Resurgence of the Airy function and other exponential integrals

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1 Introduction

1.1 Why does Borel resummation work?

- Each resummation method for asymptotic series makes some implicit assumption that allows us to reconstruct a holomorphic function from its asymptotic behavior.
- The resummation method works correctly for functions which satisfy that assumption.
- For the modified Bessel function $K_{1/3}$, Borel resummation works because the asymptotic series encodes a second-order differential equation.
  - Different aspects of this example appear in various places (Mariño, Kawai–Takei, Sauzin). We give a detailed, unified treatment.
- We can generalize this argument to all $K_{1/n}$ and their limit K_0.
- We can also generalize to all third-order exponential integrals.
  - Most of them are equivalent to the $K_{1/3}$ integral, but there's also an interesting degeneration.

1.2 Fractional derivative formula

- Theorem **??** says that for a certain class of exponential integrals

$$I(z) = \int_\Gamma e^{-zf}\ \nu,$$

  the inverse Laplace [**better to say Borel?**] transform is the $\frac{3}{2}$ derivative of $d\zeta/df$, where $f^*d\zeta = \nu$ [**check**].

1.3 Stokes phenomenon

- For Bessel functions, we can see explicitly how solutions jump when the Laplace transform angle crosses a critical value.
- The jump comes from the branch cut difference identity for hypergeometric functions.

2 The Laplace and Borel transforms

2.1 The Laplace transform

- Action on differential equations.
  - Can we find a way to prove this when the differential operator spits out a function that's not integrable around zero?
- Global picture?

2.2 The Borel transform

- Action on differential equations.
  - No inhomogeneous terms! How is this consistent with the Laplace transform's action? Is there always an inhomogeneous solution with subexponential asymptotics?

3 Third-order exponential integrals

- Reduce to
$$I(z) = \int \exp\left[-z\big(u^3 + pu + q\big)\right]\, du$$
using change of coordinate.
- When $p \neq 0$, can reduce further to
$$I(z) = p^{1/2} e^{-qz} K_{1/3}(p^{3/2} z).$$
- As p goes to zero, $I(z)$ degenerates to
$$\left(\tfrac{1}{2}\right)^{2/3} e^{-qz} \Gamma(\tfrac{1}{3}) z^{-1/3} = \left(\tfrac{1}{2}\right)^{2/3} e^{-qz} \mathcal{L}_{\zeta,0}(\zeta^{-2/3}) = \left(\tfrac{1}{2}\right)^{2/3} \mathcal{L}_{\zeta_{-q},q}(\zeta^{-2/3}).$$