

# EXPONENTIAL INTEGRALS

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## 1. INTRODUCTION

### 2. FRACTIONAL DERIVATIVES AND BOREL TRANSFORM

**Definition 2.1.** Let  $\alpha \in (0, 1)$  and  $n \in \mathbb{N}$ , then the  $n + \alpha$ -Caputo's derivative of a smooth fucntion  $f$  is defined as

$$(2.1) \quad \partial_x^{n+\alpha} f(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f^{(n+1)}(s) ds$$

In particular, this definition is well suited for the differential calculus in the convolutive model  $(\mathbb{C}[[\zeta]], *)$ . Let  $\varphi(z) := \sum_{k \geq 0} a_k z^{-k-1} \in \mathbb{C}[[z^{-1}]]$  be Gevrey 1, then assuming  $a_k = 0$  for every  $k < n$ , the Borel transform of  $z^{n+\alpha} \varphi(z)$  can be computed in two different ways:

$$(2.2) \quad \begin{aligned} \mathcal{B}(z^{n+\alpha} \varphi(z))(\zeta) &= \mathcal{B}(z^{a+n}) * \hat{\varphi}(\zeta) = \int_0^\zeta \frac{(\zeta-s)^{-1-n-\alpha}}{(-1-n-\alpha)!} \sum_{k \geq 0} \frac{a_k}{k!} s^k ds \\ &= \frac{1}{(-\alpha)!} \int_0^\zeta (\zeta-s)^{-\alpha} \sum_{k \geq 0} \frac{a_k}{(k-n-1)!} s^{k-n-1} ds = \partial_\zeta^{n+\alpha} \hat{\varphi}(\zeta) \end{aligned}$$

$$(2.3) \quad \mathcal{B}(z^{n+\alpha} \varphi(z))(\zeta) = \mathcal{B}\left(\sum_{k \geq 0} a_k z^{-k-1+n+\alpha}\right)(\zeta) = \sum_{k > n} \frac{a_k}{(k-n-\alpha)!} \zeta^{k-n-\alpha}$$

and computing the integral which defines the  $n + \alpha$ -derivative in (2.2) we get exactly the same result as (2.3).

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### 3. RESURGENCE OF EXPONENTIAL INTEGRALS

Let  $X$  be a  $N$  – dim manifold,  $f: X \rightarrow \mathbb{C}$  be a holomorphic Morse function with only simple critical points, and  $\nu \in \Gamma(X, \Omega^N)$ , and set

$$(3.1) \quad I(z) := \int_{\mathcal{C}} e^{-zf} \nu$$

where  $\mathcal{C}$  is a suitable contour such that the integral is well defined. For any Morse critical points  $x_\alpha$  of  $f$ , the saddle point approximation gives the following formal series

$$(3.2) \quad I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf} \nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)} (2\pi)^{N/2} z^{-N/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n} \quad \text{as } z \rightarrow \infty$$

where  $\mathcal{C}_\alpha$  is a steepest descent path through the critical point  $x_\alpha$ . Notice that  $f \circ \mathcal{C}_\alpha$  lies in the ray  $\zeta_\alpha + [0, \infty)$ , where  $\zeta_\alpha := f(x_\alpha)$ .

**Theorem 3.1.** *Let  $N = 1$ . Let  $\tilde{\varphi}_\alpha(z) := e^{-zf(x_\alpha)} (2\pi)^{1/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n}$  and assume  $f''(x_\alpha) \neq 0$  for every critical point  $x_\alpha$ . Then:*

- (1) *The series  $\tilde{\varphi}_\alpha$  is Gevrey-1.*
- (2) *The series  $\hat{\varphi}(\zeta) := \mathcal{B}(\tilde{\varphi})$  converges near  $\zeta = \zeta_\alpha$ .*
- (3) *For any  $\zeta$  on the ray going rightward from  $\zeta_\alpha$ , we have*

$$\begin{aligned} \hat{\varphi}_\alpha(\zeta) &= \partial_{\zeta}^{\frac{3}{2}} \int_{\mathcal{C}_\alpha(\zeta)} \nu \\ &= \Gamma\left(-\frac{1}{2}\right) \int_{\zeta_\alpha}^{\zeta} (\zeta - \zeta')^{-1/2} \partial_{\zeta'} \left( \left[ \frac{\nu}{df} \right]_{\text{start } \mathcal{C}_\alpha(\zeta')}^{\text{end } \mathcal{C}_\alpha(\zeta')} \right) d\zeta', \end{aligned}$$

where  $\mathcal{C}_\alpha(\zeta)$  is the part of  $\mathcal{C}_\alpha$  that goes through  $f^{-1}([\zeta_\alpha, \zeta])$ . Notice that  $\mathcal{C}_\alpha(\zeta)$  starts and ends in  $f^{-1}(\zeta)$ .

*Proof.* Part (1): Let us assume that locally  $v = \sum_{j \geq 0} b_j^\alpha (t - t_\alpha)^j dt$  for  $|t - t_\alpha| < \varepsilon$ . By the steepest descent method,  $I_\alpha(z)$  can be approximated as  $z \rightarrow \infty$  as

$$\begin{aligned} I_\alpha(\zeta) &\sim e^{-zf(x_\alpha)} \int_{-\varepsilon}^{\varepsilon} \sum_{n \geq 0} t^{2n} b_{2n}^\alpha e^{-zf''(x_\alpha) \frac{t^2}{2}} dt \\ &= e^{-zf(x_\alpha)} \sum_{n \geq 0} b_{2n}^\alpha \int_{-\varepsilon}^{\varepsilon} t^{2n} e^{-zf''(x_\alpha) \frac{t^2}{2}} dt \\ &= \sqrt{2\pi} e^{-zf(x_\alpha)} z^{-1/2} \sum_{n \geq 0} \frac{b_{2n}^\alpha}{(f''(x_\alpha))^{n+1/2}} (2n-1)!! z^{-n} \operatorname{Erf}\left(\frac{\sqrt{zf''(x_\alpha)}}{\sqrt{2}} \varepsilon\right) + \\ &\quad - 2e^{-zf(x_\alpha)} \sum_{n \geq 1} b_{2n}^\alpha (2n-1)!! e^{-zf''(x_\alpha) \frac{\varepsilon^2}{2}} \sum_{j=1}^n \frac{\varepsilon^{2j-1}}{(2j-1)!!} (f''(x_\alpha)z)^{n-j+1} \\ &\sim_{\varepsilon < 1} 2\varepsilon \sqrt{2\pi} e^{-zf(x_\alpha)} z^{-1/2} \sum_{n \geq 0} \frac{b_{2n}^\alpha}{(f''(x_\alpha))^{n+1/2}} (2n-1)!! z^{-n} \end{aligned}$$

therefore

$$a_{\alpha,n} := 2\varepsilon \frac{b_{2n}^\alpha}{(f''(x_\alpha))^{n+1/2}} (2n-1)!!$$

Since,  $\frac{1}{\varepsilon} = \limsup_n \sqrt[n]{b_n^\alpha}$

$$|a_{\alpha,n}| \leq C A^n n!$$

where we use  $(2n-1)!! \sim \frac{2^n}{\sqrt{\pi n}} n!$  as  $n \rightarrow \infty$ .

Part (2):

$$\begin{aligned} \hat{\varphi}_\alpha(\zeta) &= \mathcal{B}\left(e^{-zf(x_\alpha)} (2\pi)^{1/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n}\right)(\zeta) = T_{f(x_\alpha)} (2\pi)^{1/2} \left(\delta a_0 + \sum_{n \geq 0} a_{n+1} \frac{\zeta^n}{n!}\right) \\ &\quad (2\pi)^{1/2} \left(\delta(f_{x_\alpha}) a_0 + \sum_{n \geq 0} a_{n+1} \frac{(\zeta - f(x_\alpha))^n}{n!}\right) \end{aligned}$$

Since  $a_n \leq C A^n n!$ , the series  $\sum_{n \geq 0} a_{n+1} \frac{(\zeta - f(x_\alpha))^n}{n!}$  has a finite radius of convergence.

Part (3): thanks to properties of Caputo's fractional derivatives, we have that the Borel transform of  $\tilde{I}_\alpha(z) = z^{-1/2} \tilde{\varphi}_\alpha(z)$  is

$$(3.3) \quad \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \hat{I}_\alpha(\zeta) = \hat{\varphi}_\alpha(\zeta).$$

Let's recast the integral  $I_\alpha$  into the  $f$  plane. Let  $\mathcal{C}_\alpha^-$  and  $\mathcal{C}_\alpha^+$  be the parts of  $\mathcal{C}_\alpha$  that go from the past to  $x_\alpha$  and from  $x_\alpha$  to the future, respectively. As  $\zeta$  goes rightward from  $\zeta_\alpha$ , the start and end points of  $\mathcal{C}_\alpha(\zeta)$  sweep backward along  $\mathcal{C}_\alpha^-(\zeta)$  and forward along

$\mathcal{C}_a^+(\zeta)$ , respectively. Hence, we have

$$\begin{aligned} I_a(z) &= \int_{\mathcal{C}_a} e^{-zf} \nu \\ &= \int_{\zeta_a}^{\infty} e^{-z\zeta} \left[ \frac{\nu}{df} \right]_{\text{start } \mathcal{C}_a(\zeta)}^{\text{end } \mathcal{C}_a(\zeta)} d\zeta. \end{aligned}$$

Noticing that the right-hand side is a Laplace transform, we learn that

$$\hat{I}_a(\zeta) = \left[ \frac{\nu}{df} \right]_{\text{start } \mathcal{C}_a(\zeta)}^{\text{end } \mathcal{C}_a(\zeta)}.$$

Observing that

$$\partial_{\zeta} \left( \int_{\mathcal{C}_a(\zeta)} \nu \right) = \left[ \frac{\nu}{df} \right]_{\text{start } \mathcal{C}_a(\zeta)}^{\text{end } \mathcal{C}_a(\zeta)},$$

we conclude that

$$\hat{I}_a(\zeta) = \partial_{\zeta} \left( \int_{\mathcal{C}_a(\zeta)} \nu \right) d\zeta.$$

Hence

$$\begin{aligned} \hat{\varphi}_a(\zeta) &= \partial_{\zeta}^{1/2} \int_{\text{from } \zeta_a} \hat{I}_a \\ &= \partial_{\zeta}^{3/2} \int_{\mathcal{C}_a(\zeta)} \nu d\zeta. \end{aligned}$$

□

**Example 3.2 (Airy).** Let  $f(t) = \frac{t^3}{3} - t$  and

$$I(z) := \int_{\gamma} e^{-zf(t)} dt$$

where  $\gamma$  is a contour where the integral is well defined.

By the change of coordinates  $z = x^{3/2}$ ,  $I(z) = -2\pi i z^{-1/3} Ai(x)$  where

$$Ai(x) := \frac{1}{2\pi i} \int_{-\infty e^{-i\frac{\pi}{3}}}^{\infty e^{i\frac{\pi}{3}}} e^{\frac{t^3}{3} - zt} dt$$

hence  $I(z)$  solves the following ODE<sup>1</sup>

$$(3.4) \quad I''(z) - \frac{4}{9}I(z) + \frac{I'(z)}{z} - \frac{1}{9} \frac{I(z)}{z^2} = 0$$

A formal solution of (3.4) can be computed by making the following ansatz

$$(3.5) \quad \tilde{I}(z) = \sum_{k \in \mathbb{N}^2} U^k e^{-\lambda \cdot k z} z^{-\tau \cdot k} w_k(z)$$

<sup>1</sup> $Ai(x)$  solves the Airy equation  $y'' = xy$ .

with  $U^{(k_1, k_2)} = U_1^{k_1} U_2^{k_2}$  and  $U_1, U_2 \in \mathbb{C}$  are constant parameter,  $\lambda = (\frac{2}{3}, -\frac{2}{3})$ ,  $\tau = (\frac{1}{2}, \frac{1}{2})$ , and  $\tilde{w}_k(z) \in \mathbb{C}[[z^{-1}]]$ . In addition, we can check that the only non zero  $\tilde{w}_k(z)$  occurs at  $k = (1, 0)$  and  $k = (0, 1)$ , therefore

$$(3.6) \quad \tilde{I}(z) = U_1 e^{-2/3z} z^{-1/2} \tilde{w}_+(z) + U_2 e^{2/3z} z^{-1/2} \tilde{w}_-(z)$$

where from now on we denote  $\tilde{w}_+(z) = \tilde{w}_{(1,0)}(z)$  and  $\tilde{w}_-(z) = \tilde{w}_{(0,1)}(z)$ . In particular,  $\tilde{w}_+(z)$  and  $\tilde{w}_-(z)$  are formal solution of

$$(3.7) \quad \tilde{w}_+'' - \frac{4}{3} \tilde{w}_+' + \frac{5}{36} \frac{\tilde{w}_+}{z^2} = 0$$

$$(3.8) \quad \tilde{w}_-'' + \frac{4}{3} \tilde{w}_-' + \frac{5}{36} \frac{\tilde{w}_-}{z^2} = 0$$

Taking the Borel transform of (3.7), (3.8) we get

$$\zeta^2 \hat{w}_+(\zeta) + \frac{4}{3} \zeta \hat{w}_+ + \frac{5}{36} \zeta * \hat{w}_+ = 0$$

$$\zeta^2 \hat{w}_+(\zeta) + \frac{4}{3} \zeta \hat{w}_+ + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}_+(\zeta') d\zeta' = 0$$

$$\zeta^2 \hat{w}_-(\zeta) - \frac{4}{3} \zeta \hat{w}_- + \frac{5}{36} \zeta * \hat{w}_- = 0$$

$$\zeta^2 \hat{w}_-(\zeta) - \frac{4}{3} \zeta \hat{w}_- + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}_-(\zeta') d\zeta' = 0$$

and taking derivatives we get

$$\zeta(\frac{4}{3} + \zeta) \hat{w}_+'' + (\frac{8}{3} + 4\zeta) \hat{w}_+' + \frac{77}{36} \hat{w}_+ = 0$$

$$\frac{4}{3} \zeta(1 + \frac{3}{4} \zeta) \hat{w}_+'' + (\frac{8}{3} + 4\zeta) \hat{w}_+' + \frac{77}{36} \hat{w}_+ = 0$$

$$u(1-u) \hat{w}_+''(u) + (2-4u) \hat{w}_+'(u) - \frac{77}{36} \hat{w}_+(u) = 0 \quad u = -\frac{3}{4} \zeta$$

$$\zeta(-\frac{4}{3} + \zeta) \hat{w}_-'' + (-\frac{8}{3} + 4\zeta) \hat{w}_-' + \frac{77}{36} \hat{w}_- = 0$$

$$\frac{4}{3} \zeta(-1 + \frac{3}{4} \zeta) \hat{w}_-'' + (-\frac{8}{3} + 4\zeta) \hat{w}_-' + \frac{77}{36} \hat{w}_- = 0$$

$$u(1-u) \hat{w}_-''(u) + (2-4u) \hat{w}_-'(u) - \frac{77}{36} \hat{w}_-(u) = 0 \quad u = \frac{3}{4} \zeta$$

Notice that the latter equations are hypergeometric, hence a solution is given by

$$(3.9) \quad \hat{w}_+(\zeta) = c_1 {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}\zeta\right)$$

$$(3.10) \quad \hat{w}_-(\zeta) = c_2 {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{3}{4}\zeta\right)$$

for some constants  $c_1, c_2 \in \mathbb{C}$  (see DLMF 15.10.2). In addition  $\hat{w}_\pm(\zeta)$  have a log singularity respectively at  $\zeta = \mp \frac{4}{3}$ , therefore they are  $\{\mp \frac{4}{3}\}$ -resurgent functions.<sup>2</sup>

**Remark 3.3.**  $\hat{w}_+(\zeta)$  is Laplace summable along the positive real axis, and it can be analytically continued on  $\mathbb{C} \setminus -\frac{4}{3}\mathbb{R}_{\leq 0}$  with (see 15.2.3 DLMF)

$$\begin{aligned} \hat{w}_+(\zeta + i0) - \hat{w}_+(\zeta - i0) &= -\frac{36}{5} i \left(-\frac{3}{4}\zeta - 1\right)^{-1} \sum_{n \geq 0} \frac{(5/6)_n (1/6)_n}{\Gamma(n)n!} \left(1 + \frac{3}{4}\zeta\right)^n & \zeta < -\frac{4}{3} \\ &= \frac{36}{5} i \sum_{n \geq 0} \frac{(5/6)_n (1/6)_n}{\Gamma(n)n!} \left(1 + \frac{3}{4}\zeta\right)^{n-1} \\ &= -\frac{36}{5} i \left(-\frac{3}{4}\zeta - 1\right)^{-1} \left( \frac{5}{144} (4 + 3\zeta) \left(1 + {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 + \frac{3}{4}\zeta\right)\right) \right) \\ &= {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 + \frac{3}{4}\zeta\right) \\ &= i \hat{w}_-\left(\zeta + \frac{4}{3}\right) \end{aligned}$$

Analogously,  $\hat{w}_-(\zeta)$  is Laplace summable along the negative real axis, and it jumps across the branch cut  $\frac{4}{3}\mathbb{R}_{\geq 0}$  as

$$\begin{aligned} \hat{w}_-(\zeta + i0) - \hat{w}_-(\zeta - i0) &= \frac{36}{5} i \left(\frac{3}{4}\zeta - 1\right)^{-1} \sum_{n \geq 0} \frac{(5/6)_n (1/6)_n}{\Gamma(n)n!} \left(1 - \frac{3}{4}\zeta\right)^n & \zeta > \frac{4}{3} \\ &= \frac{36}{5} i \left(\frac{3}{4}\zeta - 1\right)^{-1} \left( -\frac{5}{144} (-4 + 3\zeta) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 - \frac{3}{4}\zeta\right) \right) \\ &= -i {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, 1 - \frac{3}{4}\zeta\right) \\ &= -i \hat{w}_+\left(\zeta - \frac{4}{3}\right) \end{aligned}$$

These relations manifest the resurgence property of  $\tilde{I}$ , indeed near the singularities in the Borel plane of either  $\hat{w}_+$  or  $\hat{w}_-$ ,  $\hat{w}_-$  and  $\hat{w}_+$  respectively contribute to the jump of the former solution.

Our next goal is to prove that the Borel transform of  $\tilde{I}(z)$  can be written in terms of  $1/f'(f^{-1}(\zeta))$ , namely formula (??). It is convenient to consider the two asymptotic formal solutions separately, namely we define

<sup>2</sup>The solution we find are equal to the ones in DLMF §9.7. They also agree with slide 10 of Maxim's talk for ERC and with the series (6.121) in Sauzin's book. However they do not agree with (2.16) in Mariño's Diablerets.

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$$(3.11) \quad \tilde{I}_{-1}(z) := e^{-2/3z} z^{-1/2} \tilde{w}_+(z) =: z^{-1/2} \tilde{u}_+(z)$$

$$(3.12) \quad \tilde{I}_1(z) := e^{2/3z} z^{-1/2} \tilde{w}_-(z) =: z^{-1/2} \tilde{u}_-(z)$$

In particular,  $\tilde{u}_\pm(z)$  are solutions of

$$(3.13) \quad \tilde{u}''(z) - \frac{4}{9} \tilde{u}(z) + \frac{5}{36} \frac{\tilde{u}(z)}{z^2} = 0$$

with asymptotic behaviour  $\tilde{u}_\pm(z) \sim O(e^{\pm 2/3z})$  as  $z \rightarrow \infty$ .

The Borel transforms  $\hat{u}_\pm(\zeta)$  solve the same equation

$$\begin{aligned} & \zeta^2 \hat{u} - \frac{4}{9} \hat{u} + \frac{5}{36} \zeta * \hat{u} \\ & \zeta^2 \hat{u} - \frac{4}{9} \hat{u} + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{u}(\zeta') d\zeta' \end{aligned}$$

taking derivatives is equivalent to

$$(\zeta^2 - \frac{4}{9}) \hat{u}''(\zeta) + 4\zeta \hat{u}'(\zeta) + \frac{77}{36} \hat{u}(\zeta) = 0$$

and Mathematica gives the following solutions

$$\begin{aligned} \hat{u}(\zeta) &= c_1 {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{9}{4}\zeta^2\right) + \frac{3i}{2} \zeta c_2 {}_1F_2\left(\frac{13}{12}, \frac{17}{12}, \frac{3}{2}, \frac{9}{4}\zeta^2\right) = \\ &= c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} \left( {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{1}{2}, \frac{1}{2} + \frac{3}{4}\zeta\right) \right) \quad \text{see DLMF 15.8.27} \\ &\quad + \frac{3i}{2} \zeta c_2 \left( \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{3\zeta\Gamma(-\frac{1}{2})\Gamma(2)} \right) \left( {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) - {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right) \right) \quad \text{see DLMF 15.8.28} \\ &= \left( c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} - c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}} \right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} - \frac{3}{4}\zeta\right) + \\ &\quad + \left( c_1 \frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}} + c_2 i \frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}} \right) {}_1F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{1}{2} + \frac{3}{4}\zeta\right) \end{aligned}$$

Since  $\hat{u}_+$  has a simple singularity at  $\zeta = -2/3$  and  $\hat{u}_-$  has a simple singularity at  $\zeta = 2/3$ , we have

$$\begin{aligned} \hat{u}_+(\zeta) &= C_1 T_{-2/31} F_2\left(\frac{7}{6}, \frac{11}{6}, 2, -\frac{3}{4}\zeta\right) = C_1 T_{-2/3} \hat{w}_+(\zeta) \\ \hat{u}_-(\zeta) &= C_2 T_{2/31} F_2\left(\frac{7}{6}, \frac{11}{6}, 2, \frac{3}{4}\zeta\right) = C_2 T_{2/3} \hat{w}_-(\zeta) \end{aligned}$$

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**Lemma 3.4.** *The following identity holds true*

$$(3.14) \quad {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}\zeta^2\right) = \frac{1}{1-u^2} \quad \zeta = \frac{u^3}{3} - u$$

*Proof.* From the special case of hypergeometric function (see 15.4.14 DLMF) we have the following identity:

$$\begin{aligned} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}\zeta^2\right) &= \frac{\cos(y)}{\cos(3y)} & 3y &= \arcsin\left(\frac{3}{2}\zeta\right) \\ &= \frac{\cos(y)}{\cos(2y)\cos(y) - \sin(2y)\sin(y)} \\ &= \frac{1}{\cos(2y) - 2\sin^2(y)} \\ &= \frac{1}{1 - 4\sin^2(y)} & \zeta &= 2\sin(y) - \frac{8}{3}\sin^3(y) \end{aligned}$$

Therefore, if  $u := -2\sin(y)$ , we have  $\zeta = \frac{u^3}{3} - u = f(u)$  and

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}\zeta^2\right) = \frac{1}{1-u^2} = -\frac{1}{f'(u)}$$

□

Then equations (??) is equivalent to

**Claim 3.5.** [There should be a way to predict the correct normalization of the LHS.]

$$(3.15) \quad \hat{w}_+(\zeta - 2/3) = -\frac{1}{\sqrt{\pi}} \int_{2/3}^{\zeta} (\zeta - s)^{-1/2} \partial_s \left[ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}s^2\right) \right] ds \quad \zeta \in (2/3, +\infty)$$

Using  $a = \frac{5}{3}$ ,  $b = \frac{7}{3}$ ,  $c = \frac{5}{2}$ :

$$\begin{aligned} w_1 &= \frac{\Gamma(\frac{5}{2})\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{6})} w_5 + \frac{\Gamma(\frac{5}{2})\Gamma(-\frac{2}{3})}{\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})} w_6 \\ w_3 &= \frac{1}{e^{(5/3)\pi i} - e^{(7/3)\pi i}} \left[ e^{(-1/6)\pi i} \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{6})}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})} w_5 - e^{(-5/6)\pi i} \frac{\Gamma(\frac{5}{2})\Gamma(\frac{5}{6})}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})} w_6 \right] \end{aligned}$$

Common denominators

$$\begin{aligned} w_1 &= \frac{\Gamma(\frac{5}{2})\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{6})\Gamma(\frac{1}{3})} w_5 + \frac{\Gamma(\frac{5}{2})\Gamma(-\frac{2}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})\Gamma(\frac{5}{3})} w_6 \\ w_3 &= \frac{i}{\sqrt{3}} \left[ e^{(-1/6)\pi i} \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})} w_5 - e^{(-5/6)\pi i} \frac{\Gamma(\frac{5}{2})\Gamma(\frac{5}{6})\Gamma(\frac{1}{6})}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})} w_6 \right] \end{aligned}$$



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Extract

$$w_1 = \frac{\Gamma(\frac{5}{2})\frac{2}{\sqrt{3}}\pi}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{6})\Gamma(\frac{1}{3})}w_5 - \frac{\Gamma(\frac{5}{2})\frac{2}{\sqrt{3}}\pi}{\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})\Gamma(\frac{5}{3})}w_6$$

$$w_3 = \frac{2\pi i}{\sqrt{3}}\Gamma(\frac{5}{2})\left[e^{(-1/6)\pi i}\frac{1}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}w_5 - e^{(-5/6)\pi i}\frac{1}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}w_6\right]$$

...

$$w_1 = \frac{2\pi}{\sqrt{3}}\Gamma(\frac{5}{2})\left[\frac{1}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{6})\Gamma(\frac{1}{3})}w_5 - \frac{1}{\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})\Gamma(\frac{5}{3})}w_6\right]$$

$$w_3 = \frac{2\pi}{\sqrt{3}}\Gamma(\frac{5}{2})\left[e^{(1/3)\pi i}\frac{1}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}w_5 - e^{-(1/3)\pi i}\frac{1}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}w_6\right]$$

Now subtract!

$$w_1 - w_3 = \frac{2\pi}{\sqrt{3}}\Gamma(\frac{5}{2})\left[e^{-(1/3)\pi i}\frac{1}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}w_5 + e^{-(2/3)\pi i}\frac{1}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})}w_6\right]$$

Let us study the RHS of claim (3.5). **Note that**  $x = \frac{1}{2} - \frac{3}{4}\zeta$  **and**  $y = \frac{1}{2} - \frac{3}{4}s$ .

$$\begin{aligned} & \int_{2/3}^{\zeta} (\zeta - s)^{-1/2} \partial_s \left[ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}s^2\right) \right] ds \\ &= 2 \int_{2/3}^{\zeta} (\zeta - s)^{-1/2} s {}_2F_1\left(\frac{4}{3}, \frac{5}{3}; \frac{3}{2}; \frac{9}{4}s^2\right) ds \\ &= -\frac{2}{9} \int_{2/3}^{\zeta} (\zeta - s)^{-1/2} \left[ {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; \frac{5}{2}; \frac{1}{2} - \frac{3s}{4}\right) - {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; \frac{5}{2}; \frac{1}{2} + \frac{3s}{4}\right) \right] ds && 15.8.28 \text{ DLMF} \\ &= \frac{4}{9\sqrt{3}} \int_0^x (y - x)^{-1/2} \left[ {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; \frac{5}{2}; y\right) - {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; \frac{5}{2}; 1 - y\right) \right] dy && x \in (-\infty, 0) \\ &= \frac{4}{9\sqrt{3}} \frac{2\pi}{\sqrt{3}} \Gamma(\frac{5}{2}) \int_0^x (y - x)^{-1/2} \left[ e^{-(1/3)\pi i} \frac{e^{(5/3)\pi i} y^{-5/3} {}_2F_1\left(\frac{5}{3}, \frac{1}{6}, \frac{1}{3}; \frac{1}{y}\right)}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})} \right. \\ &\quad \left. + e^{-(2/3)\pi i} \frac{e^{(7/3)\pi i} y^{-7/3} {}_2F_1\left(\frac{7}{3}, \frac{5}{6}, \frac{5}{3}; \frac{1}{y}\right)}{\Gamma(\frac{5}{3})\Gamma(\frac{5}{3})\Gamma(\frac{1}{6})} \right] dy && x \in (-\infty, 0) \end{aligned}$$

**Continue  $w_5$  and  $w_6$  counterclockwise around their singularities at zero.** Notice that  $|y - x| = y - x$  on the integration path.

$$\begin{aligned}
&= \frac{8\pi}{27} \Gamma\left(\frac{5}{2}\right) \int_0^x |y - x|^{-1/2} \left[ e^{-(1/3)\pi i} \frac{e^{(10/3)\pi i} |y|^{-5/3} {}_2F_1\left(\frac{5}{3}, \frac{1}{6}; \frac{1}{3}; \frac{1}{y}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)} \right. \\
&\quad \left. + e^{-(2/3)\pi i} \frac{e^{(14/3)\pi i} |y|^{-7/3} {}_2F_1\left(\frac{7}{3}, \frac{5}{6}; \frac{5}{3}; \frac{1}{y}\right)}{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{1}{6}\right)} \right] dy \quad x \in (-\infty, 0) \\
&= \frac{8\pi}{27} \Gamma\left(\frac{5}{2}\right) \int_0^x |y - x|^{-1/2} \left[ -\frac{|y|^{-5/3} {}_2F_1\left(\frac{5}{3}, \frac{1}{6}; \frac{1}{3}; \frac{1}{y}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)} \right. \\
&\quad \left. + \frac{|y|^{-7/3} {}_2F_1\left(\frac{7}{3}, \frac{5}{6}; \frac{5}{3}; \frac{1}{y}\right)}{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{1}{6}\right)} \right] dy \quad x \in (-\infty, 0) \\
&= \frac{8\pi}{27} \Gamma\left(\frac{5}{2}\right) \left[ -\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{5}{3}\right)} \frac{|x|^{-7/6} {}_2F_1\left(\frac{7}{6}, \frac{1}{6}; \frac{1}{3}; \frac{1}{x}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)} \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{7}{3}\right)} \frac{|x|^{-11/6} {}_2F_1\left(\frac{11}{6}, \frac{5}{6}; \frac{5}{3}; \frac{1}{x}\right)}{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{1}{6}\right)} \right] \tag{4.3} \\
&= \frac{8\pi}{27} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{5}{3}\right)} \left[ -\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)} e^{(7/6)\pi i} x^{-7/6} {}_2F_1\left(\frac{7}{6}, \frac{1}{6}; \frac{1}{3}; \frac{1}{x}\right) \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{1}{6}\right)} e^{(11/6)\pi i} x^{-11/6} {}_2F_1\left(\frac{11}{6}, \frac{5}{6}; \frac{5}{3}; \frac{1}{x}\right) \right] \\
&= \frac{8\pi}{27} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{5}{3}\right)} \left[ -\frac{\Gamma(2)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{5}{6}\right)} \frac{\Gamma\left(\frac{11}{6}\right)}{\Gamma(2)\Gamma\left(\frac{2}{3}\right)} \frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{1}{3}\right)} w_{5\left(\frac{7}{6}, \frac{11}{6}, 2\right)} \right. \\
&\quad \left. + \frac{\Gamma(2)\Gamma\left(-\frac{2}{3}\right)}{\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{1}{6}\right)} \frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma(2)\Gamma\left(-\frac{2}{3}\right)} \frac{\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{5}{3}\right)} w_{6\left(\frac{7}{6}, \frac{11}{6}, 2\right)} \right] \\
&= \frac{8\pi}{27} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{5}{3}\right)} \left[ -\frac{\Gamma(2)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{5}{6}\right)} \frac{1}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)} w_{5\left(\frac{7}{6}, \frac{11}{6}, 2\right)} \right. \\
&\quad \left. + \frac{\Gamma(2)\Gamma\left(-\frac{2}{3}\right)}{\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{1}{6}\right)} \frac{1}{\Gamma\left(-\frac{2}{3}\right)\Gamma\left(\frac{5}{3}\right)} w_{6\left(\frac{7}{6}, \frac{11}{6}, 2\right)} \right] \\
&= \frac{8\pi}{27} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{5}{3}\right)} \left[ -\frac{\Gamma(2)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{5}{6}\right)} \frac{\sin\left(\frac{2}{3}\pi\right)}{\pi} w_{5\left(\frac{7}{6}, \frac{11}{6}, 2\right)} \right. \\
&\quad \left. + \frac{\Gamma(2)\Gamma\left(-\frac{2}{3}\right)}{\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{1}{6}\right)} \frac{\sin\left(-\frac{2}{3}\pi\right)}{\pi} w_{6\left(\frac{7}{6}, \frac{11}{6}, 2\right)} \right] \\
&= \frac{8\pi}{27} \frac{\sqrt{3}}{2\pi} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{5}{3}\right)} \left[ -\frac{\Gamma(2)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{5}{6}\right)} w_{5\left(\frac{7}{6}, \frac{11}{6}, 2\right)} - \frac{\Gamma(2)\Gamma\left(-\frac{2}{3}\right)}{\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{1}{6}\right)} w_{6\left(\frac{7}{6}, \frac{11}{6}, 2\right)} \right]
\end{aligned}$$

Using DLMF 15.10.25, we see this is

$$\begin{aligned}
&= -\frac{8}{27} \frac{\sqrt{3}}{2} \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})\Gamma(\frac{7}{6})\Gamma(\frac{11}{6})}{\Gamma(\frac{7}{3})\Gamma(\frac{5}{3})} w_1(\frac{7}{6}, \frac{11}{6}, 2) \\
&= -\frac{8}{27} \frac{\sqrt{3}}{2} \frac{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\frac{1}{6}\Gamma(\frac{1}{6})\frac{5}{6}\Gamma(\frac{5}{6})}{\frac{4}{3}\frac{1}{3}\Gamma(\frac{1}{3})\frac{2}{3}\Gamma(\frac{2}{3})} w_1(\frac{7}{6}, \frac{11}{6}, 2) \\
&= -\frac{5\sqrt{3}}{6 \cdot 2^4} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} w_1(\frac{7}{6}, \frac{11}{6}, 2) \\
&= -\frac{5\sqrt{3}}{6 \cdot 2^4} \frac{\pi \sin(\frac{1}{3}\pi)}{\sin(\frac{1}{6}\pi)} w_1(\frac{7}{6}, \frac{11}{6}, 2) \\
&= -\frac{5\pi}{32} w_1(\frac{7}{6}, \frac{11}{6}, 2) \\
&= -\frac{5\pi}{32} {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; x\right) \\
&= -\frac{5\pi}{32} {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; \frac{1}{2} - \frac{3\zeta}{4}\right) \\
&= -\frac{5\pi}{32} \hat{w}_+(\zeta - 2/3)
\end{aligned}$$

This matches  $-\sqrt{\pi}$  times the LHS of claim (3.5) (**up to scaling**).

Alternate calculation with  $\frac{1}{2}$  derivative:

$$\begin{aligned}
&\int_{2/3}^{\zeta} (\zeta - s)^{-1/2} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{9}{4}s^2\right) ds \\
&= \frac{2}{3} \int_{2/3}^{\zeta} (\zeta - s)^{-1/2} \left[ {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; \frac{1}{2} - \frac{3s}{4}\right) + {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; \frac{1}{2} + \frac{3s}{4}\right) \right] ds \quad 15.8.27 \text{ DLMF} \\
&= \frac{4}{3\sqrt{3}} \int_0^x (x - y)^{-1/2} \left[ {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; 1 - y\right) + {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; y\right) \right] dy \quad x \in (-\infty, 0) \\
&= \frac{2i}{3\sqrt{3}} \int_0^x (x - y)^{-1/2} \left[ \left(1 + \frac{1}{\sqrt{3}}\right) y^{-2/3} {}_2F_1\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{3}; \frac{1}{y}\right) + \left(1 - \frac{1}{\sqrt{3}}\right) y^{-4/3} {}_2F_1\left(\frac{4}{3}, \frac{5}{6}, \frac{5}{3}; \frac{1}{y}\right) \right] dy \\
&= \frac{2i}{3\sqrt{3}} \left[ \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})} \left(1 + \frac{1}{\sqrt{3}}\right) x^{-1/6} {}_2F_1\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}; \frac{1}{x}\right) + \frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{6})}{\Gamma(\frac{4}{3})} \left(1 - \frac{1}{\sqrt{3}}\right) x^{-5/6} {}_2F_1\left(\frac{5}{6}, \frac{5}{6}, \frac{5}{3}; \frac{1}{x}\right) \right] ds \quad (4.3) \\
&= ??? \quad \zeta \in (2/3, +\infty)
\end{aligned}$$

Analogously, it can be verified for  $\hat{w}_-(\zeta + 2/3)$  for  $\zeta \in (-\infty, -2/3)$ .

**Example 3.6** (Bessel). Let  $X = \mathbb{C}^*$ ,  $f(x) = x + \frac{1}{x}$  and  $\nu = \frac{dx}{x}$ , then the critical points of  $f$  are  $x = \pm 1$  and

$$(3.16) \quad I(z) := \int_0^\infty e^{-zf(x)} \frac{dx}{x}.$$

By change of coordinates  $t = zx$

$$I(z) = \int_0^\infty e^{-z(\frac{t}{z} + \frac{z}{t})} \frac{dt}{t} = \int_0^\infty e^{-\left(t + \frac{z^2}{t}\right)} \frac{dt}{t} = 2K_0(2z) \quad |\arg z| < \frac{\pi}{4}$$

where  $K_0(z)$  is the modified Bessel function (see definition 10.32.10 DLMF). In particular, since  $K_0(z)$  solves

$$(3.17) \quad \frac{d^2}{dz^2} w(z) + \frac{1}{z} \frac{d}{dz} w(z) - w(z) = 0$$

and  $K_0(z) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-z} z^{-1/2} \sum_{k \geq 0} \frac{(1/2)_k (1/2)_k}{(-2)^k k!} z^{-k}$  as  $z \rightarrow \infty$  (see DLMF 10.40.2), then  $I(z)$  is a solution of

$$(3.18) \quad \frac{d^2}{dz^2} I(z) + \frac{1}{z} \frac{d}{dz} I(z) - 4I(z) = 0.$$

The formal integral of (3.18) is given by a two parameter formal solution  $\tilde{I}_1(z)$

$$(3.19) \quad \tilde{I}(z) = \sum_{\mathbf{k} \in \mathbb{N}^2} U^{\mathbf{k}} e^{-\mathbf{k} \cdot \lambda z} z^{-\tau \cdot \mathbf{k}} \tilde{w}_{\mathbf{k}}(z)$$

where  $\lambda = (2, -2)$ ,  $\tau = (-\frac{1}{2}, -\frac{1}{2})$ ,  $U^{\mathbf{k}} := U_1^{k_1} U_2^{k_2}$  with  $\mathbf{k} = (k_1, k_2)$  and  $U_1, U_2 \in \mathbb{C}$ , and  $\tilde{w}_{\mathbf{k}}(z) \in \mathbb{C}[[z^{-1}]]$  is a formal solution of

$$(3.20) \quad \begin{aligned} \tilde{w}_{\mathbf{k}}''(z) - 4(k_1 - k_2) \tilde{w}_{\mathbf{k}}'(z) + 4(1 - (k_1 - k_2)^2) \tilde{w}_{\mathbf{k}}(z) + \frac{(k_1 + k_2 - 1)}{z} \tilde{w}_{\mathbf{k}}'(z) + \\ - 2(k_1 - k_2) \frac{(k_1 + k_2 - 1)}{z} \tilde{w}_{\mathbf{k}}(z) + \frac{(k_1 + k_2)^2}{4z^2} \tilde{w}_{\mathbf{k}}(z) = 0 \end{aligned}$$

The only non zero  $\tilde{w}_{\mathbf{k}}(z)$  occurs for  $\mathbf{k} = (1, 0)$  and  $\mathbf{k} = (0, 1)$ , hence

$$(3.21) \quad \tilde{I}(z) = U_1 e^{-2z} z^{-1/2} \tilde{w}_{(1,0)}(z) + U_2 e^{2z} z^{-1/2} \tilde{w}_{(0,1)}(z)$$

and according to our convention, we define

$$(3.22) \quad \tilde{I}_1(z) := e^{-2z} z^{-1/2} \tilde{w}_{(1,0)}(z)$$

$$(3.23) \quad \tilde{I}_{-1}(z) := e^{2z} z^{-1/2} \tilde{w}_{(0,1)}(z).$$

We set  $\tilde{w}_{(1,0)} = \tilde{w}_+$  and  $\tilde{w}_{(0,1)} = \tilde{w}_-$ , then their Borel transforms are solutions respectively of the following equations

$$\begin{aligned} (+) \quad \zeta^2 \hat{w}_+(\zeta) + 4\zeta \hat{w}_+(\zeta) + \frac{1}{4} \zeta * \hat{w}_+(\zeta) &= 0 \\ (-) \quad \zeta^2 \hat{w}_-(\zeta) - 4\zeta \hat{w}_-(\zeta) + \frac{1}{4} \zeta * \hat{w}_-(\zeta) &= 0 \end{aligned}$$

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taking twice derivative in  $\zeta$  we get

$$\begin{aligned}
(+)\quad & (\zeta^2 + 4\zeta) \frac{d^2}{d\zeta^2} \hat{w}_+ + 4(\zeta - 1) \frac{d}{d\zeta} \hat{w}_+ + \frac{9}{4} \hat{w}_+ = 0 \\
(-)\quad & (\zeta^2 - 4\zeta) \frac{d^2}{d\zeta^2} \hat{w}_- + 4(\zeta + 1) \frac{d}{d\zeta} \hat{w}_- + \frac{9}{4} \hat{w}_- = 0 \\
(+)\quad & \xi(1 - \xi) \frac{d^2}{d\xi^2} \hat{w}_+ + (1 - 4\xi) \frac{d}{d\xi} \hat{w}_+ - \frac{9}{4} \hat{w}_+ = 0 \quad \xi = -\frac{\zeta}{4} \\
(-)\quad & \xi(1 - \xi) \frac{d^2}{d\xi^2} \hat{w}_- + (1 - 4\xi) \frac{d}{d\xi} \hat{w}_- - \frac{9}{4} \hat{w}_- = 0 \quad \xi = \frac{\zeta}{4}
\end{aligned}$$

therefore, since equation (+), (-) are hypergeometric the fundamental solution is (see DLMF 15.10.2)

$$(3.24) \quad \hat{w}_+(\zeta) = {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; -\frac{\zeta}{4}\right)$$

$$(3.25) \quad \hat{w}_-(\zeta) = {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{\zeta}{4}\right)$$

In particular, we notice that taking the series expansion of  $\hat{w}_+$  and  $\hat{w}_-$  we get numerically that

$$\begin{aligned}
\hat{w}_+(\zeta - 4) &= \frac{1}{\pi} \log(z) \hat{w}_-(z) + \phi_{\text{reg}} \\
\hat{w}_-(\zeta + 4) &= \frac{1}{\pi} \log(z) \hat{w}_+(z) + \psi_{\text{reg}}
\end{aligned}$$

and analytically (thanks to 15.2.3 DLMF)

$$\begin{aligned}
\hat{w}_+(\zeta + i0) - \hat{w}_+(\zeta - i0) &= {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; -\frac{\zeta}{4} + i0\right) - {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; -\frac{\zeta}{4} - i0\right) \quad \zeta < -4 \\
&= -8i \left(-\frac{\zeta}{4} - 1\right)^{-1} \sum_{n \geq 0} \frac{(1/2)_n (1/2)_n}{n! \Gamma(n)} \left(\frac{\zeta}{4} + 1\right)^n \\
&= 8i \sum_{n \geq 0} \frac{(1/2)_n (1/2)_n}{n! \Gamma(n)} \left(\frac{\zeta}{4} + 1\right)^{n-1} \\
&= 8i \sum_{n \geq 0} \frac{(1/2)_{n+1} (1/2)_{n+1}}{(n+1)! \Gamma(n+1)} \left(\frac{\zeta}{4} + 1\right)^n \\
&= 2i \sum_{n \geq 0} \frac{(3/2)_n (3/2)_n}{(n)! \Gamma(n+2)} \left(\frac{\zeta}{4} + 1\right)^n \\
&= 2i {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{\zeta}{4} + 1\right)
\end{aligned}$$

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$$\begin{aligned}
\hat{w}_-(\zeta + i0) - \hat{w}_-(\zeta - i0) &= {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{\zeta}{4} + i0\right) - {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{\zeta}{4} - i0\right) \quad \zeta > 4 \\
&= 8i \left(\frac{\zeta}{4} - 1\right)^{-1} \sum_{n \geq 0} \frac{(1/2)_n (1/2)_n}{n! \Gamma(n)} \left(\frac{\zeta}{4} - 1\right)^n \\
&= -8i \sum_{n \geq 0} \frac{(1/2)_n (1/2)_n}{n! \Gamma(n)} \left(1 - \frac{\zeta}{4}\right)^{n-1} \\
&= -8i \sum_{n \geq 0} (-1)^n \frac{(1/2)_{n+1} (1/2)_{n+1}}{(n+1)! \Gamma(n+1)} \left(1 - \frac{\zeta}{4}\right)^n \\
&= -2i \sum_{n \geq 0} (-1)^n \frac{(3/2)_n (3/2)_n}{(n)! \Gamma(n+2)} \left(1 - \frac{\zeta}{4}\right)^n \\
&= -2i {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; 1 - \frac{\zeta}{4}\right)
\end{aligned}$$

These are evidence of the resurgent properties of  $\tilde{I}_{\pm 1}(z)$ .

**Lemma 3.7.** *The following identity holds true*

$$(3.26) \quad {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{\zeta^2}{4}\right) = 2i \frac{u}{u^2 - 1} \quad \zeta = u + \frac{1}{u}$$

*Proof.* From 15.4.13 DLME, we have

$$\begin{aligned}
{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{\zeta^2}{4}\right) &= \frac{2}{\sqrt{4 - \zeta^2}} & y = \operatorname{arccsc}(\zeta/2) \\
&= \frac{1}{\sqrt{1 - \csc^2(y)}} \\
&= -i \tan(y) & \zeta = \frac{2}{\sin(y)}
\end{aligned}$$

therefore if  $u = \tan\left(\frac{y}{2}\right)$ , we have  $\zeta = \frac{1+u^2}{u} = f(u)$  and

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{\zeta^2}{4}\right) = 2i \frac{u}{u^2 - 1} = \frac{2i}{f'(u)u}$$

□

**Claim 3.8.**

$$(3.27) \quad \hat{w}_+(\zeta - 2) = i\pi \int_2^\zeta (\zeta - \zeta')^{-1/2} 2\zeta' {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; \frac{3}{2}; \frac{\zeta'^2}{4}\right) d\zeta' \quad \zeta \in (2, +\infty)$$

*Proof.* Let us first consider the RHS of (3.8)

$$\begin{aligned}
2\pi \int_2^\zeta (\zeta - \zeta')^{-1/2} \zeta' {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; \frac{3}{2}; \frac{\zeta^2}{4}\right) d\zeta' &= \\
&= \frac{4}{3} \int_2^\zeta (\zeta - \zeta')^{-1/2} \left[ {}_2F_1\left(2, 2; \frac{5}{2}; \frac{1}{2} + \frac{\zeta'}{4}\right) - {}_2F_1\left(2, 2; \frac{5}{2}; \frac{1}{2} - \frac{\zeta'}{4}\right) \right] d\zeta' \\
&= \frac{8}{3} \int_0^x (y - x)^{-1/2} \left[ {}_2F_1\left(2, 2; \frac{5}{2}; y\right) - {}_2F_1\left(2, 2; \frac{5}{2}; 1 - y\right) \right] dy \quad x \in (-\infty, 0) \\
&= 2\pi \int_0^x (x - y)^{-1/2} y^{-2} F\left(2, \frac{1}{2}; 1; \frac{1}{y}\right) dy \quad (4.3) \\
&= \pi^2 |x|^{-3/2} {}_2F_1\left(\frac{3}{2}, \frac{1}{2}; 1; \frac{1}{x}\right) \quad x \in (-\infty, 0) \\
&= \frac{\pi^2}{2} \left( {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{1}{2} - \frac{\zeta}{4}\right) - i {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{1}{2} + \frac{\zeta}{4}\right) \right) \quad \zeta \in (2, +\infty)
\end{aligned}$$

however  ${}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{1}{2} + \frac{\zeta}{4}\right)$  has a branch cut at  $\zeta \in (2, +\infty)$ , thus the claim holds true.  $\square$

Analogously, it can be verified for  $\hat{w}_-(\zeta + 2)$  for  $\zeta \in (-\infty, -2)$ .

#### 4. USEFUL IDENTITIES FOR GAUSS HYPERGEOMETRIC FUNCTIONS

$$\begin{aligned}
(4.1) \quad {}_2F_1(a, b; c; z) &= e^{b\pi i} \frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(a+b-c+1)\Gamma(c-b)} {}_2F_1(a, b; c; 1-z) + \\
&\quad - e^{(a+b-c)\pi i} \frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(b)\Gamma(a-b+1)} |z|^{-a} {}_2F_1\left(a, a-c+1; a-b+1; \frac{1}{z}\right)
\end{aligned}$$

$$\begin{aligned}
(4.2) \quad \int_0^x |y|^{a-\mu-1} {}_2F_1(a, b; c; y) (x-y)^{\mu-1} dy &= \frac{\Gamma(\mu)\Gamma(a-\mu)}{\Gamma(a)} |x|^{a-1} {}_2F_1(a-\mu, b; c; x) \\
&\quad x \in (-\infty, 0) \cup (0, 1), \Re a > \Re \mu > 0
\end{aligned}$$

which can be rewritten as (arXiv:1504.08144, **formula 4.8**)

$$\begin{aligned}
(4.3) \quad \int_{y>x} |y|^{-a} |x-y|^{\mu-1} {}_2F_1(a, b; c; y^{-1}) dy &= \frac{\Gamma(\mu)\Gamma(a-\mu)}{\Gamma(a)} |x|^{-a+\mu} {}_2F_1(a-\mu, b; c; x^{-1}) \\
&\quad x \in (-\infty, 0) \cup (1, \infty), \Re a > \Re \mu > 0
\end{aligned}$$

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## 5. RESURGENCE FOR DEGREE 3 POLYNOMIALS

Let  $f$  be a degree 3 polynomial, and  $t_1, t_2$  its critical points (not necessarily distinguished):

(1) if  $t_1 \neq t_2$ , then

$$I(z) = \int_{C_j} e^{-zf} dt$$

is a solution of

$$(5.1) \quad I'' + aI' + bI + c\frac{I'}{z} + \frac{d}{z}I + \frac{e}{z^2}I = 0$$

where  $a, b, c, d, e$  are determined in terms of  $f$ .

(2) if  $t_1 = t_2$ , then

$$I(z) = \int_{C_1} e^{-zf} dt$$

is a solution of a first order ODE

$$(5.2) \quad I' + \left( a_4 - \frac{a_2^3}{27a_1^2} + \frac{1}{3z} \right) I = 0$$

*Proof.* Let  $f(t) = a_1 t^3 + a_2 t^2 + a_3 t + a_4$  with  $a_1 \neq 0$ ,

$$\int_{C_j} e^{-fz} dt = \int_{C_j + \frac{a_2}{3a_1}} e^{-(a_1 t^3 + p t + q)z} dt \quad t \rightarrow t - \frac{a_2}{3a_1}$$

where  $p = a_3 - \frac{a_2^2}{3a_1}$  and  $q = a_4 - \frac{a_2 a_3}{3a_1} + \frac{2a_2^3}{27a_1^2}$ .

Case (1): if  $p \neq 0$ ,

$$\begin{aligned} I(z) &= \int t(3a_1 t^2 + p)z e^{-fz} = \int (3a_1 t^3 + p t)z e^{-fz} = \\ &= \int 2a_1 t^3 z e^{-fz} + \int (a_1 t^3 + p t + q)z e^{-fz} - q z I \\ &= 2z \int a_1 t^3 e^{-fz} - z I' - q z I \end{aligned}$$



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$$\begin{aligned}
2z \int a_1 t^3 e^{-fz} &= 2z^2 \int \frac{t^4}{4} a_1 (3a_1 t^2 + p) e^{-fz} = \frac{z^2}{2} \int (3a_1^2 t^6 + p a_1 t^4) e^{-fz} = \\
&\frac{z^2}{2} \int (3a_1^2 t^6 + 6p a_1 t^4 + 3q^2 + 3p^2 t^2 + 6p q t + 6a_1 q t^3) e^{-fz} + \\
&+ \frac{z^2}{2} \int (p a_1 t^4 - 6p a_1 t^4) e^{-fz} - \frac{z^2}{2} \int (3q^2 + 3p^2 t^2 + 6p q t + 6a_1 q t^3) e^{-fz} \\
&= \frac{3z^2}{2} I'' + \frac{3z^2}{2} q^2 I + 3q z^2 I' - \frac{z^2}{2} p \int (3a_1 t^4 + p t^2) e^{-fz} - z^2 p \int (a_1 t^4 + p t^2) e^{-fz} \\
&= \frac{3z^2}{2} I'' + \frac{3z^2}{2} q^2 I + 3q z^2 I' - \frac{5}{3} z p \int t e^{-fz} - \frac{2}{3} z^2 p^2 \int t^2 e^{-fz}
\end{aligned}$$

hence

(5.3)

$$I = -zI' - qzI + \frac{3z^2}{2} I'' + \frac{3z^2}{2} q^2 I + 3q z^2 I' - \frac{5}{3} z p \int t e^{-fz} - \frac{2}{3} z^2 p^2 \int t^2 e^{-fz}$$

(5.4)

$$\frac{3z^2}{2} \left( I'' + q^2 I + 2q I' - \frac{2}{3z} I' - \frac{2q}{3z} I - \frac{2}{3z^2} I - \frac{10}{9z} p \int t e^{-fz} dt - \frac{4}{9} p^2 \int t^2 e^{-fz} \right) = 0$$

Notice that

$$\begin{aligned}
\frac{4}{9} p^2 \int t^2 e^{-fz} &= \frac{4}{27a_1} p^2 \int (3a_1 t^2 + p) e^{-fz} - \frac{4}{27a_1} p^3 I = -\frac{4}{27a_1} p^3 I \\
-\frac{10}{9z} p \int t e^{-fz} dt &= \frac{5}{9z} \int p t e^{-fz} - \frac{5}{3z} \int (p t + q) e^{-fz} + \frac{5}{3z} q I = \\
\frac{5}{9z} \int p t e^{-fz} - \frac{5}{3z} \int (p t + q + a_1 t^3) e^{-fz} &+ \frac{5}{3z} \int a_1 t^3 e^{-fz} + \frac{5}{3z} q I = \\
&\frac{5}{9z} \int t (3a_1 t^2 + p) e^{-fz} + \frac{5}{3z} I' + \frac{5}{3z} q I \\
&= \frac{5}{9z^2} I + \frac{5}{3z} I' + \frac{5}{3z} q I
\end{aligned}$$

therefore, collecting all the contributions together we find

$$\begin{aligned}
I'' + q^2 I + 2q I' - \frac{2}{3z} I' - \frac{2q}{3z} I - \frac{2}{3z^2} I + \frac{5}{9z^2} I + \frac{5}{3z} I' + \frac{5}{3z} q I + \frac{4}{27a_1} p^3 I &= 0 \\
I'' + 2q I' + \left( \frac{4p^3}{27a_1} + q^2 \right) I + \frac{1}{z} I' + \frac{q}{z} I - \frac{1}{9z^2} I &= 0
\end{aligned}$$

Case (2): if  $p = 0$ , then integrating by part we have

$$\begin{aligned}
I(z) &= \int_{\mathcal{C}_1 + \frac{a_2}{3a_1}} e^{-(a_1 t^3 + q)z} dt \\
&= \left[ t e^{-(a_1 t^3 + q)z} \right]_{\mathcal{C}_1 + \frac{a_2}{3a_1}} + \int_{\mathcal{C}_1 + \frac{a_2}{3a_1}} 3a_1 t^3 z e^{-(a_1 t^3 + q)z} dt \\
&= 3z \int_{\mathcal{C}_1 + \frac{a_2}{3a_1}} (a_1 t^3 + q) e^{-(a_1 t^3 + q)z} dt - 3qz \int_{\mathcal{C}_1 + \frac{a_2}{3a_1}} e^{-(a_1 t^3 + q)z} dt \\
&= -3z I'(z) - 3qz I(z)
\end{aligned}$$

□

We would like to verify that for every cubic function  $f$ , the Borel transform of the exponential integral can be expressed by an hypergeometric function and hence deduce its resurgent properties in full generality. If  $p \neq 0$ ,  $I(z)$  is a solution of

$$(5.5) \quad I'' + 2qI' + \left( \frac{4p^3}{27a_1} + q^2 \right) I + \frac{1}{z} I' + \frac{q}{z} I - \frac{1}{9z^2} I = 0$$

hence a formal solution as  $z \rightarrow \infty$  is given (up to constants  $U_1, U_2 \in \mathbb{C}$ ) by

$$(5.6) \quad \tilde{I}_+(z) := U_1 e^{-(q + \sqrt{\frac{4p^3}{27a_1}})z} z^{1/2} \tilde{w}_+(z)$$

$$(5.7) \quad \tilde{I}_-(z) := U_2 e^{-(q - \sqrt{\frac{4p^3}{27a_1}})z} z^{1/2} \tilde{w}_-(z)$$

where  $\tilde{w}_{\pm}(z) \in \mathbb{C}[[z^{-1}]]$  is the formal solution of

$$(5.8) \quad \tilde{w}_{\pm}'' \mp 2\sqrt{\frac{4p^3}{27a_1}} \tilde{w}_{\pm}' + \frac{5}{36} \frac{\tilde{w}_{\pm}}{z^2} = 0$$

with  $\tilde{w}_{\pm}(z) = 1 + \sum_{k \geq 1} a_{\pm, k} z^{-k}$ .

We can now compute the Borel transform of (5.8): for  $\tilde{w}_+(z)$

$$\begin{aligned}
&\zeta^2 \hat{w} - 2\sqrt{\frac{4p^3}{27a_1}} \zeta \hat{w} + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}(\zeta') d\zeta' = 0 \\
&\zeta^2 \hat{w}' + 2\zeta \hat{w} + 2\sqrt{\frac{4p^3}{27a_1}} \hat{w} + 2\sqrt{\frac{4p^3}{27a_1}} \zeta \hat{w}' + \frac{5}{36} \int_0^\zeta \hat{w}(\zeta') = 0 \\
&\left( \zeta^2 + 2\sqrt{\frac{4p^3}{27a_1}} \zeta \right) \hat{w}'' + 4 \left( \zeta + \sqrt{\frac{4p^3}{27a_1}} \right) \hat{w}' + \frac{77}{36} \hat{w} = 0 \\
&t(1-t) \hat{w}'' + (2-4t) \hat{w}' - \frac{77}{36} \hat{w} = 0 \quad \zeta = -2t \sqrt{\frac{4p^3}{27a_1}}
\end{aligned}$$

hence

$$\hat{w}_+(\zeta) = {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; -\frac{3}{4p}\sqrt{\frac{3a_1}{p}}\zeta\right)$$

and analogously,

$$\hat{w}_-(\zeta) = {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; \frac{3}{4p}\sqrt{\frac{3a_1}{p}}\zeta\right).$$

Notice that  $\hat{w}_\pm(\zeta)$  has a branch cut singularity respectively at  $\zeta = \zeta_\pm := \pm\sqrt{\frac{16p^3}{27a_1}}$ , and thanks to the well known formulas for the analytic continuation of hypergeometric functions (see 15.2.3 DLMF), if we assume the branch cut is from  $\zeta_\pm$  to  $+\infty$

$$\begin{aligned}\hat{w}_-(\zeta + i0) - \hat{w}_-(\zeta - i0) &= \frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)} \left(\frac{\zeta}{\zeta_+} - 1\right)^{-1} \sum_{k \geq 0} \frac{(5/6)_k (1/6)_k}{\Gamma(k)k!} \left(1 - \frac{\zeta}{\zeta_+}\right)^k \quad \zeta \in (\zeta_+, +\infty) \\ &= -\frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)} \sum_{k \geq 0} \frac{(5/6)_k (1/6)_k}{\Gamma(k)k!} \left(1 - \frac{\zeta}{\zeta_+}\right)^{k-1} \\ &= -\mathbf{i} \frac{1}{\Gamma(7/6)\Gamma(11/6)} \sum_{k \geq 0} \frac{\Gamma(\frac{7}{6} + k) \Gamma(\frac{11}{6} + k)}{\Gamma(2 + k) \Gamma(k + 1)} \left(1 - \frac{\zeta}{\zeta_+}\right)^k \\ &= -\mathbf{i} {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; 1 - \frac{\zeta}{\zeta_+}\right) \\ &= -\mathbf{i} \hat{w}_+(\zeta - \zeta_+)\end{aligned}$$

Similarly, if we assume that the branch cut is from  $\zeta_\pm$  to  $-\infty$  then

$$\begin{aligned}\hat{w}_+(\zeta + i0) - \hat{w}_+(\zeta - i0) &= -\frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)} \left(\frac{\zeta}{\zeta_-} - 1\right)^{-1} \sum_{k \geq 0} \frac{(5/6)_k (1/6)_k}{\Gamma(k)k!} \left(1 - \frac{\zeta}{\zeta_-}\right)^k \quad \zeta \in (-\infty, \zeta_-) \\ &= \frac{2\pi i}{\Gamma(7/6)\Gamma(11/6)} \sum_{k \geq 0} \frac{(5/6)_k (1/6)_k}{\Gamma(k)k!} \left(1 - \frac{\zeta}{\zeta_-}\right)^{k-1} \\ &= \mathbf{i} \frac{1}{\Gamma(7/6)\Gamma(11/6)} \sum_{k \geq 0} \frac{\Gamma(\frac{7}{6} + k) \Gamma(\frac{11}{6} + k)}{\Gamma(2 + k) \Gamma(k + 1)} \left(1 - \frac{\zeta}{\zeta_-}\right)^k \\ &= \mathbf{i} {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; 1 - \frac{\zeta}{\zeta_-}\right) \\ &= \mathbf{i} \hat{w}_-(\zeta - \zeta_-)\end{aligned}$$

therefore we see that the Stokes factors are given by  $\pm i$  (as for Airy).

**I think it will be nice to add the geometric interpretation of Maxim in term of Lefschetz thimbles**

The situation is quite different if we consider the degenerate case, where we have only a singular point: indeed there is a one parameter family of solutions of

$$(5.9) \quad I'(z) + \left( \frac{1}{3z} + q \right) I(z) = 0$$

namely for  $U \in \mathbb{C}$

$$(5.10) \quad \tilde{I}(z) = U e^{-qz} z^{1/3} \tilde{w}(z) \quad \text{where } \tilde{w}(z) \in \mathbb{C}[[z^{-1}]]$$

The Borel transform of  $\tilde{w}$  is a solution of

$$(5.11) \quad \zeta \hat{w}' + \frac{\hat{w}}{3} = 0$$

hence, up to rescaling by a constant,

$$\hat{w}(\zeta) \propto \zeta^{-1/3} = {}_2F_1\left(a, \frac{1}{3}; a; 1 - \zeta\right)$$

for every  $a \in \mathbb{C}$ . In the degenerate case we get an hypergeometric function as well, but the resurgent structure is trivial, i.e.  $\hat{w}(\zeta)$  is holomorphic on the Riemann surface of  $\zeta^{1/3}$ .

**5.0.1. Alternative computation of the Borel transform of  $I$ .** Let us first compute the Borel transform of (5.1) (indeed as in the proof of Theorem 3.1 we know that (5.1) admits a formal solution which is Gevrey-1)

$$\begin{aligned} \zeta^2 \hat{I} - a\zeta \hat{I} + b\hat{I} - \int_0^\zeta \zeta' \hat{I}(\zeta') + d \int_0^\zeta \hat{I}(\zeta') - \frac{1}{9} \int_0^\zeta (\zeta - \zeta') \hat{I}(\zeta') &= 0 \\ 2\zeta \hat{I} + \zeta^2 \hat{I}' - a\hat{I} - a\zeta \hat{I}' + b\hat{I}' - \zeta \hat{I} + d\hat{I} - \frac{1}{9} \int \hat{I}(\zeta') &= 0 \\ (\zeta^2 - a\zeta + b)\hat{I}'' + (3\zeta - 2a + d)\hat{I}' + \frac{8}{9}\hat{I} &= 0 \end{aligned}$$

Now we denote by  $\lambda_1, \lambda_2$  the distinguished (we assume that  $p \neq 0$ ) roots of  $\zeta^2 - a\zeta + b$ , then

$$(5.12) \quad (\zeta - \lambda_1)(\zeta - \lambda_2)\hat{I}'' + (3\zeta - 2a + d)\hat{I}' + \frac{8}{9}\hat{I} = 0$$

$$(5.13) \quad (t + \lambda_2 - \lambda_1)t\hat{I}'' + (3t + 3\lambda_2 - 2a + d)\hat{I}' + \frac{8}{9} = 0 \quad t = \zeta - \lambda_2$$

$$(5.14) \quad s(1-s)\hat{I}'' - \left(3s + \frac{3\lambda_2 - 2a + d}{\lambda_1 - \lambda_2}\right)\hat{I}' - \frac{8}{9}\hat{I} = 0 \quad t = (\lambda_1 - \lambda_2)s$$

where (5.14) is an hypergeometric equation<sup>3</sup> and a solution is given by

(5.15)

$$\hat{I}_{\lambda_1}(\zeta; U_1, U_2) = U_1 {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; \frac{\zeta - \lambda_2}{\lambda_1 - \lambda_2}\right) + U_2 \left(\frac{\zeta - \lambda_2}{\lambda_1 - \lambda_2}\right)^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta - \lambda_2}{\lambda_1 - \lambda_2}\right)$$

which has a branch cut at  $\zeta = \lambda_1$ , where  $U_1, U_2$  are constants. Of course, reversing the role of  $\lambda_1$  and  $\lambda_2$  we find

(5.16)

$$\hat{I}_{\lambda_2}(\zeta; U_1, U_2) = U_1 {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; \frac{\zeta - \lambda_1}{\lambda_2 - \lambda_1}\right) + U_2 \left(\frac{\zeta - \lambda_1}{\lambda_2 - \lambda_1}\right)^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta - \lambda_1}{\lambda_2 - \lambda_1}\right)$$

is the Borel transform of  $\tilde{I}_{\lambda_2}(z)$  and it has a branch cut singularity at  $\zeta = \lambda_2$ . It is remarkable that the dependence on the function  $f$  is only on the location of the singularities, but it is always an hypergeometric function with the same parameters. In addition, we can compute the Stokes constants thanks to the well known formula for analytic continuation of hypergeometric (see 15.2.3 in DLMF)

$$\begin{aligned} \hat{I}_{\lambda_1}(\zeta + i0; U_1, 0) - \hat{I}_{\lambda_1}(\zeta - i0; U_1, 0) &= -U_1 \frac{2\pi i}{\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})} \left(\frac{\lambda_1 - \zeta}{\lambda_2 - \lambda_1}\right)^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta - \lambda_1}{\lambda_2 - \lambda_1}\right) \\ &= -iU_1 \frac{2\pi i}{\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})} \left(\frac{\zeta - \lambda_1}{\lambda_2 - \lambda_1}\right)^{-1/2} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta - \lambda_1}{\lambda_2 - \lambda_1}\right) \\ &= -i\hat{I}_{\lambda_2}\left(\zeta; 0, \frac{U_1}{\Gamma(2/3)\Gamma(4/3)}\right) \end{aligned}$$

<sup>3</sup>Notice that  $\lambda_{1,2} = q \pm \frac{2i}{3}p\sqrt{\frac{p}{3a_1}}$ ,  $a = 2q$  and  $d = q$ . Hence

$$\frac{2a - d - 3\lambda_2}{\lambda_1 - \lambda_2} = \frac{4q - q - 3q - 2ip\sqrt{\frac{p}{3a_1}}}{-\frac{4i}{3}p\sqrt{\frac{p}{3a_1}}} = \frac{3}{2}$$