EXPONENTIAL INTEGRALS

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Let X be a $N-\dim$ manifold, $f : X \to \mathbb{C}$ be a holomorphic Morse function with only simple critical points, and $\nu \in \Gamma(X, \Omega^N)$, and set

$$I(z) := \int_{\mathcal{C}} e^{-zf}\,\nu \tag{2.1}$$

where $\mathcal{C}$ is a suitable countur such that the integral is well defined. For any Morse cirtial points x_α of f, the saddle point approximation gives the following formal series

$$I_\alpha(z) := \int_{\mathcal{C}_\alpha} e^{-zf}\,\nu \sim \tilde{I}_\alpha := e^{-zf(x_\alpha)}(2\pi)^{N/2} z^{-N/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n}. \tag{2.2}$$

Theorem 2.1. *Let* $\tilde{\varphi}_\alpha(z) := e^{-zf(x_\alpha)}(2\pi)^{N/2} \sum_{n\geq 0} a_{\alpha,n} z^{-n}$

(1) $\tilde{\varphi}_\alpha$ *is Gevrey-1;*
(2) $\hat{\varphi}(\zeta) := \mathcal{B}(\tilde{\varphi})$ *is a germ of analytic function at* $\zeta = \zeta_\alpha = f(x_\alpha)$*;*
(3) *the following formual holds true*

$$\hat{\varphi}_\alpha(\zeta) \;=\; \partial^{N/2}_{\zeta, \text{based at}\, \zeta_\alpha} \left(\int_{f^{-1}(\zeta_\alpha)}^{f^{-1}(\zeta)} \nu \right) \;=\; \Gamma(-N/2) \int_{\zeta_\alpha}^{\zeta} (\zeta - \zeta')^{-N/2} \int_{f^{-1}(\zeta')} \frac{\nu}{df}\, d\zeta' \tag{2.3}$$

Definition 2.2. Let $\alpha \in (0,1)$, then the α-Caputo's derivative of a smooth fucntion f is defined as

$$\partial^\alpha_x f(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f'(s)\, ds \tag{2.4}$$

Example 2.3 (Airy)**.** Let $f(t) = \frac{t^3}{3} - t$ and

$$I(z) := \int_\gamma e^{-zf(t)} \mathrm{dt}$$

where γ is a countour where the integral is well defined.

By the change of coordinates $z = x^{3/2}$, $I(z) = -2\pi i z^{-1/3} Ai(x)$ where

$$Ai(x) := \frac{1}{2\pi i} \int_{-\infty e^{-i\frac{\pi}{3}}}^{\infty e^{i\frac{\pi}{3}}} e^{\frac{t^3}{3} - zt} \mathrm{dt}$$

hence $I(z)$ solves the following ODE[1]

$$I''(z) - \frac{4}{9} I(z) + \frac{I'(z)}{z} - \frac{1}{9} \frac{I(z)}{z^2} = 0 \tag{2.5}$$

A formal solution of (2.5) can be computed by making the following ansatz

$$\tilde{I}(z) = \sum_{k \in \mathbb{N}^2} U^k e^{-\lambda \cdot kz} z^{-\tau \cdot k} w_k(z) \tag{2.6}$$

with $U^{(k_1,k_2)} = U_1^{k_1} U_2^{k_2}$ and $U_1, U_2 \in \mathbb{C}$ are constant parameter, $\lambda = (\frac{2}{3}, -\frac{2}{3})$, $\tau = (\frac{1}{2}, \frac{1}{2})$, and $\tilde{w}_k(z) \in \mathbb{C}[[z^{-1}]]$. In addition, we can check that the only non zero $\tilde{w}_k(z)$ occurs at $k = (1,0)$ and $k = (0,1)$, therefore

$$\tilde{I}(z) = U_1 e^{-2/3z} z^{-1/2} \tilde{w}_+(z) + U_2 e^{2/3z} z^{-1/2} \tilde{w}_-(z) \tag{2.7}$$

where from now on we denote $\tilde{w}_+(z) = \tilde{w}_{(1,0)}(z)$ and $\tilde{w}_-(z) = \tilde{w}_{(0,1)}(z)$. In particular, $\tilde{w}_+(z)$ and $\tilde{w}_-(z)$ are formal solution of

$$\tilde{w}_+'' - \frac{4}{3} \tilde{w}_+' + \frac{5}{36} \frac{\tilde{w}_+}{z^2} = 0 \tag{2.8}$$

$$\tilde{w}_-'' + \frac{4}{3} \tilde{w}_-' + \frac{5}{36} \frac{\tilde{w}_-}{z^2} = 0 \tag{2.9}$$

Taking the Borel transform of (2.8), (2.9) we get

$$\zeta^2 \hat{w}_+(\zeta) + \frac{4}{3} \zeta \hat{w}_+ + \frac{5}{36} \zeta * \hat{w}_+ = 0$$

$$\zeta^2 \hat{w}_+(\zeta) + \frac{4}{3} \zeta \hat{w}_+ + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}_+(\zeta') d\zeta' = 0$$

$$\zeta^2 \hat{w}_-(\zeta) - \frac{4}{3} \zeta \hat{w}_- + \frac{5}{36} \zeta * \hat{w}_- = 0$$

$$\zeta^2 \hat{w}_-(\zeta) - \frac{4}{3} \zeta \hat{w}_- + \frac{5}{36} \int_0^\zeta (\zeta - \zeta') \hat{w}_-(\zeta') d\zeta' = 0$$

[1] $Ai(x)$ solves the Airy equation $y'' = xy$.

and taking derivatives we get

$$
\begin{aligned}
&\zeta(\frac{4}{3}+\zeta)\hat{w}_+''+(\frac{8}{3}+4\zeta)\hat{w}_+'+\frac{77}{36}\hat{w}_+=0\\
&\frac{4}{3}\zeta(1+\frac{3}{4}\zeta)\hat{w}_+''+(\frac{8}{3}+4\zeta)\hat{w}_+'+\frac{77}{36}\hat{w}_+=0\\
&\qquad u(1-u)\hat{w}_+''(u)+(2-4u)\hat{w}_+'(u)-\frac{77}{36}\hat{w}_+(u)=0 \qquad\qquad u=-\frac{3}{4}\zeta
\end{aligned}
$$

$$
\begin{aligned}
&\zeta(-\frac{4}{3}+\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0\\
&\frac{4}{3}\zeta(-1+\frac{3}{4}\zeta)\hat{w}_-''+(-\frac{8}{3}+4\zeta)\hat{w}_-'+\frac{77}{36}\hat{w}_-=0\\
&\qquad u(1-u)\hat{w}_-''(u)+(2-4u)\hat{w}_-'(u)-\frac{77}{36}\hat{w}(u) \qquad\qquad u=\frac{3}{4}\zeta
\end{aligned}
$$

Notice that the latter equations are hypergeometric, hence a solution is given by

$$\hat{w}_+(\zeta)=c_1\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right) \tag{2.10}$$

$$\hat{w}_-(\zeta)=c_2\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right) \tag{2.11}$$

for some constants $c_1,c_2\in\mathbb{C}$ (see DLMF 15.10.2). In addition $\hat{w}_\pm(\zeta)$ have a log singularity respectively at $\zeta=\mp\frac{4}{3}$, therefore they are $\{\mp\frac{4}{3}\}$-resurgent functions.[2]

Remark 2.4. $\hat{w}_+(\zeta)$ is Laplace summable along the positive real axis, and it can be analyticaly continued on $\mathbb{C}\setminus-\frac{4}{3}\mathbb{R}_{\leq 0}$ with (see 15.2.3 DLMF)

$$
\begin{aligned}
\hat{w}_+(\zeta+i0)-\hat{w}_+(\zeta-i0)&=-\frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\sum_{k\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1+\frac{3}{4}\zeta)^n \qquad \zeta<-\frac{4}{3}\\
&=-\frac{36}{5}i(-\frac{3}{4}\zeta-1)^{-1}\left(\frac{5}{144}(4+3\zeta)_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)\right)\\
&=\mathbf{i}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1+\frac{3}{4}\zeta\right)\\
&=\mathbf{i}\,\hat{w}_-(\zeta+\frac{4}{3})
\end{aligned}
$$

[2]The solution we find are equal to the ones in DLMF §9.7. They also agree with slide 10 of Maxim's talk for ERC and with the series (6.121) in Sauzin's book. However they do not agree with (2.16) in Mariño's Diablerets.

Anolougusly, $\hat{w}_-(\zeta)$ is Laplace summable along the negative real axis, and it jumps across the branch cut $\frac{4}{3}\mathbb{R}_{\geq 0}$ as

$$\begin{aligned}
\hat{w}_-(\zeta+i0)-\hat{w}_-(\zeta-i0) &= \frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\sum_{k\geq 0}\frac{(5/6)_n(1/6)_n}{\Gamma(n)n!}(1-\frac{3}{4}\zeta)^n \qquad \zeta>\frac{4}{3}\\
&= \frac{36}{5}i(\frac{3}{4}\zeta-1)^{-1}\left(-\frac{5}{144}(-4+3\zeta)_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)\right)\\
&= -\mathbf{i}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,1-\frac{3}{4}\zeta\right)\\
&= -\mathbf{i}\,\hat{w}_+(\zeta-\frac{4}{3})
\end{aligned}$$

These relations manifest the resurgence property of $\tilde{I}$, indeed near the singularities in the Borel plane of either $\hat{w}_+$ or $\hat{w}_-$, $\hat{w}_-$ and $\hat{w}_+$ respectively contribute to the jump of the former solution.

Our next goal is to prove that the Borel transform of $\tilde{I}(z)$ can be written in terms of $1/f'(f^{-1}(\zeta))$, namely formula (2.3). It is convenient to consider the two asymptotic formal solutions separately, namely we define

$$\tilde{I}_{-1}(z) := e^{-2/3z}z^{-1/2}\,\tilde{w}_+(z) =: z^{-1/2}\,\tilde{u}_+(z) \tag{2.12}$$

$$\tilde{I}_1(z) := e^{2/3z}z^{-1/2}\,\tilde{w}_-(z) =: z^{-1/2}\,\tilde{u}_-(z) \tag{2.13}$$

In particular, $\tilde{u}_\pm(z)$ are solutions of

$$\tilde{u}''(z)-\frac{4}{9}\tilde{u}(z)+\frac{5}{36}\frac{\tilde{u}(z)}{z^2}=0 \tag{2.14}$$

with asymptotic behaviour $\tilde{u}_\pm(z)\sim O(e^{\pm 2/3z})$ as $z\to\infty$.

The Borel transforms $\hat{u}_\pm(\zeta)$ solve the same equation

$$\zeta^2\,\hat{u}-\frac{4}{9}\hat{u}+\frac{5}{36}\zeta*\hat{u}$$

$$\zeta^2\,\hat{u}-\frac{4}{9}\hat{u}+\frac{5}{36}\int_0^\zeta(\zeta-\zeta')\hat{u}(\zeta')d\zeta'$$

taking derivatives is equivalent to

$$(\zeta^2-\frac{4}{9})\hat{u}''(\zeta)+4\zeta\,\hat{u}'(\zeta)+\frac{77}{36}\hat{u}(\zeta)=0$$

and Mathematica gives the following solutions

$$
\begin{aligned}
\hat{u}(\zeta) &= c_1\,{}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{9}{4}\zeta^2\right)+\frac{3i}{2}\zeta\, c_2\,{}_1F_2\left(\frac{13}{12},\frac{17}{12},\frac{3}{2},\frac{9}{4}\zeta^2\right)= \\
&= c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}}\left({}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{1}{2}-\frac{3}{4}\zeta\right)-{}_1F_2\left(\frac{7}{12},\frac{11}{12},\frac{1}{2},\frac{1}{2}+\frac{3}{4}\zeta\right)\right) && \text{see DLMF 15.8.27} \\
&\quad +\frac{3i}{2}\zeta\, c_2\left(\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{3\zeta\Gamma(-\frac{1}{2})\Gamma(2)}\right)\left({}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}-\frac{3}{4}\zeta\right)-{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}+\frac{3}{4}\zeta\right)\right) && \text{see DLMF 15.8.28} \\
&= \left(c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}}-c_2 i\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right){}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}-\frac{3}{4}\zeta\right)+ \\
&\quad +\left(c_1\frac{\Gamma(\frac{13}{12})\Gamma(\frac{17}{12})}{2\sqrt{\pi}}+c_2 i\frac{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}{4\sqrt{\pi}}\right){}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{1}{2}+\frac{3}{4}\zeta\right)
\end{aligned}
$$

Since $\hat{u}_+$ has a simple singularity at $\zeta=-2/3$ and $\hat{u}_-$ has a simple singularity at $\zeta=2/3$, we have

$$
\begin{aligned}
\hat{u}_+(\zeta) &= C_1 T_{-2/3}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,-\frac{3}{4}\zeta\right)=C_1 T_{-2/3}\,\hat{w}_+(\zeta) \\
\hat{u}_-(\zeta) &= C_2 T_{2/3}\,{}_1F_2\left(\frac{7}{6},\frac{11}{6},2,\frac{3}{4}\zeta\right)=C_2 T_{2/3}\,\hat{w}_-(\zeta)
\end{aligned}
$$

Claim 2.5.

$$
\hat{w}_+(\zeta-2/3)=\frac{1}{\sqrt{\pi}}\int_{-2/3}^{\zeta}(\zeta-s)^{-1/2}\frac{1}{f'(u)}ds \qquad s=f(u) \tag{2.15}
$$

Lemma 2.6. *The following identity holds true*

$$
{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right)=\frac{1}{1-u^2} \qquad \zeta=\frac{u^3}{3}-u \tag{2.16}
$$

Proof.

$$
\begin{aligned}
{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right) &= 2\cos\left(\frac{1}{3}\arcsin\left(\frac{3}{2}\zeta\right)\right)(4-9\zeta^2)^{-1/2} && \text{Mathematica [Fullsimplify]} \\
&= \frac{\cos(y)}{\cos(3y)} && 3y=\arcsin\left(\frac{3}{2}\zeta\right) \\
&= \frac{\cos(y)}{\cos(2y)\cos(y)-\sin(2y)\sin(y)} \\
&= \frac{1}{\cos(2y)-2\sin^2(y)} \\
&= \frac{1}{1-4\sin^2(y)} && \zeta=2\sin(y)-\frac{8}{3}\sin^3(y)
\end{aligned}
$$

Therefore, if $u:=-2\sin(y)$, we have $\zeta=\frac{u^3}{3}-u=f(u)$ and

$$
{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}\zeta^2\right)=\frac{1}{1-u^2}=-\frac{1}{f'(u)}
$$

$\square$

Hence claim (2.5) is equivalent to

Claim 2.7.

$$\hat{w}_+(\zeta-2/3) = -\frac{1}{\sqrt{\pi}}\int_{-2/3}^{\zeta}(\zeta-s)^{-1/2}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}s^2\right)ds \tag{2.17}$$

Let us study the RHS of claim (2.7)

$$\begin{aligned}
\frac{1}{\sqrt{\pi}}\int_{-2/3}^{\zeta}(\zeta-s)^{-1/2}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};\frac{9}{4}s^2\right)ds &= \frac{2}{\sqrt{\pi}}\int_{-2/3}^{\zeta}(\zeta-s)^{-1/2}{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};\frac{1}{2}-\frac{3s}{4}\right)ds+ \\
&\quad+\frac{2}{\sqrt{\pi}}\int_{-2/3}^{\zeta}(\zeta-s)^{-1/2}{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};\frac{1}{2}+\frac{3s}{4}\right)ds \qquad \text{15.8.27 DLMF} \\
&= \frac{4i}{\sqrt{3\pi}}\int_{1}^{\zeta'}(\zeta'-t)^{-1/2}{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};t\right)dt+ \qquad \zeta'=\frac{1}{2}-\frac{3}{4}\zeta \\
&\quad+\frac{4}{\sqrt{3\pi}}\int_{0}^{\zeta''}(\zeta''-t)^{-1/2}{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};t\right)dt \qquad \zeta''=\frac{1}{2}+\frac{3}{4}\zeta
\end{aligned}$$

By definition ${}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};t\right)=\sum_{n\geq 0}\frac{(2/3)_n(4/3)_n}{(3/2)_n\,n!}t^n$, thus

$$\int_0^{\zeta''}(\zeta''-t)^{-1/2}{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};t\right)dt = \sqrt{\pi}\sum_{n\geq 0}\frac{(2/3)_n(4/3)_n}{(3/2)_n\Gamma(3/2+n)}\zeta''^{n+\frac{1}{2}} = 2\sqrt{\zeta''}\,{}_3F_2\left(\frac{2}{3},\frac{4}{3},1;\frac{3}{2},\frac{3}{2};\zeta''\right)$$

$$\int_1^{\zeta'}(\zeta'-t)^{-1/2}{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};t\right)dt = 2\sqrt{\zeta'}\,{}_3F_2\left(\frac{2}{3},\frac{4}{3},1;\frac{3}{2},\frac{3}{2};\zeta'\right)-\int_0^{1}(\zeta'-t)^{-1/2}{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2};t\right)dt$$