Resurgence of modified Bessel functions of second kind

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1 Modified Bessel function of second kind

The modified Bessel function of the second kind $K_\nu(z)$ is a solution of the equation

$$w'' + \frac{w'}{z} - w - \frac{\nu^2}{z^2} = 0 \tag{1}$$

such that $K_\nu(z) \sim \sqrt{\pi/(2z)}e^{-z}$ as $z \to \infty$ in $|\arg z| < \frac{3\pi}{2}$[1]. It has a branch point at $z = 0$ for every $\nu \in \mathbb{C}$ and the principal branch is analytic in $\mathbb{C} \setminus (-\infty, 0]$.

1.1 Differential equation

From the general theory of ODE, the formal integral solution of (1) is two parameters family

$$\tilde{w}(z) = U_1 e^{-z} z^{-1/2} \tilde{w}_{\nu,+}(z) + U_2 e^{z} z^{-1/2} \tilde{w}_{\nu,-}(z) \tag{4}$$

where $\tilde{w}_{\nu,\pm} = \sum_{j\geq 0} a_{\pm,j} z^{-j} \in \mathbb{C}[\![z^{-1}]\!]$ are unique formal solutions of

$$\begin{aligned} \tilde{w}''_{\nu,+} - 2\tilde{w}'_{\nu,+} + \frac{\tilde{w}_{\nu,+}}{4z^2} - \frac{\nu^2}{z^2}\tilde{w}_{\nu,+} = 0 \\ \tilde{w}''_{\nu,-} + 2\tilde{w}'_{\nu,-} + \frac{\tilde{w}_{\nu,-}}{4z^2} - \frac{\nu^2}{z^2}\tilde{w}_{\nu,-} = 0 \end{aligned}$$

[1] A system of solution of Bessel equation is given by $I_\nu(z)$ and $K_\nu(z)$. In particular, their asymptotic behaviour as $z \to \infty$ is given by

$$\tilde{I}_\nu(z) = \frac{1}{\sqrt{2\pi}} e^{z} z^{-1/2} \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\nu\right)_k \left(\frac{1}{2}+\nu\right)_k}{2^k k!} z^{-k} \tag{2}$$

$$\tilde{K}_\nu(z) = \sqrt{\frac{\pi}{2}} e^{-z} z^{-1/2} \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\nu\right)_k \left(\frac{1}{2}+\nu\right)_k}{(-2)^k k!} z^{-k} \tag{3}$$

In particular, $\tilde{K}_\nu(z) = \sqrt{\frac{\pi}{2}} e^{-z} z^{-1/2} \tilde{w}_{\nu,+}(z)$ and $\tilde{I}_\nu(z) = \frac{1}{\sqrt{2\pi}} e^{z} z^{-1/2} \tilde{w}_{\nu,-}(z)$ (once we choose $a_{\pm,0} = 1$). We now compute the Borel transform of $\tilde{w}_+(z)$[2] it is a solution of

$$\begin{aligned}
&\zeta^2 \hat{w}_{\nu,+} + 2t\hat{w}_{\nu,+} + \left(\frac{1}{4} - \nu^2\right) \int_0^\zeta (\zeta - s)\hat{w}_{\nu,+}(s) ds = 0 \\
&\zeta^2 \hat{w}''_{\nu,+} + 2\zeta \hat{w}''_{+} + 4\zeta \hat{w}'_{\nu,+} + \left(\frac{9}{4} - \nu^2\right) \hat{w}_{\nu,+} = 0 \\
&t(1-t)\hat{w}''_{\nu,+} + (2-4t)\hat{w}'_{\nu,+} - \left(\frac{9}{4} - \nu^2\right) \hat{w}_{\nu,+} = 0 \qquad\qquad t = -\frac{\zeta}{2}
\end{aligned}$$

therefore $\hat{w}_{\nu,+}(\zeta)$ is an hypergeometric function

$$\hat{w}_{\nu,+}(\zeta) = {}_2F_1\left(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 2; -\frac{\zeta}{2}\right) \tag{5}$$

and it has a branch point singularities at $\zeta = -2$. By the same reasoning,

$$\hat{w}_{\nu,-}(\zeta) = {}_2F_1\left(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 2; \frac{\zeta}{2}\right) \tag{6}$$

and it has branch point at $\zeta = 2$. Thanks to explicit formula for the analytic continuation of hypergeomtric functions (see [?] 15.2.3)

$$\begin{aligned}
\hat{w}_{\nu,+}(\zeta + i0) - \hat{w}_{\nu,+}(\zeta - i0) &= \frac{2\pi i}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)} \left(-\frac{\zeta}{2} - 1\right)^{-1} {}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; 0; 1 + \frac{\zeta}{2}\right) \\
&= -\frac{2\pi i}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)} \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\nu\right)_k \left(\frac{1}{2}+\nu\right)_k}{\Gamma(k)k!} \left(1 + \frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{2\pi i}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)} \sum_{k\geq 1} \frac{\left(\frac{1}{2}-\nu\right)_k \left(\frac{1}{2}+\nu\right)_k}{\Gamma(k)k!} \left(1 + \frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{2\pi i}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)} \frac{1}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}+\nu\right)} \sum_{k\geq 1} \frac{\Gamma\left(\frac{1}{2}-\nu+k\right)\Gamma\left(\frac{1}{2}+\nu+k\right)}{\Gamma(k)k!} \left(1 + \frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{2\pi i}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)} \frac{1}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}+\nu\right)} \sum_{k\geq 0} \frac{\Gamma\left(\frac{3}{2}-\nu+k\right)\Gamma\left(\frac{3}{2}+\nu+k\right)}{\Gamma(k+1)(k+1)!} \left(1 + \frac{\zeta}{2}\right)^{k} \\
&= -\frac{2\pi i}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}+\nu\right)} {}_2F_1\left(\frac{3}{2}-\nu, \frac{3}{2}+\nu; 2; 1 + \frac{\zeta}{2}\right) \\
&= -2i\cos(\nu\pi)\hat{w}_{\nu,-}(\zeta + 2)
\end{aligned}$$

[2]We do not consider constant term of $\tilde{w}_{\nu,\pm}$, i.e. $\mathcal{B} : \mathbb{C}[\![z^{-1}]\!] \to \mathbb{C}[\zeta]$.

and for $\hat{w}_{\nu,-}(\zeta)$

$$\begin{aligned}
\hat{w}_{\nu,-}(\zeta+i0)-\hat{w}_{\nu,-}(\zeta-i0) &= -\frac{2\pi i}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)}\left(\frac{\zeta}{2}-1\right)^{-1} {}_2F_1\left(\frac{1}{2}-\nu,\frac{1}{2}+\nu;0;1-\frac{\zeta}{2}\right) \quad \zeta<2\\
&= \frac{2\pi i}{\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)}\sum_{k\geq 0}\frac{\left(\frac{1}{2}-\nu\right)_k\left(\frac{1}{2}+\nu\right)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{2}\right)^{k-1}\\
&= 2i\cos(\nu\pi)\ {}_2F_1\left(\frac{3}{2}-\nu,\frac{3}{2}+\nu;2;1-\frac{\zeta}{2}\right)\\
&= 2i\cos(\nu\pi)\hat{w}_{\nu,+}(\zeta-2)
\end{aligned}$$

In addition, the previous relations computes the Stokes constants which are funtions of ν and are given by $\pm 2i\cos(\nu\pi)$.

1.2 Exponential integral

As showed by Aaron, if $T_n(u)$ and $U_n(u)$ denote the Chebyschev polynomials [3]

$$\begin{aligned}
K_\nu(z) &= \frac{1}{2i\sin(\nu\pi)}\int_{\mathcal{C}_\alpha} e^{z\cosh(t)}\sinh(\nu t)dt & u=\cosh(\nu t)\\
&= -\frac{1}{2i\nu\sin(\nu\pi)}\int_{\mathcal{C}_\alpha} e^{zT_{\frac{1}{\nu}}(u)}du & \zeta = T_{\frac{1}{\nu}}(u)\\
&= \frac{1}{2i\sin(\nu\pi)}\int_{\mathcal{H}_\alpha} e^{-\zeta z}\frac{d\zeta}{U_{\frac{1}{\nu}-1}(u)}\\
&= -\frac{1}{2i\sin(\nu\pi)}\int_{\mathcal{H}_\alpha} e^{-\zeta z}{}_2F_1\left(\frac{1-\nu}{2},\frac{1+\nu}{2};\frac{3}{2};1-\zeta^2\right)d\zeta
\end{aligned}$$

where $\mathcal{C}_\alpha$ **has to be checked, but I guess** $\alpha=1$

Indentity 15.10.17 from [**?**] splits the integrand above into

$$\begin{aligned}
{}_2F_1\left(\frac{1-\nu}{2},\frac{1+\nu}{2};\frac{3}{2};1-\zeta^2\right) &= C_1\,{}_2F_1\left(\frac{1-\nu}{2},\frac{1+\nu}{2};\frac{1}{2};\zeta^2\right)+C_2\zeta\,{}_2F_1\left(1-\frac{\nu}{2},1+\frac{\nu}{2};\frac{3}{2};\zeta^2\right)\\
&= \tilde{C}_1\left[{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}-\frac{\zeta}{2}\right)+{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)\right]+\\
&\quad +\tilde{C}_2\left[{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}-\frac{\zeta}{2}\right)+{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)\right]
\end{aligned}$$

therefore, collecting the contrubutions together we have

$$K_\nu(z) = \frac{i}{4\sin(\nu\pi)}\int_{\mathcal{H}_\alpha} e^{-\zeta z}\left[{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}-\frac{\zeta}{2}\right)+{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)\right]d\zeta \tag{7}$$

[3] $T_n(\cos(t))=\cos(nt)$ and $U_n(\cos(t))\sin(t)=\sin((n+1)t)$.

Since ${}_2F_1\left(1-\nu, 1+\nu; \frac{3}{2}; \frac{1}{2}+\frac{\zeta}{2}\right)$ is singular at $\zeta = 1$, the inverse Laplace transform of $K_\nu(z)$ is

$$\hat{K}_\nu(\zeta) = \frac{i}{4\sin(\nu\pi)} {}_2F_1\left(1-\nu, 1+\nu; \frac{3}{2}; \frac{1}{2}-\frac{\zeta}{2}\right)$$