Resurgence of modified Bessel functions of second kind

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1 Simple resurgent functions

We are in the same assumption of Theorem 3.1. Our aim is to prove that $\hat{\varphi}_\alpha(\zeta)$ is a simple resurgent function: we expect it is a consequence of the half derivatives formula. In the proof of Theorem 3.1 we have seen

$$\hat{\varphi}_\alpha(\zeta) = \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \hat{I}_\alpha \tag{1}$$

$$\hat{I}_\alpha(\zeta) = \sum_{n\geq 0} a_n^\alpha (\zeta-\zeta_\alpha)^{n-1/2} \qquad a_n := 2^{n+1/2} b_{2n}^\alpha \tag{2}$$

$$\begin{aligned}
\hat{\varphi}_\alpha(\zeta) &= \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \left[(\zeta-\zeta_\alpha)^{-1/2} a_0^\alpha + (\zeta-\zeta_\alpha)^{1/2} \sum_{n\geq 0} a_{n+1}^\alpha (\zeta-\zeta_\alpha)^n \right] \\
&= a_0^\alpha \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \left[(\zeta-\zeta_\alpha)^{-1/2} \right] + \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \left[(\zeta-\zeta_\alpha)^{1/2} \right] \sum_{n\geq 0} a_{n+1}^\alpha (\zeta-\zeta_\alpha)^n \\
&\quad + (\zeta-\zeta_\alpha)^{1/2} \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \left[\sum_{n\geq 0} a_{n+1}^\alpha (\zeta-\zeta_\alpha)^n \right] \\
&= a_0^\alpha \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \left[(\zeta-\zeta_\alpha)^{-1/2} \right] + \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \left[(\zeta-\zeta_\alpha)^{1/2} \right] \sum_{n\geq 0} a_{n+1}^\alpha (\zeta-\zeta_\alpha)^n \\
&\quad + (\zeta-\zeta_\alpha)^{1/2} \sum_{n\geq 0} a_{n+1}^\alpha \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \left[(\zeta-\zeta_\alpha)^n \right] \\
&= a_0^\alpha \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \left[(\zeta-\zeta_\alpha)^{-1/2} \right] + \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \left[(\zeta-\zeta_\alpha)^{1/2} \right] \sum_{n\geq 0} a_{n+1}^\alpha (\zeta-\zeta_\alpha)^n \\
&\quad + (\zeta-\zeta_\alpha)^{1/2} \sum_{n\geq 0} a_{n+1}^\alpha \left[\sqrt{\pi} \tfrac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} (\zeta-\zeta_\alpha)^{n-1/2} \right] \\
&= a_0^\alpha \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \left[(\zeta-\zeta_\alpha)^{-1/2} \right] + \partial^{1/2}_{\zeta \text{ from } \zeta_\alpha} \left[(\zeta-\zeta_\alpha)^{1/2} \right] \sum_{n\geq 0} a_{n+1}^\alpha (\zeta-\zeta_\alpha)^n \\
&\quad + \sqrt{\pi} \sum_{n\geq 0} a_{n+1}^\alpha \tfrac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} (\zeta-\zeta_\alpha)^n
\end{aligned}$$

Notice that $\sum_{n\geq 0} a^\alpha_{n+1}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}(\zeta-\zeta_\alpha)^n$ has a finite radius of convergence, hence the second series is a germ of holomorphic function. We are left with the half–derivative of $(\zeta-\zeta_\alpha)^{\pm 1/2}$:

$$\begin{aligned}
\partial^{1/2}_{\zeta \text{ from } \zeta_\alpha}\left[(\zeta-\zeta_\alpha)^{1/2}\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\partial_\zeta\left[\int_{\zeta_\alpha}^{\zeta}(\zeta-\zeta')^{-1/2}(\zeta'-\zeta_\alpha)^{1/2}d\zeta'\right]\\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\partial_\zeta\left[\left[-2(\zeta-\zeta')^{1/2}(\zeta'-\zeta_\alpha)^{1/2}\right]_{\zeta_\alpha}^{\zeta}+\int_{\zeta_\alpha}^{\zeta}2(\zeta-\zeta')^{1/2}\frac{1}{2}(\zeta'-\zeta_\alpha)^{-1/2}d\zeta'\right]\\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\partial_\zeta\left[\int_{\zeta_\alpha}^{\zeta}(\zeta-\zeta')^{1/2}(\zeta'-\zeta_\alpha)^{-1/2}d\zeta'\right]\\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{\zeta_\alpha}^{\zeta}\frac{1}{2}(\zeta-\zeta')^{-1/2}(\zeta'-\zeta_\alpha)^{-1/2}d\zeta'\\
&= \frac{1}{2\Gamma\left(\frac{1}{2}\right)}\int_{\zeta_\alpha}^{\zeta}(\zeta\zeta'-\zeta\zeta_\alpha-\zeta'^2+\zeta'\zeta_\alpha)^{-1/2}d\zeta'\\
&= \frac{1}{2\Gamma\left(\frac{1}{2}\right)}\int_{\zeta_\alpha}^{\zeta}\left(\frac{(\zeta-\zeta_\alpha)^2}{4}-\left(\zeta'-\frac{\zeta+\zeta_\alpha}{2}\right)^2\right)^{-1/2}d\zeta'\\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-\frac{\zeta-\zeta_\alpha}{2}}^{\frac{\zeta-\zeta_\alpha}{2}}\frac{1}{\sqrt{\frac{(\zeta-\zeta_\alpha)^2}{4}-y^2}}dy\\
&= \frac{\sqrt{\pi}}{2}
\end{aligned}$$

$$\begin{aligned}
\partial^{1/2}_{\zeta \text{ from } \zeta_\alpha}\left[(\zeta-\zeta_\alpha)^{-1/2}\right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\partial_\zeta\left[\int_{\zeta_\alpha}^{\zeta}(\zeta-\zeta')^{-1/2}(\zeta'-\zeta_\alpha)^{-1/2}d\zeta'\right]\\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\partial_\zeta\left[\int_{\zeta_\alpha}^{\zeta}(\zeta\zeta'-\zeta\zeta_\alpha-\zeta'^2+\zeta'\zeta_\alpha)^{-1/2}d\zeta'\right]\\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\partial_\zeta\left[\int_{\zeta_\alpha}^{\zeta}\left(\frac{(\zeta-\zeta_\alpha)^2}{4}-\left(\zeta'-\frac{\zeta+\zeta_\alpha}{2}\right)^2\right)^{-1/2}d\zeta'\right]\\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\partial_\zeta\left[\int_{-\frac{\zeta-\zeta_\alpha}{2}}^{\frac{\zeta-\zeta_\alpha}{2}}\frac{1}{\sqrt{\frac{(\zeta-\zeta_\alpha)^2}{4}-y^2}}dy\right]\\
&= 0
\end{aligned}$$

Therefore, collecting all the contributions we get

$$\hat{\varphi}_\alpha(\zeta) = -\frac{4a_0^\alpha}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{(\zeta-\zeta_\alpha)\sqrt{(\zeta-\zeta_\alpha)^2-4}} - \frac{i}{\Gamma\left(\frac{1}{2}\right)} \log(2i+\sqrt{(\zeta-\zeta_\alpha)^2-4}) \sum_{n\geq 0} a_{n+1}^\alpha (\zeta-\zeta_\alpha)^n$$
$$+ \frac{i}{\Gamma\left(\frac{1}{2}\right)} \log(\zeta-\zeta_\alpha) \sum_{n\geq 0} a_{n+1}^\alpha (\zeta-\zeta_\alpha)^n + \sqrt{\pi} \sum_{n\geq 0} a_{n+1}^\alpha \tfrac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} (\zeta-\zeta_\alpha)^n$$