Airy function: Kawai+Takei vs. Mariño

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Kawai and Takei want to solve

$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) = 0.$$

They define $\psi_B(x,y)$ as the inverse Laplace transform of $\psi(x,\eta)$ with respect to η.

With $w = x\eta^{2/3}$, the equation above is equivalent to

$$\left[\left(\frac{d}{dw}\right)^2 - w\right]\psi(w\eta^{-2/3},\eta) = 0.$$

Proof: substitute back to get

$$\begin{aligned}
\left[\eta^{-4/3}\left(\frac{d}{dx}\right)^2 - \eta^{2/3}x\right]\psi(x,\eta) &= 0 \\
\left[\eta^{-4/3}\left(\frac{d}{dx}\right)^2 - \eta^{-4/3}\eta^2 x\right]\psi(x,\eta) &= 0 \\
\eta^{-4/3}\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) &= 0.
\end{aligned}$$

Hence, $\psi(w\eta^{-2/3},\eta) = k(\eta)\mathsf{Ai}(w)$ is a solution for any holomorphic function k.

1 Veronica's change of coordinates

Kawai and Takei study the WKB analysis of the equation

$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) = 0 \tag{1}$$

as $\eta \to \infty$. They define $\psi_B(x,y)$ as the inverse Laplace transform of $\psi(x,\eta)$ with respect to η. In the coordinates $t = yx^{-3/2}$ they find an explicit formula for $\psi_B(x,y)$ in terms of

Gauss hypergeometric functions:

$$\psi_{+,B}(x,y) = \frac{1}{x}\phi_+(t) = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{1}{x}s^{-1/2}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};s\right)$$
$$\psi_{-,B}(x,y) = \frac{1}{x}\phi_-(t) = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{1}{x}(1-s)^{-1/2}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};1-s\right)$$

where $s = 3t/4 + 1/2$. The same hypergeometric functions have been computed in Section ?? as the Borel transform of the formal solutions of the Airy equation

$$\left[\left(\frac{d}{dw}\right)^2 - w\right] f(w) = 0. \tag{2}$$

Although the two equations look closely related (they are equivalent by the change of coordinates $w = x\eta^{2/3}$), the Borel transform of ψ is computed with respect to $\eta x^{3/2}$ (which is the conjugate variable of t) while the Borel transform of $f(w)$ is computed with respect to w. So we need to find a different change of coordinates to explain why the Borel transforms of $\psi(x,\eta)$ and $f(w)$ are given by the same hypergeometric function.

First of all notice that if η and y are conjugate variables under Borel transform, meaning

$$\sum_{n\geq 0} a_n \eta^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n\geq 0} \frac{a_n}{n!} y^n$$

then $t = yx^{-3/2}$ is the conjugate variable of $q = \eta x^{3/2}$ up to correction by a factor of $x^{-3/2}$

$$\sum_{n\geq 0} a_n q^{-n-1} = \sum_{n\geq 0} a_n x^{-3/2(n+1)}\eta^{-n-1} \xrightarrow{\mathcal{B}} \sum_{n\geq 0} \frac{a_n x^{-3/2(n+1)}}{n!} y^n = x^{-3/2}\sum_{n\geq 0}\frac{a_n}{n!}t^n.$$

In addition, $\psi_{B,\pm}(x,y) = \frac{1}{x}\phi_\pm(t)$, therefore we expect that $\psi(x,\eta) = x^{1/2}\Phi(q)$. Assume that $\psi(x,y)$ is a solution of (1), then $\Phi(q)$ solves

$$\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi(q) = 0 \tag{3}$$

Proof.

$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]\psi(x,\eta) = 0$$

$$\left[\left(\frac{d}{dx}\right)^2 - \eta^2 x\right]x^{1/2}\Phi(q) = 0$$

$$\frac{d}{dx}\left[\frac{1}{2}x^{-1/2}\Phi + x^{1/2}\frac{d}{dx}\Phi\right] - \eta^2 x^{3/2}\Phi = 0$$

$$-\frac{1}{4}x^{-3/2}\Phi + \frac{1}{2}x^{-1/2}\frac{d}{dx}\Phi + \frac{1}{2}x^{-1/2}\frac{d}{dx}\Phi + x^{1/2}\left(\frac{d}{dx}\right)^2\Phi - \eta^2 x^{3/2}\Phi = 0$$

$$\left[x^{1/2}\left(\frac{d}{dx}\right)^2 + x^{-1/2}\frac{d}{dx} - \frac{1}{4}x^{-3/2} - \eta^2 x^{3/2}\right]\Phi = 0$$

$$\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0$$

□

Now rewrite (3) in the coordinates $q = \eta x^{3/2}$:

$$\left[\left(\frac{d}{dx}\right)^2 + x^{-1}\frac{d}{dx} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0$$

$$\left[\frac{9}{4}\eta^2 x\left(\frac{d}{dq}\right)^2 + \frac{3}{4}\eta x^{-1/2}\frac{d}{dq} + x^{-1}\cdot\frac{3}{2}\,\eta\, x^{1/2}\frac{d}{dq} - \frac{1}{4}x^{-2} - \eta^2 x\right]\Phi = 0$$

$$\left[\eta^2 x\left(\frac{d}{dq}\right)^2 + \frac{1}{3}\eta x^{-1/2}\frac{d}{dq} + \frac{2}{3}\,\eta\, x^{-1/2}\frac{d}{dq} - \frac{1}{9}x^{-2} - \frac{4}{9}\eta^2 x\right]\Phi = 0$$

$$\left[\eta^2\left(\frac{d}{dq}\right)^2 + \eta x^{-3/2}\frac{d}{dq} - \frac{1}{9}x^{-3} - \frac{4}{9}\eta^2\right]\Phi = 0$$

$$\left[\left(\frac{d}{dq}\right)^2 + \eta^{-1}x^{-3/2}\frac{d}{dq} - \frac{1}{9}\eta^{-2}x^{-3} - \frac{4}{9}\right]\Phi = 0$$

$$\left[\left(\frac{d}{dq}\right)^2 + q^{-1}\frac{d}{dq} - \frac{1}{9}q^{-2} - \frac{4}{9}\right]\Phi = 0$$

therefore $\Phi(q)$ is a solution of the transform Airy equation (see draft2).