Resurgence of modified Bessel functions of second kind

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1 Generalized Airy

Let $p \geq 0$,

$$I(z) := \int_{e^{-i\frac{\pi}{3}}\infty}^{e^{i\frac{\pi}{3}}\infty} e^{-z(\frac{t^3}{3}-t)} \frac{dt}{t^p} \tag{1}$$

Since $I(z) = z^{(p-1)/3}\mathrm{A}_1(z,p)$ which is a solution of

$$\left[\partial_z^3 - z\partial_z + (p-1)\right] \mathrm{A}_1(z,p) = 0 \tag{2}$$

it follows that $I(z)$ is a solution of

$$\partial_z^3 I - \frac{4}{9}\partial_z I + \frac{2-p}{z}\partial_z^2 I - \frac{4(1-p)}{9}\frac{I}{z} - \frac{1+3p-3p^2}{9}\frac{\partial_z I}{z^2} + \frac{3+p-3p^2-p^3}{27}\frac{I}{z^3} = 0 \tag{3}$$

In particular, the formal integral solution of (3) is a three parameter family

$$\tilde{I}(z) = \sum_{\mathbf{k}\in\mathbb{N}^3} U^{\mathbf{k}} e^{-\frac{2}{3}(k_2-k_3)z} z^{-\frac{1}{2}(k_2+k_3)-(1-p)k_1} \tilde{w}_{\mathbf{k}}(z) \tag{4}$$

where $\tilde{w}_{\mathbf{k}}(z) \in \mathbb{C}[\![z^{-1}]\!]$ is a formal solution of

$$\left[P_3(\partial,\mathbf{k}) + \frac{1}{t}P_2(\partial,\mathbf{k}) + \frac{1}{t^2}P_1(\partial,\mathbf{k}) + \frac{1}{t^3}P_0(\partial,\mathbf{k})\right]\tilde{w}_{\mathbf{k}}(z) = 0 \tag{5}$$

$$
\begin{aligned}
P_0(\partial,\mathbf{k}) &= \frac{1}{9}+\frac{1}{9}k_1-k_1^2-k_1^3+\frac{1}{18}k_2-k_1k_2-\frac{3}{2}k_1^2k_2-\frac{1}{4}k_2^2-\frac{3}{4}k_1k_2^2-\frac{1}{8}k_2^3+\frac{p^3}{3}k_1+\\
&+\frac{1}{18}k_3-k_1k_3-\frac{3}{2}k_1^2k_3-\frac{1}{2}k_2k_3-\frac{3}{2}k_1k_2k_3-\frac{3}{8}k_2^2k_3-\frac{1}{4}k_3^2-k_1^2p^3+k_1^3p^3+\\
&-\frac{3}{4}k_1k_3^2-\frac{3}{8}k_2k_3^2-\frac{1}{8}k_3^3+\frac{p}{27}-\frac{7}{9}k_1p+k_1^2p+3k_1^3p-\frac{p}{3}k_2+3k_1^2k_2p-\frac{p}{4}k_2^2+\\
&\frac{3}{4}k_1k_2^2p-\frac{p}{3}k_3+3k_1^2k_3p-\frac{p}{2}k_2k_3+\frac{3}{2}k_1k_2k_3p-\frac{p}{4}k_3^2+\frac{3}{4}k_1k_3^2p-\frac{p^2}{9}+\frac{p^2}{3}k_1+\\
&+k_1^2p^2-3k_1^3p^2-\frac{p^2}{6}k_2+k_1k_2p^2-\frac{3}{2}k_1^2k_2p^2-\frac{p^2}{6}k_3+k_1k_3p^2-\frac{3}{2}k_1^2k_3p^2-\frac{p^3}{27}\\
P_1(\partial,\mathbf{k}) &= (-\frac{1}{9}-k_1+3k_1^2-\frac{1}{2}k_2+3k_1k_2+\frac{3}{4}k_2^2-\frac{1}{2}k_3+3k_1k_3+\frac{3}{2}k_2k_3+\frac{3}{4}k_3^2-\frac{p}{3}+\\
&+3k_1p-6k_1^2p+k_2p-3k_1k_2p+k_3p-3k_1k_3p+\frac{p^2}{3}-2k_1p^2+3k_1^2p^2)\partial_z+\\
&+\frac{2}{27}k_2+\frac{2}{3}k_1k_2-2k_1^2k_2+\frac{1}{3}k_2^2-2k_1k_2^2-\frac{1}{2}k_2^3-\frac{2}{27}k_3-\frac{2}{3}k_1k_3+\\
&+2k_1^2k_3-\frac{1}{2}k_2^2k_3-\frac{1}{3}k_3^2+2k_1k_3^2+\frac{1}{2}k_2k_3^2+\frac{1}{2}k_3^3+\frac{2}{9}k_2p-2k_1k_2p\\
&+4k_1^2k_2p-\frac{2}{3}k_2^2p+2k_1k_2^2p-\frac{2}{9}k_3p+2k_1k_3p-4k_1^2k_3p+\frac{2}{3}k_3^2p-2k_1k_3^2p-\frac{2}{9}k_2p^2+\\
&+\frac{4}{3}k_1k_2p^2-2k_1^2k_2p^2+\frac{2}{9}k_3p^2-\frac{4}{3}k_1k_3p^2+2k_1^2k_3p^2\\
P_2(\partial,\mathbf{k}) &= (2-3k_1-\frac{3}{2}(k_2+k_3)-p+3k_1p)\partial_z^2+\\
&(-\frac{8}{3}k_2+4k_1k_2+2k_2^2+\frac{8}{3}k_3-4k_1k_3-2k_3^2+\frac{4}{3}k_2p-4k_1k_2p-\frac{4}{3}k_3p+4k_1k_3p)\partial_z+\\
&-\frac{4}{9}+\frac{4}{9}k_1+\frac{2}{9}k2+\frac{8}{9}k_2^2-\frac{4}{3}k_1k_2^2-\frac{2}{3}k_2^3+\frac{2}{9}k_3-\frac{16}{9}k_2k_3+\frac{8}{3}k_1k_2k_3+\frac{2}{3}k_2^2+\frac{8}{9}k_3^2-\frac{4}{3}k_1k_3^2+\\
&+\frac{2}{3}k_2k_3^2-\frac{2}{3}k_3^3+\frac{4}{9}p-\frac{4}{9}k_1p-\frac{4}{9}k_2^2p+\frac{4}{3}k_1k_2^2p+\frac{8}{9}k_2k_3p-\frac{8}{3}k_1k_2k_3p-\frac{4}{9}k_3^2p+\frac{4}{3}k_1k_3^2p\\
P_3(\partial,\mathbf{k}) &= \partial_z^3-2(k_2-k_3)\partial_z^2+\frac{4}{3}(k_2-k_3)^2\partial_z-\frac{4}{9}\partial_z+\frac{8}{27}(k_2-k_3)(1-k_2+k_3)(1+k_2-k_3)
\end{aligned}
$$

Since $\tilde{w}_{\mathbf{k}}(z)=\sum_{j\geq 0}a_{\mathbf{k},j}z^{-j}$ there only few values of $\mathbf{k}\in\mathbb{N}^3$ such that $\tilde{w}_{\mathbf{k}}\neq 0$, hence the formal integral solution of (3) reduces to

$$
\tilde{I}(z)=U_1z^{p-1}\tilde{w}_1(z)+U_2e^{-\frac{2}{3}z}z^{-1/2}\tilde{w}_2(z)+U_3e^{\frac{2}{3}z}\tilde{w}_3(z) \tag{6}
$$

where $\tilde{w}_1,\tilde{w}_2$ and $\tilde{w}_3$ are the unique (we fix $a_{k,0}=1$ for $k=1,2,3$) solutions of

$$
\left[\partial_z^3-\frac{4}{9}\partial_z-\frac{1-2p}{z}\partial_z^2+\frac{17-30p+12p^2}{9z^2}\partial_z+\frac{8}{27}\frac{p^3-6p^2+11p-6}{z^3}\right]\tilde{w}_1(z) \quad = \quad 0 \tag{7}
$$

$$\left[\partial_z^3 - 2\partial_z^2 + \frac{8}{9}\partial_z + \frac{1-2p}{2z}\partial_z^2 - \frac{2-4p}{3z}\partial_z + \frac{5+24p+12p^2}{36z^2}\partial_z + \right. \\ \left. - \frac{5+24p+12p^2}{54z^2} - \frac{1}{z^3}\left(\frac{5}{24} + \frac{59}{108}p + \frac{5}{18}p^2 + \frac{1}{27}p^3\right)\right]\tilde{w}_2(z) = 0 \quad (8)$$

$$\left[\partial_z^3 + 2\partial_z^2 + \frac{8}{9}\partial_z + \frac{1-2p}{2z}\partial_z^2 + \frac{2-4p}{3z}\partial_z + \frac{5+24p+12p^2}{36z^2}\partial_z + \right. \\ \left. + \frac{5+24p+12p^2}{54z^2} - \frac{1}{z^3}\left(\frac{5}{24} + \frac{59}{108}p + \frac{5}{18}p^2 + \frac{1}{27}p^3\right)\right]\tilde{w}_3(z) = 0 \quad (9)$$

We first study the equation (7): its unique formal solution $\tilde{w}_1(z)$ has the following coefficients (we compute them with Mathematica)

$$a_{1,2j} = \frac{\Gamma(1+3j-p)}{3^j\Gamma(1+j)\Gamma(1-p)} \qquad j \geq 0 \tag{10}$$

Notice that $a_{1,j}$ are well defined only for $p \in \mathbb{C} \setminus \{1,2,3,...\}$. However, for $p = 1,2,3$ the solution is well defined and constant $\tilde{w}_1(z) = 1$. Therefore the Borel transform of $\tilde{w}_1(z)$ is

$$\hat{w}_1(\zeta) = \begin{cases} \delta & \text{if } p = 1,2,3 \\ \delta + \zeta\frac{\Gamma(4-p)}{3\Gamma(1-p)}\,{}_3F_2\left(\frac{4-p}{3}, \frac{5-p}{3}, 2-\frac{p}{3}; \frac{3}{2}, 2; \frac{9}{4}\zeta^2\right) & \text{if } p \in \mathbb{C} \setminus \{1,2,3,...\} \end{cases} \tag{11}$$

In particular, the generalized hypergeometric series has a branch cut singularity at $\zeta = \pm\frac{3}{2}$. We expect that $\tilde{w}_1(z)$ is a simple resurgent fuction such that

$$\hat{w}_1(\zeta + \tfrac{2}{3}) = \delta + \frac{C}{2\pi i\zeta} + \frac{1}{2\pi i}\log(\zeta)\hat{w}_2(\zeta) + \text{hol. fct.}$$
$$\hat{w}_1(\zeta - \tfrac{2}{3}) = \delta + \frac{C}{2\pi i\zeta} + \frac{1}{2\pi i}\log(\zeta)\hat{w}_3(\zeta) + \text{hol. fct.}$$

Indeed, in [**D.B. Karp and E.G. Prilepkina** formula 3.1 (see also https://arxiv.org/pdf/2110.12219.pdf equation 27)] the authors compute the analytic continuation of generalized hypergeomtric functions across the branch cut

$${}_qF_{q-1}(\mathbf{a};\mathbf{b};x+i0) - {}_qF_{q-1}(\mathbf{a};\mathbf{b};x-i0) = 2\pi i\frac{\Gamma(\mathbf{b}-d+1)}{\Gamma(\mathbf{a}-d+1)}x^{d-1}G^{q,0}_{q,q}\left(d,\mathbf{b};\mathbf{a};\frac{1}{x}\right) \tag{12}$$

which specifies to

$$\frac{3}{2}\zeta\left({}_3F_2\left(\frac{4-p}{3}, \frac{5-p}{3}, 2-\frac{p}{3}; \frac{3}{2}, 2; \frac{9}{4}\zeta^2 + i0\right) - {}_3F_2\left(\frac{4-p}{3}, \frac{5-p}{3}, 2-\frac{p}{3}; \frac{3}{2}, 2; \frac{9}{4}\zeta^2 - i0\right)\right) =$$
$$= 2\pi i\frac{\Gamma(\mathbf{b}+\frac{1}{2})}{\Gamma(\mathbf{a}+\frac{1}{2})}G^{3,0}_{3,3}\left(\frac{1}{2}, \frac{3}{2}, 2; \frac{4-p}{3}, \frac{5-p}{3}, 2-\frac{p}{3}; \frac{4}{9}\zeta^{-2}\right)$$
$$= 2\pi i\frac{\Gamma(\mathbf{b}+\frac{1}{2})}{\Gamma(\mathbf{a}+\frac{1}{2})}G^{0,3}_{3,3}\left(\frac{p-1}{3}, \frac{p-2}{3}, \frac{p}{3}-1; \frac{1}{2}, -\frac{1}{2}, -1; \frac{9}{4}\zeta^2\right)$$

When $p = 1, 2, 3$ the solution is a trivial resurgent function, it is constant.

Let us now consider the formal solution $\tilde{w}_2(z)$ and $\tilde{w}_3(z)$. The recursive relation for $a_{2,j}, a_{3,j}$ have not a simple expressions as it is for $a_{1,j}$. However, we compute numerically their first coefficients:

	$j = 1$	23	45	67
$a_{2,j}$				
$a_{3,j}$				