Random Variable

A random Variable(x) is a function from Sample space (S) to set of all real no/s R.

i.e., $X : S \rightarrow R$ Where S — Domain Space
R — Range Space

* Types of Random Variable
    * Discrete Random Variable
    * Continuous Random Variable.

Discrete Random Variable

If X is a R.V which can take a finite number (or) Countably infinite number of values then X is called as a discrete R.V.

When X is a discrete R.V then the possible values of X may be assumed as $x_1, x_2 .. x_n ...$

Example

The no/s. Shown when a die is thrown and the no/s of alpha particles emitted by a radio active Source are discrete R.V.

Continuous R.V

A random variable X is said to continuous if it can take all possible values (in an interval) between certain limits.

Example

* Age of a person, height & weight of the set of persons.
* life time of a bulb. etc.

Note

* If X_1 & X_2 are R.V's & c is a Constant then cX_1, X_1+X_2, X_1X_2 & $c_1X_1+c_2X_2$ are also random variable where c_1 & c_2 are constants.

* $X_1 - X_2$ is also a r.v

* If X is a r.v then
  i) $1/X$ where $1/X = \infty$ if $X=0$
  ii) $X_+ = \text{Max}\{0, X\}$, iii) $X_- = \text{min}\{0, X\}$
  & iv) $|X|$ are also R.v's.

* If X_1 & X_2 are RV's then
  i) $\max[X_1, X_2]$ ii) $\min[X_1, X_2]$ are also Random variables.

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* If x is r.v & f is a continuous (or) increasing function then $f(x)$ is also a r.v.

Formula

Discrete R.V

① Probability mass function

i) $P(x) = \begin{cases} P(X=x_i) = p_i & \text{if } x = x_i \\ 0 & \text{if } x \neq x_i \end{cases}$

ii) $p_i \geq 0 \quad \forall i$

iii) $\sum p_i = 1$

iv)

$X=x_i$	$P(X=x_i)$
x_1	p_1
x_2	p_2
⋮	⋮
x_r	p_r
⋮	⋮

Continuous R.V

Probability density function

i) $P(x < X \leq x+dx) = f(x)\,dx$

ii) $f(x) \geq 0$

iii) $\int_R f(x)\,dx = 1$

iv) $P(a \leq X \leq b) = P(a < X < b) = \int_a^b f(x)\,dx$

v) $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$

② Cumulative Distribution function (CDF)

For discrete R.V

$$F(x) = P(X \leq x) = \sum_{\substack{j \\ x_j \leq x}} p_j$$

For Continuous R.V

$$F(x) = P(-\infty \leq X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

Discrete | Continuous

(3) $p(x_i) = F(x_i) - F(x_{i-1})$ $\quad\quad$ $\frac{d}{dx} F(x) = f(x) = F'(x)$

(4) Mean $= E(X) = \sum_i x_i p_i$ $\quad\quad$ $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$E(x^2) = \sum_i x_i^2 p_i$ $\quad\quad$ $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$Var(x) = E(x^2) - (E(x))^2 = \mu_2$ $\quad\quad$ $V(x) = E(x^2) - (E(x))^2 = \mu_2$

Properties of cdf $(F(x))$

(1) $0 \leq F(x) \leq 1$

(2) $F(x) \leq F(y)$ if $x < y$

i.e., $F(x)$ is monotonically non-decreasing function & lie between 0 & 1.

(3) $F(-\infty) = 0$ & $F(\infty) = 1$.

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Various Measures of Central tendancy, Dispersion, Skewness & kurtosis for Continuous probability Distribution & Discrete Probability distribution

Discrete | **Continuous**

① AM

$$AM = E(x) = \cancel{\int} \sum x_i p_i$$

$$AM = \int_{-\infty}^{\infty} x f(x) dx$$

If Interval is $[a, b]$

$$AM = \int_a^b x f(x) dx$$

② HM

$$HM = \sum_i \frac{1}{x_i} p_i$$

$$HM = \int_a^b \frac{1}{x} f(x) dx.$$

③ G.M

$$\log G = \cancel{\int} \sum_i \log x_i p_i$$

$$\log G = \int_a^b \log x \, f(x) dx$$

④ Moment about origin

$$M_r' = \sum_i x_i^r p_i$$

$= r^{th}$ moment about origin

$M_r' = r^{th}$ moment about origin

$$= \int_a^b x^r f(x) dx$$

r^{th} moment about the point A

$$= \mu_r' = \sum_i (x_i - A)^r p_i \qquad \mu_r' = \int_a^b (x-A)^r f(x)\,dx$$

r^{th} moment about the mean

$$\mu_r = \sum_i (x_i - \bar{x}) p_i \qquad \mu_r = \int_a^b (x-\bar{x}) f(x)\,dx$$

For $r = 1$ & 2

$$\mu_1' = \sum_i x_i p_i \qquad \mu_1' = \int_a^b x f(x)\,dx$$

$$\mu_2' = \sum_i x_i^2 p_i \qquad \mu_2' = \int_a^b x^2 f(x)\,dx$$

$$Var(x) = \mu_2 = \mu_2' - (\mu_1')^2 \qquad V(x) = \mu_2 = \mu_2' - (\mu_1')^2$$

Note (For continuous case)

① Median

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = 1/2$$

$\therefore$ By solving $\int_a^M f(x)\,dx = 1/2$ (or) $\int_M^b f(x)\,dx = 1/2$

for M, we get the value of the median.

* Mean deviation about mean

$$MD = \int_a^b |x - \text{mean}|\, f(x)\, dx$$

* Mean deviation about an Average 'A'

$$MD \text{ about } 'A' = \int_a^b |x - A|\, f(x)\, dx$$

* Quartiles & Deciles

Q_1 & Q_3 are given by the equations

$$\int_a^{Q_1} f(x)\, dx = 1/4 \quad \& \quad \int_a^{Q_3} f(x)\, dx = 3/4$$

* D_i, i^{th} decile is given by,

$$\int_a^{D_i} f(x)\, dx = i/10, \quad i = 1, 2 \ldots 9$$

* Mode: Mode is the value of x for which $f(x)$ is maximum. Mode is thus the solution of $f'(x) = 0$ & $f''(x) < 0$ provided it lies in $[a, b]$.

Note

r.v

* For discrete case, to find median, mode, M.D, Quartiles & Deciles, Mode, replace $\int$ as Σ & $f(x)$ as $p(x)$.

Problems.

① A random variable x has the following probability function

x:	0	1	2	3	4	5	6	7
$P(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

then find i) $k=?$ ii) Evaluate $P(X<6)$, $P(X\geq 6)$ & $P(0<X<5)$, iii) $P(X\leq a) > \frac{1}{2}$ find the minimum value of a.

iv) Determine the distribution function of x

v) Find the mean & Variance.

Sol

i) Since $\sum_i p_i = 1$

$$\therefore \quad k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$$

$$10k^2+9k-1=0$$

$$10(k-1/10)(k+1)=0$$

$$\therefore k = 1/10 \text{ (or) } -1 \text{ (Rejected)}$$

-10

-1×10

$\frac{-1}{10} \cdot \frac{10}{10}$

$-\frac{1}{10} \times 1$

$\therefore \quad k = 1/10$

iv) Distribution table with distribution function

x	0	1	2	3	4	5	6	7
$P(x)$	0	$1/10$	$2/10$	$2/10$	$3/10$	$1/100$	$2/100$	$17/100$
$F(x)=P(X\le x)$	0	$1/10$	$3/10$	$5/10$	$8/10$	$81/100$	$83/100$	1

ii) $P(X<6) = P(X=0)+P(X=1)+P(X=2)+P(X=3)$
$+P(X=4)+P(X=5)$

$= 1/10 + 2/10 + 2/10 + 3/10 + 1/100 = 81/100$

$P(X\ge 6) = 1-P(X<6) = 1-81/100 = 19/100$

$P(0<X<5) = P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$= 1/10 + 2/10 + 2/10 + 3/10$

$= 8/10 = 4/5$

iii) $P(X\le a) > 1/2$ when a is minimum

$P(X\le 3) = 5/10 = 0.5 = 1/2$

$P(X\le 4) = 8/10 = 0.8 > 1/2$

$\therefore \quad a = 4$

v) $\text{Mean} = E(x) = \sum x_i p_i$

$$= \frac{1}{10} + \frac{4}{10} + \frac{6}{10} + \frac{12}{10} + \frac{5}{100} + \frac{12}{100} + \frac{119}{100}$$

$$= \frac{23}{10} + \frac{136}{100} = \frac{230+136}{100}$$

$$= \frac{366}{100}$$

$$E(x^2) = \frac{1}{10} + \frac{8}{10} + \frac{18}{10} + \frac{48}{10} + \frac{25}{100} + \frac{72}{100} + \frac{833}{100}$$

$$= \sum x_i^2 p_i = \frac{75}{10} + \frac{930}{100}$$

$$= \frac{750+930}{100} = \frac{1680}{100}$$

$$\mathrm{Var}(x) = E(x^2) - \left(E(x)\right)^2$$

$$= \frac{1680}{100} - \left(\frac{366}{100}\right)^2$$

$$= 16.80 - 13.40 = 3.4$$

Problem -2

If $p(x) = \begin{cases} \frac{x}{15} & : x = 1, 2, 3, 4, 5 \\ 0 & \text{elsewhere} \end{cases}$

then find i) $P\{x = 1 \text{ (or) } 2\}$ &

ii) $P\left\{\frac{1}{2} < x < \frac{5}{2} \,\middle|\, x > 1\right\}$

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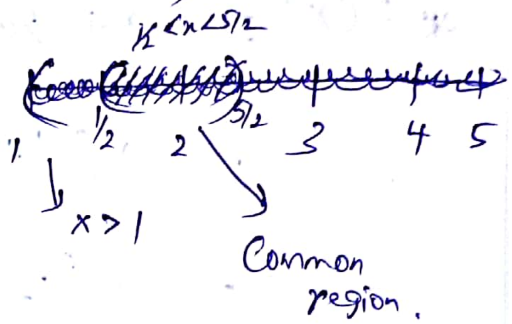
Sol

i) $P(X=1 \text{ (or) } 2) = P(X=1) + P(X=2)$

$= 1/15 + 2/15 = 1/5$

ii) $P(1/2 < x < 5/2 \mid x > 1) = \dfrac{P\{(1/2 < x < 5/2) \cap (x>1)\}}{P(x>1)}$

$= \dfrac{P\{(X=1 \text{ (or) } 2) \cap X>1\}}{P(x>1)}$



$= \dfrac{P(1/2 < x < 5/2)}{P(x>1)}$

$= \dfrac{P(X=2)}{1-P(X=1)} = \dfrac{2/15}{1-1/15} = 1/7$

(3) The probability function of an infinite discrete distribution is given by $P(X=j) = 1/2^j$ $(j=1,2\cdots\infty)$. Verify that the total probability is 1 & find the mean and variance of the distribution. Find also $P(x$ is even$)$, $P(X\geq 5)$ & $P(X$ is divisible by $3)$.

Sol

$P(X=j) = P_j = 1/2^j$

i) Verification of $\Sigma p_j = 1$.

$\sum P_i = 1/2 + 1/2^2 + \cdots \infty$ (Which is a GP with common ratio $r = 1/2$)

& $S_\infty = a/1-r$

$(\because a = 1/2, r = 1/2)$

$$= \frac{1/2}{1 - 1/2}$$

$$= 1$$

* $E(X) = a + 2a^2 + 3a^3 + \cdots \infty$ (where $a = 1/2$)

$$= a(1 + 2a + 3a^2 + \cdots \infty)$$

$$= a(1-a)^{-2} = \frac{a}{(1-a)^2} = \frac{1/2}{(1-1/2)^2} = 2$$

* $E(X^2) = \sum_j j^2 p^j$

$$= \sum_j [j(j+1) - j] a^j$$

$$= \sum_{j=1}^{\infty} j(j+1) a^j - \sum_{j=1}^{\infty} j a^j$$

$$= a(1\cdot 2 + 2\cdot 3a + 3\cdot 4a^2 + \cdots \infty) - a(1 + 2a + 3a^3 + \cdots \infty)$$

$$= a\frac{2}{(1-a)^3} - a/(1-a)^2 = 8 - 2 = 6$$

$$V(x) = E(x^2) - (E(x))^2 = 6-4 = 2$$

* $P(x \text{ is even}) = P(x=2 \text{ (or) } x=4 \text{ (or) } x=6 \text{ (or) etc})$

$$= P(x=2) + P(x=4) + \cdots + \infty$$

$$= (1/2)^2 + (1/2)^4 + (1/2)^6 + \cdots + \infty$$

$$= \frac{(1/4)}{(1-1/4)} = 1/3$$

* $P(x \geq 5) = P(x=5 \text{ (or) } x=6 \text{ (or) } x=7 \text{ (or) etc})$

$$= P(x=5) + P(x=6) + \cdots + \infty$$

$$= \frac{1/2^5}{1-1/2} = 1/16$$

* $P(x \text{ is divisible by } 3)$

$$= P(x=3 \text{ (or) } x=6 \text{ (or) } x=9 \text{ etc})$$

$$= P(x=3) + P(x=6) + \cdots$$

$$= (1/2)^3 + (1/2)^6 + (1/2)^9 + \cdots$$

$$= \frac{1/8}{1-1/8} = 1/7$$

④ If the density function of a continuous R.V X is given by,

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

i) Find the value of a

ii) Find the cdf of X

Sol

$$\int_0^3 f(x)\,dx = 1$$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$a\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^2 + \left(3ax - \frac{ax^2}{2}\right)_2^3 = 1$$

$$a\left(\tfrac{1}{2} - 0\right) + a(2-1) + \left[\left(9a - \frac{9a}{2}\right) - (6a - 2a)\right] = 1$$

$$a\left[\tfrac{1}{2} + 1 + \tfrac{9}{2} - 4\right] = 1$$

$$a(6-4) = 1$$

$$2a = 1$$

$$\boxed{a = \tfrac{1}{2}}$$

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ii) $F(x) = P(X \leq x) = 0$ when $x < 0$.

* $F(x) = \int_0^x x/2 \, dx = x^4/4$, when $0 \leq x \leq 1$

* <u>In $1 \leq x \leq 2$</u>

$$F(x) = \int_0^1 x/2 \, dx + \int_1^x 1/2 \, dx = 1/2 \left(x^2/2\right)_0^1 + 1/2 (x)_1^x$$

$$= 1/2 (1/2 - 0) + 1/2 (x-1)$$

$$= 1/4 + x/2 - 1/2 = x/2 - 1/4 \quad \text{when } 1 \leq x \leq 2$$

* <u>In $2 \leq x \leq 3$</u>

$$F(x) = \int_0^1 x/2 \, dx + \int_1^2 1/2 \, dx + \int_2^x \left(3/2 - x/2\right) dx$$

$$= 1/2 \left(x^2/2\right)_0^1 + 1/2 (x)_1^2 + 1/2 \left(3x - x^2/2\right)_2^x$$

$$= 1/2 (1/2) + 1/2 (2-1) + 1/2 \left[\left(3x - x^2/2\right) - (6-2)\right]$$

$$= 1/4 + 1/2 + 1/2 \left[3x - x^2/2\right] - 2$$

$= 3/4 + 1/2\left(3x - x^2/2\right) - 2$

$= 3/2\, x - x^2/4 - 5/2$ when $2 \le x \le 3$

&

$F(x) = 1$ when $x > 3$.

⑤ The diameter of an electric cable say X is assumed to be a continuous random variable with pdf $f(x) = kx(1-x)$, $0 \le x \le 1$

then i) find k ii) Determine a number 'b' such that $P(X<b) = P(X>b)$ iii) Find mean & var(X)

Sol

i) $\int_0^1 f(x)\,dx = 1$

$k \int_0^1 (x - x^2)\,dx = 1$

$\therefore k\left(x^2/2 - x^3/3\right)_0^1 = 1$

$k\left(1/2 - 1/3\right) = 1$

$k\left(1/6\right) = 1$

$\therefore k = 6$

$\therefore f(x) = 6x(1-x)$, $0 \le x \le 1$

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ii) $$P(X<b) = P(X>b)$$

$$\int_0^b f(x)\,dx = \int_b^1 f(x)\,dx$$

$$6\int_0^b (x-x^2)\,dx = 6\int_b^1 (x-x^2)\,dx$$

$$\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_0^b = \left(\frac{x^2}{2}-\frac{x^3}{3}\right)_b^1$$

$$\frac{b^2}{2}-\frac{b^3}{3} = \left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{b^2}{2}-\frac{b^3}{3}\right)$$

$$\frac{2b^2}{2}-\frac{2b^3}{3} = \frac{1}{6}$$

$$\frac{3b^2-2b^3}{3} = \frac{1}{6}$$

$$3b^2-2b^3 = \frac{1}{2}$$

$$3b^2-2b^3-\frac{1}{2} = 0$$

$$6b^2-4b^3-1 = 0$$

i.e; $$4b^3-6b^2+1 = 0$$

By trial & error, $b=1/2$ is a root

Since

$$\begin{array}{c|cccc} \frac{1}{2} & 4 & -6 & 0 & 1 \\ & 0 & 2 & -2 & -1 \\ \hline & 4 & -4 & -1 & \boxed{0} \end{array}$$

$\therefore$ $4b^3-6b^2+1=0$

$(b-1/2)(4b^2-4b-1)=0$

$\therefore \boxed{b=1/2}$

Mathematical Expectation

Discrete R.V

$E(X)$ = Expected value of X.

$$= \sum_{i=1}^{n} x_i p_i$$

Continuous R.V

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Two dimensional Random Variables

Let X & Y be two random variables defined on the same sample space S then the function (X, Y) that assigns a point in $R^2 (= R \times R)$ is called a two dimensional Random Variable.

Note

Let $A = \{a < X \leq b\}$ & $B = \{c < Y \leq d\}$ be two events then

$$\{a < X \leq b,\ c \leq Y \leq d\} = \{a < X \leq b\} \cap \{c < Y \leq d\}$$

$$= A \cap B$$

& $P(\{a < X \leq b,\ c \leq Y \leq d\}) = P(A \cap B)$

In 2DRV

Discrete case

(a) Joint Probability mass function

$P_{XY}(x,y)$ (or) $P(x,y)$

$= P(X=x_i, Y=y_j)$

(b) Marginal Probability function

$P(X=x_i) = P_{i.}$

$= \sum_{j=1}^{m} P_{ij} = P_{i1} + P_{i2} + \dots + P_{im}$

= Marginal probability distribution of X.

$|||^{ly}$ Marginal probability distribution of Y

$= P(Y=y_j) = P_{.j}$

$= \sum_{i=1}^{n} P_{ij} = P_{1j} + P_{2j} + \dots + P_{nj}$

(or)

$\sum_{y} P(x,y) = P(X=x_i)$

$\sum_{x} P(x,y) = P(Y=y_j)$

Continuous case

Joint pdf

$f(x,y)$

Marginal density function

Marginal density function of x

$= f(x)$

$= \int_{-\infty}^{\infty} f(x,y)\,dy$

Marginal density function of y

$= f(y)$

$= \int_{-\infty}^{\infty} f(x,y)\,dx$

Note

$f(x) = \frac{dF_X(x)}{dx}$

$f(y) = \frac{dF_Y(y)}{dy}$

Note

Cumulative distribution function for 2D R.V.

$$F(x,y) = P(X \le x, Y \le y) \quad \& \quad f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

Discrete

i) Marginal distribution fantion of X & Y respectively in 2D R.V

$$F(x) = \sum_{y} P(X \le x, Y = y)$$

&

$$F(y) = \sum_{x} P(X = x, Y \le y)$$

Continuous

i) $$F(x) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f(x,y)\,dy \right\} dx$$

$$F(y) = \int_{-\infty}^{y} \left\{ \int_{-\infty}^{\infty} f(x,y)\,dx \right\} dy$$

ii) **Cumulative distributive function**

$$F(x,y) = \sum_{y_j \le y} \sum_{x_i \le x} P_{ij}$$

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\,dx\,dy$$

iii) **Total Probability**

$$\sum_{i} \sum_{j} P_{ij} = 1$$

provided $P_{ij} \ge 0$ for all i & j

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

provided $f(x,y) \ge 0$ for all $x, y \in R$.

Discrete

iv) **Conditional probability Distribution**

$$P(X=x_i / Y=y_j)$$
$$= \frac{P(X=x_i; Y=y_j)}{P(Y=y_j)}$$
$$= \frac{P_{ij}}{P_{*j}}$$

$|||^{ly}$
$$P(Y=y_j / X=x_i)$$
$$= \frac{P(X=x_i, Y=y_j)}{P(X=x_i)}$$
$$= \frac{P_{ij}}{P_{i*}}$$

are the Conditional probability distributions of x given y & y given x respectively.

Continuous

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

$$\& \; f(y/x) = \frac{f(x,y)}{f(x)}$$

are the Conditional density functions of x given y & y given x respectively.

Discrete | **Continuous**

Condition for independent

If $p_{ij} = p_{i*} \times p_{*j} \quad \forall i,j$

then x & y are said to be independent Random Variables.

If $f(x,y) = f(x) \cdot f(y)$

then x & y are said to be independent R.V.'s

Expected Values of a Two-Dimensional Random Variable. (x,y)

Discrete

$$E(xy) = \sum\sum x_i y_j p_{ij}$$

Continuous

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E(y) = \int_{-\infty}^{\infty} y\, f(y)\,dy$$

(for continuous)

For discrete

$$E(x) = \sum x\, p(x)$$

$$E(y) = \sum y\, p(y)$$

Properties

i) $E(x+y) = E(x) + E(y)$

ii) $E(xy) \neq E(x) \cdot E(y)$ in general.

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iii) If X & Y are independent R.V's then

$$E(XY) = E(X)\cdot E(Y)$$

Conditional Expected value

Discrete

$$E(g(x,y) / Y=y_j) = \sum_i g(x_i, y_j) \cdot P(X=x_i / Y=y_j)$$

|||^{ly} $$E(g(X,Y) / X=x_i) = \sum_j g(x_i, y_j) \cdot P(Y=y_j / X=x_i)$$

Continuous

$$E(g(x,y)/y) = \int_{-\infty}^{\infty} g(x,y) \cdot f(x/y)\,dx$$

&

$$E(g(x,y)/x) = \int_{-\infty}^{\infty} g(x,y) \cdot f(y/x)\,dy$$

Sums

① The Joint Probability mass function of (X,Y) is given by $P(x,y) = k(2x+3y)$, $x=0,1,2$, $y=1,2,3$. Find all the marginal & Conditional & probability distributions. Also i) find the probability distribution of $X+Y$, ii) verify whether X & Y are independent. iii) Find Mean of X & Mean of Y, $Var(X)$ & $Var(Y)$ iv) $cov(X,Y)$, v) coefficient of correlation between X & Y.

Note:

$$\rho = \text{coefficient of correlation}$$
$$= \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y}$$

Sol

X \ Y	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

To find: $k = ?$

$$\sum_i \sum_j p_{ij} = 1$$

$$72k = 1$$

$$k = 1/72$$

Marginal Probability distribution of X

i) $P(X=0) = P_{0*} = P_{01} + P_{02} + P_{03}$

$= P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3)$

$= 18k = 18/72$

ii) $P(X=1) = P_{1*} = P_{11} + P_{12} + P_{13}$

$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$

$= 24/72$

iii) $P(X=2) = P_{2*} = P_{21} + P_{22} + P_{23}$

$= 30/72$

Marginal probability distribution of Y:

i) $P(Y=1) = 15k = 15/72$

ii) $P(Y=2) = 24k = 24/72$

iii) $P(Y=3) = 33/72$

i.e.,

X \ Y	1	2	3	Marginal distribution of X
0	$3/72$	$6/72$	$9/72$	$P(X=0) = 18/72$
1	$5/72$	$8/72$	$11/72$	$P(X=1) = 24/72$
2	$7/72$	$10/72$	$13/72$	$P(X=2) = 30/72$
Marginal distribution of Y	$P(Y=1) = 15/72$	$P(Y=2) = 24/72$	$P(Y=3) = 33/72$	$\boxed{1}$ → Total Probability

Conditional Probability Distribution:

i) Conditional Probability of X given $Y=1$:

$$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{P_{01}}{P(Y=1)}$$

$$= \frac{3/72}{15/72} = 3/15 = 1/5$$

$$P(X=1|Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{P_{11}}{P(Y=1)} = \frac{5/72}{15/72}$$

$$= 1/3$$

$$P(X=2/Y=1) = 7/15$$

ii) Conditional Probability of X given $Y=2$:

$$P(X=0|Y=2) = \frac{P(X=0, Y=2)}{P(Y=2)} = 1/4$$

$$P(X=1|Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = 8K/24K = 1/3$$

$$P(X=1|Y=3) = \frac{P(X=1, Y=3)}{P(Y=3)} = \frac{P_{13}}{P(Y=3)} = 10K/24K$$

$$= 5/12$$

iii) Conditional distribution of X given Y=3:

$$P(X=0 \mid Y=3) = \frac{9k}{33k} = 3/11$$

$$P(X=1 \mid Y=3) = \frac{11k}{33k} = 1/3$$

$$P(X=2 \mid Y=3) = \frac{13k}{33k} = 13/33$$

iv) Conditional distribution of Y given X=0:

$$P(Y=1 \mid X=0) = \frac{P(X=0, Y=1)}{P(X=0)} = \frac{3k}{18k} = 1/6$$

$$P(Y=2 \mid X=0) = \frac{6k}{18k} = 1/3$$

$$P(Y=3 \mid X=0) = \frac{9k}{18k} = 1/2$$

v) Conditional distribution of Y given X=1:

$$P(Y=1 \mid X=1) = \frac{P(X=1, Y=1)}{P(X=1)} = \frac{5k}{24k} = 5/24$$

$$P(Y=2 \mid X=1) = \frac{8k}{24k} = 1/3$$

$$P(Y=3 \mid X=1) = \frac{11k}{24k} = 11/24$$

vi) Conditional distribution of y given $x=2$:

$$P(Y=1 \mid X=2) = \frac{7k}{30k} = 7/30$$

$$P(Y=2 \mid X=2) = 10k/30k = 1/3$$

$$P(Y=3 \mid X=2) = \frac{13k}{30k} = 13/30$$

Probability distribution of $(x+y)$

$X+Y$	$P(X+Y)$
1	$P_{01} = 3/72$
2	$P_{02}+P_{11} = 11/72$
3	$P_{03}+P_{12}+P_{21} = 24/72$
4	$P_{13}+P_{22} = 21/72$
5	$P_{23} = 13/72$

ii) Verification for X & Y are independent (or) not :

Condition : $P_{ij} = P_{i*} \times P_{*j}$ for all i, j

$P_{11} = 5/72$ | $P_{1*} = P(X=1) = 11/72$

$P_{*1} = P(Y=1) = 15/72$

$P_{11} \neq P_{1*} \cdot P_{*1}$

$\therefore$ X & Y are not independent.

iii) Mean of X & Mean of Y

$$E(X) = \sum_{i=0}^{2} x_i p_i = 0 \times 18/72 + 1 \times 24/72 + 2 \times 30/72$$

$$= \frac{24}{72} + \frac{60}{72} = \frac{84}{72}$$

$$E(Y) = \sum_{j=1}^{3} y_j p_j = 1 \times P(Y=1) + 2P(Y=2) + 3P(Y=3)$$

$$= 1 \times 15/72 + 2 \times 24/72 + 3 \times 33/72$$

$$= 15/72 + 48/72 + 99/72 = \frac{162}{72}$$

iv) $COV(x, y) = E(xy) - E(x) \cdot E(y)$

$$E(xy) = \sum_i \sum_j x_i y_j p_{ij}$$

$$= 0 \times 1 \times \frac{3}{72} + 0 \times 2 \times \frac{6}{72} + 0 \times 3 \times \frac{9}{72}$$

$$+ 1 \times 1 \times \frac{5}{72} + 1 \times 2 \times \frac{8}{72} + 1 \times 3 \times \frac{11}{72}$$

$$+ 2 \times 1 \times \frac{7}{72} + 2 \times 2 \times \frac{10}{72} + 2 \times 3 \times \frac{13}{72}$$

$$= 0 + 0 + 0 + \frac{5}{72} + \frac{16}{72} + \frac{33}{72}$$

$$+ \frac{14}{72} + \frac{40}{72} + \frac{78}{72}$$

$$= \frac{206}{72}$$

v) $$\rho(x, y) = \frac{Cov(x, y)}{\sigma_x \sigma_y}$$

$\sigma_x = \sqrt{V(x)}$ where $V(x) = E(x^2) - (E(x))^2$

$\sigma_y = \sqrt{V(y)}$ where $V(y) = E(y^2) - (E(y))^2$

— Do as Exercsice.

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② The joint Pdf of a 2D R.V (X, Y) is given by $f(x,y) = \begin{cases} xy^2 + x^2/8, & 0 \le x \le 2, \ \& \ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$

then compute i) $P(X>1)$, $P(Y<1/2)$, $P(X>1 \mid Y<1/2)$, $P(Y<1/2 \mid X>1)$ &

ii) $P(X+Y \le 1)$

Sol

i) $$P(X>1) = \int_{y=0}^{1}\int_{x=1}^{2} f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_1^2 \left(xy^2 + x^2/8\right) dx\,dy = \int_0^1 \left\{ \frac{x^2}{2}y^2 + \frac{x^3}{24} \right\}_1^2 dy$$

$$= \int_0^1 \left[\left(2y^2 + 1/3\right) - \left(y^2/2 + 1/24\right)\right] dy$$

$$= \int_0^1 \left\{ 2y^2 + 8/24 - y^2/2 - 1/24 \right\} dy = \int_0^1 \left(\frac{3y^2}{2} + 7/24\right) dy$$

$$= \left[\cancel{\frac{7}{2}} \frac{y^3}{\cancel{3}} + \frac{7}{24} y^4 \right]_0^1 = \frac{1}{2} + \frac{7}{24}$$

$$= \frac{12+7}{24} = \frac{19}{24}$$

$$P(Y < 1/2) = \int_{y=0}^{1/2} \int_{x=0}^{2} \left(xy^2 + \frac{x^2}{8} \right) dx\, dy$$

$$= 1/4$$

$$P(X > 1, Y < 1/2) = \int_{y=0}^{1/2} \int_{x=1}^{2} \left(xy^2 + \frac{x^2}{8} \right) dx\, dy$$

$$= 5/24$$

$$P(X > 1 \mid Y < 1/2) = \frac{P(X > 1, Y < 1/2)}{P(Y < 1/2)}$$

$$= \frac{5/24}{1/4} = 5/6$$

$$P(Y<1/2 \mid X>1) = \frac{P(X>1, Y<1/2)}{P(X>1)}$$

$$= \frac{5/24}{19/24} = 5/19$$

$$P(X<Y) = \int_0^1 \int_0^y \left(xy^2 + x^2/8\right) dx\, dy$$

$$= 53/480$$

ii) $$P(X+Y<1) = \int_0^1 \int_0^{1-y} \left(xy^2 + x^2/8\right) dx\, dy \quad \left(\because X+Y=1 \quad X=1-Y\right)$$

$$= 13/480$$

(3) The joint Pdf of (X,Y) is given by $f(x,y) = kxy\, e^{-(x^2+y^2)}$, $x>0$, $y>0$. Find the value of k & verify whether ~~x~~ X & Y are independent.

Sol i) $\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = 1$

$K\int_0^\infty x e^{-x^2}dx \cdot \int_0^\infty y e^{-y^2} dy = 1$

$K \, 1/2 \cdot 1/2 = 1$

$K/4 = 1$

$K = 4$

$t = x^2$

$dt = 2x\,dx$

$dt/2 = x\,dx$

$\therefore$

$x:$	0	∞
$t:$	0	∞

$\int_0^\infty x e^{-x^2} dx$

$= \frac{1}{2}\int_0^\infty e^{-t} dt$

$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right)_0^\infty$

$= -\frac{1}{2}\left(e^{-\infty} - e^{0}\right)$

$= \frac{1}{2}$

ii) $f(x) = \int_{y=0}^\infty f(x,y)\,dy = 4\int_0^\infty xy e^{-x^2} \cdot e^{-y^2} dy$

$= 4 x e^{-x^2} \int_0^\infty y e^{-y^2} dy = 4 x e^{-x^2} \cdot 1/2 = 2x e^{-x^2}$

$|||^{ly}$ $f(y) = 2y e^{-y^2}$.

$\therefore f(x,y) = 4xy\, e^{-x^2} \cdot e^{-y^2}$ — (1)

$f(x) \cdot f(y) = 2x e^{-x^2} \cdot 2y\, e^{-y^2}$

$= 4xy\, e^{-x^2} \cdot e^{-y^2}$ — (2)

from (1) & (2) x & y are independent.