Resurgence of the Airy function and other exponential integrals

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1 Introduction

1.1 Why does Borel resummation work?

Borel resummation is a way of turning a formal power series

$$\varphi_\bullet = z^\sigma \left(\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \frac{\varphi_3}{z^4} + \dots \right),$$

with $\sigma \in [0, 1)$, into a function which is asymptotic to $\varphi_\bullet$ as $z \to \infty$. Different functions can be asymptotic to the same power series, and Borel resummation picks one of them, performing an implicit regularization **[arXiv:1705.03071, or maybe arXiv:1412.6614]**. When a function matches the Borel sum of its asymptotic series, we'll say it's *Borel regular*. Several familiar kinds of regularity imply Borel regularity, and shed light on why it occurs.

- **Having a good asymptotic approximation**

  Let R_N be the difference between a function and the partial sum

  $$\frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \frac{\varphi_2}{z^3} + \dots + \frac{\varphi_{N-2}}{z^{N-1}}$$

  of its asymptotic series. Watson showed a century ago that the function is Borel regular whenever there's a constant $c \in (0, \infty)$ with

  $$|R_N| \leq \frac{c^{N+1} N!}{|z|^N}$$

  over all orders N and all z in a wide enough wedge around infinity.

- **Satisfying a singular differential equation**
  - Think about conditions where this works.
  - Maybe the correct place is the setting of Ecalle's formal integral. See §5.2.2.1 of Delabaere's *Divergent Series, Summability and Resurgence III*.
  - Say there's a unique solution (up to scaling) that shrinks as you go right; everything else blows up exponentially. Then this is the only solution that can be expressed as a Laplace transform.

– If the Borel-transformed equation has a subexponential solution $\hat{f}$ which is "shifted holomorphic" (we called this having a "fractional power singularity" in `airy-resugence`), then $\mathcal{L}\hat{f}$ satisfies the original equation, because there are no boundary terms.

– Draw diagram showing formal vs. holomorphic solutions in time vs. frequency domains.

- **Being a thimble integral**

  Let X be a translation surface—a Riemann surface carrying a holomorphic 1-form ν. Suppose X is of *meromorphic type*, meaning that we got it by puncturing a compact Riemann surface $\overline{X}$ at finitely many points, and ν has a pole at each puncture. A *translation coordinate* on X is a local coordinate whose derivative is ν.

  Take another meromorphic-type translation surface B and a holomorphic Morse[1] map $f\colon \overline{X} \to \overline{B}$ that sends punctures to punctures. Suppose every singularity of B is a critical value of f. [**Typical usage of "Borel plane" seems ambiguous, so maybe we can use "Borel plane" for B and "Borel cover" for the Riemann surface of the Borel-transformed series. How to handle the Orr–Sommerfeld functions (DLMF §9.13)? We know $f = 4u^3 - 3u$ is the pullback of a translation coordinate, but we also need a puncture at $f(0)$...**] For each critical point p, let Γ_p be the ray going rightward from $f(p)$, and let ζ_p be the translation coordinate around Γ_p which vanishes at $f(p)$. These are well-defined as long as Γ_p misses the critical values of f. The preimage $f^{-1}(\Gamma_p)$ is a bunch of disjoint curves, as long as Γ_p misses the other critical values of f. The *Lefschetz thimble* Λ_p is the component of $f^{-1}(\Gamma_p)$ that goes through p, oriented so that shifting it to its left would make its projection run clockwise around Γ_p. The *thimble integral*

  $$I_p = \int_{\Lambda_p} e^{-zf^*\zeta_p}\nu$$

  is a holomorphic function on the right half-plane parameterized by z, and it turns out [**we hope**] to be Borel regular.

  [**Talk about exponential integrals and their decomposition into thimble integrals.**]

  In higher-dimensional complex manifolds, integrals over Lefschetz thimbles are still Borel regular [**"Exponential integrals, Lefschetz thimbles and linear resurgence"**][**"Exponential Integral" lectures?**]. This fact plays an important technical role in quantum mechanics, where infinite-dimensional exponential integrals are supposed to give the expectation values of observable quantities. Physicists often use Borel summation and related techniques to assign values to these integrals [**Costin & Kruskal, "On optimal truncation..."**].

  Choose a path $\gamma\colon \mathbb{R} \to X$ whose projection $f \circ \gamma$ starts out going leftward out of a puncture, ends up going rightward into a puncture, and never touches a critical value of f. Choose a translation coordinate ζ on B and continue it along $f \circ \gamma$, noting that

[1] This condition means that the critical points of f are isolated (the compactness of $\overline{X}$ guarantees this) and the 2-jet of f is non-zero at every critical point.

it may become multi-valued if $f \circ \gamma$ intersects itself. This data defines the *exponential integral*

$$I = \int_\gamma e^{-zf^*\zeta}\nu,$$

a holomorphic function on the right half-plane parameterized by z. It turns out [**we hope**] that we can get I by summing $e^{-\alpha_p z} I_p$ over various critical points—as long as none of the Γ_p run into each other. [**We get jumps at phases where the Γ_p do hit each other.**] The constants α_p are values of ζ, continued to the critical points along certain paths.

- Each resummation method for asymptotic series makes some implicit assumption that allows us to reconstruct a holomorphic function from its asymptotic behavior.
- The resummation method works correctly for functions which satisfy that assumption.
- For the modified Bessel function $K_{1/3}$, Borel resummation works because the asymptotic series encodes a second-order differential equation.
  - Different aspects of this example appear in various places (Mariño, Kawai–Takei, Sauzin). We give a detailed, unified treatment.
- We can generalize this argument to all $K_{1/n}$ and their limit K_0.
- We can also generalize to all third-order exponential integrals.
  - Most of them are equivalent to the $K_{1/3}$ integral, but there's also an interesting degeneration.

1.2 Fractional derivative formula

- Theorem **??** says that for a certain class of exponential integrals

$$I(z) = \int_\Gamma e^{-zf}\ \nu,$$

  the inverse Laplace [**better to say Borel?**] transform is the $\frac{3}{2}$ derivative of $d\zeta/df$, where $f^* d\zeta = \nu$ [**check**].
- the asymptotic expansion of $I(z)$ is a resurgent function.
- Is it always a *simple* resurgent function?
  - **Maxim belives it is in general, and indeed in our examples we get simple resurgent functions. But how to prove it in general?**

1.3 Stokes phenomenon

- For Bessel functions, we can see explicitly how solutions jump when the Laplace transform angle crosses a critical value.
- The jump comes from the branch cut difference identity for hypergeometric functions.
- Possible interpretation of the Stokes factors as intersections numbers in Morse–Novikov theory [**ask Maxim**]

2 The Laplace and Borel transforms

2.1 The Laplace transform

- Action on differential equations.
  - Can we find a way to prove this when the differential operator spits out a function that's not integrable around zero?
- Global picture?

2.2 The Borel transform

- Action on differential equations.
  - No inhomogeneous terms! How is this consistent with the Laplace transform's action? Is there always an inhomogeneous solution with subexponential asymptotics?

3 Third-order exponential integrals

- Reduce to
$$I(z) = \int \exp\left[-z\big(u^3 + pu + q\big)\right]\, du$$
  using change of coordinate.
- When $p \neq 0$, can reduce further to
$$I(z) = p^{1/2} e^{-qz} K_{1/3}(p^{3/2} z).$$
- As p goes to zero, $I(z)$ degenerates to
$$\left(\tfrac{1}{2}\right)^{2/3} e^{-qz}\Gamma(\tfrac{1}{3}) z^{-1/3} = \left(\tfrac{1}{2}\right)^{2/3} e^{-qz} \mathcal{L}_{\zeta,0}(\zeta^{-2/3}) = \left(\tfrac{1}{2}\right)^{2/3} \mathcal{L}_{\zeta_{-q},q}(\zeta^{-2/3}).$$