

EXPONENTIAL INTEGRALS

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1. INTRODUCTION

2. FRACTIONAL DERIVATIVES AND BOREL TRANSFORM

Definition 2.1. Let $\alpha \in (0, 1)$ and $n \in \mathbb{N}$, then the $n + \alpha$ -Caputo's derivative of a smooth function f is defined as

$$(2.1) \quad \partial_x^{n+\alpha} f(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f^{(n+1)}(s) ds$$

In particular, this definition is well suited for the differential calculus in the convolution model $(\mathbb{C}[[\zeta]], *)$. Let $\varphi(z) := \sum_{k \geq 0} a_k z^{-k-1} \in \mathbb{C}[[z^{-1}]]$ be Gevrey 1, then assuming $a_k = 0$ for every $k < n$, the Borel transform of $z^{n+\alpha} \varphi(z)$ can be computed in two different ways:

$$(2.2) \quad \begin{aligned} \mathcal{B}(z^{n+\alpha} \varphi(z))(\zeta) &= \mathcal{B}(z^{\alpha+n}) * \hat{\varphi}(\zeta) = \int_0^\zeta \frac{(\zeta-s)^{-1-n-\alpha}}{(-1-n-\alpha)!} \sum_{k \geq 0} \frac{a_k}{k!} s^k ds \\ &= \frac{1}{(-\alpha)!} \int_0^\zeta (\zeta-s)^{-\alpha} \sum_{k \geq 0} \frac{a_k}{(k-n-1)!} s^{k-n-1} ds = \partial_\zeta^{n+\alpha} \hat{\varphi}(\zeta) \end{aligned}$$

$$(2.3) \quad \mathcal{B}(z^{n+\alpha} \varphi(z))(\zeta) = \mathcal{B}\left(\sum_{k \geq 0} a_k z^{-k-1+n+\alpha}\right)(\zeta) = \sum_{k > n} \frac{a_k}{(k-n-\alpha)!} \zeta^{k-n-\alpha}$$

and computing the integral which defines the $n + \alpha$ -derivative we get exactly the same result as (??).

3. RESURGENCE OF EXPONENTIAL INTEGRALS

Let X be a $N - \dim$ manifold, $f: X \rightarrow \mathbb{C}$ be a holomorphic Morse function with only simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$(3.1) \quad I(z) := \int_{\mathcal{C}} e^{-zf} \nu$$

where \mathcal{C} is a suitable countur such that the integral is well defined. For any Morse cirtial points x_α of f , the saddle point approximation gives the following formal series

$$(3.2) \quad I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf} \nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)} (2\pi)^{N/2} z^{-N/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n}.$$

Theorem 3.1. *Let $N = 1$. Let $\tilde{\varphi}_\alpha(z) := e^{-zf(x_\alpha)} (2\pi)^{N/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n}$*

- (1) $\tilde{\varphi}_\alpha$ is Gevrey-1;
- (2) $\hat{\varphi}(\zeta) := \mathcal{B}(\tilde{\varphi})$ is a germ of analytic function at $\zeta = \zeta_\alpha = f(x_\alpha)$;
- (3) the following formual holds true

$$(3.3) \quad \hat{\varphi}_\alpha(\zeta) = \partial_{\zeta, \text{based at } \zeta_\alpha}^{3/2} \left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu \right) = \Gamma(-N/2) \int_{\zeta_\alpha}^{\zeta} (\zeta - \zeta')^{-N/2} \partial_{\zeta'} \left(\int_{f^{-1}(\zeta')} \frac{\nu}{df} \right) d\zeta'$$

Proof. Part (3): thanks to properties of Caputo's fractional derivatives, we have that the Borel transform of $\tilde{I}_\alpha(z) = z^{-1/2} \tilde{\varphi}_\alpha(z)$ is

$$(3.4) \quad \partial_{\zeta, \text{based at } \zeta_\alpha}^{1/2} \hat{I}_\alpha(\zeta) = \hat{\varphi}_\alpha(\zeta).$$

In addition, we notice

$$\begin{aligned} I_\alpha(z) &= \int_{\mathcal{C}_\alpha} e^{-zf} \nu & f &= \zeta \\ &= \int_{\mathcal{H}_\alpha} e^{-z\zeta} \partial_\zeta \left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu \right) d\zeta & \mathcal{H}_\alpha &\text{ is Heckel countour} \\ &=: \int_{\mathcal{H}_\alpha} e^{-\zeta z} \hat{I}_\alpha(\zeta) d\zeta \end{aligned}$$

hence

$$(3.5) \quad \hat{\varphi}_\alpha(\zeta) = \partial_{\zeta, \text{based at } \zeta_\alpha}^{1/2} \left(\partial_\zeta \left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu \right) \right) = \partial_{\zeta, \text{based at } \zeta_\alpha}^{3/2} \left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu \right)$$

□

Example 3.2 (Airy). Let $f(t) = \frac{t^3}{3} - t$ and

$$I(z) := \int_\gamma e^{-zf(t)} dt$$

where γ is a countour where the integral is well defined.

By the change of coordinates $z = x^{3/2}$, $I(z) = -2\pi i z^{-1/3} Ai(x)$ where

$$Ai(x) := \frac{1}{2\pi i} \int_{-\infty e^{-i\frac{\pi}{3}}}^{\infty e^{i\frac{\pi}{3}}} e^{\frac{t^3}{3} - zt} dt$$

hence $I(z)$ solves the following ODE¹

$$(3.6) \quad I''(z) - \frac{4}{9}I(z) + \frac{I'(z)}{z} - \frac{1}{9} \frac{I(z)}{z^2} = 0$$

A formal solution of (??) can be computed by making the following ansatz

$$(3.7) \quad \tilde{I}(z) = \sum_{k \in \mathbb{N}^2} U^k e^{-\lambda \cdot k z} z^{-\tau \cdot k} w_k(z)$$

with $U^{(k_1, k_2)} = U_1^{k_1} U_2^{k_2}$ and $U_1, U_2 \in \mathbb{C}$ are constant parameter, $\lambda = (\frac{2}{3}, -\frac{2}{3})$, $\tau = (\frac{1}{2}, \frac{1}{2})$, and $\tilde{w}_k(z) \in \mathbb{C}[[z^{-1}]]$. In addition, we can check that the only non zero $\tilde{w}_k(z)$ occurs at $k = (1, 0)$ and $k = (0, 1)$, therefore

$$(3.8) \quad \tilde{I}(z) = U_1 e^{-2/3z} z^{-1/2} \tilde{w}_+(z) + U_2 e^{2/3z} z^{-1/2} \tilde{w}_-(z)$$

where from now on we denote $\tilde{w}_+(z) = \tilde{w}_{(1,0)}(z)$ and $\tilde{w}_-(z) = \tilde{w}_{(0,1)}(z)$. In particular, $\tilde{w}_+(z)$ and $\tilde{w}_-(z)$ are formal solution of

$$(3.9) \quad \tilde{w}_+'' - \frac{4}{3} \tilde{w}_+' + \frac{5}{36} \frac{\tilde{w}_+}{z^2} = 0$$

$$(3.10) \quad \tilde{w}_-'' + \frac{4}{3} \tilde{w}_-' + \frac{5}{36} \frac{\tilde{w}_-}{z^2} = 0$$

Taking the Borel transform of (??), (??) we get

$$\begin{aligned} \zeta^2 \hat{w}_+(\zeta) + \frac{4}{3} \zeta \hat{w}_+ + \frac{5}{36} \zeta * \hat{w}_+ &= 0 \\ \zeta^2 \hat{w}_+(\zeta) + \frac{4}{3} \zeta \hat{w}_+ + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}_+(\zeta') d\zeta' &= 0 \\ \zeta^2 \hat{w}_-(\zeta) - \frac{4}{3} \zeta \hat{w}_- + \frac{5}{36} \zeta * \hat{w}_- &= 0 \\ \zeta^2 \hat{w}_-(\zeta) - \frac{4}{3} \zeta \hat{w}_- + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}_-(\zeta') d\zeta' &= 0 \end{aligned}$$

and taking derivatives we get

$$\begin{aligned} \zeta \left(\frac{4}{3} + \zeta \right) \hat{w}_+'' + \left(\frac{8}{3} + 4\zeta \right) \hat{w}_+' + \frac{77}{36} \hat{w}_+ &= 0 \\ \frac{4}{3} \zeta \left(1 + \frac{3}{4} \zeta \right) \hat{w}_+'' + \left(\frac{8}{3} + 4\zeta \right) \hat{w}_+' + \frac{77}{36} \hat{w}_+ &= 0 \\ u(1-u) \hat{w}_+''(u) + (2-4u) \hat{w}_+'(u) - \frac{77}{36} \hat{w}_+(u) &= 0 \quad u = -\frac{3}{4} \zeta \end{aligned}$$

¹ $Ai(x)$ solves the Airy equation $y'' = xy$.

$$\begin{aligned}
& \zeta\left(-\frac{4}{3} + \zeta\right)\hat{w}''_- + \left(-\frac{8}{3} + 4\zeta\right)\hat{w}'_- + \frac{77}{36}\hat{w}_- = 0 \\
& \frac{4}{3}\zeta\left(-1 + \frac{3}{4}\zeta\right)\hat{w}''_- + \left(-\frac{8}{3} + 4\zeta\right)\hat{w}'_- + \frac{77}{36}\hat{w}_- = 0 \\
& u(1-u)\hat{w}''_-(u) + (2-4u)\hat{w}'_-(u) - \frac{77}{36}\hat{w}_-(u) \qquad u = \frac{3}{4}\zeta
\end{aligned}$$

Notice that the latter equations are hypergeometric, hence a solution is given by

$$(3.11) \qquad \hat{w}_+(\zeta) = c_1 {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}\zeta\right)$$

$$(3.12) \qquad \hat{w}_-(\zeta) = c_2 {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{3}{4}\zeta\right)$$

for some constants $c_1, c_2 \in \mathbb{C}$ (see DLMF 15.10.2). In addition $\hat{w}_\pm(\zeta)$ have a log singularity respectively at $\zeta = \mp\frac{4}{3}$, therefore they are $\{\mp\frac{4}{3}\}$ -resurgent functions.²

Remark 3.3. $\hat{w}_+(\zeta)$ is Laplace summable along the positive real axis, and it can be analytically continued on $\mathbb{C} \setminus -\frac{4}{3}\mathbb{R}_{\leq 0}$ with (see 15.2.3 DLMF)

$$\begin{aligned}
\hat{w}_+(\zeta + i0) - \hat{w}_+(\zeta - i0) &= -\frac{36}{5}i\left(-\frac{3}{4}\zeta - 1\right)^{-1} \sum_{k \geq 0} \frac{(5/6)_n(1/6)_n}{\Gamma(n)n!} \left(1 + \frac{3}{4}\zeta\right)^n \qquad \zeta < -\frac{4}{3} \\
&= -\frac{36}{5}i\left(-\frac{3}{4}\zeta - 1\right)^{-1} \left(\frac{5}{144}(4 + 3\zeta)_1 {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 + \frac{3}{4}\zeta\right)\right) \\
&= \mathbf{i} {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 + \frac{3}{4}\zeta\right) \\
&= \mathbf{i}\hat{w}_-\left(\zeta + \frac{4}{3}\right)
\end{aligned}$$

Analogously, $\hat{w}_-(\zeta)$ is Laplace summable along the negative real axis, and it jumps across the branch cut $\frac{4}{3}\mathbb{R}_{\geq 0}$ as

$$\begin{aligned}
\hat{w}_-(\zeta + i0) - \hat{w}_-(\zeta - i0) &= \frac{36}{5}i\left(\frac{3}{4}\zeta - 1\right)^{-1} \sum_{k \geq 0} \frac{(5/6)_n(1/6)_n}{\Gamma(n)n!} \left(1 - \frac{3}{4}\zeta\right)^n \qquad \zeta > \frac{4}{3} \\
&= \frac{36}{5}i\left(\frac{3}{4}\zeta - 1\right)^{-1} \left(-\frac{5}{144}(-4 + 3\zeta)_1 {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 - \frac{3}{4}\zeta\right)\right) \\
&= -\mathbf{i} {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 - \frac{3}{4}\zeta\right) \\
&= -\mathbf{i}\hat{w}_+\left(\zeta - \frac{4}{3}\right)
\end{aligned}$$

These relations manifest the resurgence property of \tilde{I} , indeed near the singularities in the Borel plane of either \hat{w}_+ or \hat{w}_- , \hat{w}_- and \hat{w}_+ respectively contribute to the jump of the former solution.

²The solution we find are equal to the ones in DLMF §9.7. They also agree with slide 10 of Maxim's talk for ERC and with the series (6.121) in Sauzin's book. However they do not agree with (2.16) in Mariño's Diablerets.

Our next goal is to prove that the Borel transform of $\tilde{I}(z)$ can be written in terms of $1/f'(f^{-1}(\zeta))$, namely formula (??). It is convenient to consider the two asymptotic formal solutions separately, namely we define

$$(3.13) \quad \tilde{I}_{-1}(z) := e^{-2/3z} z^{-1/2} \tilde{w}_+(z) =: z^{-1/2} \tilde{u}_+(z)$$

$$(3.14) \quad \tilde{I}_1(z) := e^{2/3z} z^{-1/2} \tilde{w}_-(z) =: z^{-1/2} \tilde{u}_-(z)$$

In particular, $\tilde{u}_{\pm}(z)$ are solutions of

$$(3.15) \quad \tilde{u}''(z) - \frac{4}{9} \tilde{u}(z) + \frac{5}{36} \frac{\tilde{u}(z)}{z^2} = 0$$

with asymptotic behaviour $\tilde{u}_{\pm}(z) \sim O(e^{\pm 2/3z})$ as $z \rightarrow \infty$.

The Borel transforms $\hat{u}_{\pm}(\zeta)$ solve the same equation

$$\begin{aligned} & \zeta^2 \hat{u} - \frac{4}{9} \hat{u} + \frac{5}{36} \zeta * \hat{u} \\ & \zeta^2 \hat{u} - \frac{4}{9} \hat{u} + \frac{5}{36} \int_0^{\zeta} (\zeta - \zeta') \hat{u}(\zeta') d\zeta' \end{aligned}$$

taking derivatives is equivalent to

$$(\zeta^2 - \frac{4}{9}) \hat{u}''(\zeta) + 4\zeta \hat{u}'(\zeta) + \frac{77}{36} \hat{u}(\zeta) = 0$$

and Mathematica gives the following solutions

$$\begin{aligned} \hat{u}(\zeta) &= c_1 {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{9}{4}\zeta^2\right) + \frac{3i}{2} \zeta c_2 {}_1F_2\left(\frac{13}{12}, \frac{17}{12}, \frac{3}{2}, \frac{9}{4}\zeta^2\right) = \\ &= c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} \left({}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} + \frac{3}{4}\zeta\right) \right) \quad \text{see DLMF 15.8.27} \\ &\quad + \frac{3i}{2} \zeta c_2 \left(\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{3\zeta\Gamma(-\frac{1}{2})\Gamma(2)} \right) \left({}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right) \right) \quad \text{see DLMF 15.8.28} \\ &= \left(c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} - c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}} \right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) + \\ &\quad + \left(c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} + c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}} \right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right) \end{aligned}$$

Since \hat{u}_+ has a simple singularity at $\zeta = -2/3$ and \hat{u}_- has a simple singularity at $\zeta = 2/3$, we have

$$\begin{aligned}\hat{u}_+(\zeta) &= C_1 T_{-2/31} F_2\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}\zeta\right) = C_1 T_{-2/3} \hat{w}_+(\zeta) \\ \hat{u}_-(\zeta) &= C_2 T_{2/31} F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{3}{4}\zeta\right) = C_2 T_{2/3} \hat{w}_-(\zeta)\end{aligned}$$

Claim 3.4.

$$(3.16) \quad \hat{w}_+(\zeta - 2/3) = \frac{1}{\sqrt{\pi}} \int_{-2/3}^{\zeta} (\zeta - s)^{-1/2} \partial_s \left(\frac{1}{f'(u)} \right) ds \quad s = f(u)$$

Lemma 3.5. *The following identity holds true*

$$(3.17) \quad {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}\zeta^2\right) = \frac{1}{1-u^2} \quad \zeta = \frac{u^3}{3} - u$$

Proof.

$$\begin{aligned}{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}\zeta^2\right) &= 2 \cos\left(\frac{1}{3} \arcsin\left(\frac{3}{2}\zeta\right)\right) (4-9\zeta^2)^{-1/2} \quad \text{Mathematica [FullSimplify]} \\ &= \frac{\cos(y)}{\cos(3y)} \quad 3y = \arcsin\left(\frac{3}{2}\zeta\right) \\ &= \frac{\cos(y)}{\cos(2y)\cos(y) - \sin(2y)\sin(y)} \\ &= \frac{1}{\cos(2y) - 2\sin^2(y)} \\ &= \frac{1}{1-4\sin^2(y)} \quad \zeta = 2\sin(y) - \frac{8}{3}\sin^3(y)\end{aligned}$$

Therefore, if $u := -2\sin(y)$, we have $\zeta = \frac{u^3}{3} - u = f(u)$ and

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}\zeta^2\right) = \frac{1}{1-u^2} = -\frac{1}{f'(u)}$$

□

Hence claim (??) is equivalent to

Claim 3.6.

$$(3.18) \quad \hat{w}_+(\zeta - 2/3) = -\frac{1}{\sqrt{\pi}} \int_{-2/3}^{\zeta} (\zeta - s)^{-1/2} \partial_s \left[{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}s^2\right) \right] ds$$

Let us study the RHS of claim (??)

$$\begin{aligned}
& \int_{-2/3}^{\zeta} (\zeta-s)^{-1/2} \partial_s \left[{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}s^2\right) \right] ds = 2 \int_{-2/3}^{\zeta} (\zeta-s)^{-1/2} s {}_2F_1\left(\frac{4}{3}, \frac{5}{3}; \frac{3}{2}; \frac{9}{4}s^2\right) ds \\
& = -\frac{2}{9} \int_{-2/3}^{\zeta} (\zeta-s)^{-1/2} {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; \frac{5}{2}; \frac{1}{2} - \frac{3s}{4}\right) ds + \frac{2}{9} \int_{-2/3}^{\zeta} (\zeta-s)^{-1/2} {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; \frac{5}{2}; \frac{1}{2} + \frac{3s}{4}\right) ds \quad 15.8.28 \text{ DLMF} \\
& = -\frac{4i}{9\sqrt{3}} \int_1^{\zeta'} (\zeta'-t)^{-1/2} {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; \frac{5}{2}; t\right) dt + \frac{4i}{9\sqrt{3}} \int_1^{\zeta'} (\zeta'-t)^{-1/2} {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; \frac{5}{2}; 1-t\right) dt \quad \zeta' = \frac{1}{2} - \frac{3}{4}\zeta \\
& = \frac{2i}{9\sqrt{3}} \int_1^{\zeta'} (\zeta'-t)^{-1/2} {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; \frac{5}{2}; 1-t\right) dt - \frac{9i}{32} \int_1^{\zeta'} (\zeta'-t)^{-1/2} {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; -\frac{1}{2}; 1-t\right) dt \quad 15.10.21 \text{ DLMF} \\
& = \frac{2i}{9\sqrt{3}} \int_1^{\zeta'} (\zeta'-t)^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{5}{2}; 1-t\right) t^{-3/2} dt \quad 15.10.13 \text{ DLMF}
\end{aligned}$$

3.0.1. *Comparison with Aaron.* The Airy integral can be written in terms of the modified Bessel equation as

$$(3.19) \quad Ai(x) = \frac{1}{\pi\sqrt{3}} x^{1/2} K\left(\frac{2}{3}x^{3/2}\right).$$

On the other hand we have, for $z = x^{3/2}$

$$(3.20) \quad Ai(x) = -\frac{z^{1/3}}{2\pi i} I(z) = -\frac{1}{2\pi i} x^{1/2} I(x^{3/2})$$

hence

$$\begin{aligned}
-\frac{1}{2\pi i} I(x^{3/2}) &= \frac{1}{\pi\sqrt{3}} K\left(\frac{2}{3}x^{3/2}\right) \\
\frac{i}{2} I(z) &= \frac{1}{\sqrt{3}} K\left(\frac{2}{3}z\right)
\end{aligned}$$

In particular, the Borel transforms of LHS and RHS³ must be equal, i.e.

$$\begin{aligned}
\frac{i}{2}\hat{I}(\zeta) &= \frac{\sqrt{3}}{2}\hat{K}\left(\frac{3}{2}\zeta\right) \\
&= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\left(\frac{3}{2}\zeta-1\right)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{1}{2}-\frac{3}{4}\zeta\right) \\
&= \frac{\sqrt{3}}{2}(3\zeta-2)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{1}{2}-\frac{3}{4}\zeta\right) \\
&= \frac{\sqrt{3}}{2}T_{-2/3}\left((3\zeta)^{-1/2}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)\right) \\
&= \frac{1}{2}T_{-2/3}\left(\frac{1}{\sqrt{\zeta}}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)\right) \\
\hat{I}(\zeta) &= -iT_{-2/3}\left(\frac{1}{\sqrt{\zeta}}{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};-\frac{3}{4}\zeta\right)\right)
\end{aligned}$$

Example 3.7 (Bessel). Let $X = \mathbb{C}^*$, $f(x) = x + \frac{1}{x}$ and $\nu = \frac{dx}{x}$, then

$$(3.21) \quad I(z) := \int_0^\infty e^{-zf(x)} \frac{dx}{x}.$$

By change of coordinates $t = zx$

$$I(z) = \int_0^\infty e^{-z\left(\frac{t}{z} + \frac{z}{t}\right)} \frac{dt}{t} = \int_0^\infty e^{-\left(t + \frac{z^2}{t}\right)} \frac{dt}{t} = 2K_0(2z) \quad |\arg z| < \frac{\pi}{4}$$

where $K_0(z)$ is the modified Bessel function (see definition 10.32.10 DLMF). In particular, since $K_0(z)$ solves

$$(3.22) \quad \frac{d^2}{dz^2} w(z) + \frac{1}{z} \frac{d}{dz} w(z) - w(z) = 0$$

and $K_0(z) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-z} z^{-1/2} \sum_{k \geq 0} \frac{(1/2)_k (1/2)_k}{(-2)^k k!} z^{-k}$ as $z \rightarrow \infty$ (see DLMF 10.40.2), then $I(z)$ is a solution of

$$(3.23) \quad \frac{d^2}{dz^2} I(z) + \frac{1}{z} \frac{d}{dz} I(z) - 4I(z) = 0.$$

The formal integral of (??) is given by a two parameter formal solution $\tilde{I}(z)$

$$(3.24) \quad \tilde{I}(z) = \sum_{\mathbf{k} \in \mathbb{N}^2} U^{\mathbf{k}} e^{-\mathbf{k} \cdot \lambda z} z^{-\tau \cdot \mathbf{k}} \tilde{w}_{\mathbf{k}}(z)$$

³The conjugate variable of z is ζ , hence $\hat{K}(\frac{2}{3}z) = \frac{3}{2}\hat{K}(\frac{3}{2}\zeta)$. Indeed assuming $K(z) = \sum_{n \geq 0} a_n z^{-n}$, the Borel transform of $K(\frac{2}{3}z) = \sum_{n \geq 0} a_n \left(\frac{3}{2}\right)^n z^{-n}$ is by definition

$$\hat{K}\left(\frac{2}{3}z\right) = \sum_{k \geq 0} a_{n+1} \left(\frac{3}{2}\right)^{n+1} \frac{\zeta^n}{n!} = \frac{3}{2} \sum_{k \geq 0} \frac{a_{n+1}}{n!} \left(\frac{3}{2}\zeta\right)^n = \frac{3}{2}\hat{K}\left(\frac{3}{2}\zeta\right).$$

where $\lambda = (2, -2)$, $\tau = (-\frac{1}{2}, -\frac{1}{2})$, $U^k := U_1^{k_1} U_2^{k_2}$ with $k = (k_1, k_2)$ and $U_1, U_2 \in \mathbb{C}$, and $\tilde{w}_{\mathbf{k}}(z) \in \mathbb{C}[[z^{-1}]]$ is a formal solution of

$$(3.25) \quad \tilde{w}_{\mathbf{k}}''(z) - 4(k_1 - k_2)\tilde{w}_{\mathbf{k}}'(z) + 4(1 - (k_1 - k_2)^2)\tilde{w}_{\mathbf{k}}(z) + \frac{(k_1 + k_2 - 1)}{z}\tilde{w}_{\mathbf{k}}'(z) + \\ - 2(k_1 - k_2)\frac{(k_1 + k_2 - 1)}{z}\tilde{w}_{\mathbf{k}}(z) + \frac{(k_1 + k_2)^2}{4z^2}\tilde{w}_{\mathbf{k}}(z) = 0$$

The only non zero $\tilde{w}_{\mathbf{k}}(z)$ occurs for $\mathbf{k} = (1, 0)$ and $\mathbf{k} = (0, 1)$, hence

$$(3.26) \quad \tilde{I}(z) = U_1 e^{-2z} z^{-1/2} \tilde{w}_{(1,0)}(z) + U_2 e^{2z} z^{-1/2} \tilde{w}_{(0,1)}(z)$$

and according to our convention, we define

$$(3.27) \quad \tilde{I}_1(z) := e^{-2z} z^{-1/2} \tilde{w}_{(1,0)}(z)$$

$$(3.28) \quad \tilde{I}_{-1}(z) := e^{2z} z^{-1/2} \tilde{w}_{(0,1)}(z).$$

We set $\tilde{w}_{(1,0)} = \tilde{w}_+$ and $\tilde{w}_{(0,1)} = \tilde{w}_-$, then their Borel transforms are solutions respectively of the following equations

$$\begin{aligned} (+) \quad & \zeta^2 \hat{w}_+(\zeta) + 4\zeta \hat{w}_+(\zeta) + \frac{1}{4}\zeta * \hat{w}_+(\zeta) = 0 \\ (-) \quad & \zeta^2 \hat{w}_-(\zeta) - 4\zeta \hat{w}_-(\zeta) + \frac{1}{4}\zeta * \hat{w}_-(\zeta) = 0 \end{aligned}$$

taking twice derivative in ζ we get

$$\begin{aligned} (+) \quad & (\zeta^2 + 4\zeta) \frac{d^2}{d\zeta^2} \hat{w}_+ + 4(\zeta - 1) \frac{d}{d\zeta} \hat{w}_+ + \frac{9}{4} \hat{w}_+ = 0 \\ (-) \quad & (\zeta^2 - 4\zeta) \frac{d^2}{d\zeta^2} \hat{w}_- + 4(\zeta + 1) \frac{d}{d\zeta} \hat{w}_- + \frac{9}{4} \hat{w}_- = 0 \\ (+) \quad & \xi(1 - \xi) \frac{d^2}{d\xi^2} \hat{w}_+ + (1 - 4\xi) \frac{d}{d\xi} \hat{w}_+ - \frac{9}{4} \hat{w}_+ = 0 \quad \xi = -\frac{\zeta}{4} \\ (-) \quad & \xi(1 - \xi) \frac{d^2}{d\xi^2} \hat{w}_- + (1 - 4\xi) \frac{d}{d\xi} \hat{w}_- - \frac{9}{4} \hat{w}_- = 0 \quad \xi = \frac{\zeta}{4} \end{aligned}$$

therefore, since equation (+), (-) are hypergeometric the fundamental solution is (see DLMF 15.10.2)

$$(3.29) \quad \hat{w}_+(\zeta) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -\frac{\zeta}{4}\right)$$

$$(3.30) \quad \hat{w}_-(\zeta) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\zeta}{4}\right)$$