EXPONENTIAL INTEGRALS

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1. INTRODUCTION

2. FRACTIONAL DERIVATIVES AND BOREL TRANSFORM

For $\nu \in (-\infty, 1)$, the fractional integral $\partial^{\nu-1}_{x \text{ from } 0}$ is defined by

$$\partial^{\nu-1}_{x \text{ from } 0} f(x) := \frac{1}{\Gamma(1-\nu)} \int_0^x (x-x')^{-\nu} f(x')\, dx'.$$

It obeys the expected semigroup law **[Lazarević, §1.3]**

$$\partial^{\lambda}_{x \text{ from } 0} \partial^{\mu}_{x \text{ from } 0} = \partial^{\lambda+\mu}_{x \text{ from } 0} \qquad \lambda, \mu \in (-\infty, 0),$$

and agrees with ordinary repeated integration when ν is an integer **[Lazarević, equation 35]**.

For $\alpha \in (0,1)$ and integers $n \geq 0$, fractional derivatives $\partial^{n+\alpha}_{x \text{ from } 0}$ are defined by composing $\partial^{\alpha-1}_{x \text{ from } 0}$ with powers of $\frac{\partial}{\partial x}$. However, $\partial^{\alpha-1}_{x \text{ from } 0}$ and $\frac{\partial}{\partial x}$ don't commute: their commutator is an initial value operator **[check, clarify]**. Various ordering conventions give various definitions of $\partial^{n+\alpha}_{x \text{ from } 0} f(x)$, which differ by operators that act on the germ of f at zero **[Lazarević, §1.3—original source Podlubny]**. We'll use the *Riemann-Liouville* convention.

Definition 2.1. For $\alpha \in (0,1)$ and integers $n \geq 0$, the *Riemann-Liouville fractional derivative* $\partial^{n+\alpha}_{x \text{ from } 0}$ is defined by

$$\partial^{n+\alpha}_{x \text{ from } 0} := \left(\tfrac{\partial}{\partial x}\right)^{n+1} \partial^{\alpha-1}_{x \text{ from } 0}.$$

The symbol $\partial^{\mu}_{\zeta \text{ from } 0}$ is now defined for any $\mu \in \mathbb{R} \setminus \{0,1,2,3,\ldots\}$. It denotes a fractional integral when μ is negative, and a fractional derivative when μ is a positive non-integer.

The Riemann-Liouville fractional derivative is a left inverse of the fractional integral, in the sense that $\partial^{\lambda}_{x \text{ from } 0} \partial^{-\lambda}_{x \text{ from } 0} = \mathrm{Id}$ for all $\lambda \in (0,\infty)$. This extends the semigroup law:

$$\partial^{\lambda}_{x \text{ from } 0} \partial^{\mu}_{x \text{ from } 0} = \partial^{\lambda+\mu}_{x \text{ from } 0} \qquad \lambda \in \mathbb{R} \setminus \{0,1,2,3,\ldots\}, \quad \mu \in (-\infty, 0).$$

Each convention for the fractional derivative brings its own annoyances to interactions with the Borel transform.[1] The Riemann-Liouville derivative will be the least annoying for our purposes. Here's what's nice about it.

2.1. **Borel transform.** We breifly recall some basic facts on Borel transform and its properties (more details can be founded at [**?**], [**?**], [], []). The Borel transform $\mathcal{B}$ is a linear map from formal power seires in the time domain $\mathbb{C}_z$ to the space domain $\mathbb{C}_\zeta$:

$$\mathcal{B}(z^{-n-1}) := \frac{\zeta^n}{n!} \qquad , \text{if } n \geq 0$$
$$\mathcal{B}(1) := \delta$$

where δ is the element $(1,0) \in \mathbb{C} \times \mathbb{C}[[\zeta]]$[2], and $\mathcal{B}$ extends formally by linearity to

$$\mathcal{B} \colon \mathbb{C}[[z^{-1}]] \to \mathbb{C}\delta \oplus \mathbb{C}[[\zeta]]$$
$$\sum_{n\geq 0} a_n z^{-n} \to a_0 \delta + \sum_{n \geq 0} a_{n+1} \frac{\zeta^n}{n!}$$

Notice that we are at the level of formal series, so there are no convergence assumptions. However, there is a special class of formal series that behaves well under Borel transform, meaning that its Borel transform gives a holomorphic fucntion at the origin in $\mathbb{C}_\zeta$. These are the so called Gevrey-1 series: $f(z) = \sum_{n\geq 0} a_n z^{-n}$ is Gevrey-1 if there exists $A > 0$ such that $|a_n| \leq A^n n!$ for all $n \geq 0$.[3]

[1]See Remark **??** for examples.

[2]In the physical literature ([**?**][**?**]) many authors adopt a different convention, namely they define $\mathbb{B}(z^{-n}) := \frac{\zeta^n}{n!}$. However, we find more natural the mathematical convention as the Laplace transform is the inverse of $\mathcal{B}$ under suitable growth conditions.

[3]In asymptotic analysis Gevrey-k series $\sum_{n\geq 0} a_n z^{-n}$ have coefficients that grow as $(n!)^k$, i.e. there exists $A > 0$ such that $|a_n| \leq A^n (n!)^k$ for every $n \geq 0$.

Lemma 2.2. Let $f(z) = \sum_{n\geq 0} a_n z^{-n}$, $f(z)$ is Gevrey-1 if and only if $\hat{f}(\zeta) := \mathcal{B}(f) \in \mathbb{C}\{\zeta\}$.

In particular, we deduce from the lemma that the Borel transform is an isomorphism between Gevrey-1 series and $\mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$.

For what follows, it is convenient to extend the definition of $\mathcal{B}$ to fractional power of z, replacing the factorial with the Gamma function:

$$\mathcal{B}(z^{-\alpha}) := \frac{\zeta^{\alpha-1}}{\Gamma(\alpha)} \qquad \text{if } \alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$$

However, if $f(z) = \sum_{n\geq 0} a_n z^{-n}$ is a Gevrey-1 series, then the Borel transform of $\mathcal{B}(z^{-\alpha} f)(\zeta)$ is a germ of meromorphic function at the oringin: indeed

$$\begin{aligned}\mathcal{B}(z^{\alpha} f)(\zeta) &= \frac{\zeta^{\alpha-1}}{\Gamma(\alpha)} * \left(a_0 \delta + \sum_{n\geq 0} a_{n+1} \frac{\zeta^n}{n!} \right) \\ &= a_0 \frac{\zeta^{\alpha-1}}{\Gamma(\alpha)} + \sum_{n\geq 0} a_{n+1} \frac{\zeta^{n+\alpha}}{\Gamma(1+n+\alpha)}\end{aligned}$$

In particular, if $\alpha \in (1, +\infty)$, then $\mathcal{B}(z^{\alpha} f)(\zeta)$ is actually a germ of holomorphic functions on the Riemann surface of $\zeta^{\alpha-1}$, otherwise if $\alpha \in (-\infty, 1)$ then $\mathcal{B}(z^{\alpha} f)(\zeta)$ is a meromorphic functions on the Riemann surface where $\zeta^{\alpha-1}$ defines an analytic coordinate. The case of interest for usa in the following is when $\alpha = \frac{1}{2}$, hence if $a_0 \neq 0$, the Borel trasform will be merorphic at the origin and definied on the Riemann surface of the square root.

2.1.1. *Properties of the Borel transform.* We now recall some properties of the Borel transform: let $f(z) = \sum_{n\geq 0} a_n z^{-n}$ and denote by $\hat{f}(\zeta) := \mathcal{B}(f(z))$

(i) *[fractional derivative]* if $\alpha \in \mathbb{R}_{\geq 0}$, $\mathcal{B}\big(z^{\alpha} f(z)\big) = \partial_{\zeta}^{\alpha} \hat{f}(\zeta)$
(ii) *[fractional integral]* if $\alpha \in \mathbb{R}_{<0} \setminus \mathbb{Z}_{\geq 0}$, $\mathcal{B}\big(z^{\alpha} f(z)\big) = \partial_{\zeta}^{\alpha} \hat{f}(\zeta)$
(iii) if $n \in \mathbb{Z}$ and $z^n f(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$, then $\mathcal{B}(z^n f(z)) = \partial_{\zeta}^{n} \hat{f}(\zeta)$
(iv) $\mathcal{B}\big(\partial_z^n f(z)\big) = (-\zeta)^n \hat{f}(\zeta)$, for every $n \geq 0$
(v) $\mathcal{B}(f(z-c)) = e^{-c\zeta} \hat{f}(\zeta)$
(vi) if $g(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$, then $\mathcal{B}(f(z)g(z)) = \int_0^{\zeta} d\zeta' \hat{f}(\zeta - \zeta')\hat{g}(\zeta') =: \hat{f} * \hat{g}$, where $\hat{g}(\zeta) = \mathcal{B}(g(z))$.

Lemma 2.3. For any non-integer $\mu \in (0, \infty)$ and any integer $k \geq 0$,

$$\partial^{\mu}_{\zeta \text{ from } 0} \big[\mathcal{B} z^{-(k+1)}\big](\zeta) = \big[\mathcal{B} z^{\mu} z^{-(k+1)}\big](\zeta).$$

Proof. We'll show that for any $\alpha \in (0,1)$ and any integer $n \geq 0$, the claim holds with $\mu = n + \alpha$. First, evaluate

$$\begin{aligned}
\partial_{\zeta \text{ from } 0}^{\alpha-1}\left[\mathcal{B}z^{-(k+1)}\right](\zeta) &= \frac{1}{\Gamma(1-\alpha)} \int_0^\zeta (\zeta-\zeta')^{-\alpha} \frac{\zeta'^k}{\Gamma(k+1)}\, d\zeta' \\
&= \frac{1}{\Gamma(1-\alpha)\Gamma(k+1)} \int_0^1 (\zeta-\zeta t)^{-\alpha} (\zeta t)^k \zeta\, dt \\
&= \frac{\zeta^{k-(\alpha-1)}}{\Gamma(1-\alpha)\Gamma(k+1)} \int_0^1 (1-t)^{-\alpha} t^k\, dt \\
&= \frac{\zeta^{k-(\alpha-1)}}{\Gamma\big(k-(\alpha-1)+1\big)}
\end{aligned}$$

by reducing the integral to Euler's beta function **[DLMF 5.12.1]**. This establishes that

$$\left(\tfrac{\partial}{\partial\zeta}\right)^{n+1} \partial_{\zeta \text{ from } 0}^{\alpha-1}\left[\mathcal{B}z^{-(k+1)}\right](\zeta) = \frac{\zeta^{k-(n+\alpha)}}{\Gamma\big(k-(n+\alpha)+1\big)} \tag{2.1}$$

for $n = -1$. If (2.1) holds for $n = m$, it also holds for $n = m+1$, because

$$\begin{aligned}
\tfrac{\partial}{\partial\zeta}\left(\tfrac{\partial}{\partial\zeta}\right)^{m+1} \partial_{\zeta \text{ from } 0}^{\alpha-1}\left[\mathcal{B}z^{-(k+1)}\right](\zeta) &= \tfrac{\partial}{\partial\zeta}\left(\frac{\zeta^{k-(m+\alpha)}}{\big(k-(m+\alpha)\big)\Gamma\big(k-(m+\alpha)\big)}\right) \\
&= \frac{\zeta^{k-(m+1+\alpha)}}{\Gamma\big(k-(m+\alpha)\big)}
\end{aligned}$$

Hence, (2.1) holds for all $n \geq -1$, and the desired result quickly follows. The condition $\alpha \in (0,1)$ saves us from the trouble we'd run into if $k-(m+\alpha)$ were in $\mathbb{Z}_{\leq 0}$. This is how we avoid the initial value corrections that appear in ordinary derivatives of Borel transforms. $\square$

Proof. (i) follows from Lemma 2.3.

(ii) Notice that for $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$ the fractional integral $\partial_\zeta^\alpha \zeta^k = \zeta^{k-\alpha} \frac{k!}{\Gamma(k-\alpha+1)}$, hence

$$\mathcal{B}(z^\alpha f(z)) = \sum_{k\geq 0} a_k \tfrac{\zeta^{k-\alpha}}{\Gamma(k-\alpha+1)} = \sum_{k\geq 0} a_k \tfrac{\zeta^{k-\alpha}}{k!} \tfrac{k!}{\Gamma(k-\alpha+1)} = \sum_{k\geq 0} a_k \tfrac{1}{k!} \partial_\zeta^\alpha \zeta^k = \partial_\zeta^\alpha \hat{f}(\zeta).$$

(iii) $z^n f(z) = \sum_{k\geq 0} a_k z^{-(k-n)-1}$ and by assumption $k - n \geq 0$, hence by defintion

$$\mathcal{B}(z^n f(z)) = \sum_{k\geq n} a_k \tfrac{\zeta^{k-n}}{(k-n)!} = \sum_{k\geq n} a_k \tfrac{1}{k!} \tfrac{k!\zeta^{k-n}}{(k-n)!} = \sum_{k\geq n} a_k \tfrac{1}{k!} \partial_\zeta^n \zeta^k = \partial_\zeta^n \hat{f}(\zeta).$$

(vi) $\partial_z^n f(z) = \sum_{k\geq 0} a_k (-1)^n z^{-k-n-1} \frac{\Gamma(k+n+1)}{k!}$, hence

$$\mathcal{B}(z^\alpha f(z)) = \sum_{k\geq 0} a_k (-1)^n \tfrac{\zeta^{k+n}}{\Gamma(k+n+1)} \tfrac{\Gamma(k+n+1)}{k!} = (-\zeta)^n \hat{f}(\zeta).$$

(v) see Lemma 5.10 [**?**].

(vi) see Definition 5.12 and Lemma 5.14 [**?**]. $\square$

Remark 2.4. We notice that properties (i) and (ii) are special cases of property (vi), indeed we can use the convolution product

$$\begin{aligned}
\mathcal{B}(z^\alpha f(z)) &= \mathcal{B}(z^\alpha) * \hat{f}(\zeta) \\
&= \frac{\zeta^{-\alpha-1}}{\Gamma(-\alpha)} * \hat{f}(\zeta) \\
&= \int_0^\zeta \frac{(\zeta')^{-\alpha-1}}{\Gamma(-\alpha)} \sum_{k\geq 0} a_k \frac{(\zeta-\zeta')^k}{k!} d\zeta' \\
&= \sum_{k\geq 0} \frac{a_k}{k!} \frac{1}{\Gamma(-\alpha)} \int_0^1 (t\zeta)^{-\alpha-1} (\zeta - t\zeta)^k \zeta\, dt \\
&= \sum_{k\geq 0} \frac{a_k}{k!} \frac{1}{\Gamma(-\alpha)} \zeta^{k-\alpha} \int_0^1 t^{-\alpha-1}(1-t)^k dt \\
&= \sum_{k\geq 0} \frac{a_k}{k!} \frac{1}{\Gamma(-\alpha)} \zeta^{k-\alpha} \frac{\Gamma(k+1)\Gamma(-\alpha)}{\Gamma(k-\alpha+1)} \\
&= \sum_{k\geq 0} \frac{a_k}{k!} \partial_\zeta^\alpha \zeta^k \\
&= \partial_\zeta^\alpha \hat{f}(\zeta)
\end{aligned}$$

2.2. **Laplace transform. Copy from Aaron**

2.3. **Borel–Laplace summability.**

$$\begin{tikzcd}
\mathbb{C} \oplus z^{-1}\mathbb{C}[[z^{-1}]]_1 \arrow[rr, "\mathcal{B}"] & & \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\} \cap \mathcal{N} \arrow[dl, "\mathcal{L}"] \\
& s \arrow[ul, "\text{Gevrey asymptotic}"] &
\end{tikzcd}$$

2.4. **Resurgence.** Alternative to Borel–Laplace summability, J. Ecalle introduced the theory of resurgence: divergent power series that are Gevrey-1 become germs of holomorphic functions at the origin in the Borel plane. Being divergent, they have singularities in Borel plane and the aim is to investigate the type of singularities and the analytic continuation of the germ. Indeed a formal series is resurgent if it admits an *endless* analytic continuation.

Definition 2.5. A germ of analytic functions $\hat{f}$ at the origin is *endlessly continuable* on $\mathbb{C}$ if for all $L > 0$, there exists a finite set $\Omega \subset \mathbb{C}$ of singularities, such that $\hat{f}$ can be analytically continued along all paths whose length is less than L, avoiding the singularities Ω.

$$\mathbb{C} \oplus z^{-1}\mathbb{C}[[z^{-1}]]_1 \xrightarrow{\mathcal{B}} \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\} \xrightarrow{\text{analytic cont.}} \mathcal{R}_\Omega \xrightarrow{\pi} \widetilde{\mathbb{C}\setminus\Omega}$$

Among resurgent series, the class of simple resurgent series is caracterized by having a special type of singularities:

Definition 2.6. A holomorphic function $\hat{f}$ in an open disk $D \subset \mathbb{C}_\zeta$ has a simple singularity at ω, adherent to D, if there exist $\alpha \in \mathbb{C}$ and a germs of analytic functions at the origin $\hat{\Phi}(\zeta) \in \mathbb{C}\{\zeta\}$, such that

$$\hat{f}(\zeta) = \frac{\alpha}{2\pi i(\zeta - \omega)} + \frac{1}{2\pi i}\log(\zeta - \omega)\hat{\Phi}(\zeta - \omega) + \text{hol. fct.} \tag{2.2}$$

for all $\zeta \in D$ close enough to ω. The constant α is called the residuum and $\hat{\Phi}$ the minor.

The holomorphic function $\hat{\Phi}$ associated with the logarithmic singularity can be obtained by considering the analytic continuation of $\hat{f}$ across the logarithmic branch cut

$$\hat{\Phi}(\zeta) = \hat{f}(\zeta + \omega) - \hat{f}(\zeta e^{-2\pi i} + \omega) \tag{2.3}$$

where with $\hat{f}(\zeta e^{-2\pi i} + \omega)$ is the analytic continuation of $\hat{f}$ along the circular path $\omega + \zeta e^{-2\pi i t}$ with $t \in [0,1]$.

Definition 2.7 ([**?**] Definition 7)**.** A simple resurgent function is a resurgent function $c\delta + \hat{f}(\zeta)$ such that, for each $\omega \in \Omega$ and for each path γ which starts from the origin $0 \in \mathbb{C}_\zeta$, lies in $\mathbb{C} \setminus \Omega$ and has its extremity in the disc of radius π centred at ω, the branch $\text{cont}_\gamma \hat{f}$ has a simple singularity at ω.

3. Resurgence of exponential integrals

Let X be a $N-$dim manifold, $f : X \to \mathbb{C}$ be a holomorphic Morse function with only simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$I(z) := \int_{\mathcal{C}} e^{-zf}\,\nu \tag{3.1}$$

where $\mathcal{C}$ is a suitable countur such that the integral is well defined. For any Morse cirtial points x_α of f, the saddle point approximation gives the following formal series

$$I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\,\nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)}(2\pi)^{N/2} z^{-N/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n} \qquad \text{as } z \to \infty \tag{3.2}$$

where $\mathcal{C}_\alpha$ is a steepest descent path through the critical point x_α. Notice that $f \circ \mathcal{C}_\alpha$ lies in the ray $\zeta_\alpha + [0, \infty)$, where $\zeta_\alpha := f(x_\alpha)$.

Theorem 3.1. Let $N = 1$. Let $\tilde{\varphi}_\alpha(z) := e^{-zf(x_\alpha)}(2\pi)^{1/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n}$ and assume $f''(x_\alpha) \neq 0$ for every critical point x_α. Then:

(1) The series $\tilde{\varphi}_\alpha$ is Gevrey-1.

(2) The series $\hat{\varphi}_\alpha(\zeta) := \mathcal{B}(\tilde{\varphi})$ converges near $\zeta = \zeta_\alpha$.

(3) If you continue the sum of $\hat{\phi}_\alpha$ along the ray going rightward from ζ_α, and take its Laplace transform along that ray, you'll recover $z^{1/2} I_\alpha$.

(4) For any ζ on the ray going rightward from ζ_α, we have

$$\begin{aligned}\hat{\varphi}_\alpha(\zeta) &= \partial^{3/2}_{\zeta \text{ from } \zeta_\alpha}\left(\int_{\mathcal{C}_\alpha(\zeta)} \nu\right)\\ &= \left(\tfrac{\partial}{\partial\zeta}\right)^2 \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{\zeta_\alpha}^{\zeta} (\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_\alpha(\zeta')} \nu\right) d\zeta',\end{aligned}$$

where $\mathcal{C}_\alpha(\zeta)$ is the part of $\mathcal{C}_\alpha$ that goes through $f^{-1}([\zeta_\alpha, \zeta])$. Notice that $\mathcal{C}_\alpha(\zeta)$ starts and ends in $f^{-1}(\zeta)$. [**Be careful about the orientation of $\mathcal{C}_\alpha$.**]

Proof. Part (1): Let's write $\approx$ when two functions are asymptotic (at all orders around the base point [**is this the right condition?**]), and $\sim$ when a function is asymptotic to a formal power series (at the truncation order of each partial sum).

Since f is Morse, we can find a holomorphic chart τ around x_α with $\frac{1}{2}\tau^2 = f - \zeta_\alpha$. Let $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$ be the parts of $\mathcal{C}_\alpha$ that go from the past to x_α and from x_α to the future, respectively. We can arrange for τ to be valued in $(-\infty, 0]$ and $[0, \infty)$ on $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$, respectively. [**We should explicitly spell out and check the conditions that make this possible. I think we're implicitly orienting $\mathcal{C}_\alpha$ so that τ in the upper half-plane.**] Since ν is holomorphic, we can express it as a Taylor series

$$\nu = \sum_{n\ge 0} b_n^\alpha \tau^n \, d\tau$$

that converges in some disk $|\tau| < \varepsilon$.

By the steepest descent method,

$$e^{-z\zeta_\alpha} I_\alpha(z) \approx \int_{\tau\in[-\varepsilon,\varepsilon]} e^{-z\tau^2/2}\,\nu$$

as $z \to \infty$. [**I need to learn how this works! Do we get asymptoticity at all orders? —Aaron**] Plugging in the Taylor series above, we get

$$\begin{aligned}e^{-z\zeta_\alpha} I_\alpha(z) &\approx \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n\ge 0} b_n^\alpha \tau^n \, d\tau\\ &= \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n\ge 0} b_{2n}^\alpha \tau^{2n} \, d\tau.\end{aligned}$$

By the dominated convergence theorem,[4]

$$
\begin{aligned}
e^{-z\zeta_\alpha} I_\alpha(z) &\approx \sum_{n\geq 0} b_{2n}^\alpha \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2}\tau^{2n}\, d\tau \\
&= \sum_{n\geq 0} (2n-1)!!\, b_{2n}^\alpha \left[\sqrt{2\pi}\, z^{-(n+1/2)} \operatorname{erf}\!\left(\varepsilon\sqrt{z/2}\right) - 2e^{-z\varepsilon^2/2} \sum_{\substack{k\in\mathbb{N}_+ \\ k\leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{n-k+1} \right].
\end{aligned}
$$

The annoying $e^{-z\varepsilon^2/2}$ correction terms are dwarfed by their $z^{-(n+1/2)}$ counterparts when z is large. These terms are crucial, however, for the convergence of the sum. To see why, consider their absolute sum C_{exp}. When $z\in[0,\infty)$,

$$
\begin{aligned}
C_{\text{exp}} &= 2e^{-\operatorname{Re}(z)\varepsilon^2/2} \sum_{n\geq 1} (2n-1)!! \left| b_{2n}^\alpha \sum_{\substack{k\in\mathbb{N}_+ \\ k\leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{n-k+1} \right| \\
&= 2e^{-z\varepsilon^2/2} \sum_{n\geq 1} (2n-1)!! \left| b_{2n}^\alpha \right| \sum_{\substack{k\in\mathbb{N}_+ \\ k\leq n}} \frac{\varepsilon^{2k-1}}{(2k-1)!!} z^{n-k+1} \\
&\geq -2\varepsilon e^{-z\varepsilon^2/2} \sum_{n\geq 1} (2n-1)!! \left| b_{2n}^\alpha \right| z^n,
\end{aligned}
$$

which diverges for typical f and ν. **[Does it? Veronica points out that we expect b_{2n} to shrink at least as fast as $(n!)^{-1}$.]**

This argument suggests that no matter how tiny the correction terms get, we can't expect to swat them all aside. We can, however, set aside any finite set of them. **[Use Miller's proof of Watson's lemma in place of the following argument, which has a few soft spots.]** For each cutoff N, the tail

$$
\sum_{n\geq N} b_{2n}^\alpha \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2}\tau^{2n}\, d\tau
$$

[4] Notice that the sum over k is empty when $n=0$. Following convention, we extend the double factorial to all odd integers by its recurrence relation, giving $(-1)!!=1$.

For each cutoff N, the tail error **[check]**

$$\begin{aligned}
\left|\sum_{n\geq N} b_{2n}^{\alpha}\int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2}\tau^{2n}\,d\tau\right| &\leq \sum_{n\geq N}\left|b_{2n}^{\alpha}\right|\int_{-\varepsilon}^{\varepsilon} e^{-|z|\tau^2/2}\tau^{2n}\,d\tau \\
&\leq \sum_{n\geq N}\left|b_{2n}^{\alpha}\right|\int_{-\infty}^{\infty} e^{-|z|\tau^2/2}\tau^{2n}\,d\tau \\
&= \sqrt{2\pi}\sum_{n\geq N}(2n-1)!!\left|b_{2n}^{\alpha}\right||z|^{-(n+1/2)} \\
&\lesssim \sum_{n\geq N}(2n-1)!!\,\varepsilon^{-2n}|z|^{-(n+1/2)} \\
&= \varepsilon\sum_{n\geq N}(2n-1)!!\left(\varepsilon^{-1}\right)^{2n+1}\left(|z|^{-1/2}\right)^{2n+1} \\
&= \varepsilon\sum_{n\geq N}(2n-1)!!\left(\varepsilon^{-1}|z|^{-1/2}\right)^{2n+1} \\
&= \textbf{uh-oh!}
\end{aligned}$$

is in $o_{z\to\infty}(z^{-N})$ **[check]**, and the absolute sum

$$\begin{aligned}
C_{\text{exp}}^{N} = 2e^{-\text{Re}(z)\varepsilon^2/2}\sum_{n=1}^{N-1}(2n-1)!!\left|b_{2n}^{\alpha}\sum_{\substack{k\in\mathbb{N}_+\\ k\leq n}}\frac{\varepsilon^{2k-1}}{(2k-1)!!}z^{n-k+1}\right| \\
\leq 2e^{-\text{Re}(z)\varepsilon^2/2}\sum_{n=1}^{N-1}(2n-1)!!\left|b_{2n}^{\alpha}\right|\sum_{\substack{k\in\mathbb{N}_+\\ k\leq n}}\frac{\varepsilon^{2k-1}}{(2k-1)!!}|z|^{n-k+1} \\
\geq -2\varepsilon e^{-z\varepsilon^2/2}\sum_{n\geq 1}(2n-1)!!\left|b_{2n}^{\alpha}\right|z^{n},
\end{aligned}$$

is in $o_{z\to\infty}(z^{-m})$ for every m **[check]**.Hence,

$$e^{-z\zeta_\alpha}I_\alpha(z)\sim\sqrt{2\pi}\sum_{n\geq 0}(2n-1)!!\,b_{2n}^{\alpha}\,z^{-(n+1/2)}\,\text{erf}\left(\varepsilon\sqrt{z/2}\right).$$

The differences $1-\text{erf}\left(\varepsilon\sqrt{z/2}\right)$ shrink exponentially as z grows, allowing the simpler estimate

$$e^{-z\zeta_\alpha}I_\alpha(z)\sim\sqrt{2\pi}\sum_{n\geq 0}(2n-1)!!\,b_{2n}^{\alpha}\,z^{-(n+1/2)}.$$

Call the right-hand side $\tilde{I}_\alpha$. We now see that $a_{\alpha,n}=(2n-1)!!\,b_{2n}^{\alpha}$ in the statement of the theorem. **[Resolve discrepancy with previous calculation.]** Note that **[explain formally what it means to center at ζ_α]**

$$\begin{aligned}
\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha &= \sqrt{2\pi}\sum_{n\geq 0}\frac{2^n}{\sqrt{\pi}}\Gamma\left(n+\tfrac{1}{2}\right)b_{2n}^{\alpha}\,\frac{(\zeta-\zeta_\alpha)^{n-1/2}}{\Gamma\left(n+\tfrac{1}{2}\right)} \\
&= \sum_{n\geq 0}2^{n+1/2}\,b_{2n}^{\alpha}\,(\zeta-\zeta_\alpha)^{n-1/2}.
\end{aligned}$$

We know from the definition of ε that $\left|b_n^\alpha\right| \varepsilon^n \lesssim 1$. Recalling that $(2n-1)!! \approx (\pi n)^{-1/2}\, 2^n\, n!$ as $n \to \infty$, we deduce that $|a_{\alpha,n}| \lesssim \left(\frac{2}{\varepsilon^2}\right)^n\, n!$, showing that $\tilde{\varphi}_\alpha$ is Gevrey-1.

Part (2):

$$\hat{\varphi}_\alpha(\zeta) = \mathcal{B}\left(e^{-zf(x_\alpha)}(2\pi)^{1/2}\sum_{n\geq 0} a_{\alpha,n} z^{-n}\right)(\zeta) = T_{f(x_\alpha)}(2\pi)^{1/2}\left(\delta a_0 + \sum_{n\geq 0} a_{n+1}\frac{\zeta^n}{n!}\right)$$
$$(2\pi)^{1/2}\left(\delta(f_{x_\alpha})a_0 + \sum_{n\geq 0} a_{n+1}\frac{(\zeta - f(x_\alpha))^n}{n!}\right)$$

Since $a_n \leq C A^n\, n!$, the series $\sum_{n\geq 0} a_{n+1}\frac{(\zeta - f(x_\alpha))^n}{n!}$ has a finite radius of convergence.

Part (3): Let's recast the integral I_α into the f plane. As ζ goes rightward from ζ_α, the start and end points of $\mathcal{C}_\alpha(\zeta)$ sweep backward along $\mathcal{C}_\alpha^-(\zeta)$ and forward along $\mathcal{C}_\alpha^+(\zeta)$, respectively. Hence, we have

$$\begin{aligned} I_\alpha(z) &= \int_{\mathcal{C}_\alpha} e^{-zf}\, \nu \\ &= \int_{\zeta_\alpha}^{\infty} e^{-z\zeta}\left[\frac{\nu}{df}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)} d\zeta. \end{aligned}$$

Noticing that the right-hand side is a Laplace transform, we learn that

$$\hat{I}_\alpha(\zeta) = \left[\frac{\nu}{df}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)}. \tag{3.3}$$

We can rewrite our Taylor series for ν as

$$\begin{aligned} \nu &= \sum_{n\geq 0} b_n^\alpha [2(f-\zeta_\alpha)]^{n/2}\frac{df}{[2(f-\zeta_\alpha)]^{1/2}} \\ &= \sum_{n\geq 0} b_n^\alpha [2(f-\zeta_\alpha)]^{(n-1)/2}\, df, \end{aligned}$$

taking the positive branch of the square root on $\mathcal{C}_\alpha^+$ and the negative branch on $\mathcal{C}_\alpha^-$. Plugging this into our expression for $\hat{I}_\alpha$, we learn that

$$\begin{aligned} \hat{I}_\alpha(\zeta) &= \left[\sum_{n\geq 0} b_n^\alpha [2(f-\zeta_\alpha)]^{(n-1)/2}\right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)} \\ &= \sum_{n\geq 0} b_n^\alpha\Big([2(\zeta-\zeta_\alpha)]^{(n-1)/2} - (-1)^{n-1}[2(\zeta-\zeta_\alpha)]^{(n-1)/2}\Big) \\ &= \sum_{n\geq 0} 2 b_{2n}^\alpha [2(\zeta-\zeta_\alpha)]^{n-1/2} \\ &= \sum_{n\geq 0} 2^{n+1/2} b_{2n}^\alpha (\zeta-\zeta_\alpha)^{n-1/2} \\ &= \mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha. \end{aligned}$$

We already knew, from the general theory of the Borel transform, that the sum of $\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha$ would be asymptotic to $\hat{I}_\alpha$. We've now shown that the sum of $\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha$ is actually equal to $\hat{I}_\alpha$.

Theorem 2.8 tells us that

$$\begin{aligned}\mathcal{B}_{\zeta_\alpha}\tilde{I}_\alpha &:= \mathcal{B}_{\zeta_\alpha} z^{-1/2}\tilde{\varphi}_\alpha \\ &= \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha}\mathcal{B}\tilde{\varphi}_\alpha \\ &= \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha.\end{aligned}$$

It follows, from our conclusion above, that

$$\hat{I}_\alpha(\zeta) = \partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha. \tag{3.4}$$

Taking the Laplace transform of both sides and applying **the inverse of Theorem 2.8 that works for shifted analytic functions**, we see that

$$\begin{aligned}I_\alpha(z) &= \mathcal{L}_{\zeta,\zeta_\alpha}\left[\partial^{-1/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha\right] \\ &= z^{-1/2}\mathcal{L}_{\zeta,\zeta_\alpha}\hat{\varphi}_\alpha,\end{aligned}$$

as we claimed.

Part (4): Since fractional integrals form a semigroup, equation (3.4) implies that

$$\partial^{-1}_{\zeta \text{ from } \zeta_\alpha}\hat{I}_\alpha(\zeta) = \partial^{-3/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha.$$

Rewriting equation (3.3) as

$$\hat{I}_\alpha(\zeta) = \partial_\zeta\left(\int_{\mathcal{C}_\alpha(\zeta)}\nu\right),$$

we can see that

$$\partial^{-1}_{\zeta \text{ from } \zeta_\alpha}\hat{I}_\alpha(\zeta) = \int_{\mathcal{C}_\alpha(\zeta)}\nu - \int_{\mathcal{C}_\alpha(0)}\nu.$$

The initial value term vanishes, because the path $\mathcal{C}_\alpha(0)$ is a point. Hence,

$$\int_{\mathcal{C}_\alpha(\zeta)}\nu = \partial^{-3/2}_{\zeta \text{ from } \zeta_\alpha}\hat{\varphi}_\alpha(\zeta).$$

Recalling that the Riemann-Liouville fractional derivative is a left inverse of the fractional integral, we conclude that

$$\partial^{3/2}_{\zeta \text{ from } \zeta_\alpha}\left(\int_{\mathcal{C}_p(\zeta)}\nu\right) = \hat{\varphi}_p(\zeta).$$

□

Example 3.2 (Airy)**.** By definition,

$$\operatorname{Ai}(x) := \frac{1}{2\pi i}\int_{-\infty e^{-i\frac{\pi}{3}}}^{\infty e^{i\frac{\pi}{3}}} e^{t^3/3 - xt}\, dt.$$

Define $I(z)$ by the change of coordinates $z = x^{3/2}$, $I(z) = -2\pi i z^{-1/3}\operatorname{Ai}(x)$. This new function satisfies the ODE[5]

$$I''(z) - \frac{4}{9}I(z) + \frac{I'(z)}{z} - \frac{1}{9}\frac{I(z)}{z^2} = 0. \tag{3.5}$$

A formal solution of (3.5) can be computed by making the following ansatz

$$\tilde{I}(z) = \sum_{k\in\mathbb{N}^2} U^k e^{-\lambda\cdot k z} z^{-\tau\cdot k} w_k(z) \tag{3.6}$$

with $U^{(k_1,k_2)} = U_1^{k_1}U_2^{k_2}$ and $U_1, U_2 \in \mathbb{C}$ are constant parameter, $\lambda = (\frac{2}{3}, -\frac{2}{3})$, $\tau = (\frac{1}{2}, \frac{1}{2})$, and $\tilde{w}_k(z) \in \mathbb{C}[[z^{-1}]]$. In addition, we can check that the only non zero $\tilde{w}_k(z)$ occurs at $k = (1,0)$ and $k = (0,1)$, therefore

$$\tilde{I}(z) = U_1 e^{-2/3z} z^{-1/2}\tilde{w}_+(z) + U_2 e^{2/3z} z^{-1/2}\tilde{w}_-(z) \tag{3.7}$$

where from now on we denote $\tilde{w}_+(z) = \tilde{w}_{(1,0)}(z)$ and $\tilde{w}_-(z) = \tilde{w}_{(0,1)}(z)$. In particular, $\tilde{w}_+(z)$ and $\tilde{w}_-(z)$ are formal solution of

$$\tilde{w}_+'' - \frac{4}{3}\tilde{w}_+' + \frac{5}{36}\frac{\tilde{w}_+}{z^2} = 0 \tag{3.8}$$

$$\tilde{w}_-'' + \frac{4}{3}\tilde{w}_-' + \frac{5}{36}\frac{\tilde{w}_-}{z^2} = 0 \tag{3.9}$$

Taking the Borel transform of (3.8), (3.9) we get

$$\zeta^2\hat{w}_+(\zeta) + \frac{4}{3}\zeta\hat{w}_+ + \frac{5}{36}\zeta * \hat{w}_+ = 0$$

$$\zeta^2\hat{w}_+(\zeta) + \frac{4}{3}\zeta\hat{w}_+ + \frac{5}{36}\int_0^\zeta (\zeta - \zeta')\hat{w}_+(\zeta')d\zeta' = 0$$

$$\zeta^2\hat{w}_-(\zeta) - \frac{4}{3}\zeta\hat{w}_- + \frac{5}{36}\zeta * \hat{w}_- = 0$$

$$\zeta^2\hat{w}_-(\zeta) - \frac{4}{3}\zeta\hat{w}_- + \frac{5}{36}\int_0^\zeta (\zeta - \zeta')\hat{w}_-(\zeta')d\zeta' = 0$$

[5] $\operatorname{Ai}(x)$ solves the Airy equation $y'' = xy$.

and taking derivatives we get

$$\zeta(\frac{4}{3}+\zeta)\hat{w}_+''+(\frac{8}{3}+4\zeta)\hat{w}_+'+\frac{77}{36}\hat{w}_+=0$$
$$\frac{4}{3}\zeta(1+\frac{3}{4}\zeta)\hat{w}_+''+(\frac{8}{3}+4\zeta)\hat{w}_+'+\frac{77}{36}\hat{w}_+=0$$
$$u(1-u)\hat{w}_+''(u)+(2-4u)\hat{w}_+'(u)-\frac{77}{36}\hat{w}_+(u)=0 \qquad u=-\frac{3}{4}\zeta$$

$$\zeta(-\frac{4}{3}+\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0$$
$$\frac{4}{3}\zeta(-1+\frac{3}{4}\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0$$
$$u(1-u)\hat{w}_-''(u)+(2-4u)\hat{w}_-'(u)-\frac{77}{36}\hat{w}(u)=0 \qquad u=\frac{3}{4}\zeta$$

Notice that the latter equations are hypergeometric, hence a solution is given by

$$\hat{w}_+(\zeta)=c_1\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right) \tag{3.10}$$

$$\hat{w}_-(\zeta)=c_2\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right) \tag{3.11}$$

for some constants $c_1,c_2\in\mathbb{C}$ (see DLMF 15.10.2). In addition $\hat{w}_\pm(\zeta)$ have a log singularity respectively at $\zeta=\mp\frac{4}{3}$, therefore they are $\{\mp\frac{4}{3}\}$-resurgent functions.[6]

Our next goal is to prove that the Borel transform of $\tilde{I}(z)$ can be written in terms of $1/f'(f^{-1}(\zeta))$, namely formula (**??**). It is convenient to consider the two asymptotic formal solutions separately, namely we define

$$\tilde{I}_{-1}(z):=e^{-2/3z}z^{-1/2}\tilde{w}_+(z)=:z^{-1/2}\tilde{u}_+(z) \tag{3.12}$$

$$\tilde{I}_1(z):=e^{2/3z}z^{-1/2}\tilde{w}_-(z)=:z^{-1/2}\tilde{u}_-(z) \tag{3.13}$$

In particular, $\tilde{u}_\pm(z)$ are solutions of

$$\tilde{u}''(z)-\frac{4}{9}\tilde{u}(z)+\frac{5}{36}\frac{\tilde{u}(z)}{z^2}=0 \tag{3.14}$$

with asymptotic behaviour $\tilde{u}_\pm(z)\sim O(e^{\pm 2/3z})$ as $z\to\infty$.

[6]The solution we find are equal to the ones in DLMF §9.7. They also agree with slide 10 of Maxim's talk for ERC and with the series (6.121) in Sauzin's book. However they do not agree with (2.16) in Mariño's Diablerets.

The Borel transforms $\hat{u}_{\pm}(\zeta)$ solve the same equation

$$\zeta^2\hat{u} - \frac{4}{9}\hat{u} + \frac{5}{36}\zeta * \hat{u}$$

$$\zeta^2\hat{u} - \frac{4}{9}\hat{u} + \frac{5}{36}\int_0^{\zeta}(\zeta-\zeta')\hat{u}(\zeta')d\zeta'$$

taking derivatives is equivalent to

$$(\zeta^2 - \frac{4}{9})\hat{u}''(\zeta) + 4\zeta\hat{u}'(\zeta) + \frac{77}{36}\hat{u}(\zeta) = 0$$

and Mathematica gives the following solutions

$$\hat{u}(\zeta) = c_1\,{}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{9}{4}\zeta^2\right) + \frac{3i}{2}\zeta c_2\,{}_1F_2\left(\frac{13}{12},\frac{17}{12},\frac{3}{2},\frac{9}{4}\zeta^2\right) =$$

$$= c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}}\left({}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{1}{2}-\frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{1}{2}+\frac{3}{4}\zeta\right)\right) \qquad \text{see DLMF 15.8.27}$$

$$+\frac{3i}{2}\zeta c_2\left(\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{3\zeta\Gamma(-\frac{1}{2})\Gamma(2)}\right)\left({}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}-\frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}+\frac{3}{4}\zeta\right)\right) \qquad \text{see DLMF 15.8.28}$$

$$= \left(c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} - c_2 i\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right){}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}-\frac{3}{4}\zeta\right) +$$

$$+\left(c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} + c_2 i\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right){}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}+\frac{3}{4}\zeta\right)$$

Since $\hat{u}_+$ has a simple singularity at $\zeta = -2/3$ and $\hat{u}_-$ has a simple singularity at $\zeta = 2/3$, we have

$$\hat{u}_+(\zeta) = C_1 T_{-\frac{2}{3}}\,{}_2F_1\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right) = C_1 T_{-\frac{2}{3}}\;\hat{w}_+(\zeta)$$

$$\hat{u}_-(\zeta) = C_2 T_{\frac{2}{3}}\,{}_2F_1\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right) = C_2 T_{\frac{2}{3}}\;\hat{w}_-(\zeta)$$

Lemma 3.3. The following identity holds true

$$ {}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right) = \frac{1}{1-u^2} \qquad \zeta = \frac{u^3}{3} - u \tag{3.15}$$

Proof. From the special case of hypergeometric function (see 15.4.14 DLMF) we have the following identity:

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right) = \frac{\cos(y)}{\cos(3y)} \qquad 3y = \arcsin\left(\frac{3}{2}\zeta\right)$$

$$= \frac{\cos(y)}{\cos(2y)\cos(y) - \sin(2y)\sin(y)}$$

$$= \frac{1}{\cos(2y) - 2\sin^2(y)}$$

$$= \frac{1}{1-4\sin^2(y)} \qquad \zeta = 2\sin(y) - \frac{8}{3}\sin^3(y)$$

Therefore, if $u := -2\sin(y)$, we have $\zeta = \frac{u^3}{3} - u = f(u)$ and

$$ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}\zeta^2\right) = \frac{1}{1-u^2} = -\frac{1}{f'(u)} $$

□

Then equations (**??**) is equivalent to: **[Now that we've switched to the Riemann-Liouville derivative, the claim that was referenced here no longer involves ν/df. The calculation above is still useful, but it should go somewhere else.]**

Claim 3.4. **[There should be a way to predict the correct normalization of the RHS.]**

$$ \partial^{3/2}_{\zeta \text{ from } 2/3}\left(\int_{\mathcal{C}_+(\zeta)} \nu\right) = i\,\tfrac{\sqrt{\pi}}{8}\,\tfrac{5}{12}\,\hat{w}_+(\zeta - 2/3), $$

consistent with Theorem 3.1 (4).

Proof. With the substitution $t = -2ux^{1/2}$, we can rewrite the Airy integral as

$$ \mathrm{Ai}(x) = x^{1/2}\,\frac{i}{\pi}\int_{x^{-1/2}\mathcal{C}_+} \exp\left[-\tfrac{2}{3}x^{3/2}\left(4u^3 - 3u\right)\right] du, $$

where $\mathcal{C}_+$ is the path $\theta \mapsto \cosh(\theta - \frac{2}{3}\pi i)$. When $x \in [0,\infty)$, this leads to the expression

$$ I_+(z) = \int_{\mathcal{C}_+} \exp\left[-\tfrac{2}{3}z\left(4u^3 - 3u\right)\right] du. $$

In our general picture of exponential integrals, $f = \frac{2}{3}(4u^3 - 3u)$ and $\nu = du$. Hence,

$$ \begin{aligned} \int_{\mathcal{C}_+(\zeta)} \nu &= \int_{\mathcal{C}_+(\zeta)} du \\ &= u\Big|_{\text{start}\,\mathcal{C}_+(\zeta)}^{\text{end}\,\mathcal{C}_+(\zeta)}. \end{aligned} $$

Since $4u^3 - 3u$ is the third Chebyshev polynomial, and cosh is 2π-periodic in the imaginary direction, the start and end points of $\mathcal{C}_+(\zeta)$ are characterized by

$$ \begin{aligned} u &= \cosh(\mp\theta - \tfrac{2}{3}\pi i) \\ \zeta &= \tfrac{2}{3}\cosh(3\theta), \end{aligned} $$

so

$$\begin{aligned}\int_{\mathcal{C}_+(\zeta)} \nu &= \cosh(\theta - \tfrac{2}{3}\pi i) - \cosh(-\theta - \tfrac{2}{3}\pi i)\\ &= \big[\cosh(\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(\theta)\sinh(-\tfrac{2}{3}\pi i)\big]\\ &\quad - \big[\cosh(-\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(-\theta)\sinh(-\tfrac{2}{3}\pi i)\big]\\ &= 2\sinh(\theta)\sinh(-\tfrac{2}{3}\pi i)\\ &= -i\sqrt{3}\,\sinh(\theta)\end{aligned}$$

with $\frac{3}{2}\zeta = \cosh(3\theta)$. Let $\xi = \frac{1}{2}(1-\frac{3}{2}\zeta)$, and notice that $\xi = -\sinh(\frac{3}{2}\theta)^2$ at the start and end points. The identity **[DLMF 15.4.16]**

$$\sinh(\theta) = \tfrac{2}{3}\sinh(\tfrac{3}{2}\theta)\,F\big(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};-\sinh(\tfrac{3}{2}\theta)^2\big)$$

then shows us that

$$\frac{i}{\sqrt{3}}\int_{\mathcal{C}_+(\zeta)} \nu = \tfrac{2}{3}(-\xi)^{1/2}\,F\big(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};\xi\big).$$

Now we can evaluate the half-integral of $\int_{\mathcal{C}_+}\nu$ using Bateman's fractional integral formula for hypergeometric functions [**Koornwinder, §4.1**].

$$\begin{aligned}\partial^{-1/2}_{\zeta\text{ from }2/3}\left(\int_{\mathcal{C}_+(\zeta)}\nu\right) &= \frac{1}{\Gamma\big(\frac{1}{2}\big)}\int_{2/3}^{\zeta}(\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_+(\zeta')}\nu\right)d\zeta'\\ &= \frac{1}{\Gamma\big(\frac{1}{2}\big)}\int_0^{\xi}\tfrac{\sqrt{3}}{2}(\xi'-\xi)^{-1/2}\Big[-i\sqrt{3}\,\tfrac{2}{3}(-\xi)^{1/2}\,F\big(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};\xi\big)\Big]\big(-\tfrac{4}{3}\,d\xi'\big)\\ &= -i\frac{4}{3}\frac{\Gamma\big(\frac{3}{2}\big)}{\Gamma(2)}(-\xi)\,F\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big)\\ &= i\frac{2}{3}\sqrt{\pi}\,\xi\,F\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big).\end{aligned}$$

Finally, we differentiate twice using **[DLMF 15.5.4]** and **[DLMF 15.5.1]**.

$$\begin{aligned}\partial^{3/2}_{\zeta\text{ from }2/3}\left(\int_{\mathcal{C}_+(\zeta)}\nu\right) &= \big(-\tfrac{3}{4}\tfrac{\partial}{\partial\xi}\big)^2\left[i\frac{2}{3}\sqrt{\pi}\,\xi\,F\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big)\right]\\ &= i\,\tfrac{3\sqrt{\pi}}{8}\big(\tfrac{\partial}{\partial\xi}\big)^2\big[\xi\,F\big(\tfrac{1}{6},\tfrac{5}{6};2;\xi\big)\big]\\ &= i\,\tfrac{3\sqrt{\pi}}{8}\,\tfrac{\partial}{\partial\xi}\big[F\big(\tfrac{1}{6},\tfrac{5}{6};1;\xi\big)\big]\\ &= i\,\tfrac{\sqrt{\pi}}{8}\,\tfrac{5}{12}\,F\big(\tfrac{7}{6},\tfrac{11}{6};2;\xi\big).\end{aligned}$$

[Check comparison with Mariño's result more carefully?] □

Analogously, it can be verified for $\hat{w}_-(\zeta+2/3)$ for $\zeta\in(-\infty,-2/3)$.

Claim 3.5.

$$\partial^{3/2}_{\zeta \text{ from } -2/3}\left(\int_{\mathcal{C}_-(\zeta)} \nu\right) = -\tfrac{\sqrt{\pi}}{8}\,\tfrac{5}{12}\,\hat{w}_-(\zeta+2/3),$$

consistent with Theorem 3.1 (4).

Proof. As before, with the substitution $t = -2ux^{1/2}$, we can rewrite the Airy integral as

$$\mathrm{Ai}(x) = x^{1/2}\,\frac{i}{\pi}\int_{x^{-1/2}\mathcal{C}_-} \exp\left[-\tfrac{2}{3}x^{3/2}\left(4u^3-3u\right)\right] du,$$

where $\mathcal{C}_-$ is the path $\theta \mapsto -\cosh(\theta - \tfrac{2}{3}\pi i)$. When $x \in [0,\infty)$, this leads to the expression

$$L_-(z) = \int_{\mathcal{C}_-} \exp\left[-\tfrac{2}{3}z\left(4u^3-3u\right)\right] du.$$

In our general picture of exponential integrals, $f = \frac{2}{3}(4u^3-3u)$ and $\nu = du$. Hence,

$$\begin{aligned}\int_{\mathcal{C}_-(\zeta)} \nu &= \int_{\mathcal{C}_-(\zeta)} du \\ &= u\Big|_{\mathrm{start}\,\mathcal{C}_-(\zeta)}^{\mathrm{end}\,\mathcal{C}_-(\zeta)}.\end{aligned}$$

The start and end points of $\mathcal{C}_-(\zeta)$ are characterized by

$$\begin{aligned} u &= -\cosh(\mp\theta - \tfrac{2}{3}\pi i) \\ \zeta &= -\tfrac{2}{3}\cosh(3\theta),\end{aligned}$$

so

$$\begin{aligned}\int_{\mathcal{C}_-(\zeta)} \nu &= -\cosh(\theta - \tfrac{2}{3}\pi i) + \cosh(-\theta - \tfrac{2}{3}\pi i) \\ &= -\left[\cosh(\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(\theta)\sinh(-\tfrac{2}{3}\pi i)\right] \\ &\quad + \left[\cosh(-\theta)\cosh(-\tfrac{2}{3}\pi i) + \sinh(-\theta)\sinh(-\tfrac{2}{3}\pi i)\right] \\ &= 2\sinh(\theta)\sinh(\tfrac{2}{3}\pi i) \\ &= i\sqrt{3}\,\sinh(\theta)\end{aligned}$$

with $\frac{3}{2}\zeta = -\cosh(3\theta)$. Let $\xi = \frac{1}{2}(1+\frac{3}{2}\zeta)$, and notice that $\xi = -\sinh(\frac{3}{2}\theta)^2$ at the start and end points. The identity **[DLMF 15.4.16]**

$$\sinh(\theta) = \tfrac{2}{3}\sinh(\tfrac{3}{2}\theta)\,F\left(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};-\sinh(\tfrac{3}{2}\theta)^2\right)$$

then shows us that

$$-\frac{i}{\sqrt{3}}\int_{\mathcal{C}_-(\zeta)} \nu = \tfrac{2}{3}(-\xi)^{1/2}\,F\left(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};\xi\right).$$

Now we can evaluate the half-integral of $\int_{\mathcal{C}_-} \nu$ using Bateman's fractional integral formula for hypergeometric functions **[Koornwinder, §4.1]**.

$$\begin{aligned}
\partial_{\zeta \text{ from } -2/3}^{-1/2}\left(\int_{\mathcal{C}_-(\zeta)} \nu\right) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-2/3}^{\zeta}(\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_-(\zeta')} \nu\right) d\zeta' \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{\xi} \tfrac{\sqrt{3}}{2}(\xi-\xi')^{-1/2}\Big[i\sqrt{3}\,\tfrac{2}{3}(-\xi')^{1/2}\,F\left(\tfrac{1}{6},\tfrac{5}{6};\tfrac{3}{2};\xi'\right)\Big]\left(\tfrac{4}{3}\,d\xi'\right) \\
&= \frac{4}{3}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}(-\xi)\,F\left(\tfrac{1}{6},\tfrac{5}{6};2;\xi\right) \\
&= -\frac{2}{3}\sqrt{\pi}\,\xi\,F\left(\tfrac{1}{6},\tfrac{5}{6};2;\xi\right).
\end{aligned}$$

Finally, we differentiate twice using **[DLMF 15.5.4]** and **[DLMF 15.5.1]**.

$$\begin{aligned}
\partial_{\zeta \text{ from } 2/3}^{3/2}\left(\int_{\mathcal{C}_+(\zeta)} \nu\right) &= \left(\tfrac{3}{4}\tfrac{\partial}{\partial\xi}\right)^2\left[-\frac{2}{3}\sqrt{\pi}\,\xi\,F\left(\tfrac{1}{6},\tfrac{5}{6};2;\xi\right)\right] \\
&= -\tfrac{3\sqrt{\pi}}{8}\left(\tfrac{\partial}{\partial\xi}\right)^2\left[\xi\,F\left(\tfrac{1}{6},\tfrac{5}{6};2;\xi\right)\right] \\
&= -\tfrac{3\sqrt{\pi}}{8}\tfrac{\partial}{\partial\xi}\left[F\left(\tfrac{1}{6},\tfrac{5}{6};1;\xi\right)\right] \\
&= -\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\,F\left(\tfrac{7}{6},\tfrac{11}{6};2;\xi\right).
\end{aligned}$$

[Check comparison with Mariño's result more carefully?] □

In particular, from the previous computations we get the right normalization constants for $\hat{w}_\pm(\zeta)$, which allows to compute the Stokes factors from the analytic continuation of $\hat{w}_\pm(\zeta)$ at the branch cut: from [**?**] 15.2.3

$$\begin{aligned}
\left(i\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\right)[\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0)] &= \left(i\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\right)\left(-\tfrac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\sum_{n\geq 0}\tfrac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\tfrac{3}{4}\zeta)^n\right) \qquad \zeta<-\tfrac{4}{3} \\
&= \left(i\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\right)\tfrac{36}{5}i\sum_{n\geq 0}\tfrac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\tfrac{3}{4}\zeta)^{n-1} \\
&= -\left(i\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\right)\tfrac{36}{5}i(-\tfrac{3}{4}\zeta-1)^{-1}\left(\tfrac{5}{144}(4+3\zeta)\left(1+{}_1F_2\left(\tfrac{7}{6},\tfrac{11}{6},2,1+\tfrac{3}{4}\zeta\right)\right)\right) \\
&= \left(i\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\right)i\;{}_1F_2\left(\tfrac{7}{6},\tfrac{11}{6},2,1+\tfrac{3}{4}\zeta\right) \\
&= \left(-\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\right){}_1F_2\left(\tfrac{7}{6},\tfrac{11}{6},2,1+\tfrac{3}{4}\zeta\right) \\
&= +\mathbf{1}\,\hat{w}_-(\zeta+\tfrac{4}{3})
\end{aligned}$$

Anolougusly, $\hat{w}_-(\zeta)$ is Laplace summable along the negative real axis, and it jumps across the branch cut $\frac{4}{3}\mathbb{R}_{\geq 0}$ as

$$\left(-\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\right)[\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0)] = \left(-\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\right)\left(-\tfrac{36}{5}i(\tfrac{3}{4}\zeta-1)^{-1}\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1-\tfrac{3}{4}\zeta)^n\right) \qquad \zeta > \tfrac{4}{3}$$

$$= -\left(-\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\right)\tfrac{36}{5}i(\tfrac{3}{4}\zeta-1)^{-1}\left(-\tfrac{5}{144}(-4+3\zeta)_2F_1\left(\tfrac{7}{6},\tfrac{11}{6},2,1-\tfrac{3}{4}\zeta\right)\right)$$

$$= i\left(-\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\right){}_2F_1\left(\tfrac{7}{6},\tfrac{11}{6},2,1-\tfrac{3}{4}\zeta\right)$$

$$= -\left(i\tfrac{\sqrt{\pi}}{8}\tfrac{5}{12}\right){}_2F_1\left(\tfrac{7}{6},\tfrac{11}{6},2,1-\tfrac{3}{4}\zeta\right)$$

$$= -\mathbf{1}\,\hat{w}_+(\zeta-\tfrac{4}{3})$$

These relations manifest the resurgence property of $\tilde{I}_{\pm 1}$, indeed near the singularities in the Borel plane of either $\hat{w}_+$ or $\hat{w}_-$, $\hat{w}_-$ and $\hat{w}_+$ respectively contribute to the jump of the former solution.

Remark 3.6. $\hat{w}_+(\zeta)$ is Laplace summable along the positive real axis, and it can be analyticaly continued on $\mathbb{C}\setminus -\frac{4}{3}\mathbb{R}_{\leq 0}$ with (see 15.2.3 DLMF)

$$\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0) = -\frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\frac{3}{4}\zeta)^n \qquad \zeta < -\frac{4}{3}$$

$$= \frac{36}{5}i\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\frac{3}{4}\zeta)^{n-1}$$

$$= -\frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\left(\frac{5}{144}(4+3\zeta)\left(1+{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)\right)\right)$$

$$= \mathbf{i}\;{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)$$

$$= \mathbf{i}\,\hat{w}_-(\zeta+\frac{4}{3})$$

Anolougusly, $\hat{w}_-(\zeta)$ is Laplace summable along the negative real axis, and it jumps across the branch cut $\frac{4}{3}\mathbb{R}_{\geq 0}$ as

$$\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0) = \frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\sum_{n\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1-\frac{3}{4}\zeta)^n \qquad \zeta > \frac{4}{3}$$

$$= \frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\left(-\frac{5}{144}(-4+3\zeta)_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)\right)$$

$$= -\mathbf{i}\pi\;{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)$$

$$= -\mathbf{i}\,\hat{w}_+(\zeta-\frac{4}{3})$$

These relations manifest the resurgence property of $\tilde{I}$, indeed near the singularities in the Borel plane of either $\hat{w}_+$ or $\hat{w}_-$, $\hat{w}_-$ and $\hat{w}_+$ respectively contribute to the jump of the former solution.

Example 3.7 (Bessel)**.** Let $X=\mathbb{C}^*$, $f(x)=x+\frac{1}{x}$ and $\nu=\frac{dx}{x}$, then the ciritcal points of f are $x=\pm 1$ and

$$I(z):=\int_0^\infty e^{-zf(x)}\frac{dx}{x}. \tag{3.16}$$

Let $\pi\colon\tilde{\mathbb{C}}\to\mathbb{C}^*$ be the universal cover of $\mathbb{C}^*$, where $\pi(u)=e^u$, then on $\tilde{\mathbb{C}}$, $I(z)$ turns into

$$I(z)=\int_{-\infty}^\infty e^{-2z\cosh(u)}du=2\int_0^\infty e^{-2z\cosh(u)}du=2K_0(2z)\qquad |\arg z|<\frac{\pi}{2}$$

where $K_0(z)$ is the modified Bessel function (see definition 10.32.10 DLMF). In particular, since $K_0(z)$ solves

$$\frac{d^2}{dz^2}w(z)+\frac{1}{z}\frac{d}{dz}w(z)-w(z)=0 \tag{3.17}$$

and $K_0(z)\sim\left(\frac{\pi}{2}\right)^{1/2}e^{-z}z^{-1/2}\sum_{k\geq 0}\frac{(1/2)_k(1/2)_k}{(-2)^k k!}z^{-k}$ as $z\to\infty$ (see DLMF 10.40.2), then $I(z)$ is a solution of

$$\frac{d^2}{dz^2}I(z)+\frac{1}{z}\frac{d}{dz}I(z)-4I(z)=0. \tag{3.18}$$

The formal integral of (3.18) is given by a two parameter formal solution $\tilde{I}(z)$

$$\tilde{I}(z)=\sum_{\mathbf{k}\in\mathbb{N}^2}U^k e^{-\mathbf{k}\cdot\lambda z}z^{-\tau\cdot\mathbf{k}}\tilde{w}_{\mathbf{k}}(z) \tag{3.19}$$

where $\lambda=(2,-2)$, $\tau=(-\frac{1}{2},-\frac{1}{2})$, $U^k:=U_1^{k_1}U_2^{k_2}$ with $k=(k_1,k_2)$ and $U_1,U_2\in\mathbb{C}$, and $\tilde{w}_{\mathbf{k}}(z)\in\mathbb{C}[[z^{-1}]]$ is a formal solution of

$$\begin{aligned}\tilde{w}_{\mathbf{k}}''(z)-4(k_1-k_2)\tilde{w}_{\mathbf{k}}'(z)+4(1-(k_1-k_2)^2)\tilde{w}_{\mathbf{k}}(z)+\frac{(k_1+k_2-1)}{z}\tilde{w}_{\mathbf{k}}'(z)+\\ -2(k_1-k_2)\frac{(k_1+k_2-1)}{z}\tilde{w}_{\mathbf{k}}(z)+\frac{(k_1+k_2)^2}{4z^2}\tilde{w}_{\mathbf{k}}(z)=0\end{aligned} \tag{3.20}$$

The only non zero $\tilde{w}_{\mathbf{k}}(z)$ occurs for $\mathbf{k}=(1,0)$ and $\mathbf{k}=(0,1)$, hence

$$\tilde{I}(z)=U_1e^{-2z}z^{-1/2}\tilde{w}_{(1,0)}(z)+U_2e^{2z}z^{-1/2}\tilde{w}_{(0,1)}(z) \tag{3.21}$$

and we define

$$\tilde{I}_1(z):=e^{-2z}z^{-1/2}\tilde{w}_{(1,0)}(z) \tag{3.22}$$

$$\tilde{I}_{-1}(z):=e^{2z}z^{-1/2}\tilde{w}_{(0,1)}(z). \tag{3.23}$$

FIGURE 1. Integration path in the u-plane and in the Borel plane $\zeta = 2\cosh(u)$.

We set $\tilde{w}_{(1,0)} = \tilde{w}_+$ and $\tilde{w}_{(0,1)} = \tilde{w}_-$, then their Borel transforms are solutions respectively of the following equations

$$\zeta^2 \hat{w}_+(\zeta) + 4\zeta \hat{w}_+(\zeta) + \frac{1}{4}\zeta * \hat{w}_+(\zeta) = 0 \tag{3.24}$$

$$\zeta^2 \hat{w}_-(\zeta) - 4\zeta \hat{w}_-(\zeta) + \frac{1}{4}\zeta * \hat{w}_-(\zeta) = 0 \tag{3.25}$$

taking twice derivative in ζ we get respectively for $\hat{w}_+(\zeta)$ and $\hat{w}_-(\zeta)$

$$(\zeta^2 + 4\zeta)\frac{d^2}{d\zeta^2}\hat{w}_+ + 4(\zeta - 1)\frac{d}{d\zeta}\hat{w}_+ + \frac{9}{4}\hat{w}_+ = 0$$

$$(+) \quad \xi(1-\xi)\frac{d^2}{d\xi^2}\hat{w}_+ + (1-4\xi)\frac{d}{d\xi}\hat{w}_+ - \frac{9}{4}\hat{w}_+ = 0 \qquad \xi = -\frac{\zeta}{4}$$

$$(\zeta^2 - 4\zeta)\frac{d^2}{d\zeta^2}\hat{w}_- + 4(\zeta + 1)\frac{d}{d\zeta}\hat{w}_+ + \frac{9}{4}\hat{w}_- = 0$$

$$(-) \quad \xi(1-\xi)\frac{d^2}{d\xi^2}\hat{w}_- + (1-4\xi)\frac{d}{d\xi}\hat{w}_- - \frac{9}{4}\hat{w}_- = 0 \qquad \xi = \frac{\zeta}{4}$$

Since equation (+), (−) are hypergeometric, the fundamental solutions are respectively (see DLMF 15.10.2)

$$\hat{w}_+(\zeta) = c_1 \, {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; -\frac{\zeta}{4}\right) \tag{3.26}$$

$$\hat{w}_-(\zeta) = c_2 \, {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{\zeta}{4}\right) \tag{3.27}$$

Now we show that the fractional derivative formula holds in this example: first we write a parametrization of the integration path in the Borel plane. Let $\mathcal{C}_p = \mathbb{R}$, $p \in \pi\mathbb{Z}$, be the contour of integration in $\tilde{\mathbb{C}}$ which is parametrized as

$$\begin{aligned} \mathcal{C}_p : \mathbb{R} &\to \tilde{\mathbb{C}} \\ \theta &\to \theta + ip \end{aligned}$$

Then in the Borel plane $\mathbb{C}_\zeta$, where $\zeta = 2\cosh(u)$, the path $\mathcal{C}_0$ is parametrized as

$$\begin{aligned} \mathcal{C}_0(\zeta) : \mathbb{R} &\to \mathbb{C}_\zeta \\ \theta &\to 2\cosh(\theta) \end{aligned}$$

Check Aaron notation

(3.28) $$\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu) = \int_{\mathcal{C}_0(\zeta)} du = \left[u\right]_{\mathrm{start}\mathcal{C}_0(\zeta)}^{\mathrm{end}\mathcal{C}_0(\zeta)} = 2\operatorname{arc}\cosh\left(\frac{\zeta}{2}\right)$$

We can now write $\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu)$ as an hypergeometric function thanks to identity 14.4.4 [**?**]: set $\xi = \frac{1}{2}\left(\frac{1}{2}\zeta - 1\right) = \frac{1}{2}(\cosh(\theta) - 1) = -\sinh^2\left(\frac{\theta}{2}\right)$, then

$$\begin{aligned}
\sinh\left(\frac{\theta}{2}\right) {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\sinh^2\left(\frac{\theta}{2}\right)\right) &= i\frac{\theta}{2} \\
(-\xi)^{1/2} {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-\xi\right) &= i\frac{\theta}{2} \\
&= \frac{i}{2}\operatorname{arcosh}\left(\frac{\zeta}{2}\right) \\
&= \frac{i}{4}\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu)
\end{aligned}$$

The 3/2-derivative of $\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu)$ can be computed as follows: we compute the $-1/2$-derivatve and then we differentiate twice

$$\begin{aligned}
\partial_\zeta^{-1/2}\left(\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu)\right) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_2^\zeta (\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu)\right) d\zeta' \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^\xi \frac{1}{2}(\xi-\xi')^{-1/2}(-4i)(-\xi')^{1/2} {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-\xi'\right) 4d\xi' \\
&= -8i(i)\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\xi\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};2,-\xi\right) \\
&= 4\sqrt{\pi}\xi\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};2,-\xi\right)
\end{aligned}$$

$$\begin{aligned}
\partial_\zeta^{3/2}\left(\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu)\right) &= \partial_\zeta^2\left(4\sqrt{\pi}\xi\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};2,-\xi\right)\right) \\
&= \frac{1}{16}\partial_\xi^2\left(4\sqrt{\pi}\xi\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};2,-\xi\right)\right) && \partial_\xi = 4\partial_\zeta \\
&= -\frac{\sqrt{\pi}}{4}\Gamma\left(\frac{3}{2}\right)\partial_\xi\left({}_2F_1\left(\frac{1}{2},\frac{1}{2};1,-\xi\right)\right) && \textbf{DLMF}15.5.4 \\
&= \frac{\sqrt{\pi}}{16} {}_2F_1\left(\frac{3}{2},\frac{3}{2};2,-\xi\right) \\
&= \frac{\sqrt{\pi}}{16} {}_2F_1\left(\frac{3}{2},\frac{3}{2};2,\frac{1}{2}-\frac{\zeta}{4}\right)
\end{aligned}$$

Analougusly, it can be verified for $\hat{w}_{-}(\zeta+2)$ for $\zeta \in (-\infty,-2)$. The path $\mathcal{C}_\pi$ is parametrized as

$$\begin{aligned}\mathcal{C}_\pi : \mathbb{R} &\to \mathbb{C}_\zeta \\ \theta &\to 2\cosh(\theta - i\pi) = -2\cosh(\theta)\end{aligned}$$

$$\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu) = \int_{\mathcal{C}_\pi(\zeta)} du = \left[u\right]_{\text{start}\mathcal{C}_\pi(\zeta)}^{\text{end}\mathcal{C}_\pi(\zeta)} = 2\operatorname{arc}\cosh\left(-\frac{\zeta}{2}\right) \tag{3.29}$$

We can now write $\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu)$ as an hypergeometric function thanks to identity 15.4.4 [**?**]: set $\xi = \frac{1}{2}\left(\frac{1}{2}\zeta+1\right) = \frac{1}{2}(-\cosh(\theta)+1) = \sinh^2\left(\frac{\theta}{2}\right)$, then

$$\begin{aligned}\sinh\left(\frac{\theta}{2}\right) {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\sinh^2\left(\frac{\theta}{2}\right)\right) &= i\frac{\theta}{2} \\ (\xi)^{1/2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\xi\right) &= i\frac{\theta}{2} \\ &= \frac{i}{2}\operatorname{arcosh}\left(-\frac{\zeta}{2}\right) \\ &= \frac{i}{4}\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu)\end{aligned}$$

The 3/2-derivative of $\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu)$ is computed in two steps: first we compute the $-1/2$-derivatve and then we differentiate twice

$$\begin{aligned}\partial_\zeta^{-1/2}\left(\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu)\right) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-2}^{\zeta} (\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu)\right) d\zeta' \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^{\xi} \frac{1}{2}(\xi-\xi')^{-1/2}(-4i)\xi'^{1/2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\xi'\right)4d\xi' \\ &= -8i\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\xi\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2,\xi\right)\end{aligned}$$

$$
\begin{aligned}
\partial_\zeta^{3/2}\left(\int_{\mathcal{C}_0(\zeta)}\pi^*(\nu)\right) &= \partial_\zeta^2\left(-8i\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\xi\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2,\xi\right)\right)\\
&= \frac{1}{16}\partial_\xi^2\left(-8i\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\xi\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2,\xi\right)\right) && \partial_\xi = 4\partial_\zeta\\
&= -\frac{i}{2}\Gamma\left(\frac{3}{2}\right)\partial_\xi\left({}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\xi\right)\right) && \textbf{DLMF}15.5.4\\
&= -\frac{i}{8}\Gamma\left(\frac{3}{2}\right){}_2F_1\left(\frac{3}{2},\frac{3}{2};2,\xi\right)\\
&= -\frac{i\sqrt{\pi}}{16}{}_2F_1\left(\frac{3}{2},\frac{3}{2};2,\frac{1}{2}+\frac{\zeta}{4}\right)
\end{aligned}
$$

Once we have determined the right constants c_1, c_2 we can compute the Stokes constants. First, we notice that taking the series expansion of $\hat{w}_+$ and $\hat{w}_-$ at the critical point we get numerically that

$$
\begin{aligned}
\hat{w}_+(\zeta-4) &= \frac{1}{\pi}\log(z)\hat{w}_-(z)+\phi_{\mathrm{reg}}\\
\hat{w}_-(\zeta+4) &= \frac{1}{\pi}\log(z)\hat{w}_+(z)+\psi_{\mathrm{reg}}
\end{aligned}
$$

and analytically (thanks to 15.2.3 DLMF)

Let us redefine $\hat{w}_+(\zeta) := -i\frac{\sqrt{\pi}}{16}{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{1}{2}-\frac{\zeta}{4}\right)$ and $\hat{w}_-(\zeta) := \frac{\sqrt{\pi}}{16}{}_2F_1\left(\frac{3}{2},\frac{3}{2};2;\frac{\zeta}{4}+\frac{1}{2}\right)$. From equation 15.2.3 in [**?**]

$$
\begin{aligned}
\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0) &= -\tfrac{i\sqrt{\pi}}{16}\left({}_2F_1\left(\tfrac{3}{2},\tfrac{3}{2};2;-\tfrac{\zeta}{4}+i0\right)-{}_2F_1\left(\tfrac{3}{2},\tfrac{3}{2};2;-\tfrac{\zeta}{4}-i0\right)\right) && \zeta<-4\\
&= -8i\left(\tfrac{\sqrt{\pi}}{16}\right)\left(-\tfrac{\zeta}{4}-1\right)^{-1}\sum_{n\geq 0}\tfrac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\tfrac{\zeta}{4}+1\right)^n\\
&= 8i\left(\tfrac{\sqrt{\pi}}{16}\right)\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\tfrac{\zeta}{4}+1\right)^{n-1}\\
&= 8i\left(\tfrac{\sqrt{\pi}}{16}\right)\sum_{n\geq 0}\tfrac{(1/2)_{n+1}(1/2)_{n+1}}{(n+1)!\Gamma(n+1)}\left(\tfrac{\zeta}{4}+1\right)^n\\
&= 2i\left(\tfrac{\sqrt{\pi}}{16}\right)\sum_{n\geq 0}\tfrac{(3/2)_n(3/2)_n}{(n)!\Gamma(n+2)}\left(\tfrac{\zeta}{4}+1\right)^n\\
&= -\mathbf{2}\left(-\tfrac{i\sqrt{\pi}}{16}\right){}_2F_1\left(\tfrac{3}{2},\tfrac{3}{2};2;\tfrac{\zeta}{4}+1\right)\\
&= -\mathbf{2}\,\hat{w}_-(\zeta+2)
\end{aligned}
$$

$$\begin{aligned}
\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0) &= \left(-\tfrac{i\sqrt{\pi}}{16}\right)\left[{}_2F_1\left(\tfrac{3}{2},\tfrac{3}{2};2;\tfrac{\zeta}{4}+i0\right)-{}_2F_1\left(\tfrac{3}{2},\tfrac{3}{2};2;\tfrac{\zeta}{4}-i0\right)\right] \qquad \zeta>4\\
&= -8i\left(-\tfrac{i\sqrt{\pi}}{16}\right)\left(\tfrac{\zeta}{4}-1\right)^{-1}\sum_{n\geq 0}\tfrac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(\tfrac{\zeta}{4}-1\right)^n\\
&= 8i\left(-\tfrac{i\sqrt{\pi}}{16}\right)\sum_{n\geq 0}\tfrac{(1/2)_n(1/2)_n}{n!\Gamma(n)}\left(1-\tfrac{\zeta}{4}\right)^{n-1}\\
&= 8i\left(-\tfrac{i\sqrt{\pi}}{16}\right)\sum_{n\geq 0}(-1)^n\frac{(1/2)_{n+1}(1/2)_{n+1}}{(n+1)!\Gamma(n+1)}\left(1-\tfrac{\zeta}{4}\right)^n\\
&= 2i\left(-\tfrac{i\sqrt{\pi}}{16}\right)\sum_{n\geq 0}(-1)^n\tfrac{(3/2)_n(3/2)_n}{(n)!\Gamma(n+2)}\left(1-\tfrac{\zeta}{4}\right)^n\\
&= +\mathbf{2}\left(\tfrac{\sqrt{\pi}}{16}\right){}_2F_1\left(\frac{3}{2},\frac{3}{2};2;1-\frac{\zeta}{4}\right)\\
&= +\mathbf{2}\,\hat{w}_+(\zeta-2)
\end{aligned}$$

These are evidence of the resurgent properties of $\tilde{I}_{\pm 1}(z)$. **With the correct normalization, the latter identities show that the Stokes constants can be computed via Alien calculus and they are equal to** ± 2**.**

4. Useful identities for Gauss hypergeomtric functions

(4.1)
$$ {}_2F_1(a,b;c;z) = e^{b\pi i}\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(a+b-c+1)\Gamma(c-b)}{}_2F_1(a,b;c;1-z)+ \\ -e^{(a+b-c)\pi i}\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(b)\Gamma(a-b+1)}|z|^{-a}{}_2F_1\left(a,a-c+1;a-b+1;\frac{1}{z}\right)$$

(4.2)
$$\int_0^x |y|^{a-\mu-1}{}_2F_1(a,b;c,y)|x-y|^{\mu-1}dy = \frac{\Gamma(\mu)\Gamma(a-\mu)}{\Gamma(a)}|x|^{a-1}{}_2F_1(a-\mu,b;c;x)$$
$$x\in(-\infty,0)\cup(0,1),\ \Re a>\Re\mu>0$$

which can be rewritten as (`arXiv:1504.08144`**, formula 4.8)**

(4.3)
$$\int_{y>x}|y|^{-a}|x-y|^{\mu-1}{}_2F_1(a,b;c;y^{-1})dy = \frac{\Gamma(\mu)\Gamma(a-\mu)}{\Gamma(a)}|x|^{-a+\mu}{}_2F_1(a-\mu,b;c;x^{-1})$$
$$x\in(-\infty,0)\cup(1,\infty),\ \Re a>\Re\mu>0$$

5. Resurgence for degree 3 polynomials

Let f be a degree 3 polynomial, and t_1,t_2 its critical points (not necessarily distinguished):

(1) if $t_1 \neq t_2$, then

$$I(z) = \int_{\mathcal{C}_j} e^{-zf}\, dt$$

is a solution of

$$I'' + aI' + bI + c\frac{I'}{z} + \frac{d}{z}I + \frac{e}{z^2}I = 0 \tag{5.1}$$

where a, b, c, d, e are determined in terms of f.

(2) if $t_1 = t_2$, then

$$I(z) = \int_{\mathcal{C}_1} e^{-zf}\, dt$$

is a solution of a first order ODE

$$I' + \left(a_4 - \frac{a_2^3}{27a_1^2} + \frac{1}{3z}\right)I = 0 \tag{5.2}$$

Proof. Let $f(t) = a_1t^3 + a_2t^2 + a_3t + a_4$ with $a_1 \neq 0$,

$$\int_{\mathcal{C}_j} e^{-fz}\, dt = \int_{\mathcal{C}_j + \frac{a_2}{3a_1}} e^{-(a_1t^3+pt+q)z}\, dt \qquad\qquad t \to t - \frac{a_2}{3a_1}$$

where $p = a_3 - \frac{a_2^2}{3a_1}$ and $q = a_4 - \frac{a_2a_3}{3a_1} + \frac{2a_2^3}{27a_1^2}$.

Case (1): if $p \neq 0$,

$$\begin{aligned} I(z) = \int t(3a_1t^2+p)ze^{-fz} = \int (3a_1t^3+pt)ze^{-ft} = \\ \int 2a_1t^3ze^{-fz} + \int (a_1t^3+pt+q)ze^{-fz} - qzI \\ 2z\int a_1t^3e^{-fz} - zI' - qzI \end{aligned}$$

$$\begin{aligned} 2z\int a_1t^3e^{-fz} = 2z^2\int \frac{t^4}{4}a_1(3a_1t^2+p)e^{-fz} = \frac{z^2}{2}\int (3a_1^2t^6+pa_1t^4)e^{-fz} = \\ \frac{z^2}{2}\int (3a_1^2t^6+6pa_1t^4+3q^2+3p^2t^2+6pqt+6a_1qt^3)e^{-fz} + \\ +\frac{z^2}{2}\int (pa_1t^4-6pa_1t^4)e^{-fz} - \frac{z^2}{2}\int (3q^2+3p^2t^2+6pqt+6a_1qt^3)e^{-fz} \\ = \frac{3z^2}{2}I'' + \frac{3z^2}{2}q^2I + 3qz^2I' - \frac{z^2}{2}p\int (3a_1t^4+pt^2)e^{-fz} - z^2p\int (a_1t^4+pt^2)e^{-fz} \\ = \frac{3z^2}{2}I'' + \frac{3z^2}{2}q^2I + 3qz^2I' - \frac{5}{3}zp\int te^{-fz} - \frac{2}{3}z^2p^2\int t^2e^{-fz} \end{aligned}$$

hence

(5.3)

$$I = -zI' - qzI + \frac{3z^2}{2}I'' + \frac{3z^2}{2}q^2I + 3qz^2I' - \frac{5}{3}zp\int te^{-fz} - \frac{2}{3}z^2p^2\int t^2e^{-fz}$$

(5.4)

$$\frac{3z^2}{2}\left(I'' + q^2I + 2qI' - \frac{2}{3z}I' - \frac{2q}{3z}I - \frac{2}{3z^2}I - \frac{10}{9z}p\int te^{-fz}dt - \frac{4}{9}p^2\int t^2e^{-fz}\right) = 0$$

Notice that

$$\frac{4}{9}p^2\int t^2e^{-fz} = \frac{4}{27a_1}p^2\int(3a_1t^2+p)e^{-fz} - \frac{4}{27a_1}p^3I = -\frac{4}{27a_1}p^3I$$

$$\begin{aligned}
-\frac{10}{9z}p\int te^{-fz}dt = \frac{5}{9z}\int pte^{-fz} - \frac{5}{3z}\int(pt+q)e^{-fz} + \frac{5}{3z}qI = \\
\frac{5}{9z}\int pte^{-fz} - \frac{5}{3z}\int(pt+q+a_1t^3)e^{-fz} + \frac{5}{3z}\int a_1t^3e^{-fz} + \frac{5}{3z}qI = \\
\frac{5}{9z}\int t(3a_1t^2+p)e^{-fz} + \frac{5}{3z}I' + \frac{5}{3z}qI \\
= \frac{5}{9z^2}I + \frac{5}{3z}I' + \frac{5}{3z}qI
\end{aligned}$$

thefore, collecting all the contributions together we find

$$\begin{aligned}
I'' + q^2I + 2qI' - \frac{2}{3z}I' - \frac{2q}{3z}I - \frac{2}{3z^2}I + \frac{5}{9z^2}I + \frac{5}{3z}I' + \frac{5}{3z}qI + \frac{4}{27a_1}p^3I = 0 \\
I'' + 2qI' + \left(\frac{4p^3}{27a_1} + q^2\right)I + \frac{1}{z}I' + \frac{q}{z}I - \frac{1}{9z^2}I = 0
\end{aligned}$$

Case (2): if $p = 0$, then integrating by part we have

$$\begin{aligned}
I(z) &= \int_{\mathcal{C}_1 + \frac{a_2}{3a_1}} e^{-(a_1t^3+q)z}dt \\
&= \left[te^{-(a_1t^3+q)z}\right]_{\mathcal{C}_1 + \frac{a_2}{3a_1}} + \int_{\mathcal{C}_1 + \frac{a_2}{3a_1}} 3a_1t^3ze^{-(a_1t^3+q)z}dt \\
&= 3z\int_{\mathcal{C}_1 + \frac{a_2}{3a_1}}(a_1t^3+q)e^{-(a_1t^3+q)z}dt - 3qz\int_{\mathcal{C}_1 + \frac{a_2}{3a_1}} e^{-(a_1t^3+q)z}dt \\
&= -3zI'(z) - 3qzI(z)
\end{aligned}$$

□

We would like to verify that for every cubic function f, the Borel transform of the exponential integral can be expressed by an hypergeometric function and hence deduce its resurgent properties in full generality. If $p \neq 0$, $I(z)$ is a solution of

$$I'' + 2qI' + \left(\frac{4p^3}{27a_1} + q^2\right) I + \frac{1}{z} I' + \frac{q}{z} I - \frac{1}{9z^2} I = 0 \tag{5.5}$$

hence a formal solution as $z \to \infty$ is given (up to constants $U_1, U_2 \in \mathbb{C}$) by

$$\tilde{I}_+(z) := U_1 e^{-(q+\sqrt{\frac{4p^3}{27a_1}})z} z^{1/2} \tilde{w}_+(z) \tag{5.6}$$

$$\tilde{I}_-(z) := U_2 e^{-(q-\sqrt{\frac{4p^3}{27a_1}})z} z^{1/2} \tilde{w}_-(z) \tag{5.7}$$

where $\tilde{w}_\pm(z) \in \mathbb{C}[[z^{-1}]]$ is the formal solution of

$$\tilde{w}''_\pm \mp 2\sqrt{\frac{4p^3}{27a_1}}\, \tilde{w}'_\pm + \frac{5}{36} \frac{\tilde{w}_\pm}{z^2} = 0 \tag{5.8}$$

with $\tilde{w}_\pm(z) = 1 + \sum_{k \geq 1} a_{\pm,k} z^{-k}$.

We can now compute the Borel transform of (5.8): for $\tilde{w}_+(z)$

$$\zeta^2 \hat{w} - 2\sqrt{\frac{4p^3}{27a_1}} \zeta \hat{w} + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}(\zeta') d\zeta' = 0$$

$$\zeta^2 \hat{w}' + 2\zeta \hat{w} + 2\sqrt{\frac{4p^3}{27a_1}} \hat{w} + 2\sqrt{\frac{4p^3}{27a_1}} \zeta \hat{w}' + \frac{5}{36} \int_0^\zeta \hat{w}(\zeta') = 0$$

$$\left(\zeta^2 + 2\sqrt{\frac{4p^3}{27a_1}} \zeta\right) \hat{w}'' + 4\left(\zeta + \sqrt{\frac{4p^3}{27a_1}}\right) \hat{w}' + \frac{77}{36} \hat{w} = 0$$

$$t(1-t)\hat{w}'' + (2-4t)\hat{w}' - \frac{77}{36} \hat{w} = 0 \qquad \zeta = -2t\sqrt{\frac{4p^3}{27a_1}}$$

hence

$$\hat{w}_+(\zeta) = c_1\, {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; -\frac{3}{4p}\sqrt{\frac{3a_1}{p}} \zeta\right)$$

and analougusly,

$$\hat{w}_-(\zeta) = c_2\, {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; \frac{3}{4p}\sqrt{\frac{3a_1}{p}} \zeta\right).$$

Notice that $\hat{w}_\pm(\zeta)$ has a branch cut singularity respectively at $\zeta = \zeta_\pm := \pm\sqrt{\frac{16p^3}{27a_1}}$, and thanks to the well known formulas for the analytic continuation of hypergeometric functions (see 15.2.3 DLMF), if we assume the branch cut is from $\zeta_\pm$ to $+\infty$

$$
\begin{aligned}
\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0) &= c_2\frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\left(\frac{\zeta}{\zeta_+}-1\right)^{-1}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_+}\right)^k \quad \zeta\in(\zeta_+,+\infty)\\
&= -c_2\frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_+}\right)^{k-1}\\
&= -i\,c_2\frac{1}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{\Gamma\left(\frac{7}{6}+k\right)\Gamma\left(\frac{11}{6}+k\right)}{\Gamma(2+k)\Gamma(k+1)}\left(1-\frac{\zeta}{\zeta_+}\right)^k\\
&= -i\,c_2\,{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;1-\frac{\zeta}{\zeta_+}\right)\\
&= -i\frac{c_2}{c_1}\hat{w}_+(\zeta-\zeta_+)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0) &= c_1\frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\left(\frac{\zeta}{\zeta_-}-1\right)^{-1}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_-}\right)^k \quad \zeta\in(-\infty,\zeta_-)\\
&= -c_1\frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{(5/6)_k(1/6)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{\zeta_-}\right)^{k-1}\\
&= -i\,c_1\frac{1}{\Gamma(7/6)\Gamma(11/6)}\sum_{k\geq 0}\frac{\Gamma\left(\frac{7}{6}+k\right)\Gamma\left(\frac{11}{6}+k\right)}{\Gamma(2+k)\Gamma(k+1)}\left(1-\frac{\zeta}{\zeta_-}\right)^k\\
&= -i\,c_1\,{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;1-\frac{\zeta}{\zeta_-}\right)\\
&= -i\frac{c_1}{c_2}\hat{w}_-(\zeta-\zeta_-)
\end{aligned}
$$

therefore we see that the Stokes factors are given by $\pm i$ (as for Airy).

I think it will be nice to add the geometric interpretation of Maxim in term of Lefschetz thimbles

The situation is quite different if we consider the degenerate case, where we have only a singular point: indeed there is a one parameter family of solutions of

$$I'(z)+\left(\frac{1}{3z}+q\right)I(z)=0 \tag{5.9}$$

namely for $U\in\mathbb{C}$

$$\tilde{I}(z)=U\,e^{-qz}z^{1/3}\tilde{w}(z) \qquad \text{where } \tilde{w}(z)\in\mathbb{C}[\![z^{-1}]\!]. \tag{5.10}$$

The Borel transform of $\tilde{w}$ is a solution of

$$\zeta\,\hat{w}'+\frac{\hat{w}}{3}=0 \tag{5.11}$$

hence, up to rescaling by a constant,

$$\hat{w}(\zeta) \propto \zeta^{-1/3} = {}_2F_1\left(a, \frac{1}{3}; a; 1-\zeta\right)$$

for every $a \in \mathbb{C}$. In the degenerate case we get an hypergeometric function as well, but the resurgent structure is trivial, i.e. $\hat{w}(\zeta)$ is holomorphic on the Riemann surface of $\zeta^{1/3}$.

5.0.1. *Alternative computation of the Borel transform of I.* Let us first compute the Borel transform of (5.1) (indeed as in the proof of Theorem 3.1 we know that (5.1) admits a formal solution which is Gevrey-1)

$$\zeta^2 \hat{I} - a\zeta \hat{I} + b\hat{I} - \int_0^\zeta \zeta' \hat{I}(\zeta') + d\int_0^\zeta \hat{I}(\zeta') - \frac{1}{9}\int_0^\zeta (\zeta-\zeta')\hat{I}(\zeta') = 0$$

$$2\zeta\hat{I} + \zeta^2\hat{I}' - a\hat{I} - a\zeta\hat{I}' + b\hat{I}' - \zeta\hat{I} + d\hat{I} - \frac{1}{9}\int \hat{I}(\zeta') = 0$$

$$(\zeta^2 - a\zeta + b)\hat{I}'' + (3\zeta - 2a + d)\hat{I}' + \frac{8}{9}\hat{I} = 0$$

Now we denote by λ_1, λ_2 the distinguished (we assume that $p \neq 0$) roots of $\zeta^2 - a\zeta + b$, then

$$(\zeta-\lambda_1)(\zeta-\lambda_2)\hat{I}'' + (3\zeta - 2a + d)\hat{I}' + \frac{8}{9}\hat{I} = 0 \tag{5.12}$$

$$(t + \lambda_2 - \lambda_1)t\hat{I}'' + (3t + 3\lambda_2 - 2a + d)\hat{I}' + \frac{8}{9} = 0 \qquad t = \zeta - \lambda_2 \tag{5.13}$$

$$s(1-s)\hat{I}'' - \left(3s + \frac{3\lambda_2 - 2a + d}{\lambda_1 - \lambda_2}\right)\hat{I}' - \frac{8}{9}\hat{I} = 0 \qquad t = (\lambda_1 - \lambda_2)s \tag{5.14}$$

where (5.14) is an hypergeometric equation[7] and a solution is given by

(5.15)

$$\hat{I}_{\lambda_1}(\zeta; U_1, U_2) = U_1\, {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; \frac{\zeta-\lambda_2}{\lambda_1-\lambda_2}\right) + U_2\left(\frac{\zeta-\lambda_2}{\lambda_1-\lambda_2}\right)^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta-\lambda_2}{\lambda_1-\lambda_2}\right)$$

which has a branch cut at $\zeta = \lambda_1$, where U_1, U_2 are constants. Of course, reversing the role of λ_1 and λ_2 we find

[7]Notice that $\lambda_{1,2} = q \pm \frac{2i}{3} p\sqrt{\frac{p}{3a_1}}$, $a = 2q$ and $d = q$. Hence

$$\frac{2a - d - 3\lambda_2}{\lambda_1 - \lambda_2} = \frac{4q - q - 3q - 2ip\sqrt{\frac{p}{3a_1}}}{-\frac{4i}{3}p\sqrt{\frac{p}{3a_1}}} = \frac{3}{2}$$

(5.16)

$$\hat{I}_{\lambda_2}(\zeta,;U_1,U_2) = U_1 {}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right) + U_2\left(\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)^{-1/2} {}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)$$

is the Borel transform of $\tilde{I}_{\lambda_2}(z)$ and it has a branch cut singularity at $\zeta = \lambda_2$. It is remarkable that the dependece on the function f is only on the location of the singularities, but it is always an hypergeometric function with the same parameters. In addition, we can compute the Stokes constants thanks to the well known formula for analyitic continuation of hypergeometric (see 15.2.3 in DLMF)

$$\begin{aligned}
\hat{I}_{\lambda_1}(\zeta+i0;U_1,0) - \hat{I}_{\lambda_1}(\zeta-i0;U_1,0) &= -U_1\frac{2\pi i}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}\left(\frac{\lambda_1-\zeta}{\lambda_2-\lambda_1}\right)^{-1/2} {}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right) \\
&= -\mathbf{i}U_1\frac{2\pi i}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}\left(\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right)^{-1/2} {}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{\zeta-\lambda_1}{\lambda_2-\lambda_1}\right) \\
&= -\mathbf{i}\hat{I}_{\lambda_2}\left(\zeta;0,\frac{U_1}{\Gamma(2/3)\Gamma(4/3)}\right)
\end{aligned}$$