GENERAL THIMBLES INTEGRALS

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1. PROOF OF BOREL REGULARITY

We are going to prove Theorem 5.1 `draft2`. Let X be a $N-$dim manifold, $f: X \to \mathbb{C}$ be a holomorphic Morse function with simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$(1.1) \qquad I(z) := \int_{\mathcal{C}} e^{-zf} \, \nu$$

where $\mathcal{C}$ is a suitable contour such that the integral is well defined. Indeed, $I(z)$ represents a pairing between a relative homology class $\mathcal{C} \in H_N^B(X, zf)$ and a cohomology class $\nu \in H_{dR}^N(X, zf)$ (see Section 1.3.1 Thimble integrals in the introduction). Let us restrict to one dimensional X. For any Morse critical points x_α[1] of f, the saddle point approximation allows to compute the asymptotic expansion of $I_\alpha(z)$

$$(1.2) \quad I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf} \, \nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)} \sqrt{2\pi} z^{-1/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n} \qquad \text{as } \mathrm{Re}(ze^{i\theta}) \to \infty$$

where $\mathcal{C}_\alpha$ is a steepest descent path through the critical point x_α and θ is chosen such that $f(x_\beta) \notin f(x_\alpha) + [0, e^{i\theta}\infty)$ for $\beta \neq \alpha$[2]. Notice that $f \circ \mathcal{C}_\alpha$ lies in the ray $\zeta_\alpha + [0, e^{i\theta}\infty)$, where $\zeta_\alpha := f(x_\alpha)$.

Theorem 1.1. Let $N = 1$. Let $I_\alpha(z)$ defined as in (1.2) for every critical point x_α. Then $\tilde{I}_\alpha$ is Borel regular for $\mathrm{Re}(ze^{i\theta}) > 0$:

(1) The series $\tilde{I}_\alpha(z) = e^{-zf(x_\alpha)} \sqrt{2\pi} z^{-1/2} \sum_{n \geq 0} a_{\alpha,n} z^{-n}$ is Gevrey-1.
(2) The series $\tilde{\iota}_\alpha(\zeta) := \mathcal{B}(\tilde{I}_\alpha)$ converges near $\zeta = \zeta_\alpha$.
(3) If you continue the sum of $\tilde{\iota}_\alpha$ along the ray going rightward from ζ_α in the direction θ, and take its Laplace transform along that ray, you'll recover I_α.

Remark 1.2.   (1) We may drop the assumption of non degenerate critical points for f, however the asymptotic expansion of $I_\alpha(z)$ will depend on the order m such that $f^{(m)}(x_\alpha) \neq 0$ and $f^{(j)}(x_\alpha) = 0$ for every $j = 1, ..., m-1$ (see [Zorich] Theorem 1 Section 19.2.5).

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[1] By Morse critical points we mean non–degenerate isolated critical points.
[2] Such a θ exists because f has a finite number of critical points.

(2) in [Malgrange74] (see also Chapter 5 of [Mistergard Phd thesis] for a general review), the author computes the asymptotic expansion of exponential integrals for $N > 1$ which get logarithmic terms like

$$\tilde{I}(z) = \sum_{j \in A} \sum_{n \geq 0} \sum_{q=0}^{N-1} a_{n,q,j} z^{-n-j} (\log z)^q,$$

for $A \subset \mathbb{Q}_{\geq 0}$ finite. Due to the presence of logarithmic terms, the definition of Borel transform has to be further extended (see [Mistergard phd] Definition pag 5) and the study of Borel regularity becomes more involved.

Proof. Part ([1]): Since f is Morse, we can find a holomorphic chart τ around x_α with $\frac{1}{2}\tau^2 = f - \zeta_\alpha$. Let $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$ be the parts of $\mathcal{C}_\alpha$ that go from the past to x_α and from x_α to the future, respectively. We can arrange for τ to be valued in $(-\infty e^{i\theta}, 0]$ and $[0, e^{i\theta}\infty)$ on $\mathcal{C}_\alpha^-$ and $\mathcal{C}_\alpha^+$, respectively. [**We should explicitly spell out and check the conditions that make this possible. I think we're implicitly orienting** $\mathcal{C}_\alpha$ **so that** τ **in the upper half-plane.**] Since ν is holomorphic, we can express it as a Taylor series

$$\nu = \sum_{n \geq 0} b_n^\alpha \tau^n \, d\tau$$

that converges in some disk $|\tau| < \varepsilon$.

In coordinates τ the integral $I_\alpha(z)$ can be approximated as

$$I_\alpha(z) \sim e^{-z\zeta_\alpha} \int_{\tau \in [-\varepsilon, \varepsilon]} e^{-z\tau^2/2} \nu$$

as $\text{Re}(z e^{i\theta}) \to \infty$ (see Lemma 1 in Section 19.2.2 Zorich). [**I need to learn how this works! Do we get asymptoticity at all orders? —Aaron**] Plugging in the Taylor series above, we get

$$I_\alpha(z) \sim e^{-z\zeta_\alpha} \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n \geq 0} b_n^\alpha \tau^n \, d\tau$$

$$= e^{-z\zeta_\alpha} \int_{-\varepsilon}^{\varepsilon} e^{-z\tau^2/2} \sum_{n \geq 0} b_{2n}^\alpha \tau^{2n} \, d\tau$$

$$= 2 e^{-z\zeta_\alpha} \int_0^{\varepsilon} e^{-z\tau^2/2} \sum_{n \geq 0} b_{2n}^\alpha \tau^{2n} \, d\tau.$$

By Watson's Lemma (see Lemma 4 Section 19.2.2 Zorich)

$$I_\alpha(z) \sim e^{-z\zeta_\alpha} \sum_{n \geq 0} b_{2n}^\alpha \Gamma\left(n + \tfrac{1}{2}\right) 2^{n+1/2} z^{-n-1/2}$$

$$= e^{-z\zeta_\alpha} \sqrt{2\pi} \sum_{n \geq 0} b_{2n}^\alpha (2n-1)!! z^{-n-1/2}$$

Call the right-hand side $\tilde{I}_\alpha$. We now see that $a_{\alpha,n} = (2n-1)!! \, b_{2n}^\alpha$ in the statement of the theorem. We know from the definition of ε that $\left| b_n^\alpha \right| \varepsilon^n \lesssim 1$. Recalling that $(2n-1)!! \sim (\pi n)^{-1/2} 4^n \, n!$ as $n \to \infty$, we deduce that $|a_{\alpha,n}| \lesssim \left(\frac{4}{\varepsilon^2} \right)^n n!$, showing that $\tilde{I}_\alpha$ is Gevrey-1.

Part (2): note that **[explain formally what it means to center at ζ_α]**

$$\tilde{\iota}_\alpha := \mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha = \sqrt{2\pi} \sum_{n \geq 0} (2n-1)!! \, b_{2n}^\alpha \, \frac{(\zeta - \zeta_\alpha)^{n-1/2}}{\Gamma\left(n + \frac{1}{2}\right)}$$

Since $(2n-1)!! = \pi^{-1/2} 2^n \Gamma\left(n + \frac{1}{2}\right)$ and $|b_n^\alpha| \epsilon^n \lesssim 1$, then $\tilde{\iota}_\alpha(\zeta)$ has a finite radius of convergence.

Part (3): Let's recast the integral I_α into the f plane. As ζ goes rightward from ζ_α, the start and end points of $\mathcal{C}_\alpha(\zeta)$ sweep backward along $\mathcal{C}_\alpha^-(\zeta)$ and forward along $\mathcal{C}_\alpha^+(\zeta)$, respectively. Hence, we have

$$\begin{aligned}
I_\alpha(z) &= \int_{\mathcal{C}_\alpha} e^{-zf} \, \nu \\
&= \int_{\mathcal{H}_\alpha} e^{-z\zeta} \left(\int_{f^{-1}(\zeta)} \frac{\nu}{df} \right) d\zeta \\
&= \int_{\zeta_\alpha}^{e^{i\theta} \infty} e^{-z\zeta} \left[\frac{\nu}{df} \right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)} d\zeta.
\end{aligned}$$

where $\mathcal{H}_\alpha$ is the Hanckel contour through the point ζ_α (see Figure [?]) with ends in the θ direction. Noticing that the last integral is a Laplace transform for the initial choice of θ, we learn that

(1.3)
$$\hat{\iota}_\alpha(\zeta) = \left[\frac{\nu}{df} \right]_{\text{start}\,\mathcal{C}_\alpha(\zeta)}^{\text{end}\,\mathcal{C}_\alpha(\zeta)}.$$

In Ecalle's formalism, $\int_{f^{-1}(\zeta)} \frac{\nu}{df}$ and $\hat{\iota}_\alpha$ are respectively a major and a minor of the singularity $\overset{\triangledown}{\iota}_\alpha$ and they differ by an holomorphic function (we will see this in the examples Section Airy, Bessel).

We can rewrite our Taylor series for ν as

$$\begin{aligned}
\nu &= \sum_{n \geq 0} b_n^\alpha [2(f - \zeta_\alpha)]^{n/2} \frac{df}{[2(f - \zeta_\alpha)]^{1/2}} \\
&= \sum_{n \geq 0} b_n^\alpha [2(f - \zeta_\alpha)]^{(n-1)/2} \, df,
\end{aligned}$$

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taking the positive branch of the square root on $\mathcal{C}_\alpha^+$ and the negative branch on $\mathcal{C}_\alpha^-$. Plugging this into our expression for $\hat{\iota}_\alpha$, we learn that

$$
\begin{aligned}
\hat{\iota}_\alpha(\zeta) &= \left[\sum_{n \geq 0} b_n^\alpha [2(f - \zeta_\alpha)]^{(n-1)/2} \right]_{\mathrm{start}\,\mathcal{C}_\alpha(\zeta)}^{\mathrm{end}\,\mathcal{C}_\alpha(\zeta)} \\
&= \sum_{n \geq 0} b_n^\alpha \Big([2(\zeta - \zeta_\alpha)]^{(n-1)/2} - (-1)^{n-1}[2(\zeta - \zeta_\alpha)]^{(n-1)/2} \Big) \\
&= \sum_{n \geq 0} 2 b_{2n}^\alpha [2(\zeta - \zeta_\alpha)]^{n-1/2} \\
&= \sum_{n \geq 0} 2^{n+1/2} b_{2n}^\alpha (\zeta - \zeta_\alpha)^{n-1/2} \\
&= \mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha.
\end{aligned}
$$

We have now shown that the sum of $\mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha$ is actually equal to $\hat{\iota}_\alpha$ as $\zeta \in \zeta_\alpha + [0, e^{i\theta}\infty)$.

$\square$

Remark 1.3. Different choices of admissible θ correspond to different choices of thimbles $[\mathcal{C}_\alpha] \in H_N^B(X, zf)$, but the Borel transform of $\tilde{I}_\alpha$ does not depend on θ. However, if $\theta_* := \arg(\zeta_\alpha - \zeta_\beta)$ and $\theta_\pm := \theta_* \pm \delta$ for small δ, then $I_\alpha(z)$ jumps on the intersection between $\mathrm{Re}(e^{i\theta_+}z) > 0$ and $\mathrm{Re}(e^{i\theta_-}z) > 0$. This is known as the Stokes phenomenon (see Section resurgence thimbles integrals).

2. $3/2$ DERIVATIVE FORMULA

In Theorem 1.1 we have seen that the asymptotic behaviour of $I_\alpha(z)$ has a fractional power contribution namely $\tilde{I}_\alpha(z) = e^{-z\zeta_\alpha} z^{-1/2} \sqrt{2\pi} \sum_{n \geq 0} a_{\alpha,n} z^{-n}$, hence we have used the extended notion of Borel transform to deal with fractional powers. Now we will focus on the formal series $\tilde{\Phi}_\alpha(z) := e^{-z\zeta_\alpha} \sqrt{2\pi} \sum_{n \geq 0} a_{\alpha,n} z^{-n} = z^{1/2} \tilde{I}_\alpha(z)$ which does not contain any fractional power and we prove a fractional derivative formula which relates the Borel transform s $\hat{\varphi}_\alpha(\zeta)$ and $\hat{\iota}_\alpha(\zeta)$. Moreover we show that the $\hat{\varphi}_\alpha(\zeta)$ depends on v and df as well as $\hat{\iota}_\alpha(\zeta)$ does.

Corollary 2.1. Under the same assumptions of Theorem 1.1, for any ζ on the ray going rightward from ζ_α in the direction of θ, we have

$$
(2.1) \quad \hat{\varphi}_\alpha(\zeta) = \partial_{\zeta\,\mathrm{from}\,\zeta_\alpha}^{3/2} \left(\int_{\mathcal{C}_\alpha(\zeta)} v \right) = \left(\frac{\partial}{\partial \zeta} \right)^2 \frac{1}{\Gamma(\frac{1}{2})} \int_{\zeta_\alpha}^\zeta (\zeta - \zeta')^{-1/2} \left(\int_{\mathcal{C}_\alpha(\zeta')} v \right) d\zeta',
$$

where $\mathcal{C}_\alpha(\zeta)$ is the part of $\mathcal{C}_\alpha$ that goes through $e^{-i\theta} f^{-1}([\zeta_\alpha, \zeta])$. Notice that $\mathcal{C}_\alpha(\zeta)$ starts and ends in $e^{-i\theta} f^{-1}(\zeta)$. [**Be careful about the orientation of $\mathcal{C}_\alpha$.**]

Proof. Theorem **??** tells us that

$$\mathcal{B}_{\zeta_\alpha} \tilde{I}_\alpha = \mathcal{B}_{\zeta_\alpha} z^{-1/2} \tilde{\varphi}_\alpha$$

$$= \partial_{\zeta \text{ from } \zeta_\alpha}^{-1/2} \mathcal{B} \tilde{\varphi}_\alpha$$

$$= \partial_{\zeta \text{ from } \zeta_\alpha}^{-1/2} \hat{\varphi}_\alpha.$$

It follows, from the proof of part 3 of Theorem 1.1, that

$$(2.2) \qquad\qquad \hat{\iota}_\alpha(\zeta) = \partial_{\zeta \text{ from } \zeta_\alpha}^{-1/2} \hat{\varphi}_\alpha.$$

Since fractional integrals form a semigroup, equation (2.2) implies that

$$\partial_{\zeta \text{ from } \zeta_\alpha}^{-1} \hat{\iota}_\alpha(\zeta) = \partial_{\zeta \text{ from } \zeta_\alpha}^{-3/2} \hat{\varphi}_\alpha.$$

Rewriting equation (1.3) as

$$\hat{\iota}_\alpha(\zeta) = \partial_\zeta \left(\int_{\mathcal{C}_\alpha(\zeta)} \nu \right),$$

we can see that

$$\partial_{\zeta \text{ from } \zeta_\alpha}^{-1} \hat{\iota}_\alpha(\zeta) = \int_{\mathcal{C}_\alpha(\zeta)} \nu - \int_{\mathcal{C}_\alpha(0)} \nu.$$

The initial value term vanishes, because the path $\mathcal{C}_\alpha(0)$ is a point. Hence,

$$\int_{\mathcal{C}_\alpha(\zeta)} \nu = \partial_{\zeta \text{ from } \zeta_\alpha}^{-3/2} \hat{\varphi}_\alpha(\zeta).$$

Recalling that the Riemann-Liouville fractional derivative is a left inverse of the fractional integral, we conclude that

$$\partial_{\zeta \text{ from } \zeta_\alpha}^{3/2} \left(\int_{\mathcal{C}_\alpha(\zeta)} \nu \right) = \hat{\varphi}_\alpha(\zeta).$$

$$\square$$

2.1. **Singularities.** From equation (2.2) we see that singularities of $\hat{\iota}_\alpha(\zeta)$ in the Borel plane comes from either poles of ν or zeros of df. Instead, the fractional derivatives formula tells that singularities of $\hat{\varphi}_\alpha$ are given by convolutions of $\zeta^{-1/2}/\Gamma(1/2)$ with $\hat{\iota}_\alpha$. Since $\zeta^{-1/2}/\Gamma(1/2)$ is singular at $\zeta = 0$ the set of singularities of $\hat{\varphi}_\alpha(\zeta)$ is exactly the same as the one of $\hat{\iota}_\alpha(\zeta)$. However, the type of singularities will change and we expect $\hat{\varphi}_\alpha(\zeta)$ to have only simple singularities.

3. CONTOUR ARGUMENT