RESURGENCE OF THE AIRY FUNCTION AND OTHER EXPONENTIAL INTEGRALS

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1. HYPERGEOMETRIC FUNCTIONS AS BOREL TRANSFORM OF SECOND ORDER ODE (*series normales de* I*er ordre*)

Let us consider the following linear second order ODE

$$\left[P(\tfrac{\partial}{\partial z})+\tfrac{1}{z}Q(\tfrac{\partial}{\partial z})+\tfrac{1}{z}R(\tfrac{1}{z})\right]f(z)=0 \tag{1.1}$$

with $\deg P = 2$, $\deg Q = 1$ and $R = O(\frac{1}{z})$. We denote by α_1,α_2 the roots of $P(\lambda)$ and we assume they are distinct. Furthermore we assume $\tau_j := \frac{Q(-\alpha_j)}{P'(-\alpha_j)} \neq 0$. The latter assumption guarantees the formal solution $\tilde{f}$ being slight, while the former assumption implies there will be two independent solutions.

Under the previous assumptions we prove that the Borel transformed solution $\hat{f}(\zeta_j)$ is a Gauss hypergeometric function, $\zeta_j = \zeta-\alpha_j$.

Proposition 1.1. Let $P(\lambda)=\lambda^2+a_1\lambda+a_0$, $Q(\lambda)=b_1\lambda+b_0$ and $R(\frac{1}{z})=\frac{c_1}{z}$ satisfying the previous assumptions. Then

$$\hat{f}(\zeta_1)=\left(1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)^{c-1}{}_2F_1\left(a,b;c;1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right) \tag{1.2}$$

$$\hat{f}(\zeta_2)=\left(1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right)^{c-1}{}_2F_1\left(a,b;c;1-\frac{\zeta_2}{\alpha_1-\alpha_2}\right) \tag{1.3}$$

where the coefficients a,b,c depend on the parameter of P,Q,R.

Proof. We start by taking the Borel transform of (1.1):

(1.4)

$$(\zeta^2-a_1\zeta+a_0)\hat{f}(\zeta)+\int_0^\zeta b_1(-\zeta')\hat{f}(\zeta')d\zeta'+b_0\int_0^\zeta \hat{f}(\zeta')d\zeta'+c_1\int_0^\zeta(\zeta-\zeta')\hat{f}(\zeta')d\zeta'=0$$

then we differentiate twice in order to have a differential equation which can be easier recognized as a hypergeometric equation. Since $\tilde{F}$ is slight and locally integrable at 0 by assumption, Proposition 1 `Resurgent Airy doc by Aaron` tells we are not loosing information taking derivatives, and that $\hat{f}(\zeta)$ is a solution of (1.4) if and only if it is a solution of (1.5)

$$\left[(\zeta^2-a_1\zeta+a_0)\partial_\zeta^2+(4\zeta-b_1\zeta-2a_1+b_0)\partial_\zeta+(c_1+2-b_1)\right]\hat{f}(\zeta)=0 \tag{1.5}$$

We introduce some notation to simplify the computations, we denote by $\beta_1=4-b_1$, $\beta_0=b_0-2a_1$, $\gamma=c_1+2-b_1$ so (1.5) turns into

$$\left[(\zeta-\alpha_1)(\zeta-\alpha_2)\partial_\zeta^2+(\beta_1\zeta+\beta_0)\partial_\zeta+\gamma\right]\hat{f}(\zeta)=0$$

We consider the following change of coordinates $\zeta=\alpha_2-(\alpha_2-\alpha_1)\xi$[1]

$$\left[(\alpha_2-(\alpha_2-\alpha_1)\xi-\alpha_1)(\alpha_2-(\alpha_2-\alpha_1)\xi-\alpha_2)(\alpha_1-\alpha_2)^{-2}\partial_\xi^2+(\beta_1(\alpha_2-(\alpha_2-\alpha_1)\xi)+\beta_0)(\alpha_1-\alpha_2)^{-1}\partial_\xi+\gamma\right]\hat{f}(\xi)=0$$
$$\left[(\alpha_2-\alpha_1)(1-\xi)(\alpha_1-\alpha_2)\xi(\alpha_1-\alpha_2)^{-2}\partial_\xi^2+(\beta_1\alpha_2-\beta_1(\alpha_2-\alpha_1)\xi+\beta_0)(\alpha_1-\alpha_2)^{-1}\partial_\xi+\gamma\right]\hat{f}(\xi)=0$$
$$\left[-(1-\xi)\xi\partial_\xi^2+((\beta_1\alpha_2+\beta_0)(\alpha_1-\alpha_2)^{-1}+\beta_1\xi)\partial_\xi+\gamma\right]\hat{f}(\xi)=0$$
$$\left[(1-\xi)\xi\partial_\xi^2-((\beta_1\alpha_2+\beta_0)(\alpha_1-\alpha_2)^{-1}+\beta_1\xi)\partial_\xi-\gamma\right]\hat{f}(\xi)=0$$

The latter equation is an hypergeometric equation of parameters

$$\begin{aligned}
&C=(\beta_1\alpha_2+\beta_0)(\alpha_2-\alpha_1)^{-1}\\
&A+B+1=\beta_1=4-b_1\Rightarrow A+B=3-b_1\\
&AB=\gamma=c_1+2-b_1
\end{aligned}$$

and a solution is given by

$$\begin{aligned}
\hat{f}(\xi)&=\xi^{1-C}\,{}_2F_1(A-C+1,B-C+1;2-C;\xi)\\
&=\left(\frac{\alpha_2-\zeta}{\alpha_2-\alpha_1}\right)^{1-C}{}_2F_1\left(A-C+1,B-C+1;2-C;\frac{\alpha_2-\zeta}{\alpha_2-\alpha_1}\right)\\
&=\left(1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)^{1-C}{}_2F_1\left(A-C+1,B-C+1;2-C;1-\frac{\zeta_1}{\alpha_2-\alpha_1}\right)
\end{aligned}$$

□

[1] $\partial_\zeta=(\alpha_1-\alpha_2)^{-1}\partial_\xi$ and $\partial_\zeta^2=(\alpha_1-\alpha_2)^{-2}\partial_\xi^2$