Resurgence of modified Bessel functions of second kind

Veronica Fantini

March 10, 2022

1 Modified Bessel function of second kind

The modified Bessel function of the second kind $K_\mu(z)$ is a solution of the equation

$$w'' + \frac{w'}{z} - w - \frac{\mu^2}{z^2} = 0 \tag{1}$$

such that $K_\mu(z) \sim \sqrt{\pi/(2z)}e^{-z}$ as $z \to \infty$ in $|\arg z| < \frac{3\pi}{2}$[1]. It has a branch point at $z = 0$ for every $\mu \in \mathbb{C}$ and the principal branch is analytic in $\mathbb{C} \setminus (-\infty, 0]$.

1.1 Differential equation

From the general theory of ODE, the formal integral solution of (1) is two parameters family

$$\tilde{w}(z) = U_1 e^{-z} z^{-1/2} \tilde{w}_{\mu,+}(z) + U_2 e^{z} z^{-1/2} \tilde{w}_{\mu,-}(z) \tag{4}$$

where $\tilde{w}_{\mu,\pm} = \sum_{j\geq 0} a_{\pm,j} z^{-j} \in \mathbb{C}[\![z^{-1}]\!]$ are unique formal solutions of

$$\begin{aligned}
\tilde{w}''_{\mu,+} - 2\tilde{w}'_{\nu,+} + \frac{\tilde{w}_{\mu,+}}{4z^2} - \frac{\mu^2}{z^2}\tilde{w}_{\mu,+} = 0\\
\tilde{w}''_{\mu,-} + 2\tilde{w}'_{\mu,-} + \frac{\tilde{w}_{\mu,-}}{4z^2} - \frac{\mu^2}{z^2}\tilde{w}_{\mu,-} = 0
\end{aligned}$$

[1] A system of solution of Bessel equation is given by $I_\mu(z)$ and $K_\mu(z)$. In particular, their asymptotic behaviour as $z \to \infty$ is given by

$$\tilde{I}_\mu(z) = \frac{1}{\sqrt{2\pi}} e^{z} z^{-1/2} \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\mu\right)_k \left(\frac{1}{2}+\mu\right)_k}{2^k k!} z^{-k} \tag{2}$$

$$\tilde{K}_\mu(z) = \sqrt{\frac{\pi}{2}} e^{-z} z^{-1/2} \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\mu\right)_k \left(\frac{1}{2}+\mu\right)_k}{(-2)^k k!} z^{-k} \tag{3}$$

In particular, $\tilde{K}_\mu(z) = \sqrt{\frac{\pi}{2}} e^{-z} z^{-1/2} \tilde{w}_{\mu,+}(z)$ and $\tilde{I}_\mu(z) = \frac{1}{\sqrt{2\pi}} e^{z} z^{-1/2} \tilde{w}_{\mu,-}(z)$ (once we choose $a_{\pm,0} = 1$). We now compute the Borel transform of $\tilde{w}_+(z)^2$ it is a solution of

$$\begin{aligned}
&\zeta^2 \hat{w}_{\mu,+} + 2t\hat{w}_{\mu,+} + \left(\frac{1}{4} - \nu^2\right) \int_0^\zeta (\zeta - s)\hat{w}_{\nu,+}(s)ds = 0 \\
&\zeta^2 \hat{w}''_{\mu,+} + 2\zeta \hat{w}''_{+} + 4\zeta \hat{w}'_{\mu,+} + \left(\frac{9}{4} - \mu^2\right) \hat{w}_{\mu,+} = 0 \\
&t(1-t)\hat{w}''_{\mu,+} + (2-4t)\hat{w}'_{\mu,+} - \left(\frac{9}{4} - \mu^2\right) \hat{w}_{mu,+} = 0 \qquad\qquad t = -\frac{\zeta}{2}
\end{aligned}$$

therefore $\hat{w}_{\mu,+}(\zeta)$ is an hypergeometric function

$$\hat{w}_{\mu,+}(\zeta) = c_{\mu,+2}F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; -\frac{\zeta}{2}\right) \tag{5}$$

and it has a branch point singularities at $\zeta = -2$. By the same reasoning,

$$\hat{w}_{\mu,-}(\zeta) = c_{\mu,-2}F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; \frac{\zeta}{2}\right) \tag{6}$$

and it has branch point at $\zeta = 2$.

1.2 Exponential integral

Let $X = \mathbb{C}^*$, $f(x) = x + \frac{1}{x}$ and for every $\mu \in [0, +\infty)$ let $\nu = x^{\mu-1} + \frac{1}{x^{\mu+1}} dx$, then

$$I(z; m) := \int_0^\infty e^{-zf} \nu \tag{7}$$

In particulr, on the universal cover $\pi\colon \tilde{C} \to \mathbb{C}^*$ setting $x = e^u$

$$I(\frac{z}{2}; \mu) = 2\int_{-\infty}^\infty e^{-z\cosh(u)} \cosh(\mu u) du = 4K_\mu(z) \quad |\arg(z)| < \pi/2 \tag{8}$$

where $K_\mu(z)$ is the second kind modified Bessel function with parameter μ.

Add definition of $\tilde{I}_\pm$

It is worth mentioning that $I(z;\mu)$ differs from $I(z;0)$ only in $\pi^*(\nu)$ while $\pi^*(f)$ stays the same for every $\mu \in [0, \infty)$. Hence we can adapt part of the argument used in Bessel example (see), and apply the 3/2-derivative formula: let $\zeta = \cosh(u)$

$$\begin{aligned}
\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu) &= \int_{\mathcal{C}_0(\zeta)} \cosh(\mu u) du \\
&= \frac{1}{\mu}\Big[\sinh(\mu u)\Big]_{\mathrm{start}\mathcal{C}_0(\zeta)}^{\mathrm{end}\mathcal{C}_0(\zeta)} \\
&= \frac{1}{\mu}\left(\sinh\left(\mu \operatorname{acosh}(\zeta)\right) - \sinh\left(-\mu \operatorname{acosh}(\zeta)\right)\right) \\
&= \frac{2}{\mu}\sinh\left(\mu \operatorname{acosh}(\zeta)\right)
\end{aligned}$$

[2] We do not consider constant term of $\tilde{w}_{\mu,\pm}$, i.e. $\mathcal{B} : \mathbb{C}[\![z^{-1}]\!] \to \mathbb{C}[\zeta]$.

The we set $\xi = \frac{1}{2}(\zeta - 1)$, thanks to identity 15.4.16 **DLMF**

$$
\begin{aligned}
\sinh(\tau){}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; -\sinh^2(\tau)\right) &= \frac{1}{2\mu}\sinh(2\mu\tau) \\
\xi^{1/2}{}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; -\xi\right) &= \frac{1}{2\mu}\sinh(2\mu\tau) && \sinh^2(\tau) = \xi \\
\xi^{1/2}{}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; -\xi\right) &= \frac{1}{2\mu}\sinh(\mu \operatorname{acosh}(\zeta)) && \cosh(2\tau) = \zeta \\
&= \frac{1}{4}\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu)
\end{aligned}
$$

Thus we take 3/2-derivative based at $\zeta = 1$

$$
\begin{aligned}
\partial_\zeta^{3/2}\left(\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu)\right) &= \partial_\zeta^2\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^\zeta (\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_0(\zeta')} \pi^*(\nu)\right) d\zeta'\right) \\
&= 4\partial_\zeta^2\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^\xi \frac{1}{\sqrt{2}}(\xi-\xi')^{-1/2}(\xi')^{1/2}{}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; -\xi'\right) 2\ d\xi'\right] \\
&= \frac{8}{\sqrt{2}}\partial_\zeta^2\left[\Gamma\left(\frac{3}{2}\right)\xi\ {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; 2; -\xi\right)\right] \\
&= \frac{8}{\sqrt{2}}\frac{\sqrt{\pi}}{2}\frac{1}{4}\partial_\xi^2\left[\xi\ {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; 2; -\xi\right)\right] \\
&= -\frac{\sqrt{\pi}}{\sqrt{2}}\partial_\xi\ {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; 1; -\xi\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{2}}\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\mu\right){}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; -\xi\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{2}}\frac{\pi}{\cos(\mu\pi)}{}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; \frac{1}{2}-\frac{\zeta}{2}\right)
\end{aligned}
$$

Let us now consider the integral whose asymptotic behvaior is given in terms of $\hat{w}_{\mu,-}(z)$:

I have to check the correct form of the integral I which correspond to the path C_π. I suspect a scaling factor of $\cos(\pi\mu)$ that will adjust the Stokes factor computations.

set $\zeta = -\cosh(u)$,

$$
\begin{aligned}
\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu) &= \int_{\mathcal{C}_\pi(\zeta)} \cosh(\mu u) du \\
&= \frac{1}{\mu}\Big[\sinh(\mu u)\Big]_{\mathrm{start}\mathcal{C}_\pi(\zeta)}^{\mathrm{end}\mathcal{C}_\pi(\zeta)} \\
&= \frac{1}{\mu}\left(\sinh\left(\mu \operatorname{acosh}(-\zeta)\right) - \sinh\left(-\mu \operatorname{acosh}(-\zeta)\right)\right) \\
&= \frac{2}{\mu}\sinh\left(\mu \operatorname{acosh}(-\zeta)\right)
\end{aligned}
$$

The we set $\xi = \frac{1}{2}(\zeta+1)$, thanks to identity 15.4.16 **DLMF**

$$
\begin{aligned}
\sinh(\tau)_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; -\sinh^2(\tau)\right) &= \frac{1}{2\mu}\sinh(2\mu\tau) \\
(-\xi)^{1/2}{}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; \xi\right) &= \frac{1}{2\mu}\sinh(2\mu\tau) && \sinh^2(\tau) = -\xi \\
(-\xi)^{1/2}{}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; \xi\right) &= \frac{1}{2\mu}\sinh(\mu \ \operatorname{acosh}(-\zeta)) && \cosh(2\tau) = -\zeta \\
&= \frac{1}{4}\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu)
\end{aligned}
$$

Thus we take 3/2-derivative based at $\zeta = -1$

$$
\begin{aligned}
\partial_\zeta^{3/2}\left(\int_{\mathcal{C}_\pi(\zeta)} \pi^*(\nu)\right) &= \partial_\zeta^2\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{\zeta}(\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_\pi(\zeta')} \pi^*(\nu)\right) d\zeta'\right) \\
&= 4\partial_\zeta^2\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^{\xi}\frac{1}{\sqrt{2}}(\xi-\xi')^{-1/2}(-\xi')^{1/2}{}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; \frac{3}{2}; \xi'\right) 2 \ d\xi'\right) \\
&= -i\frac{8}{\sqrt{2}}\partial_\zeta^2\left(\Gamma\left(\tfrac{3}{2}\right)(-\xi) \ {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; 2; \xi\right)\right) \\
&= i\frac{8}{\sqrt{2}}\frac{\sqrt{\pi}}{2}\frac{1}{4}\partial_\xi^2\left(\xi \ {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; 2; \xi\right)\right) \\
&= i\frac{\sqrt{\pi}}{\sqrt{2}}\partial_\xi \ {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; 1; \xi\right) \\
&= i\frac{\sqrt{\pi}}{\sqrt{2}}\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\mu\right){}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; \xi\right) \\
&= i\frac{\sqrt{\pi}}{\sqrt{2}}\frac{\pi}{\cos(\mu\pi)}{}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; \frac{1}{2}-\frac{\zeta}{2}\right)
\end{aligned}
$$

1.3 Stokes factors

Thanks to explicit formula for the analytic continuation of hypergeomtric functions (see [?] 15.2.3) and using the constants prescribed by the fractional derivative formula we are able ot compute the Stokes constants: set $\hat{w}_{+,\mu}(\zeta) = \frac{\pi\sqrt{\pi}}{\cos(\mu\pi)}\,{}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}-\mu; 2, 1-\frac{\zeta}{2}\right)$ and $\hat{w}_{-,\mu}(\zeta) := i\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)}\,{}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}-\mu; 2, 1+\frac{\zeta}{2}\right)$

$$\begin{aligned}
\hat{w}_{\mu,+}(\zeta+i0) - \hat{w}_{\mu,+}(\zeta-i0) &= \frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \left(-\frac{\zeta}{2}-1\right)^{-1} {}_2F_1\left(\frac{1}{2}+\mu, \frac{1}{2}-\mu; 0; 1+\frac{\zeta}{2}\right) \quad \zeta > -2 \\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\mu\right)_k \left(\frac{1}{2}+\mu\right)_k}{\Gamma(k)k!} \left(1+\frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \sum_{k\geq 1} \frac{\left(\frac{1}{2}-\mu\right)_k \left(\frac{1}{2}+\mu\right)_k}{\Gamma(k)k!} \left(1+\frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \frac{1}{\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\mu\right)} \cdot \\
&\quad \cdot \sum_{k\geq 1} \frac{\Gamma\left(\frac{1}{2}-\mu+k\right)\Gamma\left(\frac{1}{2}+\mu+k\right)}{\Gamma(k)k!} \left(1+\frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \frac{1}{\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\mu\right)} \cdot \\
&\quad \cdot \sum_{k\geq 0} \frac{\Gamma\left(\frac{3}{2}-\mu+k\right)\Gamma\left(\frac{3}{2}+\mu+k\right)}{\Gamma(k+1)(k+1)!} \left(1+\frac{\zeta}{2}\right)^{k} \\
&= -\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\mu\right)} \; {}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; 1+\frac{\zeta}{2}\right) \\
&= -2\pi i\frac{\sqrt{\pi}}{2} \; {}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; 1+\frac{\zeta}{2}\right) \\
&= -\mathbf{2}\cos(\pi\mu)\hat{w}_{-,\mu}(\zeta+2)
\end{aligned}$$

and for $\hat{w}_{\mu,-}(\zeta)$

$$\begin{aligned}
\hat{w}_{\mu,-}(\zeta+i0) - \hat{w}_{\mu,-}(\zeta-i0) &= i\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \left(\frac{\zeta}{2}-1\right)^{-1} {}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu; 0; 1-\frac{\zeta}{2}\right) \quad \zeta < 2 \\
&= -i\frac{\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)} \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\mu\right)_k \left(\frac{1}{2}+\mu\right)_k}{\Gamma(k)k!} \left(1-\frac{\zeta}{2}\right)^{k-1} \\
&= -2i\cos(\mu\pi)\,\frac{i\pi\sqrt{\pi}}{\sqrt{2}\cos(\mu\pi)} \; {}_2F_1\left(\frac{3}{2}-\mu, \frac{3}{2}+\mu; 2; 1-\frac{\zeta}{2}\right) \\
&= +\mathbf{2}\cos(\mu\pi)\hat{w}_{\mu,+}(\zeta-2)
\end{aligned}$$

Therefore we have shown that Stokes constants are independent on μ and equal to ± 2.