

EXPONENTIAL INTEGRALS

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2. RESURGENCE OF EXPONENTIAL INTEGRALS

Let X be a $N - \dim$ manifold, $f : X \rightarrow \mathbb{C}$ be a holomorphic Morse function with only simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$(2.1) \quad I(z) := \int_{\mathcal{C}} e^{-zf} \nu$$

where \mathcal{C} is a suitable contour such that the integral is well defined. For any Morse critical points x_α of f , the saddle point approximation gives the following formal series

$$(2.2) \quad I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf} \nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)} (2\pi)^{N/2} z^{-N/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n}.$$

Theorem 2.1. Let $\tilde{\varphi}_\alpha(z) := e^{-zf(x_\alpha)} (2\pi)^{N/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n}$

- (1) $\tilde{\varphi}_\alpha$ is Gevrey-1;
- (2) $\hat{\varphi}(\zeta) := \mathcal{B}(\tilde{\varphi})$ is a germ of analytic function at $\zeta = \zeta_\alpha = f(x_\alpha)$;
- (3) the following formal holds true

$$(2.3) \quad \hat{\varphi}_\alpha(\zeta) = \partial_{\zeta, \text{based at } \zeta_\alpha}^{N/2} \left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu \right) = \Gamma(-N/2) \int_{\zeta_\alpha}^{\zeta} (\zeta - \zeta')^{-N/2} \int_{f^{-1}(\zeta')} \frac{\nu}{df} d\zeta'$$

Definition 2.2. Let $\alpha \in (0, 1)$, then the α -Caputo's derivative of a smooth function f is defined as

$$(2.4) \quad \partial_x^\alpha f(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f'(s) ds$$

Example 2.3 (Airy). Let $f(t) = \frac{t^3}{3} - t$ and

$$I(z) := \int_{\gamma} e^{-zf(t)} dt$$

where γ is a contour where the integral is well defined.

By the change of coordinates $z = x^{3/2}$, $I(z) = -2\pi i z^{-1/3} Ai(x)$ where

$$Ai(x) := \frac{1}{2\pi i} \int_{-\infty e^{-i\frac{\pi}{3}}}^{\infty e^{i\frac{\pi}{3}}} e^{\frac{t^3}{3} - zt} dt$$

hence $I(z)$ solves the following ODE¹

$$(2.5) \quad I''(z) - \frac{4}{9}I(z) + \frac{I'(z)}{z} - \frac{1}{9} \frac{I(z)}{z^2} = 0$$

A formal solution of (2.5) can be computed by making the following ansatz

$$(2.6) \quad \tilde{I}(z) = \sum_{k \in \mathbb{N}^2} U^k e^{-\lambda \cdot k z} z^{-\tau \cdot k} w_k(z)$$

with $U^{(k_1, k_2)} = U_1^{k_1} U_2^{k_2}$ and $U_1, U_2 \in \mathbb{C}$ are constant parameter, $\lambda = (\frac{2}{3}, -\frac{2}{3})$, $\tau = (\frac{1}{2}, \frac{1}{2})$, and $\tilde{w}_k(z) \in \mathbb{C}[[z^{-1}]]$. In addition, we can check that the only non zero $\tilde{w}_k(z)$ occurs at $k = (1, 0)$ and $k = (0, 1)$, therefore

$$(2.7) \quad \tilde{I}(z) = U_1 e^{-2/3z} z^{-1/2} \tilde{w}_+(z) + U_2 e^{2/3z} z^{-1/2} \tilde{w}_-(z)$$

where from now on we denote $\tilde{w}_+(z) = \tilde{w}_{(1,0)}(z)$ and $\tilde{w}_-(z) = \tilde{w}_{(0,1)}(z)$. In particular, $\tilde{w}_+(z)$ and $\tilde{w}_-(z)$ are formal solution of

$$(2.8) \quad \tilde{w}_+'' - \frac{4}{3}\tilde{w}_+' + \frac{5}{36} \frac{\tilde{w}_+}{z^2} = 0$$

$$(2.9) \quad \tilde{w}_-'' + \frac{4}{3}\tilde{w}_-'' + \frac{5}{36} \frac{\tilde{w}_-}{z^2} = 0$$

Taking the Borel transform of (2.8), (2.9) we get

$$\begin{aligned} \zeta^2 \hat{w}_+(\zeta) + \frac{4}{3}\zeta \hat{w}_+ + \frac{5}{36} \zeta * \hat{w}_+ &= 0 \\ \zeta^2 \hat{w}_+(\zeta) + \frac{4}{3}\zeta \hat{w}_+ + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}_+(\zeta') d\zeta' &= 0 \\ \zeta^2 \hat{w}_-(\zeta) - \frac{4}{3}\zeta \hat{w}_- + \frac{5}{36} \zeta * \hat{w}_- &= 0 \\ \zeta^2 \hat{w}_-(\zeta) - \frac{4}{3}\zeta \hat{w}_- + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}_-(\zeta') d\zeta' &= 0 \end{aligned}$$

¹ $Ai(x)$ solves the Airy equation $y'' = xy$.

and taking derivatives we get

$$\begin{aligned}\zeta\left(\frac{4}{3} + \zeta\right)\hat{w}_+'' + \left(\frac{8}{3} + 4\zeta\right)\hat{w}_+' + \frac{77}{36}\hat{w}_+ &= 0 \\ \frac{4}{3}\zeta\left(1 + \frac{3}{4}\zeta\right)\hat{w}_+'' + \left(\frac{8}{3} + 4\zeta\right)\hat{w}_+' + \frac{77}{36}\hat{w}_+ &= 0 \\ u(1-u)\hat{w}_+''(u) + (2-4u)\hat{w}_+'(u) - \frac{77}{36}\hat{w}_+(u) &= 0 \quad u = -\frac{3}{4}\zeta\end{aligned}$$

$$\begin{aligned}\zeta\left(-\frac{4}{3} + \zeta\right)\hat{w}_-'' + \left(-\frac{8}{3} + 4\zeta\right)\hat{w}_-' + \frac{77}{36}\hat{w}_- &= 0 \\ \frac{4}{3}\zeta\left(-1 + \frac{3}{4}\zeta\right)\hat{w}_-'' + \left(-\frac{8}{3} + 4\zeta\right)\hat{w}_-' + \frac{77}{36}\hat{w}_- &= 0 \\ u(1-u)\hat{w}_-''(u) + (2-4u)\hat{w}_-'(u) - \frac{77}{36}\hat{w}_-(u) &= 0 \quad u = \frac{3}{4}\zeta\end{aligned}$$

Notice that the latter equations are hypergeometric, hence a solution is given by

$$(2.10) \quad \hat{w}_+(\zeta) = c_1 {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}\zeta\right)$$

$$(2.11) \quad \hat{w}_-(\zeta) = c_2 {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{3}{4}\zeta\right)$$

for some constants $c_1, c_2 \in \mathbb{C}$ (see DLMF 15.10.2). In addition $\hat{w}_\pm(\zeta)$ have a log singularity respectively at $\zeta = \mp\frac{4}{3}$, therefore they are $\{\mp\frac{4}{3}\}$ -resurgent functions.²

Remark 2.4. $\hat{w}_+(\zeta)$ is Laplace summable along the positive real axis, and it can be analytically continued on $\mathbb{C} \setminus -\frac{4}{3}\mathbb{R}_{\geq 0}$ with

$$\begin{aligned}\hat{w}_+(\zeta + i0) - \hat{w}_+(\zeta - i0) &= \frac{36}{5}i\left(-\frac{3}{4}\zeta - 1\right)^{-1} \sum_{k \geq 0} \frac{(5/6)_n (1/6)_n}{\Gamma(n)n!} \left(1 + \frac{3}{4}\zeta\right)^n \quad \zeta < -\frac{4}{3} \\ &= \frac{36}{5}i\left(-\frac{3}{4}\zeta - 1\right)^{-1} \left(\frac{5}{144}(4 + 3\zeta)_1 {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 + \frac{3}{4}\zeta\right)\right) \\ &= -\mathbf{i} {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 + \frac{3}{4}\zeta\right) \\ &= -\mathbf{i}\hat{w}_-\left(\zeta + \frac{4}{3}\right)\end{aligned}$$

²The solution we find are equal to the ones in DLMF §9.7. They also agree with slide 10 of Maxim's talk for ERC and with the series (6.121) in Sauzin's book. However they do not agree with (2.16) in Mariño's Diablerets.

Anolougsly, $\hat{w}_-(\zeta)$ is Laplace summable along the negative real axis, and it jumps across the branch cut $\frac{4}{3}\mathbb{R}_{\geq 0}$ as

$$\begin{aligned}\hat{w}_-(\zeta + i0) - \hat{w}_-(\zeta - i0) &= -\frac{36}{5}i\left(\frac{3}{4}\zeta - 1\right)^{-1} \sum_{k \geq 0} \frac{(5/6)_n (1/6)_n}{\Gamma(n)n!} \left(1 - \frac{3}{4}\zeta\right)^n & \zeta > \frac{4}{3} \\ &= -\frac{36}{5}i\left(\frac{3}{4}\zeta - 1\right)^{-1} \left(-\frac{5}{144}(-4 + 3\zeta)_1 F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 - \frac{3}{4}\zeta\right)\right) \\ &= i {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 - \frac{3}{4}\zeta\right) \\ &= i\hat{w}_+(\zeta - \frac{4}{3})\end{aligned}$$

These relations manifest the resurgence property of \tilde{I} , indeed near the singularities in the Borel plane of either \hat{w}_+ or \hat{w}_- , \hat{w}_- and \hat{w}_+ respectively contribute to the jump of the former solution.

Our next goal is to prove that the Borel transform of $\tilde{I}(z)$ can be written in terms of $1/f'(f^{-1}(\zeta))$, namely formula (2.3). First of all we define $\tilde{u}(z) := z^{1/2}\tilde{I}(z)$ which is a solution of $\hat{\varphi}_1(\zeta)$ and $\hat{\varphi}_2(\zeta)$:

$$(2.12) \quad \tilde{u}''(z) - \frac{4}{9}\tilde{u}(z) + \frac{5}{36} \frac{\tilde{u}(z)}{z^2} = 0$$

As a consequence, the Borel transform $\hat{u}(\zeta)$ solves

$$\begin{aligned}\zeta^2 \hat{u} - \frac{4}{9} \hat{u} + \frac{5}{36} \zeta * \hat{u} \\ \zeta^2 \hat{u} - \frac{4}{9} \hat{u} + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{u}(\zeta') d\zeta' \\ \text{taking derivatives is equivalent to} \\ (\zeta^2 - \frac{4}{9}) \hat{u}''(\zeta) + 4\zeta \hat{u}'(\zeta) + \frac{77}{36} \hat{u}(\zeta) = 0\end{aligned}$$

and Mathematica gives the following solutions

$$\begin{aligned}\hat{u}(\zeta) &= c_1 {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{9}{4}\zeta^2\right) + \frac{3i}{2}\zeta c_2 {}_1F_2\left(\frac{13}{12}, \frac{17}{12}, \frac{3}{2}, \frac{9}{4}\zeta^2\right) = \\ &= c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} \left({}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} + \frac{3}{4}\zeta\right)\right) & \text{see DLMF 15.8.27} \\ &\quad + \frac{3i}{2}\zeta c_2 \left(\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{3\zeta\Gamma(-\frac{1}{2})\Gamma(2)}\right) \left({}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right)\right) & \text{see DLMF 15.8.28} \\ &= \left(c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} - c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) + \\ &\quad + \left(c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} + c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right)\end{aligned}$$

$$\begin{aligned}\hat{u}(\zeta) &= C_1 T_{-2/31} F_2\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}\zeta\right) + C_2 T_{2/31} F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{3}{4}\zeta\right) \\ &= C_1 T_{-2/3} \hat{w}_+(\zeta) + C_2 T_{2/3} \hat{w}_-(\zeta)\end{aligned}$$

Using properties of Caputo's 1/2-derivative we would like to express $\hat{w}_\pm(\zeta)$ as the 1/2-derivatives of other hypergeometric series. From the integral representation of hypergeometric functions (see DLMF 15.6.1)

$$\begin{aligned}{}_2F_1\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}\zeta\right) &= \frac{3}{5\pi} \int_0^1 t^{5/6} (1-t)^{-5/6} \left(1 + \frac{3}{4}\zeta t\right)^{-7/6} dt & |u| < \frac{4}{3} \\ &= \frac{3}{5\pi} \zeta^{-1} \int_0^\zeta u^{5/6} (\zeta - u)^{-5/6} \left(1 + \frac{3}{4}u\right)^{-7/6} du & u = \zeta t \\ &= \frac{3}{5\pi} \int_0^\zeta (\zeta - u)^{-1/2} \left(\zeta^{-1/2} \left(1 - \frac{u}{\zeta}\right)^{-1/3} \left(\frac{u}{\zeta}\right)^{5/6} \left(1 + \frac{3}{4}u\right)^{-7/6}\right) du \\ &= \frac{3}{5\sqrt{\pi}} \frac{1}{\Gamma(1/2)} \int_0^\zeta (\zeta - u)^{-1/2} \left(\zeta^{-1/2} \left(1 - \frac{u}{\zeta}\right)^{-1/3} \left(\frac{u}{\zeta}\right)^{5/6} \left(1 + \frac{3}{4}u\right)^{-7/6}\right) du\end{aligned}$$

For formula (2.3) to hold true we must have

$$(2.13) \quad \frac{3}{5\sqrt{\pi}} \left(\zeta^{-1/2} \left(1 - \frac{u}{\zeta}\right)^{-1/3} \left(\frac{u}{\zeta}\right)^{5/6} \left(1 + \frac{3}{4}u\right)^{-7/6} \right) = \frac{1}{f'(f^{-1}(u))}$$

however, $\frac{1}{f'(f^{-1}(u))}$ does not depend on ζ , thus (2.13) is false, and the Airy exponential integral is a counter example of the formula (2.3).

2.1. Second tentative formula. Let us assume that the correct formula that replaces (2.3) in Theorem 2.1 is

$$(2.14) \quad \hat{I}_\alpha(\zeta) = \partial_{\zeta, \text{based at } \zeta_\alpha}^{N/2} \left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} v \right) = \Gamma(-N/2) \int_{\zeta_\alpha}^\zeta (\zeta - \zeta')^{-N/2} \int_{f^{-1}(\zeta')} \frac{v}{df} d\zeta'$$

then we compute \hat{I} using properties of the Borel transform of a product:

$$\begin{aligned}\hat{I}(\zeta) &= \mathcal{B}(z^{-1/2}) * \hat{u}(\zeta) \\ &= \frac{1}{\Gamma(1/2)} \int_0^\zeta (\zeta - s)^{-1/2} \hat{u}(s) ds \\ &= \frac{1}{\Gamma(1/2)} \int_0^\zeta (\zeta - s)^{-1/2} (T_{-2/3} \hat{w}_+(s) + T_{2/3} \hat{w}_-(s)) ds\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(1/2)} \int_0^\zeta (\zeta - s)^{-1/2} \hat{w}_+(s - 2/3) + \frac{1}{\Gamma(1/2)} \int_0^\zeta (\zeta - s)^{-1/2} \hat{w}_-(s + 2/3) ds \\
&= \frac{1}{\Gamma(1/2)} \int_{-2/3}^{\zeta-2/3} (\zeta - \frac{2}{3} - s)^{-1/2} \hat{w}_+(s) + \frac{1}{\Gamma(1/2)} \int_{2/3}^{\zeta+2/3} (\zeta + \frac{2}{3} - s)^{-1/2} \hat{w}_-(s) ds \\
&= \frac{C_1}{\Gamma(1/2)} \int_{-2/3}^{\zeta-2/3} (\zeta - \frac{2}{3} - s)^{-1/2} {}_2F_1\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}s\right) ds + \frac{C_2}{\Gamma(1/2)} \int_{2/3}^{\zeta+2/3} (\zeta + \frac{2}{3} - s)^{-1/2} {}_2F_1\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{3}{4}s\right) ds \\
&= \frac{C_3}{\Gamma(1/2)} \int_{-2/3}^{\zeta-2/3} (\zeta - \frac{2}{3} - s)^{-\frac{1}{2}} \frac{\partial}{\partial s} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, -\frac{3}{4}s\right) ds + \frac{C_4}{\Gamma(1/2)} \int_{2/3}^{\zeta+2/3} (\zeta + \frac{2}{3} - s)^{-\frac{1}{2}} \frac{\partial}{\partial s} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, \frac{3}{4}s\right) ds \\
&= C_3 \partial_{\zeta, \text{based at } -2/3}^{1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, -\frac{3}{4}\zeta\right) + C_4 \partial_{\zeta, \text{based at } 2/3}^{1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, \frac{3}{4}\zeta\right)
\end{aligned}$$

However, ${}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1, \mp \frac{3}{4}\zeta\right)$ are not algebraic, hence they can not be equal to $\frac{1}{f'(f^{-1}(\zeta))}$ as it is in in Theorem 2.1 part II (2.14).

2.1.1. *Comparison with Aaron.* The Airy integral can be written in terms of the modified Bessel equation as

$$(2.15) \quad \text{Ai}(x) = \frac{1}{\pi\sqrt{3}} x^{1/2} K\left(\frac{2}{3}x^{3/2}\right).$$

On the other hand we have, for $z = x^{3/2}$

$$(2.16) \quad \text{Ai}(x) = -\frac{z^{1/3}}{2\pi i} I(z) = -\frac{1}{2\pi i} x^{1/2} I(x^{3/2})$$

hence

$$\begin{aligned}
-\frac{1}{2\pi i} I(x^{3/2}) &= \frac{1}{\pi\sqrt{3}} K\left(\frac{2}{3}x^{3/2}\right) \\
\frac{i}{2} I(z) &= \frac{1}{\sqrt{3}} K\left(\frac{2}{3}z\right)
\end{aligned}$$

In particular, the Borel transforms of LHS and RHS³ must be equal, i.e.

$$\begin{aligned}
\frac{i}{2}\hat{I}(\zeta) &= \frac{\sqrt{3}}{2}\hat{K}\left(\frac{3}{2}\zeta\right) \\
&= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\left(\frac{3}{2}\zeta-1\right)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{1}{2}-\frac{3}{4}\zeta\right) \\
&= \frac{\sqrt{3}}{2}(3\zeta-2)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{1}{2}-\frac{3}{4}\zeta\right) \\
&= \frac{\sqrt{3}}{2}T_{-2/3}\left((3\zeta)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)\right) \\
&= \frac{1}{2}T_{-2/3}\left(\frac{1}{\sqrt{\zeta}}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)\right) \\
\hat{I}(\zeta) &= -iT_{-2/3}\left(\frac{1}{\sqrt{\zeta}}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)\right)
\end{aligned}$$

In addition, since $\hat{I}(\zeta) = C_3\partial_{\zeta}^{1/2}$ based at $-2/3$ ${}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)$ is a solution of the Borel transform of equation (2.5); thus

$$\begin{aligned}
C_3\partial_{\zeta}^{1/2} \text{ based at } -2/3 {}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right) &= -iT_{-2/3}\left(\frac{1}{\sqrt{\zeta}}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)\right) \\
C_3\partial_{\zeta}^{1/2} {}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right) &= -i\frac{1}{\sqrt{\zeta}}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)
\end{aligned}$$

Equation 5 in Aaron is

$$\begin{aligned}
&\left[z^2\left(\frac{\partial}{\partial z}+1\right)^2+z\left(\frac{\partial}{\partial z}+1\right)-\left[\left(\frac{1}{3}\right)^2+z^2\right]\right]\kappa=0 \\
&\left[z^2\left(\left(\frac{\partial}{\partial z}\right)^2+2\frac{\partial}{\partial z}+1\right)+z\left(\frac{\partial}{\partial z}+1\right)-\left[\left(\frac{1}{3}\right)^2+z^2\right]\right]\kappa=0 \\
&\left[z^2\left(\left(\frac{\partial}{\partial z}\right)^2+2\frac{\partial}{\partial z}\right)+z\frac{\partial}{\partial z}-\left[\left(\frac{1}{3}\right)^2+z^2-z^2-z\right]\right]\kappa=0 \\
&\left[z^2\left(\frac{\partial}{\partial z}\right)^2+(2z^2+z)\frac{\partial}{\partial z}-\left[\left(\frac{1}{3}\right)^2+z^2-z^2-z\right]\right]\kappa=0 \\
&\left[z^2\left(\frac{\partial}{\partial z}\right)^2+(2z^2+z)\frac{\partial}{\partial z}+\left(z-\frac{1}{9}\right)\right]\kappa=0,
\end{aligned}$$

which can be written as

$$(2.17) \quad \left[\frac{\partial^2}{\partial z^2}+2\frac{\partial}{\partial z}+\frac{1}{z}\frac{\partial}{\partial z}+\frac{1}{z}-\frac{1}{9z^2}\right]\kappa=0$$

³The conjugate variable of z is ζ , hence $\hat{K}(\frac{2}{3}z) = \frac{3}{2}\hat{K}(\frac{3}{2}\zeta)$. Indeed assuming $K(z) = \sum_{n \geq 0} a_n z^{-n}$, the Borel transform of $K(\frac{2}{3}z) = \sum_{n \geq 0} a_n \left(\frac{3}{2}\right)^n z^{-n}$ is by definition

$$\hat{K}\left(\frac{2}{3}z\right) = \sum_{k \geq 0} a_{n+1} \left(\frac{3}{2}\right)^{n+1} \frac{\zeta^n}{n!} = \frac{3}{2} \sum_{k \geq 0} \frac{a_{n+1}}{n!} \left(\frac{3}{2}\zeta\right)^n = \frac{3}{2}\hat{K}\left(\frac{3}{2}\zeta\right).$$

Its Borel transform is

$$\zeta^2 \hat{\kappa} - 2\zeta \hat{\kappa} + 1 * (-\zeta \hat{\kappa}) + \textcolor{red}{1} * \hat{\kappa} - \frac{1}{9} \zeta * \hat{\kappa} = 0$$

$$\zeta^2 \hat{\kappa} - 2\zeta \hat{\kappa} - \int_0^\zeta s \hat{\kappa}(s) ds + \textcolor{red}{1} * \hat{\kappa} - \frac{1}{9} \int_0^\zeta (\zeta - s) \hat{\kappa}(s) ds = 0$$

taking derivatives once

$$2\zeta \hat{\kappa} + \zeta^2 \hat{\kappa}' - 2\hat{\kappa} - 2\zeta \hat{\kappa}' - \zeta \hat{\kappa} + \textcolor{red}{\hat{\kappa}} - \frac{1}{9} \int_0^\zeta \hat{\kappa}(s) ds = 0$$

$$\zeta \hat{\kappa} + \zeta^2 \hat{\kappa}' - \textcolor{red}{\hat{\kappa}} - 2\zeta \hat{\kappa}' - \frac{1}{9} \int_0^\zeta \hat{\kappa}(s) ds = 0$$

taking derivatives once again

$$\hat{\kappa} + \zeta \hat{\kappa}' + 2\zeta \hat{\kappa}' + \zeta^2 \hat{\kappa}'' - \textcolor{red}{\hat{\kappa}'} - 2\hat{\kappa}' - 2\zeta \hat{\kappa}'' - \frac{1}{9} \hat{\kappa} = 0$$

$$(\zeta^2 - 2\zeta) \hat{\kappa}'' + (3\zeta - \textcolor{red}{3}) \hat{\kappa}' + \frac{8}{9} \hat{\kappa} = 0$$

The last equation is the hypergeomtric equation

$$\begin{aligned} \left(4\left(\frac{\zeta}{2}\right)^2 - 4\frac{\zeta}{2}\right) \hat{\kappa}'' + (6\frac{\zeta}{2} - 3) \hat{\kappa}' + \frac{8}{9} \hat{\kappa} &= 0 \\ \frac{\zeta}{2} \left(\frac{\zeta}{2} - 1\right) (4\hat{\kappa}'') + (3\frac{\zeta}{2} - \frac{3}{2}) (2\hat{\kappa}') + \left(\frac{2}{3}\right) \left(\frac{4}{3}\right) \hat{\kappa} &= 0, \end{aligned}$$

and a solution is given by

$$(2.18) \quad \hat{\kappa}(\zeta) = {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \textcolor{red}{\frac{3}{2}}; \frac{\zeta}{2}\right)$$

However, at pag. 6 in Aaron there is a different solution, namely

$$\begin{aligned} \hat{\kappa}(\zeta - 1) &= {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; \frac{1}{2} + \frac{\zeta}{2}\right) \\ \hat{\kappa}(\zeta) &= {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; 1 + \frac{\zeta}{2}\right) \end{aligned}$$