums at zeroes of λ when we're trying to solve an ODE.
- Illustrate with detailed treatments of several examples.
  - Some have been discussed many times, using different approaches and conventions. We'll try to give an idea of how all these different treatments fit together. • The Airy function. • The anharmonic oscillator (Bender–Wu, Schiappa).
  - Others haven't been discussed much.
- To understand, for example, the Ecalle formalism, you can start with these toy examples, which illustrate what's going on, but can also be studied in a more elementary way. [How do we work this into the introduction?]
- The examples give a place to compare more complicated formalisms like the Picard-Lefshetz (Morse theory) or Ecalle formalisms? [How do we work this into the introduction?]

1.2 Why does Borel resummation work?

Borel resummation is a way of turning a formal power series

$$\tilde{\varphi} = z^\sigma \left(\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \frac{\varphi_3}{z^4} + \dots \right),$$

with $\sigma \in [0, 1)$, into a function which is asymptotic to $\tilde{\varphi}$ as $z \to \infty$. Different functions can be asymptotic to the same power series, and Borel resummation picks one of them, performing an implicit regularization **[arXiv:1705.03071, or maybe arXiv:1412.6614]**. When a function matches the Borel sum of its asymptotic series, we'll say it's *Borel regular*. Several familiar kinds of regularity imply Borel regularity, and shed light on why it occurs.

- **Having a good asymptotic approximation**

  Let R_N be the difference between a function and the partial sum

  $$\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \dots + \frac{\varphi_{N-2}}{z^{N-1}}$$

of its asymptotic series. Watson showed a century ago that the function is Borel regular whenever there's a constant $c \in (0, \infty)$ with

$$|R_N| \leq \frac{c^{N+1} N!}{|z|^N}$$

over all orders N and all z in a wide enough wedge around infinity.

- **Satisfying a singular differential equation**
  - Think about conditions where this works.
  - Maybe the correct place is the setting of Ecalle's formal integral. See §5.2.2.1 of Delabaere's *Divergent Series, Summability and Resurgence III*.
  - Say there's a unique solution (up to scaling) that shrinks as you go right; everything else blows up exponentially. Then this is the only solution that can be expressed as a Laplace transform.
  - If the Borel-transformed equation has a subexponential solution $\hat{f}$ which is "shifted holomorphic" (we called this having a "fractional power singularity" in `airy-resurgence`), then $\mathcal{L}\hat{f}$ satisfies the original equation, because there are no boundary terms.
  - Draw diagram showing formal vs. holomorphic solutions in time vs. frequency domains.

- **Being a thimble integral**

  Let X be a translation surface—a Riemann surface carrying a holomorphic 1-form ν. Suppose X is of *meromorphic type*, meaning that we got it by puncturing a compact Riemann surface $\overline{X}$ at finitely many points, and ν has a pole at each puncture. A *translation coordinate* on X is a local coordinate whose derivative is ν.

  Take another meromorphic-type translation surface B and a holomorphic Morse[1] map $f \colon \overline{X} \to \overline{B}$ that sends punctures to punctures [actually, don't require this; the Orr–Sommerfeld integrals, for example, don't satisfy it]. Suppose every singularity of B is a critical value of f. **[Typical usage of "Borel plane" seems ambiguous, so maybe we can use "Borel plane" for B and "Borel cover" for the Riemann surface of the Borel-transformed series. How to handle the Orr–Sommerfeld functions (DLMF §9.13)? We know $f = 4u^3 - 3u$ is the pullback of a translation coordinate, but we also need a puncture at $f(0)$...]** For each critical point p, let Γ_p be the ray going rightward from $f(p)$, and let ζ_p be the translation coordinate around Γ_p which vanishes at $f(p)$. These are well-defined as long as Γ_p misses the critical values of f. The preimage $f^{-1}(\Gamma_p)$ is a bunch of disjoint curves, as long as Γ_p misses the other critical values of f. The *Lefschetz thimble* Λ_p is the component of $f^{-1}(\Gamma_p)$ that goes through p, oriented so that shifting it to its left would make its projection run clockwise around Γ_p. The *thimble integral*

$$I_p = \int_{\Lambda_p} e^{-z f^* \zeta_p} \nu$$

[1]This condition means that the critical points of f are isolated (the compactness of $\overline{X}$ guarantees this) and the 2-jet of f is non-zero at every critical point.

is a holomorphic function on the right half-plane parametrized by z, and it turns out [**we hope**] to be Borel regular.

[**Talk about exponential integrals and their decomposition into thimble integrals.**]

In higher-dimensional complex manifolds, integrals over Lefschetz thimbles are still Borel regular [**"Exponential integrals, Lefschetz thimbles and linear resurgence"**][**"Exponential Integral" lectures?**]. This fact plays an important technical role in quantum mechanics, where infinite-dimensional exponential integrals are supposed to give the expectation values of observable quantities. Physicists often use Borel summation and related techniques to assign values to these integrals [**Costin & Kruskal, "On optimal truncation..."**].

Choose a path $\gamma\colon \mathbb{R} \to X$ whose projection $f \circ \gamma$ starts out going leftward out of a puncture, ends up going rightward into a puncture, and never touches a critical value of f. Choose a translation coordinate ζ on B and continue it along $f \circ \gamma$, noting that it may become multi-valued if $f \circ \gamma$ intersects itself. This data defines the *exponential integral*

$$I = \int_\gamma e^{-z f^* \zeta} \nu,$$

a holomorphic function on the right half-plane parametrized by z. It turns out [**we hope**] that we can get I by summing $e^{-\alpha_p z} I_p$ over various critical points—as long as none of the Γ_p run into each other. [**We get jumps at phases where the Γ_p do hit each other.**] The constants α_p are values of ζ, continued to the critical points along certain paths.

- Each resummation method for asymptotic series makes some implicit assumption that allows us to reconstruct a holomorphic function from its asymptotic behaviour.
- The resummation method works correctly for functions which satisfy that assumption.
- For the modified Bessel function $K_{1/3}$, Borel resummation works because the asymptotic series encodes a second-order differential equation.
  - Different aspects of this example appear in various places (Mariño, Kawai–Takei, Sauzin). We give a detailed, unified treatment.
- We can generalize this argument to all K_ν with $\nu \in \mathbb{Q}$.
- We can also generalize to all third-order exponential integrals.
  - Most of them are equivalent to the $K_{1/3}$ integral, but there's also an interesting degeneration.

1.3 Fractional derivative formula

- Theorem **??** says that for a certain class of exponential integrals

$$I(z) = \int_\Gamma e^{-zf}\ \nu,$$

  the inverse Laplace [**better to say Borel?**] transform is the $\frac{3}{2}$ derivative of $d\zeta/df$, where $f^* d\zeta = \nu$ [**check**].

- the asymptotic expansion of $I(z)$ is a resurgent function.

- Is it always a *simple* resurgent function?

  - **Maxim belies it is in general, and indeed in our examples we get simple resurgent functions. But how to prove it in general?**

1.4 Stokes phenomenon

- For Bessel functions, we can see explicitly how solutions jump when the Laplace transform angle crosses a critical value.

- The jump comes from the branch cut difference identity for hypergeometric functions.

- Possible interpretation of the Stokes factors as intersections numbers in Morse–Novikov theory [**ask Maxim**]

2 The Laplace and Borel transforms

2.1 The Laplace transform

- Action on differential equations.

  - Can we find a way to prove this when the differential operator spits out a function that's not integrable around zero?

- Global picture?

2.2 The Borel transform

- Action on differential equations.

  - No inhomogeneous terms! How is this consistent with the Laplace transform's action? Is there always an inhomogeneous solution with subexponential asymptotics?

3 Third-order exponential integrals

- Reduce to
$$I(z) = \int \exp\left[-z\big(u^3 + pu + q\big)\right] du$$
using change of coordinate.

- When $p \neq 0$, can reduce further to
$$I(z) = p^{1/2} e^{-qz} K_{1/3}(p^{3/2} z).$$

- As p goes to zero, $I(z)$ degenerates to
$$\left(\tfrac{1}{2}\right)^{2/3} e^{-qz} \Gamma(\tfrac{1}{3}) z^{-1/3} = \left(\tfrac{1}{2}\right)^{2/3} e^{-qz} \mathcal{L}_{\zeta,0}(\zeta^{-2/3}) = \left(\tfrac{1}{2}\right)^{2/3} \mathcal{L}_{\zeta_{-q},q}(\zeta^{-2/3}).$$

Outline

Title: Borel regularity and Resurgence of Exponential Integrals

1. introduction
   - Borel regularity
     - what does it mean being Borel regular?
     - when does it happen?
       - Recall Watson condition (old)
       - State new Borel regularity results
         - Linear, homogeneous ODE with regular singularity at 0 and irregular singularity at infinity [P.Ramis ?, Loday-Richaud ?, big idea in `airy-resurgence`]
         - Thimble integral [`draft2`]
   - what are exponential integrals? has to be done
     - motivation
       - In the classical theory of special functions, exponential integrals are often used to express solutions of linear differential and difference equations.
       - In physics ??
       - Geometrically they represent a Poincaré pairing (as explained by Kontsevich in **IHES lectures**).
   - What is the class of ODEs that we study? has to be done
   - State results about resurgence of exponential integrals and Stokes phenomena
     - Thimbles integrals [Kontsevich]: geometric computation of Stokes constants has to be done
     - ODE and fractional derivative formula [`draft2`]

- if hypergeometric functions appear in a large class of examples: integral formulas for hypergeometric functions has to be done

2. Formalism for Laplace transform [`draft2`, "The geometry of the Laplace transform"]
   (a) Analytic
      i. Introduction
      ii. Brief revew of translation surfaces (we can refer to this from the introduction if we need to)
      iii. The Laplace transform of a holomorphic function
         A. Over an ordinary point
         B. Over a branch point
         C. Differential equation
      iv. Relating differential equations in the frequency domain to integral equations in the position domain
   (b) Formal
      i. Laplace transform of a formal series
      ii. Borel transform
      iii. Relating differential equations in the frequency variable to integral equations in the position variable

3. Review of integral equations
   - Existence of solutions
   - Fractional integrals and derivatives
   - Going between integral and differential equations (slight functions)

4. General cases
   (a) Borel regularity
      - General ODE of the form

$$\left[P\left(\tfrac{\partial}{\partial z}\right) + z^{-1}Q\left(\tfrac{\partial}{\partial z}\right) + z^{-2}R(z^{-1})\right]\Phi = 0,$$

        where P is a polynomial, Q is a polynomial of one degree lower, and R is an entire function [see `airy-resurgence` and written notes]
        - More generally, for P of degree n, we should be able to handle

$$\left[P\left(\tfrac{\partial}{\partial z}\right) + z^{-1}Q_1\left(\tfrac{\partial}{\partial z}\right) + z^{-2}Q_2\left(\tfrac{\partial}{\partial z}\right) + \ldots + z^{-(n-1)}Q_{n-1}\left(\tfrac{\partial}{\partial z}\right) + z^{-n}R(z^{-1})\right]\Phi = 0,$$

          where Q_k has degree $n-k$. has to be done
          * We want the most general ODE with a regular singularity at $z=0$ and its only other singularity, typically irregular, at $z=\infty$. has to be done
          * The singularity at ∞ should only be regular for an Euler equation. has to be done

- Show that we can find a slight solution at each critical value.
- Show that $\hat{\iota} = \tilde{\iota}$, where:
  * $I = \mathcal{L}\iota$
  * $\hat{\iota}$ is the Taylor expansion of ι
  * $\tilde{I}$ is the asymptotic series of I
  * $\tilde{\iota} = \mathcal{B}\tilde{I}$
  * Idea: Show that $\hat{\iota}$ and $\tilde{\iota}$ have matching asymptotics at $\zeta = 0$. Since they both satisfy the position-domain integral equation, they must coincide.

- General thimble integral (conditions?)
  - Proof of Borel regularity
  - 3/2-derivative formula
  - Contour argument

(b) Resurgence

- Explain how Borel regularity relates resurgence of formal series to resurgence of holomorphic functions in the position domain. think more about what we're trying to say here
- Relate to Ecalle's formalism and the alien derivative
- Stokes factors
  - For ODEs
  - For thimble integrals

5. Examples make sure each example contains a computation of the Borel transform, so we can see it matches

(a) The Airy example

- $I(z)$ is a solution of a linear ODE. We explicitly find its Borel transform, knowing the nature of singularities and the asymptotic behaviour of a basis of solution for the ODE [`airy-resurgence`]
- Compute Stokes constants
  - Using fractional derivative formula and Borel transform computation [`draft2`]
  - Using Picard-Lefschetz theory (Pham, Kontsevich, etc.)
- Comparison with the literature has to be done
  - Mariño
  - Sauzin
  - Kontsevich slides
  - Kawai–Takei? [might take too long to understand well enough]

(b) The Airy–Lucas examples

- Compute Borel transform [`airy-resurgence`]
- Compute Stokes constants has to be done

(c) Bessel 0 (it is different because we have infinite cover)

- Compute Stokes constants [`draft2`]

(d) Bessel μ (follows from Bessel 0)

- Compute Stokes constants [`modified Bessel`]

(e) The generalized Airy example

(f) The vibrating beam example

- In addition to the simple example, maybe we can do an example where the equation on the spatial domain includes fractional integrals, since Andy is interested in that sort of thing

References

[1] Eric Delabaere. *Divergent series, summability and resurgence. III*, volume 2155 of *Lecture Notes in Mathematics*. Springer, [Cham], 2016. Resurgent methods and the first Painlevé equation, With prefaces by Jean-Pierre Ramis, Michèle Loday-Richaud, Claude Mitschi and David Sauzin.