

# Resurgence of modified Bessel functions of second kind

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## 1 Modified Bessel function of second kind

The modified Bessel function of the second kind  $K_\mu(z)$  is a solution of the equation

$$w'' + \frac{w'}{z} - w - \frac{\mu^2}{z^2} = 0 \quad (1)$$

such that  $K_\mu(z) \sim \sqrt{\pi/(2z)}e^{-z}$  as  $z \rightarrow \infty$  in  $|\arg z| < \frac{3\pi}{2}$ <sup>1</sup>. It has a branch point at  $z = 0$  for every  $\mu \in \mathbb{C}$  and the principal branch is analytic in  $\mathbb{C} \setminus (-\infty, 0]$ .

### 1.1 Differential equation

From the general theory of ODE, the formal integral solution of (1) is two parameters family

$$\tilde{w}(z) = U_1 e^{-z} z^{-1/2} \tilde{w}_{\mu,+}(z) + U_2 e^z z^{-1/2} \tilde{w}_{\mu,-}(z) \quad (4)$$

where  $\tilde{w}_{\mu,\pm} = \sum_{j \geq 0} a_{\pm,j} z^{-j} \in \mathbb{C}[[z^{-1}]]$  are unique formal solutions of

$$\begin{aligned} \tilde{w}_{\mu,+}'' - 2\tilde{w}_{\mu,+}' + \frac{\tilde{w}_{\mu,+}}{4z^2} - \frac{\mu^2}{z^2} \tilde{w}_{\mu,+} &= 0 \\ \tilde{w}_{\mu,-}'' + 2\tilde{w}_{\mu,-}' + \frac{\tilde{w}_{\mu,-}}{4z^2} - \frac{\mu^2}{z^2} \tilde{w}_{\mu,-} &= 0 \end{aligned}$$

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<sup>1</sup>A system of solution of Bessel equation is given by  $I_\mu(z)$  and  $K_\mu(z)$ . In particular, their asymptotic behaviour as  $z \rightarrow \infty$  is given by

$$\tilde{I}_\mu(z) = \frac{1}{\sqrt{2\pi}} e^z z^{-1/2} \sum_{k \geq 0} \frac{(\frac{1}{2} - \mu)_k (\frac{1}{2} + \mu)_k}{2^k k!} z^{-k} \quad (2)$$

$$\tilde{K}_\mu(z) = \sqrt{\frac{\pi}{2}} e^{-z} z^{-1/2} \sum_{k \geq 0} \frac{(\frac{1}{2} - \mu)_k (\frac{1}{2} + \mu)_k}{(-2)^k k!} z^{-k} \quad (3)$$

In particular,  $\tilde{K}_\mu(z) = \sqrt{\frac{\pi}{2}}e^{-z}z^{-1/2}\tilde{w}_{\mu,+}(z)$  and  $\tilde{I}_\mu(z) = \frac{1}{\sqrt{2\pi}}e^z z^{-1/2}\tilde{w}_{\mu,-}(z)$  (once we choose  $a_{\pm,0} = 1$ ). We now compute the Borel transform of  $\tilde{w}_+(z)^2$  it is a solution of

$$\begin{aligned}\zeta^2 \hat{w}_{\mu,+} + 2t \hat{w}_{\mu,+} + \left(\frac{1}{4} - \nu^2\right) \int_0^\zeta (\zeta - s) \hat{w}_{\nu,+}(s) ds &= 0 \\ \zeta^2 \hat{w}_{\mu,+}'' + 2\zeta \hat{w}_{\mu,+}' + 4\zeta \hat{w}_{\mu,+}' + \left(\frac{9}{4} - \mu^2\right) \hat{w}_{\mu,+} &= 0 \\ t(1-t) \hat{w}_{\mu,+}'' + (2-4t) \hat{w}_{\mu,+}' - \left(\frac{9}{4} - \mu^2\right) \hat{w}_{\mu,+} &= 0 \quad t = -\frac{\zeta}{2}\end{aligned}$$

therefore  $\hat{w}_{\mu,+}(\zeta)$  is an hypergeometric function

$$\hat{w}_{\mu,+}(\zeta) = {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; -\frac{\zeta}{2}\right) \quad (5)$$

and it has a branch point singularities at  $\zeta = -2$ . By the same reasoning,

$$\hat{w}_{\mu,-}(\zeta) = {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; \frac{\zeta}{2}\right) \quad (6)$$

and it has branch point at  $\zeta = 2$ . Thanks to explicit formula for the analytic continuation of hypergeometric functions (see [?] 15.2.3)

$$\begin{aligned}\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} (\hat{w}_{\mu,+}(\zeta + i0) - \hat{w}_{\mu,+}(\zeta - i0)) &= \frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \left(-\frac{\zeta}{2} - 1\right)^{-1} {}_2F_1\left(\frac{1}{2} + \mu, \frac{1}{2} - \mu; 0; 1 + \frac{\zeta}{2}\right) \quad \zeta > \\ &= -\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \sum_{k \geq 0} \frac{(\frac{1}{2} - \mu)_k (\frac{1}{2} + \mu)_k}{\Gamma(k)k!} \left(1 + \frac{\zeta}{2}\right)^{k-1} \\ &= -\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \sum_{k \geq 1} \frac{(\frac{1}{2} - \mu)_k (\frac{1}{2} + \mu)_k}{\Gamma(k)k!} \left(1 + \frac{\zeta}{2}\right)^{k-1} \\ &= -\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \frac{1}{\Gamma(\frac{1}{2} - \mu)\Gamma(\frac{1}{2} + \mu)} \cdot \\ &\quad \cdot \sum_{k \geq 1} \frac{\Gamma(\frac{1}{2} - \mu + k)\Gamma(\frac{1}{2} + \mu + k)}{\Gamma(k)k!} \left(1 + \frac{\zeta}{2}\right)^{k-1} \\ &= -\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \frac{1}{\Gamma(\frac{1}{2} - \mu)\Gamma(\frac{1}{2} + \mu)} \cdot \\ &\quad \cdot \sum_{k \geq 0} \frac{\Gamma(\frac{3}{2} - \mu + k)\Gamma(\frac{3}{2} + \mu + k)}{\Gamma(k+1)(k+1)!} \left(1 + \frac{\zeta}{2}\right)^k \\ &= -\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{1}{2} - \mu)\Gamma(\frac{1}{2} + \mu)} {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; 1 + \frac{\zeta}{2}\right) \\ &= -2i\sqrt{\pi}\hat{w}_{\mu,-}(\zeta + 2)\end{aligned}$$

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<sup>2</sup>We do not consider constant term of  $\tilde{w}_{\mu,\pm}$ , i.e.  $\mathcal{B} : \mathbb{C}[[z^{-1}]] \rightarrow \mathbb{C}[\zeta]$ .

and for  $\hat{w}_{\mu,-}(\zeta)$

$$\begin{aligned}
\hat{w}_{\mu,-}(\zeta + i0) - \hat{w}_{\mu,-}(\zeta - i0) &= -\frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \left(\frac{\zeta}{2} - 1\right)^{-1} {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; 0; 1 - \frac{\zeta}{2}\right) \quad \zeta < 2 \\
&= \frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \sum_{k \geq 0} \frac{(\frac{1}{2} - \mu)_k (\frac{1}{2} + \mu)_k}{\Gamma(k)k!} \left(1 - \frac{\zeta}{2}\right)^{k-1} \\
&= 2i \cos(\nu\pi) {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; 1 - \frac{\zeta}{2}\right) \\
&= 2i \cos(\mu\pi) \hat{w}_{\mu,+}(\zeta - 2)
\end{aligned}$$

In addition, the previous relations computes the Stokes constants which are funtions of  $\nu$  and are given by  $\pm 2i \cos(\mu\pi)$ .

## 1.2 Exponential integral

Let  $X = \mathbb{C}^*$ ,  $f(x) = x + \frac{1}{x}$  and for every  $\mu \in [0, +\infty)$  let  $\nu = x^{\mu-1} + \frac{1}{x^{\mu+1}} dx$ , then

$$I(z; m) := \int_0^\infty e^{-zf} \nu \quad (7)$$

In particulr, on the universal cover  $\pi: \tilde{C} \rightarrow \mathbb{C}^*$  setting  $x = e^u$

$$I\left(\frac{z}{2}; \mu\right) = 2 \int_{-\infty}^\infty e^{-z \cosh(u)} \cosh(\mu u) du = 4K_\mu(z) \quad |\arg(z)| < \pi/2 \quad (8)$$

where  $K_\mu(z)$  is the second kind modified Bessel function with parameter  $\mu$ . It is worth mentioning that  $I(z; \mu)$  differs from  $I(z; 0)$  only in  $\pi^*(\nu)$  while  $\pi^*(f)$  stays the same for every  $\mu \in [0, \infty)$ . Hence we can adapt part of the argument used in Bessel example (see ), and verify the 3/2-derivative formula: let  $\zeta = \cosh(u)$

$$\begin{aligned}
\int_{\mathcal{C}_0(\zeta)} \pi^*(\nu) &= \int_{\mathcal{C}_0(\zeta)} \cosh(\mu u) du \\
&= \frac{1}{\mu} \left[ \sinh(\mu u) \right]_{\text{start } \mathcal{C}_0(\zeta)}^{\text{end } \mathcal{C}_0(\zeta)} \\
&= \frac{1}{\mu} (\sinh(\mu \text{acosh}(\zeta)) - \sinh(-\mu \text{acosh}(\zeta))) \\
&= \frac{2}{\mu} \sinh(\mu \text{acosh}(\zeta))
\end{aligned}$$

The we set  $\xi = \frac{1}{2}(\zeta - 1)$ , thanks to identity 15.4.16 **DLMF**

$$\begin{aligned}
\sinh(\tau) {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; -\sinh^2(\tau)\right) &= \frac{1}{2\mu} \sinh(2\mu\tau) \\
\xi^{1/2} {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; -\xi\right) &= \frac{1}{2\mu} \sinh(2\mu\tau) & \sinh^2(\tau) &= \xi \\
\xi^{1/2} {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; -\xi\right) &= \frac{1}{2\mu} \sinh(\mu \operatorname{acosh}(\zeta)) & \cosh(2\tau) &= \zeta \\
&= \frac{1}{4} \int_{\mathcal{C}_0(\zeta)} \pi^*(\nu)
\end{aligned}$$

Thus we take 3/2-derivative based at  $\zeta = 1$

$$\begin{aligned}
\partial_\zeta^{3/2} \left( \int_{\mathcal{C}_0(\zeta)} \pi^*(\nu) \right) &= \partial_\zeta^2 \left( \frac{1}{\Gamma(\frac{1}{2})} \int_1^\zeta (\zeta - \zeta')^{-1/2} \left( \int_{\mathcal{C}_0(\zeta')} \pi^*(\nu) \right) d\zeta' \right) \\
&= 4\partial_\zeta^2 \left( \frac{1}{\Gamma(\frac{1}{2})} \int_0^\xi \frac{1}{2} (\xi - \xi')^{-1/2} \xi^{1/2} {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; \frac{3}{2}; -\xi'\right) 4 d\xi' \right) \\
&= 8\partial_\zeta^2 \left( \Gamma\left(\frac{3}{2}\right) \xi {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; 2; -\xi\right) \right) \\
&= 4\sqrt{\pi} \frac{1}{4} \partial_\xi^2 \left( \xi {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; 2; -\xi\right) \right) \\
&= -\sqrt{\pi} \partial_\xi {}_2F_1\left(\frac{1}{2} - \mu, \frac{1}{2} + \mu; 1; -\xi\right) \\
&= \sqrt{\pi} \Gamma\left(\frac{1}{2} - \mu\right) \Gamma\left(\frac{1}{2} + \mu\right) {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; -\xi\right) \\
&= \frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; \frac{1}{2} - \frac{\zeta}{2}\right) \\
&= \frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \hat{w}_{\mu,+}(\zeta - 1)
\end{aligned}$$