Resurgence of the Airy function

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December 14, 2021

1 The Laplace transform

1.1 Analytic version

1.1.1 Regularity and decay properties

Take two copies $\mathbb{R}$ and $\hat{\mathbb{R}}$ of the real line, with standard coordinates z and ζ respectively. The Laplace transform in ζ turns a function $\hat{\varphi}$ on $\hat{\mathbb{R}}_{\zeta>0}$ into a function $\mathcal{L}_\zeta\hat{\varphi}$ on $\mathbb{R}_{z>0}$, defined by the integral

$$\mathcal{L}_\zeta\hat{\varphi} = \int_0^\infty e^{-z\zeta}\,\hat{\varphi}\,d\zeta.$$

For $a \in [0,\infty]$, recall that recall that $O_{\zeta\to a}(g)$ is the space of functions φ on $\hat{\mathbb{R}}_{\zeta>0}$ with $|\varphi| \lesssim g$ in some neighborhood of a. A function is *subexponential* if it's in $O_{\zeta\to\infty}(e^{c\zeta})$ for all $c > 0$. Let $\mathcal{E}_\zeta$ be the space of subexponential functions on $\hat{\mathbb{R}}_{\zeta>0}$ which are L^1 both locally and around $\zeta = 0$. If $\hat{\varphi}$ is in $\mathcal{E}_\zeta$, then $\varphi = \mathcal{L}_\zeta\hat{\varphi}$ is well-defined, and it extends to a holomorphic function on the right half-plane $\hat{\mathbb{C}}_{\Re(z)>0}$ [1, §5.6]. If $\hat{\varphi}$ is in $O_{\zeta\to 0}(1)$, then φ is in $O_{z\to\infty}(z^{-1})$ [2, equation 1.8].[1] More generally, if $\hat{\varphi}$ is in $O_{\zeta\to 0}(\zeta^\alpha)$, with $\alpha > -1$, then φ is in $O_{z\to\infty}\big(z^{-(\alpha+1)}\big)$.

1.1.2 Action on differential operators

When $\hat{\varphi} \in \mathcal{E}_\zeta$, we can use differentiation under the integral to show that [2, Theorem 1.34]

$$\mathcal{L}_\zeta(\zeta^n\hat{\varphi}) = \left(-\tfrac{\partial}{\partial z}\right)^n \mathcal{L}_\zeta\hat{\varphi}. \tag{1}$$

When $\hat{\varphi}$ is n times differentiable, its nth derivative is in $\mathcal{E}$, and its zeroth through $(n-1)$st derivatives extend continuously to zero, integration by parts gives the formula

$$\begin{aligned} \mathcal{L}_\zeta\left(\tfrac{\partial}{\partial\zeta}\right)^n\hat{\varphi} &= z^n\mathcal{L}_\zeta\hat{\varphi} - \Big[\hat{\varphi}\,z^{n-1} + \hat{\varphi}'\,z^{n-2} + \hat{\varphi}''\,z^{n-3} + \ldots + \hat{\varphi}^{(n-1)}\Big]_{\zeta=0} \\ &= z^n\,\mathcal{L}\left(\hat{\varphi} - \Big[\hat{\varphi} + \hat{\varphi}'\,\zeta + \tfrac{\hat{\varphi}''}{2!}\,\zeta^2 + \ldots + \tfrac{\hat{\varphi}^{(n-1)}}{(n-1)!}\,\zeta^{n-1}\Big]_{\zeta=0}\right). \end{aligned} \tag{2}$$

[1] The argument cited still works in our generality. For holomorphic $\hat{\varphi}$, one can also use Equation 1.5 of *Borel-Laplace Transform and Asymptotic Theory* **(Sternin & Shatalov)**.

Note that if a function's derivative is subexponential, so is the function itself.[2]

1.2 Algebraic version

1.2.1 Definition

Let $\mathcal{P}_\zeta$ be the vector space spanned by ζ^α for $\alpha \in \mathbb{R} \smallsetminus \mathbb{Z}_{<0}$. Note that $\mathcal{P}_\zeta \cap \mathcal{E}_\zeta$ is $\mathcal{P}_\zeta^{>-1}$, the subspace spanned by ζ^α with $\alpha > -1$. Since

$$\mathcal{L}_\zeta(\zeta^\alpha) = \Gamma(\alpha+1)\, z^{-(\alpha+1)}$$

for all $\alpha > -1$, let's use the same formula to extend $\mathcal{L}_\zeta$ to all of $\mathcal{P}_\zeta$. This defines $\mathcal{L}_\zeta$ consistently on $\mathcal{E}_\zeta + \mathcal{P}_\zeta$.

1.2.2 Action on differential operators

Observe that

$$\mathcal{L}_\zeta(\zeta^{\alpha+1}) = -\tfrac{\partial}{\partial z}\, \mathcal{L}_\zeta(\zeta^\alpha)$$

for $\alpha \neq -1$. This extends identity 1 to all of $\mathcal{P}_\zeta$.

Observe that

$$\mathcal{L}_\zeta \tfrac{\partial}{\partial \zeta}(\zeta^\alpha) = \begin{cases} z\, \mathcal{L}_\zeta(\zeta^\alpha) & \alpha \neq 0 \\ 0 & \alpha = 0, \end{cases}$$

and that $0 = z\, \mathcal{L}_\zeta(1) - 1$. This recovers identity 2 for any function in $\mathcal{P}_\zeta$ whose nth derivative is in $\mathcal{P}_\zeta^{>-1}$. Although the functions in $\mathcal{P}_\zeta^{<0}$ are singular at zero, let's pretend they vanish at zero. With that convention, formula 2 extends to all of $\mathcal{P}_\zeta$.

Now we have the results of Section 1.1.2 for all functions in $\mathcal{E}_\zeta + \mathcal{P}_\zeta$. Identity 2 is particularly simple when $\hat{\varphi}$ has a *fractional power singularity* at $\zeta = 0$. By this, I mean that $\hat{\varphi}$ can be written as $\hat{\varphi}_{\text{frac}} + \hat{\varphi}_{\text{reg}}$, where $\hat{\varphi}_{\text{frac}} \in \mathcal{P}_\zeta$ has only non-integer exponents, and the zeroth through $(n-1)$st derivatives of $\hat{\varphi}_{\text{reg}} \in \mathcal{E}_\zeta$ vanish at zero. Under this condition, all the initial value terms in the identity vanish, leaving

$$\mathcal{L}_\zeta \left(\tfrac{\partial}{\partial \zeta}\right)^n \hat{\varphi} = z^n \mathcal{L}_\zeta \hat{\varphi}.$$

[2] Say $f' \in O_{\zeta \to \infty}(e^{c\zeta})$. Then

$$\left| \int_0^Z f'\, d\zeta \right| \leq \int_0^Z |f'|\, d\zeta \lesssim \int_0^Z e^{c\zeta}\, d\zeta = \tfrac{1}{c}(e^{cZ} - 1) \lesssim e^{cZ}.$$

Now we know the integral on the left-hand side converges, implying that f extends continuously to zero, with $|f - f_{\zeta=0}| \lesssim e^{c\zeta}$.

1.3 Change of coordinates

Define a new coordinate ζ_a on $\hat{\mathbb{R}}$ so that $\zeta = a + \zeta_a$. From the calculation

$$\begin{aligned}
\mathcal{L}_\zeta \hat{\varphi} &= \int_0^\infty e^{-z\zeta}\, \hat{\varphi}\, d\zeta \\
&= \int_0^\infty e^{-z(a+\zeta_a)}\, \hat{\varphi}\, d\zeta_a \\
&= e^{-az} \int_0^\infty e^{-z\zeta_a}\, \hat{\varphi}\, d\zeta_a \\
&= e^{-az} \mathcal{L}_{\zeta_a} \hat{\varphi},
\end{aligned}$$

we learn that

$$\mathcal{L}_{\zeta_a} \hat{\varphi} = e^{az} \mathcal{L}_\zeta \hat{\varphi}.$$

Define new coordinates x and ξ on $\mathbb{R}$ and $\hat{\mathbb{R}}$, respectively, so that $\zeta = b\xi$ and $z\, d\zeta = x\, d\xi$. Explicitly, $z = b^{-1}x$. From the calculation

$$\begin{aligned}
\mathcal{L}_\zeta \hat{\varphi} &= \int_0^\infty e^{-z\zeta}\, \hat{\varphi}\, d\zeta \\
&= \int_0^\infty e^{-x\xi}\, \hat{\varphi}\, b\, d\xi \\
&= b \mathcal{L}_\xi \hat{\varphi},
\end{aligned}$$

we learn that

$$\mathcal{L}_\xi \hat{\varphi} = b^{-1} \mathcal{L}_\zeta \hat{\varphi}.$$

2 The Airy equation

2.1 Basics

The Airy equation is

$$\left[\left(\tfrac{\partial}{\partial y} \right)^2 - y \right] \psi = 0. \tag{3}$$

One solution is given by the Airy function,

$$\operatorname{Ai}(y) = \frac{1}{2\pi i} \int_{\Gamma_0} \exp\left(\tfrac{1}{3} t^3 - yt \right)\, dt,$$

where Γ_0 is a path that comes from ∞ at $-60°$ and goes to ∞ at $60°$. With the substitution $t = -2uy^{1/2}$, we can rewrite the Airy integral as

$$\operatorname{Ai}(y) = y^{1/2}\, \frac{1}{\pi i} \int_{-y^{-1/2}\Gamma_0} \exp\left[-\tfrac{2}{3} y^{3/2} \left(4u^3 - 3u \right) \right]\, du.$$

We've rescaled the contour by a factor of two, but it still approaches ∞ in the desired way. Note that $4u^3 - 3u$ is the third Chebyshev polynomial.

2.2 Rewriting as a modified Bessel equation

We can distill the most interesting part of the Airy function by writing

$$\operatorname{Ai}(y) = \tfrac{1}{\pi\sqrt{3}}\, y^{1/2}\, K\big(\tfrac{2}{3}y^{3/2}\big),$$

where

$$K(z) = \frac{\sqrt{3}}{i} \int_{-z^{-1/3}\Gamma_0} \exp\left[-z\left(4u^3 - 3u\right)\right]\, du.$$

Saying that Ai satisfies the Airy equation is equivalent to saying that K satisfies the modified Bessel equation

$$\left[z^2\big(\tfrac{\partial}{\partial z}\big)^2 + z\tfrac{\partial}{\partial z} - \big[\big(\tfrac{1}{3}\big)^2 + z^2\big]\right]\varphi = 0. \tag{4}$$

In fact, K is the modified Bessel function $K_{1/3}$ [3, equation 9.6.1].

As we'll see in Section **??**, K is in $O_{z\to\infty}(e^{-z})$. It'll be helpful to pull out the exponential decay factor and work instead with the function κ defined by $K = e^{-z}\kappa$. Saying that K satisfies equation 4 is equivalent to saying that κ satisfies the equation

$$\left[z^2\big(\tfrac{\partial}{\partial z} + 1\big)^2 + z\big(\tfrac{\partial}{\partial z} + 1\big) - \big[\big(\tfrac{1}{3}\big)^2 + z^2\big]\right]\varphi = 0. \tag{5}$$

2.3 Asymptotic analysis

From [3], equations 10.40.2 and 10.17.1, we get the asymptotic series

$$\kappa \sim \big(\tfrac{\pi}{2}\big)^{1/2}\left[z^{-1/2} - \frac{(\frac{1}{6})_1(\frac{5}{6})_1}{2^1\cdot 1!}\, z^{-3/2} + \frac{(\frac{1}{6})_2(\frac{5}{6})_2}{2^2\cdot 2!}\, z^{-5/2} - \frac{(\frac{1}{6})_3(\frac{5}{6})_3}{2^3\cdot 3!}\, z^{-7/2} + \dots\right] \tag{6}$$

2.4 Going to the spatial domain

2.4.1 A good try at $\zeta = 0$

Let's try to find a function $\hat{K}$ with $K = \mathcal{L}_\zeta \hat{K}$, which is unique if it exists [2, Theorem 1.23]. If a function $\hat{\varphi}$ satisfies the equation

$$\left[(\zeta^2 - 1)\big(\tfrac{\partial}{\partial\zeta}\big)^2 + 3\zeta\tfrac{\partial}{\partial\zeta} + \big[1 - \big(\tfrac{1}{3}\big)^2\big]\right]\hat{\varphi} = 0, \tag{7}$$

its Laplace transform $\varphi = \mathcal{L}_\zeta\hat{\varphi}$ satisfies the equation

$$\begin{aligned}
\left[\big(-\tfrac{\partial}{\partial z}\big)^2 - 1\right]\Big(z^2\varphi - \big[\hat{\varphi}\, z + \hat{\varphi}'\big]_{\zeta=0}\Big) + 3\big(-\tfrac{\partial}{\partial z}\big)\big[z\varphi - \hat{\varphi}\big]_{\zeta=0} + \big[1 - \big(\tfrac{1}{3}\big)^2\big]\varphi &= 0\\
\big(\tfrac{\partial}{\partial z}\big)^2\big[z^2\varphi\big] - \Big(z^2\varphi - \big[\hat{\varphi}\, z + \hat{\varphi}'\big]_{\zeta=0}\Big) - 3\big(\tfrac{\partial}{\partial z}\big)\big[z\varphi\big] + \big[1 - \big(\tfrac{1}{3}\big)^2\big]\varphi &= 0\\
\left[2 + 4z\tfrac{\partial}{\partial z} + z^2\big(\tfrac{\partial}{\partial z}\big)^2\right]\varphi - \Big(z^2\varphi - \big[\hat{\varphi}\, z + \hat{\varphi}'\big]_{\zeta=0}\Big) - 3\Big[1 + z\tfrac{\partial}{\partial z}\Big]\varphi + \big[1 - \big(\tfrac{1}{3}\big)^2\big]\varphi &= 0,
\end{aligned}$$

which simplifies to

$$\left[z^2\big(\tfrac{\partial}{\partial z}\big)^2 + z\tfrac{\partial}{\partial z} - \big[\big(\tfrac{1}{3}\big)^2 + z^2\big]\right]\varphi = -\big[\hat{\varphi}\, z + \hat{\varphi}'\big]_{\zeta=0}. \tag{8}$$

Since we want $\mathcal{L}_\zeta\hat{K}$ to satisfy equation 4, which is the homogeneous version of equation 8, we might guess that $\hat{K}$ is a solution of equation 7 that vanishes through first order at $\zeta = 0$. Unfortunately, this would force $\hat{K}$ to be zero.

2.4.2 Success at $\zeta = 1$

Define a new coordinate ζ_1 on $\hat{\mathbb{R}}$ so that $\zeta = 1 + \zeta_1$. Since

$$\begin{aligned} \mathcal{L}_{\zeta_1} \hat{K} &= e^z \mathcal{L}_\zeta \hat{K} \\ &= e^z K \\ &= \kappa, \end{aligned}$$

we want $\mathcal{L}_{\zeta_1} \hat{K}$ to satisfy equation 5. Rewrite equation 7 as

$$\left[\zeta_1(\zeta_1 + 2)\left(\tfrac{\partial}{\partial \zeta_1}\right)^2 + 3(\zeta_1 + 1)\tfrac{\partial}{\partial \zeta_1} + \left[1 - \left(\tfrac{1}{3}\right)^2\right]\right] \hat{\varphi} = 0. \tag{9}$$

If $\hat{\varphi}$ satisfies equation 9, $\mathcal{L}_{\zeta_1} \hat{\varphi}$ will satisfy an inhomogeneous version of equation 5, analogous to equation 8. This time, though, there's a trick we can use to zero out the inhomogeneity. Equation 9 has a regular singularity at $\zeta_1 = 0$, and one solution (up to scaling) is a holomorphic multiple of $\zeta_1^{-1/2}$. That solution has a fractional power singularity at $\zeta_1 = 0$, as defined in Section 1.2.2, so its Laplace transform in ζ_1 satisfies equation 5.

Following this plan, let's find $\hat{K}$ explicitly. Defining another coordinate ξ on $\hat{\mathbb{R}}$ so that $\zeta_1 = -2\xi$, we can rewrite equation 9 as the hypergeometric equation

$$\left[\xi(1 - \xi)\left(\tfrac{\partial}{\partial \xi}\right)^2 + 3(\tfrac{1}{2} - \xi)\tfrac{\partial}{\partial \xi} - \left[1 - \left(\tfrac{1}{3}\right)^2\right]\right] \hat{\varphi} = 0. \tag{10}$$

The hypergeometric function

$$\hat{f}_1 = F\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; \xi\right)$$

satisfies equation 10 by definition. It's not the solution we want, though, because it's holomorphic around $\xi = 0$. Formula 15.10.12 from [3] gives another solution,

$$\hat{f}_0 = \xi^{-1/2} F\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; \xi\right),$$

which is a holomorphic multiple of $\xi^{-1/2}$ near $\xi = 0$. By the argument above, $f_0 = \mathcal{L}_{\zeta_1} \hat{f}_0$ satisfies equation 5. This suggests that a constant multiple of $\hat{f}_0$ is our desired $\hat{K}$. The power series [3, equation 15.2.1]

$$\hat{f}_0 = \xi^{-1/2} + \frac{(\frac{1}{6})_1 (\frac{5}{6})_1}{(\frac{1}{2})_1 \ 1!} \xi^{1/2} + \frac{(\frac{1}{6})_2 (\frac{5}{6})_2}{(\frac{1}{2})_2 \ 2!} \xi^{3/2} + \frac{(\frac{1}{6})_3 (\frac{5}{6})_3}{(\frac{1}{2})_3 \ 3!} \xi^{5/2} + \ldots$$

converges near $\xi = 0$, showing that

$$\hat{f}_0 \in \xi^{-1/2} + O_{\xi \to 0}(\xi^{1/2}).$$

In terms of ζ_1, we have

$$\hat{f}_0 \in -i\sqrt{2}\, \zeta_1^{-1/2} + O_{\zeta_1 \to 0}(\zeta_1^{1/2}).$$

Using the decay properties from Section 1.1.1, we deduce that

$$f_0 \in -i\sqrt{2\pi}\, z^{-1/2} + O_{z \to \infty}(z^{-3/2}).$$

Since we know that f_0 satisfies equation 5, this confirms that f_0 is a constant multiple of κ, which is the only subexponential solution of equation 5 (up to scaling). Comparing with series 6, we see that $\kappa = i\, \frac{1}{2}\, f_0$. We conclude that $\kappa = \mathcal{L}_{\zeta_1} \hat{K}$ for

$$\hat{K} = \tfrac{1}{\sqrt{2}}\, \zeta_1^{-1/2} F\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; -\tfrac{1}{2}\zeta_1\right).$$

3 Sketches

3.1 Contour argument

The hypergeometric function

$$F\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; \xi\right)$$

satisfies equation 10 by definition. It has a singularity of type $(1-\xi)^{-1/2}$ at $\xi = 1$, and continues holomorphically to the rest of $\hat{\mathbb{C}}$.

Based on the calculations in the other sheet, we should have

$$K(z) = i\frac{2}{\sqrt{3}} \int_{\text{Hankel at 1}} e^{-z\zeta} F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\right) d\zeta.$$

This is because the hypergeometric function in the integrand happens to be algebraic. It can be expressed in terms of Legendre functions and rearranged to get

$$F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\right) = \frac{1}{4u^2 - 1}$$

with $\zeta = -4u^3 + 3u$.

The quadratic transformation identity DLMF 15.8.27 then shows that

$$F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\right) \propto F\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; \xi\right) + F\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; 1-\xi\right),$$

so we have

$$K(z) \propto \int_{\text{Hankel at 1}} e^{-z\zeta} \left[F\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; \xi\right) + F\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; 1-\xi\right)\right] d\zeta.$$

The first term in the integrand is regular on $\zeta \in [1, \infty)$, so it integrates to zero. The second term looks like $(1-\zeta)^{-1/2}$ times a function which is regular for $\zeta \in [1, \infty)$, so integrating it along a Hankel contour is the same as integrating it times two times a direct contour, with its value continued from below the singularity. Hence,

$$K(z) \propto \int_1^\infty e^{-z\zeta} F\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; 1-\xi\right) d\zeta.$$

With $\zeta = 1 + \tilde{\zeta}$, we get $1 - \xi = 1 + \frac{1}{2}\tilde{\zeta}$, so

$$K(z) = \int_0^\infty e^{-z(1+\tilde{\zeta})} F\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; 1 + \tfrac{1}{2}\tilde{\zeta}\right) d\tilde{\zeta}$$

$$e^z K(z) = \int_0^\infty e^{-z\tilde{\zeta}} F\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; 1 + \tfrac{1}{2}\tilde{\zeta}\right) d\tilde{\zeta}$$

In other words, $\kappa(z) = e^z K(z)$ is the Laplace transform of $\hat{\kappa}(\tilde{\zeta}) = F\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; 1 + \frac{1}{2}\tilde{\zeta}\right)$. We've confirmed numerically that $\hat{\kappa}(\tilde{\zeta})$ seems to have a singularity of type $\tilde{\zeta}^{-1/2}$ at $\tilde{\zeta} = 0$, as expected. **Can we see this analytically?**

Using the DLMF identities 15.10.12 and 15.10.14, we can see that the functions

$$\xi^{-1/2} F\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; \xi\right)$$

$$(1-\xi)^{-1/2} F\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; 1-\xi\right)$$

also satisfy equation 10. Numerically, these functions appear to sum to a constant multiple of $F\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \zeta^2\right)$.

3.2 Correspondence with Mariño's series

Let $f_1(z)$ be the holomorphic function corresponding to Mariño's formal power series $\varphi_1(z^{-1})$. The formal power series corresponding to f will be written in the variable z.

$$
\begin{aligned}
\mathrm{Ai}(x) &= \tfrac{1}{2\sqrt{\pi}} x^{-1/4} e^{-z} \varphi_1\big(\tfrac{2}{3} z^{-1}\big) \\
&= \tfrac{1}{2\sqrt{\pi}} x^{-1/4} e^{-z} f_1\big(\tfrac{3}{2} z\big) \\
\mathrm{Ai}(x) &= \frac{1}{\pi\sqrt{3}} x^{1/2} K\big(\tfrac{2}{3} x^{3/2}\big)
\end{aligned}
$$

Putting together,

$$
\begin{aligned}
\tfrac{1}{2\sqrt{\pi}} x^{-1/4} e^{-z} f_1\big(\tfrac{3}{2} z\big) &= \frac{1}{\pi\sqrt{3}} x^{1/2} K\big(\tfrac{2}{3} x^{3/2}\big) \\
\tfrac{\sqrt{3\pi}}{2}\ x^{-3/4} e^{-z} f_1\big(\tfrac{3}{2} z\big) &= K\big(\tfrac{2}{3} x^{3/2}\big) \\
\tfrac{\sqrt{3\pi}}{2}\ \big(\tfrac{3}{2} z\big)^{-1/2} e^{-z} f_1\big(\tfrac{3}{2} z\big) &= K(z) \\
\sqrt{\tfrac{\pi}{2}}\ z^{-1/2} e^{-z} f_1\big(\tfrac{3}{2} z\big) &= K(z) \\
\sqrt{\tfrac{\pi}{2}}\ \big[\mathcal{L}^{-1} z^{-1/2}\big] * \big[\mathcal{L}^{-1} f_1\big(\tfrac{3}{2} z\big)\big](\zeta - 1) &= \hat{K}(\zeta) \\
\sqrt{\tfrac{\pi}{2}}\ \Big[\Gamma\big(-\tfrac{1}{2}\big)^{-1} \zeta^{-1/2}\Big] * \tfrac{2}{3} \hat{f}_1\big[\tfrac{2}{3}(\zeta - 1)\big] &= \hat{K}(\zeta) \\
-\tfrac{1}{3\sqrt{2}}\ \Big[\zeta^{-1/2}\Big] * \hat{f}_1\big[\tfrac{2}{3}(\zeta - 1)\big] &= \hat{K}(\zeta)
\end{aligned}
$$

Notice that if the hypergeometric differentiation formula holds for fractional derivatives,

$$
\big(\tfrac{\partial}{\partial \xi}\big)^{1/2} F\big(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; \xi\big) \propto F\big(\tfrac{7}{6}, \tfrac{11}{6}; 2; \xi\big)
$$

References

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