

Resurgence of the Airy function

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1 The Laplace transform

1.1 Analytic version

1.1.1 Regularity and decay properties

Take two copies \mathbb{R} and $\hat{\mathbb{R}}$ of the real line, with standard coordinates z and ζ respectively. The Laplace transform in ζ turns a function $\hat{\varphi}$ on $\hat{\mathbb{R}}_{>0}$ into a function $\mathcal{L}_\zeta \hat{\varphi}$ on $\mathbb{R}_{z>0}$, defined by the integral

$$\mathcal{L}_\zeta \hat{\varphi} = \int_0^\infty e^{-z\zeta} \hat{\varphi} d\zeta.$$

For $a \in [0, \infty]$, recall that $O_{\zeta \rightarrow a}(g)$ is the space of functions φ on $\hat{\mathbb{R}}_{>0}$ with $|\varphi| \lesssim g$ in some neighborhood of a . A function is *subexponential* if it's in $O_{\zeta \rightarrow \infty}(e^{c\zeta})$ for all $c > 0$. Let \mathcal{E}_ζ be the space of subexponential functions on $\hat{\mathbb{R}}_{>0}$ which are L^1 both locally and around $\zeta = 0$. If $\hat{\varphi}$ is in \mathcal{E}_ζ , then $\varphi = \mathcal{L}_\zeta \hat{\varphi}$ is well-defined, and it extends to a holomorphic function on the right half-plane $\mathbb{C}_{\operatorname{Re}(z)>0}$ [1, §5.6]. If $\hat{\varphi}$ is in $O_{\zeta \rightarrow 0}(1)$, then φ is in $O_{z \rightarrow \infty}(z^{-1})$ [2, equation 1.8].¹ More generally, if $\hat{\varphi}$ is in $O_{\zeta \rightarrow 0}(\zeta^\alpha)$, with $\alpha > -1$, then φ is in $O_{z \rightarrow \infty}(z^{-(\alpha+1)})$.

1.1.2 Action on differential operators

When $\hat{\varphi} \in \mathcal{E}_\zeta$, we can use differentiation under the integral to show that [2, Theorem 1.34]

$$\mathcal{L}_\zeta(\zeta^n \hat{\varphi}) = \left(-\frac{\partial}{\partial z}\right)^n \mathcal{L}_\zeta \hat{\varphi}. \quad (1)$$

When $\hat{\varphi}$ is n times differentiable, its n th derivative is in \mathcal{E} , and its zeroth through $(n-1)$ st derivatives extend continuously to zero, integration by parts gives the formula

$$\begin{aligned} \mathcal{L}_\zeta \left(\frac{\partial}{\partial \zeta} \right)^n \hat{\varphi} &= z^n \mathcal{L}_\zeta \hat{\varphi} - \left[\hat{\varphi} z^{n-1} + \hat{\varphi}' z^{n-2} + \hat{\varphi}'' z^{n-3} + \dots + \hat{\varphi}^{(n-1)} \right]_{\zeta=0} \\ &= z^n \mathcal{L} \left(\hat{\varphi} - \left[\hat{\varphi} + \hat{\varphi}' \zeta + \frac{\hat{\varphi}''}{2!} \zeta^2 + \dots + \frac{\hat{\varphi}^{(n-1)}}{(n-1)!} \zeta^{n-1} \right]_{\zeta=0} \right). \end{aligned} \quad (2)$$

¹The argument cited still works in our generality. For holomorphic $\hat{\varphi}$, one can also use Equation 1.5 of *Borel-Laplace Transform and Asymptotic Theory* (Sternin & Shatalov).

Note that if a function's derivative is subexponential, so is the function itself.²

1.2 Algebraic version

1.2.1 Definition

Let \mathcal{P}_ζ be the vector space spanned by ζ^α for $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{<0}$. Note that $\mathcal{P}_\zeta \cap \mathcal{E}_\zeta$ is $\mathcal{P}_\zeta^{>-1}$, the subspace spanned by ζ^α with $\alpha > -1$. Since

$$\mathcal{L}_\zeta(\zeta^\alpha) = \Gamma(\alpha + 1) z^{-(\alpha+1)}$$

for all $\alpha > -1$, let's use the same formula to extend \mathcal{L}_ζ to all of \mathcal{P}_ζ . This defines \mathcal{L}_ζ consistently on $\mathcal{E}_\zeta + \mathcal{P}_\zeta$.

1.2.2 Action on differential operators

Observe that

$$\mathcal{L}_\zeta(\zeta^{\alpha+1}) = -\frac{\partial}{\partial z} \mathcal{L}_\zeta(\zeta^\alpha)$$

for $\alpha \neq -1$. This extends identity 1 to all of \mathcal{P}_ζ .

Observe that

$$\mathcal{L}_\zeta \frac{\partial}{\partial \zeta}(\zeta^\alpha) = \begin{cases} z \mathcal{L}_\zeta(\zeta^\alpha) & \alpha \neq 0 \\ 0 & \alpha = 0, \end{cases}$$

and that $0 = z \mathcal{L}_\zeta(1) - 1$. This recovers identity 2 for any function in \mathcal{P}_ζ whose n th derivative is in $\mathcal{P}_\zeta^{>-1}$. Although the functions in $\mathcal{P}_\zeta^{<0}$ are singular at zero, let's pretend they vanish at zero. With that convention, formula 2 extends to all of \mathcal{P}_ζ .

Now we have the results of Section 1.1.2 for all functions in $\mathcal{E}_\zeta + \mathcal{P}_\zeta$. Identity 2 is particularly simple when $\hat{\varphi}$ has a *fractional power singularity* at $\zeta = 0$. By this, I mean that $\hat{\varphi}$ can be written as $\hat{\varphi}_{\text{frac}} + \hat{\varphi}_{\text{reg}}$, where $\hat{\varphi}_{\text{frac}} \in \mathcal{P}_\zeta$ has only non-integer exponents, and the zeroth through $(n-1)$ st derivatives of $\hat{\varphi}_{\text{reg}} \in \mathcal{E}_\zeta$ vanish at zero. Under this condition, all the initial value terms in the identity vanish, leaving

$$\mathcal{L}_\zeta \left(\frac{\partial}{\partial \zeta} \right)^n \hat{\varphi} = z^n \mathcal{L}_\zeta \hat{\varphi}.$$

²Say $f' \in O_{\zeta \rightarrow \infty}(e^{c\zeta})$. Then

$$\left| \int_0^Z f' d\zeta \right| \leq \int_0^Z |f'| d\zeta \lesssim \int_0^Z e^{c\zeta} d\zeta = \frac{1}{c}(e^{cZ} - 1) \lesssim e^{cZ}.$$

Now we know the integral on the left-hand side converges, implying that f extends continuously to zero, with $|f - f_{\zeta=0}| \lesssim e^{c\zeta}$.

1.3 Change of coordinates

Define a new coordinate ζ_a on $\hat{\mathbb{R}}$ so that $\zeta = a + \zeta_a$. From the calculation

$$\begin{aligned}\mathcal{L}_\zeta \hat{\varphi} &= \int_0^\infty e^{-z\zeta} \hat{\varphi} d\zeta \\ &= \int_0^\infty e^{-z(a+\zeta_a)} \hat{\varphi} d\zeta_a \\ &= e^{-az} \int_0^\infty e^{-z\zeta_a} \hat{\varphi} d\zeta_a \\ &= e^{-az} \mathcal{L}_{\zeta_a} \hat{\varphi},\end{aligned}$$

we learn that

$$\mathcal{L}_{\zeta_a} \hat{\varphi} = e^{az} \mathcal{L}_\zeta \hat{\varphi}.$$

Define new coordinates x and ξ on \mathbb{R} and $\hat{\mathbb{R}}$, respectively, so that $\zeta = b\xi$ and $z d\zeta = x d\xi$. Explicitly, $z = b^{-1}x$. From the calculation

$$\begin{aligned}\mathcal{L}_\zeta \hat{\varphi} &= \int_0^\infty e^{-z\zeta} \hat{\varphi} d\zeta \\ &= \int_0^\infty e^{-x\xi} \hat{\varphi} b d\xi \\ &= b \mathcal{L}_\xi \hat{\varphi},\end{aligned}$$

we learn that

$$\mathcal{L}_\xi \hat{\varphi} = b^{-1} \mathcal{L}_\zeta \hat{\varphi}.$$

2 The Airy equation

2.1 Basics

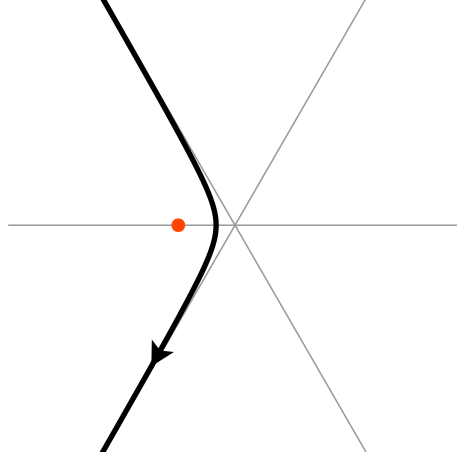
The Airy equation is

$$\left[\left(\frac{\partial}{\partial y} \right)^2 - y \right] \psi = 0. \quad (3)$$

One solution is given by the Airy function,

$$\text{Ai}(y) = \frac{1}{2\pi i} \int_\Gamma \exp \left(-\frac{1}{3}t^3 + yt \right) dt,$$

where Γ is a path that comes from ∞ at 120° and goes to ∞ at -120° .



The contour Γ in the u plane.

With the substitution $t = 2uy^{1/2}$, we can rewrite the Airy integral as

$$\text{Ai}(y) = y^{1/2} \frac{1}{\pi i} \int_{y^{-1/2}\Gamma} \exp \left[-\frac{2}{3}y^{3/2} (4u^3 - 3u) \right] du.$$

We've rescaled the contour by a factor of two, but it still approaches ∞ in the desired way. Note that $4u^3 - 3u$ is the third Chebyshev polynomial.

2.2 Rewriting as a modified Bessel equation

We can distill the most interesting part of the Airy function by writing

$$\text{Ai}(y) = \frac{1}{\pi\sqrt{3}} y^{1/2} K\left(\frac{2}{3}y^{3/2}\right),$$

where

$$K(z) = \frac{\sqrt{3}}{i} \int_{z^{-1/3}\Gamma} \exp \left[-z (4u^3 - 3u) \right] du. \quad (4)$$

Saying that Ai satisfies the Airy equation is equivalent to saying that K satisfies the modified Bessel equation

$$\left[z^2 \left(\frac{\partial}{\partial z} \right)^2 + z \frac{\partial}{\partial z} - \left[\left(\frac{1}{3} \right)^2 + z^2 \right] \right] \varphi = 0. \quad (5)$$

In fact, K is the modified Bessel function $K_{1/3}$ [3, equation 9.6.1].

As we'll see in Section ??, K is in $O_{z \rightarrow \infty}(e^{-z})$. It'll be helpful to pull out the exponential decay factor and work instead with the function κ defined by $K = e^{-z}\kappa$. Saying that K satisfies equation 5 is equivalent to saying that κ satisfies the equation

$$\left[z^2 \left(\frac{\partial}{\partial z} + 1 \right)^2 + z \left(\frac{\partial}{\partial z} + 1 \right) - \left[\left(\frac{1}{3} \right)^2 + z^2 \right] \right] \varphi = 0. \quad (6)$$

2.3 Asymptotic analysis

From [3], equations 10.40.2 and 10.17.1, we get the asymptotic series

$$\kappa \sim \left(\frac{\pi}{2}\right)^{1/2} \left[z^{-1/2} - \frac{(\frac{1}{6})_1(\frac{5}{6})_1}{2^1 \cdot 1!} z^{-3/2} + \frac{(\frac{1}{6})_2(\frac{5}{6})_2}{2^2 \cdot 2!} z^{-5/2} - \frac{(\frac{1}{6})_3(\frac{5}{6})_3}{2^3 \cdot 3!} z^{-7/2} + \dots \right] \quad (7)$$

2.4 Going to the spatial domain

2.4.1 A good try at $\zeta = 0$

Let's try to find a function \hat{K} with $K = \mathcal{L}_\zeta \hat{K}$, which is unique if it exists [2, Theorem 1.23]. If a function $\hat{\varphi}$ satisfies the equation

$$\left[(\zeta^2 - 1) \left(\frac{\partial}{\partial \zeta} \right)^2 + 3\zeta \frac{\partial}{\partial \zeta} + \left[1 - \left(\frac{1}{3} \right)^2 \right] \right] \hat{\varphi} = 0, \quad (8)$$

its Laplace transform $\varphi = \mathcal{L}_\zeta \hat{\varphi}$ satisfies the equation

$$\begin{aligned} \left[\left(-\frac{\partial}{\partial z} \right)^2 - 1 \right] \left(z^2 \varphi - [\hat{\varphi} z + \hat{\varphi}']_{\zeta=0} \right) + 3 \left(-\frac{\partial}{\partial z} \right) [z \varphi - \hat{\varphi}]_{\zeta=0} + \left[1 - \left(\frac{1}{3} \right)^2 \right] \varphi &= 0 \\ \left(\frac{\partial}{\partial z} \right)^2 [z^2 \varphi] - \left(z^2 \varphi - [\hat{\varphi} z + \hat{\varphi}']_{\zeta=0} \right) - 3 \left(\frac{\partial}{\partial z} \right) [z \varphi] + \left[1 - \left(\frac{1}{3} \right)^2 \right] \varphi &= 0 \\ \left[2 + 4z \frac{\partial}{\partial z} + z^2 \left(\frac{\partial}{\partial z} \right)^2 \right] \varphi - \left(z^2 \varphi - [\hat{\varphi} z + \hat{\varphi}']_{\zeta=0} \right) - 3 \left[1 + z \frac{\partial}{\partial z} \right] \varphi + \left[1 - \left(\frac{1}{3} \right)^2 \right] \varphi &= 0, \end{aligned}$$

which simplifies to

$$\left[z^2 \left(\frac{\partial}{\partial z} \right)^2 + z \frac{\partial}{\partial z} - \left[\left(\frac{1}{3} \right)^2 + z^2 \right] \right] \varphi = -[\hat{\varphi} z + \hat{\varphi}']_{\zeta=0}. \quad (9)$$

Since we want $\mathcal{L}_\zeta \hat{K}$ to satisfy equation 5, which is the homogeneous version of equation 9, we might guess that \hat{K} is a solution of equation 8 that vanishes through first order at $\zeta = 0$. Unfortunately, this would force \hat{K} to be zero.

2.4.2 Success at $\zeta = 1$

Define a new coordinate ζ_1 on $\hat{\mathbb{R}}$ so that $\zeta = 1 + \zeta_1$. Since

$$\begin{aligned} \mathcal{L}_{\zeta_1} \hat{K} &= e^z \mathcal{L}_\zeta \hat{K} \\ &= e^z K \\ &= \kappa, \end{aligned}$$

we want $\mathcal{L}_{\zeta_1} \hat{K}$ to satisfy equation 6. Rewrite equation 8 as

$$\left[\zeta_1 (\zeta_1 + 2) \left(\frac{\partial}{\partial \zeta_1} \right)^2 + 3(\zeta_1 + 1) \frac{\partial}{\partial \zeta_1} + \left[1 - \left(\frac{1}{3} \right)^2 \right] \right] \hat{\varphi} = 0. \quad (10)$$

If $\hat{\varphi}$ satisfies equation 10, $\mathcal{L}_{\zeta_1} \hat{\varphi}$ will satisfy an inhomogeneous version of equation 6, analogous to equation 9. This time, though, there's a trick we can use to zero out the inhomogeneity. Equation 10 has a regular singularity at $\zeta_1 = 0$, and one solution (up to scaling) is a

holomorphic multiple of $\zeta_1^{-1/2}$. That solution has a fractional power singularity at $\zeta_1 = 0$, as defined in Section 1.2.2, so its Laplace transform in ζ_1 satisfies equation 6.

Following this plan, let's find \hat{K} explicitly. Defining another coordinate ξ on $\hat{\mathbb{R}}$ so that $\zeta_1 = -2\xi$, we can rewrite equation 10 as the hypergeometric equation

$$\left[\xi(1-\xi) \left(\frac{\partial}{\partial \xi} \right)^2 + 3 \left(\frac{1}{2} - \xi \right) \frac{\partial}{\partial \xi} - \left[1 - \left(\frac{1}{3} \right)^2 \right] \right] \hat{\varphi} = 0. \quad (11)$$

The hypergeometric function

$$\hat{g}_1 = F\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; \xi\right)$$

satisfies equation 11 by definition. It's not the solution we want, though, because it's holomorphic around $\xi = 0$. Formula 15.10.12 from [3] gives another solution,

$$\hat{f}_0 = \xi^{-1/2} F\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \xi\right),$$

which is a holomorphic multiple of $\xi^{-1/2}$ near $\xi = 0$. By the argument above, $f_0 = \mathcal{L}_{\zeta_1} \hat{f}_0$ satisfies equation 6. This suggests that a constant multiple of \hat{f}_0 is our desired \hat{K} . The power series [3, equation 15.2.1]

$$\hat{f}_0 = \xi^{-1/2} + \frac{(\frac{1}{6})_1 (\frac{5}{6})_1}{(\frac{1}{2})_1 1!} \xi^{1/2} + \frac{(\frac{1}{6})_2 (\frac{5}{6})_2}{(\frac{1}{2})_2 2!} \xi^{3/2} + \frac{(\frac{1}{6})_3 (\frac{5}{6})_3}{(\frac{1}{2})_3 3!} \xi^{5/2} + \dots$$

converges near $\xi = 0$, showing that

$$\hat{f}_0 \in \xi^{-1/2} + O_{\xi \rightarrow 0}(\xi^{1/2}).$$

In terms of ζ_1 , we have

$$\hat{f}_0 \in -i\sqrt{2} \zeta_1^{-1/2} + O_{\zeta_1 \rightarrow 0}(\zeta_1^{1/2}).$$

Using the decay properties from Section 1.1.1, we deduce that

$$f_0 \in -i\sqrt{2\pi} z^{-1/2} + O_{z \rightarrow \infty}(z^{-3/2}).$$

Since we know that f_0 satisfies equation 6, this confirms that f_0 is a constant multiple of κ , which is the only subexponential solution of equation 6 (up to scaling). Comparing with series 7, we see that $\kappa = i \frac{1}{2} f_0$. We conclude that $\kappa = \mathcal{L}_{\zeta_1} \hat{K}$ for

$$\hat{K} = \frac{1}{\sqrt{2}} \zeta_1^{-1/2} F\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; -\frac{1}{2}\zeta_1\right).$$

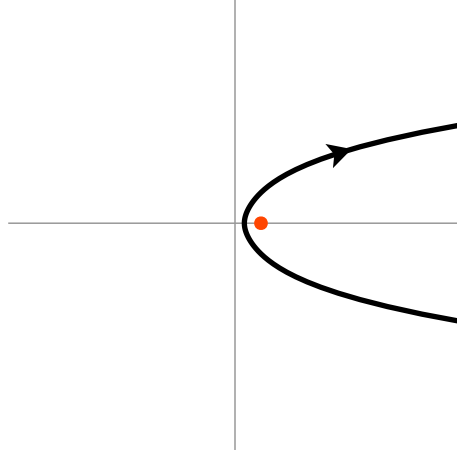
3 Sketches

3.1 Contour argument

We can recast integral 4 into $\hat{\mathbb{C}}$ by setting $\zeta = 4u^3 - 3u$. Projecting $z^{-1/3}\Gamma$ to a contour γ_z in $\hat{\mathbb{C}}$ and choosing the branch of u that lifts γ_z back to $z^{-1/3}\Gamma$, we have

$$K(z) = \frac{1}{i\sqrt{3}} \int_{\gamma_z} e^{-z\zeta} \frac{d\zeta}{4u^2 - 1}. \quad (12)$$

For $z \in (0, \infty)$, the contour $\gamma(z)$ runs clockwise around $[1, \infty)$, as shown below. Let's assume $z \in (0, \infty)$ for the rest of the section. **[Our conclusions should probably hold whenever $\text{Re}(z) > 0$.]**



The contour γ_1 in $\hat{\mathbb{C}}$.

It happens³ that for our desired branch of u ,

$$\frac{1}{4u^2 - 1} = F\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \zeta^2\right),$$

so we can rewrite integral 12 as

$$K(z) = \frac{1}{i\sqrt{3}} \int_{\gamma_z} e^{-z\zeta} F\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \zeta^2\right) d\zeta.$$

This gives us an alternate route to the conclusion of Section 2.4, which we'll follow below.

In addition to the solutions \hat{g}_1 and \hat{f}_0 from Section 2.4.2, equation 11 has the solutions

$$\begin{aligned} \hat{g}_0 &= F\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; 1 - \xi\right) \\ \hat{f}_1 &= (1 - \xi)^{-1/2} F\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; 1 - \xi\right), \end{aligned}$$

given by formulas 15.10.13 and 15.10.14 from [3]. These four solutions are related by identities 15.10.18, and 15.10.22 from [3]:

$$\begin{aligned} \hat{f}_1 &= \frac{1}{\sqrt{3}} \hat{g}_1 + \frac{1}{2} \hat{f}_0 \\ \hat{f}_0 &= \frac{1}{\sqrt{3}} \hat{g}_0 + \frac{1}{2} \hat{f}_1. \end{aligned}$$

Summing these identities, we see that

$$\hat{g}_1 + \hat{g}_0 = \frac{\sqrt{3}}{2} (\hat{f}_1 + \hat{f}_0).$$

The quadratic transformation identity 15.8.27 from [3] shows [verified numerically] that⁴

$$\begin{aligned} F\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \zeta^2\right) &= \frac{1}{3} (\hat{g}_1 + \hat{g}_0) \\ &= \frac{1}{2\sqrt{3}} (\hat{f}_1 + \hat{f}_0) \end{aligned}$$

³How to verify this? This hypergeometric function can be written in terms of Legendre functions, but I don't know where to go from there. **Veronica:** Look at "Special cases" section in [3], e.g. 15.4.14!

⁴Note that $2\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}) = 2\Gamma(\frac{1}{2})\frac{1}{2}\Gamma(\frac{1}{2}) = \pi$ and $[\Gamma(\frac{5}{6})\Gamma(\frac{7}{6})]^{-1} = [\Gamma(\frac{5}{6})\frac{1}{6}\Gamma(\frac{1}{6})]^{-1} = \frac{6\sin(\frac{1}{6}\pi)}{\pi} = \frac{3}{\pi}$.

so we have

$$K(z) = \frac{1}{6i} \int_{\gamma_z} e^{-z\zeta} (\hat{f}_1 + \hat{f}_0) d\zeta.$$

The solution \hat{f}_1 is regular on $\zeta \in [1, \infty)$, so it integrates to zero. The solution \hat{f}_0 looks like $(1 - \zeta)^{-1/2}$ times a function which is regular on $\zeta \in [1, \infty)$, so integrating it along γ_z is the same as integrating it negative-twice along the contour that goes rightward from 1, with its value continued from above the singularity. Hence,

$$\begin{aligned} K(z) &= \frac{i}{3} \int_1^\infty e^{-z\zeta} \hat{f}_0 d\zeta \\ &= \frac{i}{3} \mathcal{L}_{\zeta_1} \hat{f}_0. \end{aligned}$$

[Constant conflicts with Section 2.4]

3.2 Correspondence with Mariño's series

Let $f_1(z)$ be the holomorphic function corresponding to Mariño's formal power series $\varphi_1(z^{-1})$. The formal power series corresponding to f will be written in the variable z .

$$\begin{aligned} \text{Ai}(x) &= \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-z} \varphi_1\left(\frac{2}{3}z^{-1}\right) \\ &= \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-z} f_1\left(\frac{3}{2}z\right) \\ \text{Ai}(x) &= \frac{1}{\pi\sqrt{3}} x^{1/2} K\left(\frac{2}{3}x^{3/2}\right) \end{aligned}$$

Putting together,

$$\begin{aligned} \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-z} f_1\left(\frac{3}{2}z\right) &= \frac{1}{\pi\sqrt{3}} x^{1/2} K\left(\frac{2}{3}x^{3/2}\right) \\ \frac{\sqrt{3\pi}}{2} x^{-3/4} e^{-z} f_1\left(\frac{3}{2}z\right) &= K\left(\frac{2}{3}x^{3/2}\right) \\ \frac{\sqrt{3\pi}}{2} \left(\frac{3}{2}z\right)^{-1/2} e^{-z} f_1\left(\frac{3}{2}z\right) &= K(z) \\ \sqrt{\frac{\pi}{2}} z^{-1/2} e^{-z} f_1\left(\frac{3}{2}z\right) &= K(z) \\ \sqrt{\frac{\pi}{2}} \left[\mathcal{L}^{-1} z^{-1/2}\right] * \left[\mathcal{L}^{-1} f_1\left(\frac{3}{2}z\right)\right](\zeta - 1) &= \hat{K}(\zeta) \\ \sqrt{\frac{\pi}{2}} \left[\Gamma\left(-\frac{1}{2}\right)^{-1} \zeta^{-1/2}\right] * \frac{2}{3} \hat{f}_1\left[\frac{2}{3}(\zeta - 1)\right] &= \hat{K}(\zeta) \\ -\frac{1}{3\sqrt{2}} \left[\zeta^{-1/2}\right] * \hat{f}_1\left[\frac{2}{3}(\zeta - 1)\right] &= \hat{K}(\zeta) \end{aligned}$$

Notice that if the hypergeometric differentiation formula holds for fractional derivatives,

$$\left(\frac{\partial}{\partial \xi}\right)^{1/2} F\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2}; \xi\right) \propto F\left(\frac{7}{6}, \frac{11}{6}; 2; \xi\right)$$

References

- [1] C. Mitschi and D. Sauzin, *Divergent Series, Summability and Resurgence I: Monodromy and Resurgence*. No. 2153 in Lecture Notes in Mathematics. 2016.
- [2] J. L. Schiff, *The Laplace Transform: Theory and Applications*. Springer, 1999.
- [3] “NIST digital library of mathematical functions.” <http://dlmf.nist.gov>, release 1.1.3 of 2021-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, editors.