Resurgence of the Airy function

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1 The Laplace transform

1.1 Analytic version

1.1.1 Regularity and decay properties

Take two copies $\mathbb{R}$ and $\hat{\mathbb{R}}$ of the real line, with standard coordinates z and ζ respectively. The Laplace transform in ζ turns a function $\hat{\varphi}$ on $\hat{\mathbb{R}}_{\zeta>0}$ into a function $\mathcal{L}_\zeta \hat{\varphi}$ on $\mathbb{R}_{z>0}$, defined by the integral

$$\mathcal{L}_\zeta \hat{\varphi} = \int_0^\infty e^{-z\zeta}\, \hat{\varphi}\, d\zeta.$$

For $a \in [0, \infty]$, recall that recall that $O_{\zeta \to a}(g)$ is the space of functions φ on $\hat{\mathbb{R}}_{\zeta>0}$ with $|\varphi| \lesssim g$ in some neighborhood of a. A function is *subexponential* if it's in $O_{\zeta\to\infty}(e^{c\zeta})$ for all $c > 0$. Let $\mathcal{E}_\zeta$ be the space of subexponential functions on $\hat{\mathbb{R}}_{\zeta>0}$ which are L^1 both locally and around $\zeta = 0$. If $\hat{\varphi}$ is in $\mathcal{E}_\zeta$, then $\varphi = \mathcal{L}_\zeta \hat{\varphi}$ is well-defined, and it extends to a holomorphic function on the right half-plane $\mathbb{C}_{\mathrm{Re}(z)>0}$ [1, §5.6]. If $\hat{\varphi}$ is in $O_{\zeta\to 0}(1)$, then φ is in $O_{z\to\infty}(z^{-1})$ [2, equation 1.8].[1] More generally, if $\hat{\varphi}$ is in $O_{\zeta\to 0}(\zeta^\alpha)$, with $\alpha > -1$, then φ is in $O_{z\to\infty}\big(z^{-(\alpha+1)}\big)$.

1.1.2 Action on differential operators

When $\hat{\varphi} \in \mathcal{E}_\zeta$, we can use differentiation under the integral to show that [2, Theorem 1.34]

$$\mathcal{L}_\zeta(\zeta^n \hat{\varphi}) = \big(-\tfrac{\partial}{\partial z}\big)^n \mathcal{L}_\zeta \hat{\varphi}. \tag{1}$$

When $\hat{\varphi}$ is n times differentiable, its nth derivative is in $\mathcal{E}$, and its zeroth through $(n-1)$st derivatives extend continuously to zero, integration by parts gives the formula

$$\begin{aligned}\mathcal{L}_\zeta \big(\tfrac{\partial}{\partial \zeta}\big)^n \hat{\varphi} &= z^n \mathcal{L}_\zeta \hat{\varphi} - \Big[\hat{\varphi}\, z^{n-1} + \hat{\varphi}'\, z^{n-2} + \hat{\varphi}''\, z^{n-3} + \ldots + \hat{\varphi}^{(n-1)}\Big]_{\zeta=0} \\ &= z^n\, \mathcal{L}\left(\hat{\varphi} - \Big[\hat{\varphi} + \hat{\varphi}'\,\zeta + \tfrac{\hat{\varphi}''}{2!}\,\zeta^2 + \ldots + \tfrac{\hat{\varphi}^{(n-1)}}{(n-1)!}\,\zeta^{n-1}\Big]_{\zeta=0}\right).\end{aligned} \tag{2}$$

[1] The argument cited still works in our generality. For holomorphic $\hat{\varphi}$, one can also use Equation 1.5 of *Borel-Laplace Transform and Asymptotic Theory* **(Sternin & Shatalov)**.

Note that if a function's derivative is subexponential, so is the function itself.[2]

1.2 Algebraic version

1.2.1 Definition

Let $\mathcal{P}_\zeta$ be the vector space spanned by ζ^α for $\alpha \in \mathbb{R} \smallsetminus \mathbb{Z}_{<0}$. Note that $\mathcal{P}_\zeta \cap \mathcal{E}_\zeta$ is $\mathcal{P}_\zeta^{>-1}$, the subspace spanned by ζ^α with $\alpha > -1$. Since

$$\mathcal{L}_\zeta(\zeta^\alpha) = \Gamma(\alpha+1)\, z^{-(\alpha+1)}$$

for all $\alpha > -1$, let's use the same formula to extend $\mathcal{L}_\zeta$ to all of $\mathcal{P}_\zeta$. This defines $\mathcal{L}_\zeta$ consistently on $\mathcal{E}_\zeta + \mathcal{P}_\zeta$.

1.2.2 Action on differential operators

Observe that

$$\mathcal{L}_\zeta(\zeta^{\alpha+1}) = -\tfrac{\partial}{\partial z}\, \mathcal{L}_\zeta(\zeta^\alpha)$$

for $\alpha \neq -1$. This extends identity 1 to all of $\mathcal{P}_\zeta$.

Observe that

$$\mathcal{L}_\zeta \tfrac{\partial}{\partial \zeta}(\zeta^\alpha) = \begin{cases} z\, \mathcal{L}_\zeta(\zeta^\alpha) & \alpha \neq 0 \\ 0 & \alpha = 0, \end{cases}$$

and that $0 = z\, \mathcal{L}_\zeta(1) - 1$. This recovers identity 2 for any function in $\mathcal{P}_\zeta$ whose nth derivative is in $\mathcal{P}_\zeta^{>-1}$. Although the functions in $\mathcal{P}_\zeta^{<0}$ are singular at zero, let's pretend they vanish at zero. With that convention, formula 2 extends to all of $\mathcal{P}_\zeta$.

Now we have the results of Section 1.1.2 for all functions in $\mathcal{E}_\zeta + \mathcal{P}_\zeta$. Identity 2 is particularly simple when $\hat{\varphi}$ has a *fractional power singularity* at $\zeta = 0$. By this, I mean that $\hat{\varphi}$ can be written as $\hat{\varphi}_{\text{frac}} + \hat{\varphi}_{\text{reg}}$, where $\hat{\varphi}_{\text{frac}} \in \mathcal{P}_\zeta$ has only non-integer exponents, and the zeroth through $(n-1)$st derivatives of $\hat{\varphi}_{\text{reg}} \in \mathcal{E}_\zeta$ vanish at zero. Under this condition, all the initial value terms in the identity vanish, leaving

$$\mathcal{L}_\zeta \left(\tfrac{\partial}{\partial \zeta}\right)^n \hat{\varphi} = z^n \mathcal{L}_\zeta \hat{\varphi}.$$

[2] Say $f' \in O_{\zeta\to\infty}(e^{c\zeta})$. Then

$$\left|\int_0^Z f'\, d\zeta\right| \leq \int_0^Z |f'|\, d\zeta \lesssim \int_0^Z e^{c\zeta}\, d\zeta = \tfrac{1}{c}(e^{cZ} - 1) \lesssim e^{cZ}.$$

Now we know the integral on the left-hand side converges, implying that f extends continuously to zero, with $|f - f_{\zeta=0}| \lesssim e^{c\zeta}$.

1.3 Change of coordinates

Define a new coordinate ζ_a on $\hat{\mathbb{R}}$ so that $\zeta = a + \zeta_a$. From the calculation

$$\begin{aligned}\mathcal{L}_\zeta \hat{\varphi} &= \int_0^\infty e^{-z\zeta}\, \hat{\varphi}\, d\zeta \\ &= \int_0^\infty e^{-z(a+\zeta_a)}\, \hat{\varphi}\, d\zeta_a \\ &= e^{-az} \int_0^\infty e^{-z\zeta_a}\, \hat{\varphi}\, d\zeta_a \\ &= e^{-az} \mathcal{L}_{\zeta_a} \hat{\varphi},\end{aligned}$$

we learn that

$$\mathcal{L}_{\zeta_a} \hat{\varphi} = e^{az} \mathcal{L}_\zeta \hat{\varphi}.$$

Define new coordinates x and ξ on $\mathbb{R}$ and $\hat{\mathbb{R}}$, respectively, so that $\zeta = b\xi$ and $z\, d\zeta = x\, d\xi$. Explicitly, $z = b^{-1}x$. From the calculation

$$\begin{aligned}\mathcal{L}_\zeta \hat{\varphi} &= \int_0^\infty e^{-z\zeta}\, \hat{\varphi}\, d\zeta \\ &= \int_0^\infty e^{-x\xi}\, \hat{\varphi}\, b\, d\xi \\ &= b \mathcal{L}_\xi \hat{\varphi},\end{aligned}$$

we learn that

$$\mathcal{L}_\xi \hat{\varphi} = b^{-1} \mathcal{L}_\zeta \hat{\varphi}.$$

2 The Airy equation

2.1 Basics

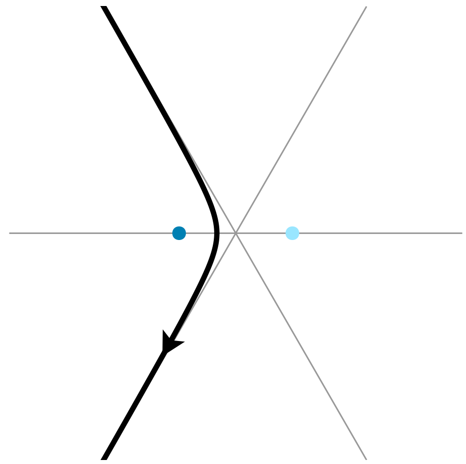
The Airy equation is

$$\left[\left(\tfrac{\partial}{\partial y} \right)^2 - y \right] \psi = 0. \tag{3}$$

One solution is given by the Airy function,

$$\mathrm{Ai}(y) = \frac{i}{2\pi} \int_\Gamma \exp\left(-\tfrac{1}{3} t^3 + yt \right)\, dt,$$

where Γ is a path that comes from ∞ at $120°$ and goes to ∞ at $-120°$.



The contour Γ in the u plane.

With the substitution $t = 2uy^{1/2}$, we can rewrite the Airy integral as

$$\mathrm{Ai}(y) = y^{1/2}\ \frac{i}{\pi} \int_{y^{-1/2}\Gamma} \exp\left[-\tfrac{2}{3} y^{3/2} \left(4u^3 - 3u\right)\right]\, du.$$

We've rescaled the contour by a factor of two, but it still approaches ∞ in the desired way. Note that $4u^3 - 3u$ is the third Chebyshev polynomial.

2.2 Rewriting as a modified Bessel equation

We can distill the most interesting part of the Airy function by writing

$$\mathrm{Ai}(y) = \tfrac{1}{\pi\sqrt{3}}\, y^{1/2}\, K\big(\tfrac{2}{3} y^{3/2}\big),$$

where

$$K(z) = i\sqrt{3} \int_{z^{-1/3}\Gamma} \exp\left[-z\left(4u^3 - 3u\right)\right]\, du. \tag{4}$$

Saying that Ai satisfies the Airy equation is equivalent to saying that K satisfies the modified Bessel equation

$$\left[z^2\big(\tfrac{\partial}{\partial z}\big)^2 + z\tfrac{\partial}{\partial z} - \left[\big(\tfrac{1}{3}\big)^2 + z^2\right]\right]\varphi = 0. \tag{5}$$

In fact, K is the modified Bessel function $K_{1/3}$ [3, equation 9.6.1].

As we'll see in Section **??**, K is in $O_{z\to\infty}(e^{-z})$. It'll be helpful to pull out the exponential decay factor and work instead with the function κ defined by $K = e^{-z}\kappa$. Saying that K satisfies equation 5 is equivalent to saying that κ satisfies the equation

$$\left[z^2\big(\tfrac{\partial}{\partial z} + 1\big)^2 + z\big(\tfrac{\partial}{\partial z} + 1\big) - \left[\big(\tfrac{1}{3}\big)^2 + z^2\right]\right]\varphi = 0. \tag{6}$$

2.3 Asymptotic analysis

From [3], equations 10.40.2 and 10.17.1, we get the asymptotic series

$$\kappa \sim \left(\tfrac{\pi}{2}\right)^{1/2}\left[z^{-1/2} - \frac{(\frac{1}{6})_1(\frac{5}{6})_1}{2^1 \cdot 1!}\, z^{-3/2} + \frac{(\frac{1}{6})_2(\frac{5}{6})_2}{2^2 \cdot 2!}\, z^{-5/2} - \frac{(\frac{1}{6})_3(\frac{5}{6})_3}{2^3 \cdot 3!}\, z^{-7/2} + \dots\right] \tag{7}$$

2.4 Going to the spatial domain

2.4.1 A good try at $\zeta = 0$

Let's try to find a function $\hat{K}$ with $K = \mathcal{L}_\zeta \hat{K}$, which is unique if it exists [2, Theorem 1.23]. If a function $\hat{\varphi}$ satisfies the equation

$$\left[(\zeta^2 - 1)\left(\tfrac{\partial}{\partial\zeta}\right)^2 + 3\zeta\tfrac{\partial}{\partial\zeta} + \left[1 - \left(\tfrac{1}{3}\right)^2\right]\right]\hat{\varphi} = 0, \tag{8}$$

its Laplace transform $\varphi = \mathcal{L}_\zeta \hat{\varphi}$ satisfies the equation

$$\begin{aligned}
\left[\left(-\tfrac{\partial}{\partial z}\right)^2 - 1\right]\left(z^2\varphi - \left[\hat{\varphi}\, z + \hat{\varphi}'\right]_{\zeta=0}\right) + 3\left(-\tfrac{\partial}{\partial z}\right)\left[z\varphi - \hat{\varphi}\right]_{\zeta=0} + \left[1 - \left(\tfrac{1}{3}\right)^2\right]\varphi &= 0 \\
\left(\tfrac{\partial}{\partial z}\right)^2\left[z^2\varphi\right] - \left(z^2\varphi - \left[\hat{\varphi}\, z + \hat{\varphi}'\right]_{\zeta=0}\right) - 3\left(\tfrac{\partial}{\partial z}\right)\left[z\varphi\right] + \left[1 - \left(\tfrac{1}{3}\right)^2\right]\varphi &= 0 \\
\left[2 + 4z\tfrac{\partial}{\partial z} + z^2\left(\tfrac{\partial}{\partial z}\right)^2\right]\varphi - \left(z^2\varphi - \left[\hat{\varphi}\, z + \hat{\varphi}'\right]_{\zeta=0}\right) - 3\left[1 + z\tfrac{\partial}{\partial z}\right]\varphi + \left[1 - \left(\tfrac{1}{3}\right)^2\right]\varphi &= 0,
\end{aligned}$$

which simplifies to

$$\left[z^2\left(\tfrac{\partial}{\partial z}\right)^2 + z\tfrac{\partial}{\partial z} - \left[\left(\tfrac{1}{3}\right)^2 + z^2\right]\right]\varphi = -\left[\hat{\varphi}\, z + \hat{\varphi}'\right]_{\zeta=0}. \tag{9}$$

Since we want $\mathcal{L}_\zeta \hat{K}$ to satisfy equation 5, which is the homogeneous version of equation 9, we might guess that $\hat{K}$ is a solution of equation 8 that vanishes through first order at $\zeta = 0$. Unfortunately, this would force $\hat{K}$ to be zero.

2.4.2 Success at $\zeta = 1$

Define a new coordinate ζ_1 on $\hat{\mathbb{R}}$ so that $\zeta = 1 + \zeta_1$. Since

$$\begin{aligned}
\mathcal{L}_{\zeta_1}\hat{K} &= e^z \mathcal{L}_\zeta \hat{K} \\
&= e^z K \\
&= \kappa,
\end{aligned}$$

we want $\mathcal{L}_{\zeta_1}\hat{K}$ to satisfy equation 6. Rewrite equation 8 as

$$\left[\zeta_1(\zeta_1 + 2)\left(\tfrac{\partial}{\partial\zeta_1}\right)^2 + 3(\zeta_1 + 1)\tfrac{\partial}{\partial\zeta_1} + \left[1 - \left(\tfrac{1}{3}\right)^2\right]\right]\hat{\varphi} = 0. \tag{10}$$

If $\hat{\varphi}$ satisfies equation 10, $\mathcal{L}_{\zeta_1}\hat{\varphi}$ will satisfy an inhomogeneous version of equation 6, analogous to equation 9. This time, though, there's a trick we can use to zero out the inhomogeneity. Equation 10 has a regular singularity at $\zeta_1 = 0$, and one solution (up to scaling) is a holomorphic multiple of $\zeta_1^{-1/2}$. That solution has a fractional power singularity at $\zeta_1 = 0$, as defined in Section 1.2.2, so its Laplace transform in ζ_1 satisfies equation 6.

Following this plan, let's find $\hat{K}$ explicitly. Defining another coordinate ξ on $\hat{\mathbb{R}}$ so that $\zeta_1 = -2\xi$, we can rewrite equation 10 as the hypergeometric equation

$$\left[\xi(1-\xi)\left(\tfrac{\partial}{\partial\xi}\right)^2 + 3(\tfrac{1}{2} - \xi)\tfrac{\partial}{\partial\xi} - \left[1 - \left(\tfrac{1}{3}\right)^2\right]\right]\hat{\varphi} = 0. \tag{11}$$

The hypergeometric function

$$\hat{g}_1 = F\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; \xi\right)$$

satisfies equation 11 by definition. It's not the solution we want, though, because it's holomorphic around $\xi = 0$. Formula 15.10.12 from [3] gives another solution,

$$\hat{f}_0 = \xi^{-1/2} F\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; \xi\right),$$

which is a holomorphic multiple of $\xi^{-1/2}$ near $\xi = 0$. By the argument above, $f_0 = \mathcal{L}_{\zeta_1}\hat{f}_0$ satisfies equation 6. This suggests that a constant multiple of $\hat{f}_0$ is our desired $\hat{K}$. The power series [3, equation 15.2.1]

$$\hat{f}_0 = \xi^{-1/2} + \frac{(\frac{1}{6})_1(\frac{5}{6})_1}{(\frac{1}{2})_1\ 1!}\,\xi^{1/2} + \frac{(\frac{1}{6})_2(\frac{5}{6})_2}{(\frac{1}{2})_2\ 2!}\,\xi^{3/2} + \frac{(\frac{1}{6})_3(\frac{5}{6})_3}{(\frac{1}{2})_3\ 3!}\,\xi^{5/2} + \ldots$$

converges near $\xi = 0$, showing that

$$\hat{f}_0 \in \xi^{-1/2} + O_{\xi\to 0}(\xi^{1/2}).$$

In terms of ζ_1, we have

$$\hat{f}_0 \in -i\sqrt{2}\,\zeta_1^{-1/2} + O_{\zeta_1\to 0}(\zeta_1^{1/2}).$$

Using the decay properties from Section 1.1.1, we deduce that

$$f_0 \in -i\sqrt{2\pi}\,z^{-1/2} + O_{z\to\infty}(z^{-3/2}).$$

Since we know that f_0 satisfies equation 6, this confirms that f_0 is a constant multiple of κ, which is the only subexponential solution of equation 6 (up to scaling). Comparing with series 7, we see that $\kappa = \frac{i}{2}\,f_0$. We conclude that $\kappa = \mathcal{L}_{\zeta_1}\hat{K}$ for

$$\hat{K} = \tfrac{1}{\sqrt{2}}\,\zeta_1^{-1/2} F\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; -\tfrac{1}{2}\zeta_1\right).$$

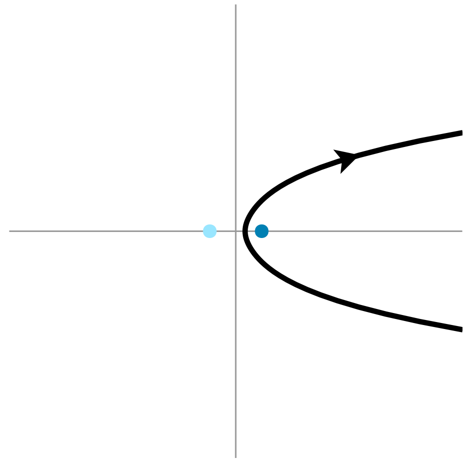
3 Sketches

3.1 Contour argument

We can recast integral 4 into $\hat{\mathbb{C}}$ by setting $\zeta = 4u^3 - 3u$. Projecting $z^{-1/3}\Gamma$ to a contour γ_z in $\hat{\mathbb{C}}$ and choosing the branch of u that lifts γ_z back to $z^{-1/3}\Gamma$, we have

$$K = \frac{i}{\sqrt{3}} \int_{\gamma_z} e^{-z\zeta} \frac{d\zeta}{4u^2 - 1}. \tag{12}$$

For $z \in (0, \infty)$, the contour γ_z runs clockwise around $[1, \infty)$, as shown below. Let's assume $z \in (0, \infty)$ for the rest of the section. **[Our conclusions should probably hold whenever** $\operatorname{Re}(z) > 0$**.]**



The contour γ_1 in $\hat{\mathbb{C}}$.

It happens[3] that for our desired branch of u,

$$\frac{1}{4u^2 - 1} = -F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\right),$$

so we can rewrite integral 12 as

$$K = \frac{1}{i\sqrt{3}} \int_{\gamma_z} e^{-z\zeta} F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\right) \, d\zeta.$$

This gives us an alternate route to the conclusion of Section 2.4, which we'll follow below.

In addition to the solutions $\hat{g}_1$ and $\hat{f}_0$ from Section 2.4.2, equation 11 has the solutions

$$\begin{aligned} \hat{g}_0 &= \phantom{(1-\xi)^{-1/2}} F\left(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; 1-\xi\right) \\ \hat{f}_1 &= (1-\xi)^{-1/2} F\left(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{1}{2}; 1-\xi\right), \end{aligned}$$

given by formulas 15.10.13 and 15.10.14 from [3].

The quadratic transformation identity 15.8.27 from [3] shows **[verified numerically]** that[4]

$$F\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; \zeta^2\right) = \tfrac{1}{3}(\hat{g}_1 + \hat{g}_0),$$

so we have

$$K = \frac{1}{i\,3\sqrt{3}} \int_{\gamma_z} e^{-z\zeta} (\hat{g}_1 + \hat{g}_0) \, d\zeta.$$

The solution $\hat{g}_1$ is holomorphic on $\zeta \in [1, \infty)$, so it integrates to zero. The solution $\hat{g}_0$, in contrast, is non-meromorphic at $\zeta = 1$. Along the branch cut $\zeta \in [1, \infty)$, its above-minus-

[3]How to verify this? This hypergeometric function can be written in terms of Legendre functions, but I don't know where to go from there. **Veronica:** Look at "Special cases" section in [3], e.g. 15.4.14!

[4]Note that $2\Gamma(\tfrac{1}{2})\Gamma(\tfrac{3}{2}) = 2\Gamma(\tfrac{1}{2})\,\tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \pi$ and $\left[\Gamma(\tfrac{5}{6})\Gamma(\tfrac{7}{6})\right]^{-1} = \left[\Gamma(\tfrac{5}{6})\,\tfrac{1}{6}\Gamma(\tfrac{1}{6})\right]^{-1} = \frac{6\sin(\frac{1}{6}\pi)}{\pi} = \frac{3}{\pi}$.

below difference is $-\frac{3\sqrt{3}}{2}\hat{f}_0$, as given[5] by equation 15.2.3 from [3]. Hence,

$$
\begin{aligned}
K &= \frac{i}{2}\int_1^\infty e^{-z\zeta}\hat{f}_0\,d\zeta \\
e^z K &= \frac{i}{2}\int_1^\infty e^{-z(\zeta-1)}\hat{f}_0\,d\zeta \\
\kappa &= \tfrac{i}{2}\mathcal{L}_{\zeta_1}\hat{f}_0,
\end{aligned}
$$

just as we found in Section 2.4.2.

3.2 Another solution

Section 3.1 associates the solution K of equation 5 with the solution $\hat{g}_0$ of equation 11, which contributes the pole at $\zeta = 1$ of

$$\frac{du}{d\zeta} = \frac{1}{4u^2-1} = \tfrac{1}{3}(\hat{g}_1 + \hat{g}_0).$$

The solution $\hat{g}_1$, which contributes the pole at $\zeta = -1$, is associated with another solution of equation 5.

To express this other solution as a Laplace transform, following the method of Section 2.4.2, we would use the solution

$$\hat{f}_1 = (1-\xi)^{-1/2}F\big(\tfrac{1}{6},\tfrac{5}{6};\tfrac{1}{2};1-\xi\big)$$

of equation 11, given by formula 15.10.14 from [3]. This is the only solution, up to scale, which has a fractional power singularity at $\zeta = -1$.

In summary, the contour integration method of solving equation 5 is associated with the basis

$$
\begin{aligned}
\hat{g}_1 &= F\big(\tfrac{2}{3},\tfrac{4}{3};\tfrac{3}{2};\xi\big) \\
\hat{g}_0 &= F\big(\tfrac{2}{3},\tfrac{4}{3};\tfrac{3}{2};1-\xi\big)
\end{aligned}
$$

of solutions for equation 11, given by formulas 15.10.11 and 15.10.13 from [3]. These solutions contribute the poles at $\xi = 1$ and $\xi = 0$, respectively, of a generic solution.

The Laplace transformation method of solving equation 5, on the other hand, is associated with the basis

$$
\begin{aligned}
\hat{f}_1 &= (1-\xi)^{-1/2}F\big(\tfrac{1}{6},\tfrac{5}{6};\tfrac{1}{2};1-\xi\big) \\
\hat{f}_0 &= \xi^{-1/2}F\big(\tfrac{1}{6},\tfrac{5}{6};\tfrac{1}{2};\xi\big)
\end{aligned}
$$

given by formulas 15.10.14 and 15.10.12 from [3]. These solutions, up to scale, are the only ones with fractional power singularities.

Identities 15.10.18, and 15.10.22 from [3] give the change of basis

$$
\begin{aligned}
\hat{f}_1 &= \tfrac{1}{\sqrt{3}}\hat{g}_1 + \tfrac{1}{2}\hat{f}_0 \\
\hat{f}_0 &= \tfrac{1}{\sqrt{3}}\hat{g}_0 + \tfrac{1}{2}\hat{f}_1.
\end{aligned}
$$

[5] Note that $\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})^{-1} = \frac{1}{2}$ and $\big[\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})\big]^{-1} = \big[\Gamma(\frac{2}{3})\,\frac{1}{3}\Gamma(\frac{1}{3})\big]^{-1} = \frac{3\sin(\frac{1}{3}\pi)}{\pi} = \frac{3\sqrt{3}}{2\pi}$.

Summing these identities, we see that

$$\hat{g}_1 + \hat{g}_0 = \tfrac{\sqrt{3}}{2}\,(\hat{f}_1 + \hat{f}_0),$$

giving the alternate decomposition

$$\frac{du}{d\zeta} = \tfrac{1}{2\sqrt{3}}\,(\hat{f}_1 + \hat{f}_0).$$

3.3 Correspondence with Mariño's series

Let $f_1(z)$ be the holomorphic function corresponding to Mariño's formal power series $\varphi_1(z^{-1})$. The formal power series corresponding to f will be written in the variable z.

$$\begin{aligned}
\mathrm{Ai}(x) &= \tfrac{1}{2\sqrt{\pi}} x^{-1/4} e^{-z} \varphi_1\big(\tfrac{2}{3} z^{-1}\big) \\
&= \tfrac{1}{2\sqrt{\pi}} x^{-1/4} e^{-z} f_1\big(\tfrac{3}{2} z\big) \\
\mathrm{Ai}(x) &= \frac{1}{\pi\sqrt{3}} x^{1/2} K(\tfrac{2}{3} x^{3/2})
\end{aligned}$$

Putting together,

$$\begin{aligned}
\tfrac{1}{2\sqrt{\pi}} x^{-1/4} e^{-z} f_1\big(\tfrac{3}{2} z\big) &= \frac{1}{\pi\sqrt{3}} x^{1/2} K(\tfrac{2}{3} x^{3/2}) \\
\tfrac{\sqrt{3\pi}}{2}\; x^{-3/4} e^{-z} f_1\big(\tfrac{3}{2} z\big) &= K(\tfrac{2}{3} x^{3/2}) \\
\tfrac{\sqrt{3\pi}}{2}\; \big(\tfrac{3}{2} z\big)^{-1/2} e^{-z} f_1\big(\tfrac{3}{2} z\big) &= K(z) \\
\sqrt{\tfrac{\pi}{2}}\; z^{-1/2} e^{-z} f_1\big(\tfrac{3}{2} z\big) &= K(z) \\
\sqrt{\tfrac{\pi}{2}}\; \big[\mathcal{L}^{-1} z^{-1/2}\big] * \big[\mathcal{L}^{-1} f_1\big(\tfrac{3}{2} z\big)\big](\zeta - 1) &= \hat{K}(\zeta) \\
\sqrt{\tfrac{\pi}{2}}\; \Big[\Gamma\big(-\tfrac{1}{2}\big)^{-1} \zeta^{-1/2}\Big] * \tfrac{2}{3} \hat{f}_1\big[\tfrac{2}{3}(\zeta - 1)\big] &= \hat{K}(\zeta) \\
-\tfrac{1}{3\sqrt{2}}\; \Big[\zeta^{-1/2}\Big] * \hat{f}_1\big[\tfrac{2}{3}(\zeta - 1)\big] &= \hat{K}(\zeta)
\end{aligned}$$

Notice that if the hypergeometric differentiation formula holds for fractional derivatives,

$$\big(\tfrac{\partial}{\partial\xi}\big)^{1/2} F\big(\tfrac{2}{3}, \tfrac{4}{3}; \tfrac{3}{2}; \xi\big) \propto F\big(\tfrac{7}{6}, \tfrac{11}{6}; 2; \xi\big)$$

References

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[3] "NIST digital library of mathematical functions." http://dlmf.nist.gov, release 1.1.3 of 2021-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, editors.