

Resurgence of modified Bessel functions of second kind

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1 Simple resurgent functions

We are in the same assumption of Theorem 3.1: our aim is to prove that $\hat{\varphi}_\alpha(\zeta)$ is a simple resurgent function: we expect it is a consequence of the half derivatives formula. In the proof of Theorem 3.1 we have seen

$$\hat{\varphi}_\alpha(\zeta) = \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \hat{I}_\alpha \quad (1)$$

$$\hat{I}_\alpha(\zeta) = \sum_{n \geq 0} a_n^\alpha (\zeta - \zeta_\alpha)^{n-1/2} \quad a_n := 2^{n+1/2} b_{2n}^\alpha \quad (2)$$

$$\begin{aligned} \hat{\varphi}_\alpha(\zeta) &= \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \left[(\zeta - \zeta_\alpha)^{-1/2} a_0^\alpha + (\zeta - \zeta_\alpha)^{1/2} \sum_{n \geq 0} a_{n+1}^\alpha (\zeta - \zeta_\alpha)^n \right] \\ &= a_0^\alpha \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \left[(\zeta - \zeta_\alpha)^{-1/2} \right] + \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \left[(\zeta - \zeta_\alpha)^{1/2} \right] \sum_{n \geq 0} a_{n+1}^\alpha (\zeta - \zeta_\alpha)^n \\ &\quad + (\zeta - \zeta_\alpha)^{1/2} \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \left[\sum_{n \geq 0} a_{n+1}^\alpha (\zeta - \zeta_\alpha)^n \right] \\ &= a_0^\alpha \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \left[(\zeta - \zeta_\alpha)^{-1/2} \right] + \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \left[(\zeta - \zeta_\alpha)^{1/2} \right] \sum_{n \geq 0} a_{n+1}^\alpha (\zeta - \zeta_\alpha)^n \\ &\quad + (\zeta - \zeta_\alpha)^{1/2} \sum_{n \geq 0} a_{n+1}^\alpha \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} [(\zeta - \zeta_\alpha)^n] \\ &= a_0^\alpha \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \left[(\zeta - \zeta_\alpha)^{-1/2} \right] + \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \left[(\zeta - \zeta_\alpha)^{1/2} \right] \sum_{n \geq 0} a_{n+1}^\alpha (\zeta - \zeta_\alpha)^n \\ &\quad + (\zeta - \zeta_\alpha)^{1/2} \sum_{n \geq 0} a_{n+1}^\alpha \left[\sqrt{\pi} \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} (\zeta - \zeta_\alpha)^{n-1/2} \right] \\ &= a_0^\alpha \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \left[(\zeta - \zeta_\alpha)^{-1/2} \right] + \partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} \left[(\zeta - \zeta_\alpha)^{1/2} \right] \sum_{n \geq 0} a_{n+1}^\alpha (\zeta - \zeta_\alpha)^n \\ &\quad + \sqrt{\pi} \sum_{n \geq 0} a_{n+1}^\alpha \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} (\zeta - \zeta_\alpha)^n \end{aligned}$$

Notice that $\sum_{n \geq 0} a_{n+1}^\alpha \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} (\zeta - \zeta_\alpha)^n$ has a finite radius of convergence, hence the second series is a germ of holomorphic function. We are left with the half-derivative of $(\zeta - \zeta_\alpha)^{\pm 1/2}$:

$$\begin{aligned}
\partial_{\zeta \text{ from } \zeta_\alpha}^{1/2} [(\zeta - \zeta_\alpha)^{1/2}] &= \frac{1}{\Gamma(\frac{1}{2})} \partial_\zeta \left[\int_{\zeta_\alpha}^\zeta (\zeta - \zeta')^{-1/2} (\zeta' - \zeta_\alpha)^{1/2} d\zeta' \right] \\
&= \frac{1}{\Gamma(\frac{1}{2})} \partial_\zeta \left[\left[-2(\zeta - \zeta')^{1/2} (\zeta' - \zeta_\alpha)^{1/2} \right]_{\zeta_\alpha}^\zeta + \int_{\zeta_\alpha}^\zeta 2(\zeta - \zeta')^{1/2} \frac{1}{2} (\zeta' - \zeta_\alpha)^{-1/2} d\zeta' \right] \\
&= \frac{1}{\Gamma(\frac{1}{2})} \partial_\zeta \left[\int_{\zeta_\alpha}^\zeta (\zeta - \zeta')^{1/2} (\zeta' - \zeta_\alpha)^{-1/2} d\zeta' \right] \\
&= \frac{1}{\Gamma(\frac{1}{2})} \int_{\zeta_\alpha}^\zeta \frac{1}{2} (\zeta - \zeta')^{-1/2} (\zeta' - \zeta_\alpha)^{-1/2} d\zeta' \\
&= \frac{1}{2\Gamma(\frac{1}{2})} \int_{\zeta_\alpha}^\zeta (\zeta \zeta' - \zeta \zeta_\alpha - \zeta'^2 + \zeta' \zeta_\alpha)^{-1/2} d\zeta' \\
&= \frac{1}{2\Gamma(\frac{1}{2})} \int_{\zeta_\alpha}^\zeta \left(\frac{(\zeta - \zeta_\alpha)^2}{4} - \left(\zeta' - \frac{\zeta + \zeta_\alpha}{2} \right)^2 \right)^{-1/2} d\zeta' \\
&= \frac{1}{2\Gamma(\frac{1}{2})} \int_{-\frac{\zeta - \zeta_\alpha}{2}}^{\frac{\zeta - \zeta_\alpha}{2}} \frac{1}{\sqrt{\frac{(\zeta - \zeta_\alpha)^2}{4} - y^2}} dy \\
&= \frac{1}{2\Gamma(\frac{1}{2})} \left[\arcsin \left(\frac{4y}{(\zeta - \zeta_\alpha)^2} \right) \right]_{-\frac{\zeta - \zeta_\alpha}{2}}^{\frac{\zeta - \zeta_\alpha}{2}} \\
&= \frac{1}{\Gamma(\frac{1}{2})} \arcsin \left(\frac{2}{\zeta - \zeta_\alpha} \right) \\
&= \frac{i}{\Gamma(\frac{1}{2})} \left(\log(\zeta - \zeta_\alpha) - \log(2i + \sqrt{(\zeta - \zeta_\alpha)^2 - 4}) \right)
\end{aligned}$$

$$\begin{aligned}
\partial_{\zeta}^{1/2} \Big|_{\text{from } \zeta_{\alpha}} \left[(\zeta - \zeta_{\alpha})^{-1/2} \right] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \partial_{\zeta} \left[\int_{\zeta_{\alpha}}^{\zeta} (\zeta - \zeta')^{-1/2} (\zeta' - \zeta_{\alpha})^{-1/2} d\zeta' \right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \partial_{\zeta} \left[\int_{\zeta_{\alpha}}^{\zeta} (\zeta \zeta' - \zeta \zeta_{\alpha} - \zeta'^2 + \zeta' \zeta_{\alpha})^{-1/2} d\zeta' \right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \partial_{\zeta} \left[\int_{\zeta_{\alpha}}^{\zeta} \left(\frac{(\zeta - \zeta_{\alpha})^2}{4} - \left(\zeta' - \frac{\zeta + \zeta_{\alpha}}{2} \right)^2 \right)^{-1/2} d\zeta' \right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \partial_{\zeta} \left[\int_{-\frac{\zeta - \zeta_{\alpha}}{2}}^{\frac{\zeta - \zeta_{\alpha}}{2}} \frac{1}{\sqrt{\frac{(\zeta - \zeta_{\alpha})^2}{4} - y^2}} dy \right] \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \partial_{\zeta} \left[\left[\arcsin \left(\frac{4y}{(\zeta - \zeta_{\alpha})^2} \right) \right]_{-\frac{\zeta - \zeta_{\alpha}}{2}}^{\frac{\zeta - \zeta_{\alpha}}{2}} \right] \\
&= \frac{2}{\Gamma\left(\frac{1}{2}\right)} \partial_{\zeta} \left[\arcsin \left(\frac{2}{\zeta - \zeta_{\alpha}} \right) \right] \\
&= -\frac{4}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{(\zeta - \zeta_{\alpha}) \sqrt{(\zeta - \zeta_{\alpha})^2 - 4}}
\end{aligned}$$

Therefore, collecting all the contributions we get

$$\begin{aligned}
\hat{\varphi}_{\alpha}(\zeta) &= -\frac{4a_0^{\alpha}}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{(\zeta - \zeta_{\alpha}) \sqrt{(\zeta - \zeta_{\alpha})^2 - 4}} - \frac{i}{\Gamma\left(\frac{1}{2}\right)} \log(2i + \sqrt{(\zeta - \zeta_{\alpha})^2 - 4}) \sum_{n \geq 0} a_{n+1}^{\alpha} (\zeta - \zeta_{\alpha})^n \\
&\quad + \frac{i}{\Gamma\left(\frac{1}{2}\right)} \log(\zeta - \zeta_{\alpha}) \sum_{n \geq 0} a_{n+1}^{\alpha} (\zeta - \zeta_{\alpha})^n + \sqrt{\pi} \sum_{n \geq 0} a_{n+1}^{\alpha} \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} (\zeta - \zeta_{\alpha})^n
\end{aligned}$$