Resurgence of modified Bessel functions of second kind

Veronica Fantini

February 23, 2022

1 Modified Bessel function of second kind

The modified Bessel function of the second kind $K_\mu(z)$ is a solution of the equation

$$w'' + \frac{w'}{z} - w - \frac{\mu^2}{z^2} = 0 \tag{1}$$

such that $K_\mu(z) \sim \sqrt{\pi/(2z)}e^{-z}$ as $z \to \infty$ in $|\arg z| < \frac{3\pi}{2}$[1]. It has a branch point at $z = 0$ for every $\mu \in \mathbb{C}$ and the principal branch is analytic in $\mathbb{C} \setminus (-\infty, 0]$.

1.1 Differential equation

From the general theory of ODE, the formal integral solution of (1) is two parameters family

$$\tilde{w}(z) = U_1 e^{-z} z^{-1/2} \tilde{w}_{\mu,+}(z) + U_2 e^{z} z^{-1/2} \tilde{w}_{\mu,-}(z) \tag{4}$$

where $\tilde{w}_{\mu,\pm} = \sum_{j\geq 0} a_{\pm,j} z^{-j} \in \mathbb{C}[\![z^{-1}]\!]$ are unique formal solutions of

$$\begin{aligned} \tilde{w}''_{\mu,+} - 2\tilde{w}'_{\nu,+} + \frac{\tilde{w}_{\mu,+}}{4z^2} - \frac{\mu^2}{z^2}\tilde{w}_{\mu,+} = 0 \\ \tilde{w}''_{\mu,-} + 2\tilde{w}'_{\mu,-} + \frac{\tilde{w}_{\mu,-}}{4z^2} - \frac{\mu^2}{z^2}\tilde{w}_{\mu,-} = 0 \end{aligned}$$

[1] A system of solution of Bessel equation is given by $I_\mu(z)$ and $K_\mu(z)$. In particular, their asymptotic behaviour as $z \to \infty$ is given by

$$\tilde{I}_\mu(z) = \frac{1}{\sqrt{2\pi}} e^{z} z^{-1/2} \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\mu\right)_k \left(\frac{1}{2}+\mu\right)_k}{2^k k!} z^{-k} \tag{2}$$

$$\tilde{K}_\mu(z) = \sqrt{\frac{\pi}{2}} e^{-z} z^{-1/2} \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\mu\right)_k \left(\frac{1}{2}+\mu\right)_k}{(-2)^k k!} z^{-k} \tag{3}$$

In particular, $\tilde{K}_\mu(z) = \sqrt{\frac{\pi}{2}} e^{-z} z^{-1/2} \tilde{w}_{\mu,+}(z)$ and $\tilde{I}_\mu(z) = \frac{1}{\sqrt{2\pi}} e^{z} z^{-1/2} \tilde{w}_{\mu,-}(z)$ (once we choose $a_{\pm,0} = 1$). We now compute the Borel transform of $\tilde{w}_+(z)$[2] it is a solution of

$$\begin{aligned}
&\zeta^2 \hat{w}_{\mu,+} + 2t\hat{w}_{\mu,+} + \left(\frac{1}{4} - \nu^2\right) \int_0^\zeta (\zeta - s)\hat{w}_{\nu,+}(s)ds = 0 \\
&\zeta^2 \hat{w}''_{\mu,+} + 2\zeta \hat{w}''_{+} + 4\zeta \hat{w}'_{\mu,+} + \left(\frac{9}{4} - \mu^2\right) \hat{w}_{\mu,+} = 0 \\
&t(1-t)\hat{w}''_{\mu,+} + (2-4t)\hat{w}'_{\mu,+} - \left(\frac{9}{4} - \mu^2\right) \hat{w}_{mu,+} = 0 \qquad\qquad t = -\frac{\zeta}{2}
\end{aligned}$$

therefore $\hat{w}_{\mu,+}(\zeta)$ is an hypergeometric function

$$\hat{w}_{\mu,+}(\zeta) = {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; -\frac{\zeta}{2}\right) \tag{5}$$

and it has a branch point singularities at $\zeta = -2$. By the same reasoning,

$$\hat{w}_{\mu,-}(\zeta) = {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; \frac{\zeta}{2}\right) \tag{6}$$

and it has branch point at $\zeta = 2$. Thanks to explicit formula for the analytic continuation of hypergeomtric functions (see [?] 15.2.3)

$$\begin{aligned}
\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \left(\hat{w}_{\mu,+}(\zeta + i0) - \hat{w}_{\mu,+}(\zeta - i0)\right) &= \frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \left(-\frac{\zeta}{2} - 1\right)^{-1} {}_2F_1\left(\frac{1}{2} + \mu, \frac{1}{2} - \mu; 0; 1 + \frac{\zeta}{2}\right) \quad \zeta > \\
&= -\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \sum_{k\geq 0} \frac{\left(\frac{1}{2} - \mu\right)_k \left(\frac{1}{2} + \mu\right)_k}{\Gamma(k)k!} \left(1 + \frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \sum_{k\geq 1} \frac{\left(\frac{1}{2} - \mu\right)_k \left(\frac{1}{2} + \mu\right)_k}{\Gamma(k)k!} \left(1 + \frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \frac{1}{\Gamma\left(\frac{1}{2} - \mu\right)\Gamma\left(\frac{1}{2} + \mu\right)} \cdot \\
&\qquad \cdot \sum_{k\geq 1} \frac{\Gamma\left(\frac{1}{2} - \mu + k\right)\Gamma\left(\frac{1}{2} + \mu + k\right)}{\Gamma(k)k!} \left(1 + \frac{\zeta}{2}\right)^{k-1} \\
&= -\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma(\frac{3}{2} - \mu)\Gamma(\frac{3}{2} + \mu)} \frac{1}{\Gamma\left(\frac{1}{2} - \mu\right)\Gamma\left(\frac{1}{2} + \mu\right)} \cdot \\
&\qquad \cdot \sum_{k\geq 0} \frac{\Gamma\left(\frac{3}{2} - \mu + k\right)\Gamma\left(\frac{3}{2} + \mu + k\right)}{\Gamma(k+1)(k+1)!} \left(1 + \frac{\zeta}{2}\right)^{k} \\
&= -\frac{\pi\sqrt{\pi}}{\cos(\mu\pi)} \frac{2\pi i}{\Gamma\left(\frac{1}{2} - \mu\right)\Gamma\left(\frac{1}{2} + \mu\right)} \; {}_2F_1\left(\frac{3}{2} - \mu, \frac{3}{2} + \mu; 2; 1 + \frac{\zeta}{2}\right) \\
&= -2i\sqrt{\pi}\hat{w}_{\mu,-}(\zeta + 2)
\end{aligned}$$

[2]We do not consider constant term of $\tilde{w}_{\mu,\pm}$, i.e. $\mathcal{B} : \mathbb{C}[\![z^{-1}]\!] \to \mathbb{C}[\zeta]$.

and for $\hat{w}_{\mu,-}(\zeta)$

$$\begin{aligned}
\hat{w}_{\mu,-}(\zeta+i0)-\hat{w}_{\mu,-}(\zeta-i0) &= -\frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)}\left(\frac{\zeta}{2}-1\right)^{-1}{}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;0;1-\frac{\zeta}{2}\right)\quad \zeta<2\\
&= \frac{2\pi i}{\Gamma(\frac{3}{2}-\mu)\Gamma(\frac{3}{2}+\mu)}\sum_{k\geq 0}\frac{\left(\frac{1}{2}-\mu\right)_k\left(\frac{1}{2}+\mu\right)_k}{\Gamma(k)k!}\left(1-\frac{\zeta}{2}\right)^{k-1}\\
&= 2i\cos(\nu\pi)\ {}_2F_1\left(\frac{3}{2}-\mu,\frac{3}{2}+\mu;2;1-\frac{\zeta}{2}\right)\\
&= 2i\cos(\mu\pi)\hat{w}_{\mu,+}(\zeta-2)
\end{aligned}$$

In addition, the previous relations computes the Stokes constants which are funtions of ν and are given by $\pm 2i\cos(\mu\pi)$.

1.2 Exponential integral

Let $X=\mathbb{C}^*$, $f(x)=x+\frac{1}{x}$ and for every $\mu\in[0,+\infty)$ let $\nu=x^{\mu-1}+\frac{1}{x^{\mu+1}}dx$, then

$$I(z;m):=\int_0^\infty e^{-zf}\nu \tag{7}$$

In particulr, on the universal cover $\pi\colon\tilde{C}\to\mathbb{C}^*$ setting $x=e^u$

$$I(\frac{z}{2};\mu)=2\int_{-\infty}^{\infty}e^{-z\cosh(u)}\cosh(\mu u)du=4K_\mu(z)\quad |\arg(z)|<\pi/2 \tag{8}$$

where $K_\mu(z)$ is the second kind modified Bessel function with parameter μ. It is worth mentioning that $I(z;\mu)$ differs from $I(z;0)$ only in $\pi^*(\nu)$ while $\pi^*(f)$ stays the same for every $\mu\in[0,\infty)$. Hence we can adapt part of the argument used in Bessel example (see), and verify the 3/2-derivative formula: let $\zeta=\cosh(u)$

$$\begin{aligned}
\int_{\mathcal{C}_0(\zeta)}\pi^*(\nu) &= \int_{\mathcal{C}_0(\zeta)}\cosh(\mu u)du\\
&= \frac{1}{\mu}\Big[\sinh(\mu u)\Big]_{\mathrm{start}\mathcal{C}_0(\zeta)}^{\mathrm{end}\mathcal{C}_0(\zeta)}\\
&= \frac{1}{\mu}\left(\sinh\left(\mu\,\mathrm{acosh}\left(\zeta\right)\right)-\sinh\left(-\mu\,\mathrm{acosh}\left(\zeta\right)\right)\right)\\
&= \frac{2}{\mu}\sinh\left(\mu\,\mathrm{acosh}\left(\zeta\right)\right)
\end{aligned}$$

The we set $\xi=\frac{1}{2}\left(\zeta-1\right)$, thanks to identity 15.4.16 **DLMF**

$$
\begin{aligned}
\sinh(\tau)\,{}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;\frac{3}{2};-\sinh^2(\tau)\right) &= \frac{1}{2\mu}\sinh(2\mu\tau) \\
\xi^{1/2}\,{}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;\frac{3}{2};-\xi\right) &= \frac{1}{2\mu}\sinh(2\mu\tau) && \sinh^2(\tau)=\xi \\
\xi^{1/2}\,{}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;\frac{3}{2};-\xi\right) &= \frac{1}{2\mu}\sinh(\mu\ \mathrm{acosh}\,(\zeta)) && \cosh(2\tau)=\zeta \\
&= \frac{1}{4}\int_{\mathcal{C}_0(\zeta)}\pi^*(\nu)
\end{aligned}
$$

Thus we take 3/2-derivative based at $\zeta=1$

$$
\begin{aligned}
\partial_\zeta^{3/2}\left(\int_{\mathcal{C}_0(\zeta)}\pi^*(\nu)\right) &= \partial_\zeta^2\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^\zeta(\zeta-\zeta')^{-1/2}\left(\int_{\mathcal{C}_0(\zeta')}\pi^*(\nu)\right)d\zeta'\right) \\
&= 4\partial_\zeta^2\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^\xi\frac{1}{2}(\xi-\xi')^{-1/2}\xi^{1/2}\,{}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;\frac{3}{2};-\xi'\right)4\ d\xi'\right) \\
&= 8\partial_\zeta^2\left(\Gamma\left(\frac{3}{2}\right)\xi\ {}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;2;-\xi\right)\right) \\
&= 4\sqrt{\pi}\frac{1}{4}\partial_\xi^2\left(\xi\ {}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;2;-\xi\right)\right) \\
&= -\sqrt{\pi}\partial_\xi\ {}_2F_1\left(\frac{1}{2}-\mu,\frac{1}{2}+\mu;1;-\xi\right) \\
&= \sqrt{\pi}\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\mu\right){}_2F_1\left(\frac{3}{2}-\mu,\frac{3}{2}+\mu;2;-\xi\right) \\
&= \frac{\pi\sqrt{\pi}}{\cos(\mu\pi)}{}_2F_1\left(\frac{3}{2}-\mu,\frac{3}{2}+\mu;2;\frac{1}{2}-\frac{\zeta}{2}\right) \\
&= \frac{\pi\sqrt{\pi}}{\cos(\mu\pi)}\hat{w}_{\mu,+}(\zeta-1)
\end{aligned}
$$

As showed by Aaron, if $T_n(u)$ and $U_n(u)$ denote the Chebyschev polynomials [3]

[3] $T_n(\cos(t))=\cos(nt)$ and $U_n(\cos(t))\sin(t)=\sin((n+1)t)$.

$$
\begin{aligned}
K_\nu(z) &= \frac{1}{2i\sin(\nu\pi)}\int_{\mathcal{C}_\alpha} e^{z\cosh(t)}\sinh(\nu t)dt && u=\cosh(\nu t)\\
&= -\frac{1}{2i\nu\sin(\nu\pi)}\int_{\mathcal{C}_\alpha} e^{zT_{\frac{1}{\nu}}(u)}du && \zeta = T_{\frac{1}{\nu}}(u)\\
&= \frac{1}{2i\sin(\nu\pi)}\int_{\mathcal{H}_\alpha} e^{-\zeta z}\frac{d\zeta}{U_{\frac{1}{\nu}-1}(u)}\\
&= -\frac{1}{2i\sin(\nu\pi)}\int_{\mathcal{H}_\alpha} e^{-\zeta z}{}_2F_1\left(\frac{1-\nu}{2},\frac{1+\nu}{2};\frac{3}{2};1-\zeta^2\right)d\zeta
\end{aligned}
$$

where $\mathcal{C}_\alpha$ **has to be checked, but I guess** $\alpha = 1$

Indentity 15.10.17 from [**?**] splits the integrand above into

$$
\begin{aligned}
{}_2F_1\left(\frac{1-\nu}{2},\frac{1+\nu}{2};\frac{3}{2};1-\zeta^2\right) &= C_1\,{}_2F_1\left(\frac{1-\nu}{2},\frac{1+\nu}{2};\frac{1}{2};\zeta^2\right)+C_2\zeta\,{}_2F_1\left(1-\frac{\nu}{2},1+\frac{\nu}{2};\frac{3}{2};\zeta^2\right)\\
&= \tilde{C}_1\left[{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}-\frac{\zeta}{2}\right)+{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)\right]+\\
&\quad+\tilde{C}_2\left[{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}-\frac{\zeta}{2}\right)+{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)\right]
\end{aligned}
$$

therefore, collecting the contrubutions together we have

$$
K_\nu(z) = \frac{i}{4\sin(\nu\pi)}\int_{\mathcal{H}_\alpha} e^{-\zeta z}\left[{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}-\frac{\zeta}{2}\right)+{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)\right]d\zeta \tag{9}
$$

Since ${}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}+\frac{\zeta}{2}\right)$ is singular at $\zeta=1$, the inverse Laplace transform of $K_\nu(z)$ is

$$
\hat{K}_\nu(\zeta) = \frac{i}{4\sin(\nu\pi)}{}_2F_1\left(1-\nu,1+\nu;\frac{3}{2};\frac{1}{2}-\frac{\zeta}{2}\right)
$$