Mathematical Expectation & it's Properties ①

Discrete case

* $E(x) = \sum_i x_i p_i$

* $E(x^r) = \sum_i x_i^r p_i$

* "r^{th} moment about origin" $= \mu_r'$

Continuous Case

$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$

$E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$

$= \mu_r'$

* **Properties**

① $E(c) = c$

② $E(x+y) = E(x) + E(y)$ ($\because$ Addition Theorem)

③ $E\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} E(x_i)$

④ **Multiplication Theorem**

* If x & y are independent then $E(xy) = E(x).E(y)$
* In General,

$$E\left(\prod_{i=1}^{n} x_i\right) = \prod_{i=1}^{n} E(x_i)$$

⑤ If x is a random Variable & 'a' is a constant then

i) $E(a\psi(x)) = a\,E(\psi(x))$

ii) $E(\psi(x)+a) = E(\psi(x)) + a$

Corollary

i) If $\psi(x) = x$ then $E(ax) = aEx$

& $E(x+a) = E(x) + a$

ii) $E(ax+b) = aE(x) + b$

iii) $E(g(x)) = g(E(x))$

Note

i) $E(1/x) \neq \frac{1}{E(x)}$

iii) $E(\log x) \neq \log(E(x))$

ii) $E(x^{1/2}) \neq (E(x))^{1/2}$

iv) $E(x^2) \neq (E(x))^2$

v) $E\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i E(x_i)$

vi) If $x \geq 0$ then $E(x) \geq 0$

vii) Let x & y be two random variables $y \leq x$ then $E(y) \leq E(x)$

viii) $|E(x)| \leq E(|x|)$

Properties

① If x is a random variable then

$V(ax+b) = a^2 V(x)$ Since $V(b) = 0$

(2) If $b=0$ then $V(ax)=a^2V(x)$

$\Rightarrow$ Variance is not independent of change of Scale.

(3) If $a=0$ then $V(b)=0$

$\Rightarrow$ Variance of Constant is zero

(4) ~~If~~ $a=1$, $V(x+b)=V(x)$

$\Rightarrow$ variance is independent of change of origin.

Covariance

* If X & Y are two random Variables then covariance between them is defined as

$$Cov(x,y) = E(xy) - E(x) \cdot E(y)$$

Note

(1) If X & Y are Independent then

$E(xy) = E(x)E(y)$

$\therefore \quad Cov(X,Y) = 0$

(2) $Cov(ax, by) = ab\ Cov(x,y)$

③ $cov(x+a, x+b) = cov(x,y)$

④ $cov\left(\frac{x-\bar{x}}{\sigma_x}, \frac{y-\bar{y}}{\sigma_y}\right) = \frac{1}{\sigma_x \sigma_y} cov(x,y)$

⑤ $cov(ax+b, cx+d) = ac\, cov(x,y)$

⑥ $cov(ax+by, cx+dy) = ac\sigma_x^2 + bd\sigma_y^2 + (ad+bc)\, cov(x,y)$

Correlation Coefficient: (Karl Pearson's)

The correlation coefficient between the variables x & y is defined as,

$$\rho_{xy} = \frac{cov(x,y)}{\sigma_x \sigma_y}$$

Note

$$Var(ax \pm by) = a^2 var(x) + b^2 var(y) \pm 2ab\, cov(x,y)$$

Problems

① Let x be a random variable with the following probability distribution

x	:	-3	6	9
$P(X=x)$	:	$1/6$	$1/2$	$1/3$

then find i) $E(x)$ ii) $E(x^2)$ iii) $E(2x+1)^2$

Sol

$$E(x) = \sum_i x_i p_i = -3 \times 1/6 + 6 \times 1/2 + 9 \times 1/3$$
$$= 11/2$$

$$E(x^2) = \sum_i x_i^2 p_i = 9 \times 1/6 + 36 \times 1/2 + 81 \times 1/3$$
$$= 93/2$$

$$E(2x+1)^2 = E(4x^2 + 4x + 1) = 4E(x^2) + 4E(x) + 1$$
$$= 4(93/2) + 4 \times 11/2 + 1 = 209$$

Note

$$Var(x) = E(x^2) - (E(x))^2$$
$$= 93/2 - \frac{121}{4}$$
$$= \frac{186 - 121}{4} = \frac{45}{4}$$

$$V(2x \pm 3) = 4V(x) = 4(45/4) = 45$$

Moment Generating function (MGF)

The MGF of a random variable X (about origin) having the probability function $f(x)$ is given by

$$M_X(t) = E(e^{tx}) = \begin{cases} \sum_x e^{tx} f(x) & (\text{for Discrete case}) \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & (\text{for continuous case}) \end{cases}$$

Remarks

* $M_X(t) = E(e^{tx})$

$$= E\left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots + \frac{t^r x^r}{r!} + \cdots\right)$$

$$= 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$\therefore \mu_r' = r^{th}$ moment about the origin

$= \text{Coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)$

& $$M_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}$$

& $M_r' = E[(X-a)^r]$ is the r^{th} moment about $x=a$

Some limitation of MGF

① A random variable X may have no moments although mgf exists.

② A random variable X can have mgf & some moments, yet the mgf does not generate the moments.

Properties of MGF

① $M_{cX}(t) = M_X(ct)$ where c is a constant.

② If $X_1, X_2 \dots X_n$ are independent random variables then the MGF of their sum is equal to product of their respective MGFs.

i.e., $M_{X_1+X_2+\dots+X_n}(t) = M_{X_1}(t) M_{X_2}(t) \dots M_{X_n}(t)$

(3) Effect of change of origin & scale on MGF.

Sums

(1) Let x be the random variable with $P(X=r) = q^{r-1} \cdot p$, $r = 1, 2 \ldots$ then find the

i) MGF of X

ii) mean & variance of X.

Sol

$$M_X(t) = E(e^{tx}) = \sum_{r=1}^{\infty} e^{tr} P(X=r)$$

$$= \sum_{r=1}^{\infty} e^{tr} q^{r-1} \cdot p = p/q \sum_{r=1}^{\infty} (qe^t)^r$$

$$= \cancel{p/q} (\cancel{q}e^t) \sum_{r=1}^{\infty} (qe^t)^{r-1}$$

$$= pe^t \{1 + qe^t + (qe^t)^2 + \ldots\} = pe^t (1-qe^t)^{-1}$$

$$M_X(t) = \frac{pe^t}{1-qe^t}$$

$$M_X'(t) = \frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2} = \frac{pe^t(1-\cancel{qe^t}+\cancel{qe^t})}{(1-qe^t)^2}$$

$$M_x'(t) = \frac{pe^t}{(1-qe^t)^2}$$

$$M_x''(t) = \frac{(1-qe^t)^2 pe^t - 2pe^t(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$= \frac{pe^t \cancel{(1-qe^t)}(1-qe^t+2qe^t)}{(1-qe^t)^4}$$

$$= \frac{pe^t(1+qe^t)}{(1-qe^t)^3}$$

$$\therefore M_x'(0) = \frac{p}{(1-q)^2} = p/p^2 = 1/p = \mu_1' = \text{mean}$$

$$M_x''(0) = \frac{p(1+q)}{(1-q)^3} = \frac{p(1+q)}{p^3}$$

$$= \frac{1+q}{p^2} = \mu_2'$$

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \frac{1+q}{p^2} - 1/p^2$$

$$= \frac{q}{p^2}$$

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② Find the MGF of a R.V x whose Pdf is given by

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$ & also find

its mean & variance

Sol

Let $M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)\, dx$

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} e^{-x}\, dx$$

$$= \int_0^\infty e^{-\{1-t\}x}\, dx = \left\{ \frac{e^{-(1-t)x}}{-(1-t)} \right\}_0^\infty$$

$$= \frac{-1}{1-t}\left(e^{-\infty} - e^{0}\right)$$

$$= \frac{1}{1-t} \quad \text{if } |t| < 1$$

$$M_x(t) = \frac{1}{1-t} = (1-t)^{-1} = 1 + t + t^2 + t^3 + \cdots + t^r + \cdots$$

$$= 1 + t/1! + 2!\, t^2/2! + 3!\, t^3/3! + \cdots + r!\, t^r/r! \cdots$$

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$E(x) = \mu_1' =$ Coefficient of $\frac{t^1}{1!}\Big|_0 = 1 =$ mean

$E(x^2) = \mu_2' =$ Coefficient of $\frac{t^2}{2!}\Big|_0 = 2$

$\mu_2 =$ Variance $= \mu_2' - \mu_1' = 2 - 1 = 1$

(2) Find the MGF of a R.V X whose pdf is given by

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$ & find its mean & variance (If it possible)

Sol

Since $|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{tx} e^{-|x|}\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} e^{tx} \cdot e^{x}\, dx + \frac{1}{2}\int_{0}^{\infty} e^{tx} e^{-x}\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} e^{(1+t)x}\, dx + \frac{1}{2}\int_{0}^{\infty} e^{-(1-t)x}\, dx$$

$$= \frac{1}{2}\left\{\left(\frac{e^{(1+t)x}}{(1+t)}\right)_{-\infty}^{0} + \left(\frac{e^{-(1-t)x}}{-(1-t)}\right)_{0}^{\infty}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{1+t} + \frac{1}{1-t}\right]$$

$$= \frac{1}{2}\left(\frac{2}{1-t^2}\right) = \frac{1}{1-t^2}$$

* Mean & Variance – H.W

Characteristic function

$$\phi_X(t) = E(e^{itx}) = \begin{cases} \int e^{itx} f(x)\,dx & \text{for Continuous} \\ \sum_x e^{itx} p(x) & \text{for discrete.} \end{cases}$$

is known as the characteristic function of the random variable X & where $\phi_X(t)$ is a complex valued function of real variable 't'.

Remarks

i) $\phi_X(t)$ always exists only when $|\phi(t)| \le 1$

ii) $\phi_X(0) = 1$