EE16A HW2

1 Powers of Nilpotent Matrices

2 Elementary Matrices

2A — Switch rows 1 and 3

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E$$

— Multiply row 3 by -5

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E$$

— Adding 3x row 2 to row 4

Subtracting row 2 from row 1

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} = E$$

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2 Elementary Matrices

2B

$$A = \left[\begin{array}{cccc|c} 1 & -2 & 0 & -5 & 16 \\ 0 & 1 & 0 & 3 & -7 \\ -2 & -3 & 1 & -6 & 9 \\ 0 & 1 & 0 & 2 & -5 \end{array}\right] \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R1+2R2 \\ \\ 2R1+R3 \\ R2-R4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 & -7 \\ 0 & -7 & 1 & -16 & 41 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \quad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad \begin{array}{l} R1-R4 \\ R2-3R3 \\ 7R2+R3 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 5 & -8 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 3 \\ 2 & 7 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad \begin{array}{l} \\ \\ R3-5R4 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \quad \boxed{\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 3 \\ 2 & 2 & 1 & 5 \\ 0 & 1 & 0 & -1 \end{bmatrix} = A^{-1}}$$

$$EA = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 3 \\ 2 & 2 & 1 & 5 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ -2 & -3 & 1 & -6 \\ 0 & 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \checkmark$$

$E = A^{-1}$

3A This transmission can recover from erasure of up to either a, b, or c so at least one copy of a, b, or c remains. It won't be able to handle erasures of both A's, B's or C's or erasures of more than 4 characters at all. For example (? b ? a ? c) is still salvageable, however (? b c ? b c) is not.

3B $k_1 = \vec{v}_1^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_i a + \beta_i b + \gamma_i c$

$$\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_1 & \gamma_2 \\ \alpha_3 & \beta_2 & \gamma_3 \\ \alpha_4 & \beta_3 & \gamma_4 \\ \alpha_5 & \beta_4 & \gamma_5 \\ \alpha_6 & \beta_5 & \gamma_6 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \alpha_1 a + \beta_1 b + \gamma_1 c \\ \alpha_2 a + \beta_2 b + \gamma_2 c \\ \alpha_3 a + \beta_3 b + \gamma_3 c \\ \alpha_4 a + \beta_4 b + \gamma_4 c \\ \alpha_5 a + \beta_5 b + \gamma_5 c \\ \alpha_6 a + \beta_6 b + \gamma_6 c \end{bmatrix}$$

3C
$\vec{v}_1^T = [1, 0, 0]$ $\quad \vec{v}_1 \rightarrow a$
$\vec{v}_2^T \; [0, 1, 0]$ $\quad \vec{v}_2 \rightarrow b$
$\vec{v}_3^T \; [0, 0, 1]$ $\quad \vec{v}_3 \rightarrow c$
$\vec{v}_4^T \; [1, 0, 0]$ $\quad \vec{v}_4 \rightarrow a$
$\vec{v}_5^T \; [0, 1, 0]$ $\quad \vec{v}_5 \rightarrow b$
$\vec{v}_6 \; [0, 0, 1]$ $\quad \vec{v}_6 \rightarrow c$

3D $\vec{v}_1^T = [1,0,0]$
$\vec{v}_2^T = [0,1,0]$
$\vec{v}_3^T = [0,0,1]$
$v_4^T = [1,1,0]$
$\vec{v}_5^T = [1,0,1]$
$\vec{v}_6^T = [0,1,1]$
$\vec{v}_7^T = [1,1,1]$

As long as Bob can get any 3 unique vectors out of the 7, the message is still recoverable using a system of linear equations. This is because there are 3 variables. Ultimately, the more vectors, the easier it is to recover.

3E $[7\ ?\ ?\ 3\ 4\ ?\ ?]$

$$\begin{array}{ccc|c}
a & b & c & \\
\end{array}$$

$$\left[\begin{array}{ccc|c}
1 & 0 & 0 & 7 \\
0 & 1 & 0 & ? \\
0 & 0 & 1 & ? \\
1 & 1 & 0 & 3 \\
1 & 0 & 1 & 4 \\
0 & 1 & 1 & ? \\
1 & 1 & 1 & ?
\end{array}\right]
\begin{array}{l}
\\ \\ \\ R4-R1 \\ R5-R1 \\ \\ \\
\end{array}$$

$$\begin{array}{ccc|c}
a & b & c & \\
\end{array}$$

$$\left[\begin{array}{ccc|c}
1 & 0 & 0 & 7 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & -4 \\
0 & 0 & 1 & -3 \\
0 & 1 & 1 & ? \\
1 & 1 & 1 & ?
\end{array}\right]$$

$a = 7,\ b = -4,\ c = -3$

3F Alice should choose vectors from part D because she has a higher chance of avoiding erasures. As long as she is able to get to Bob one of the 5 vectors he hasn't seen, she's good, and the message should be translated. If she used strategy A, it has a higher chance of redundancy, as she won't know which code Bob

4 Show It

If $\{\vec{v}_1 \dots \vec{v}_k\}$ is a set of k linearly independent vectors in $\mathbb{R}^n$. Show that for any $n \times n$ matrix A, the set $\{A\vec{v}_1, A\vec{v}_2, \dots A\vec{v}_k\}$ is a set of linearly dependent vectors.

$$\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j \quad \leftarrow \text{need to prove this}$$

Using Definition of Linear independence, we can translate this to

$$A\vec{v}_k = \sum_{l \neq k} \beta_l (A\vec{v}_l)$$

Multiply both sides by Matrix A

$$A\vec{v}_1 = A\left(\sum_{j=1} \alpha_j \vec{v}_j\right)$$

$$A\left(\sum_{j=1} \alpha_j \vec{v}_j\right) = \sum_{j=1} A(\alpha_j \vec{v}_j) = \sum_{j=1} \alpha_j (A\vec{v}_j).$$

This works for any matrix A, and this will always result in a set of linearly dependent

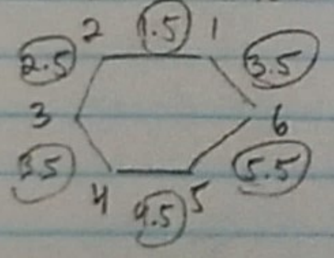
5 Figuring out the Tips

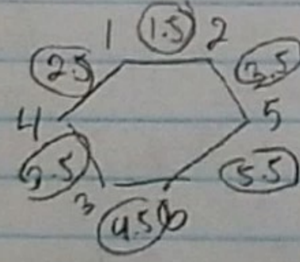
5A

$$\cancel{2}\cancel{\tfrac{1}{2}}(T_1 + T_2) = P1\,2 \qquad 2P_1 = \cancel{T_1} + \cancel{T_2}$$
$$\cancel{2}\cancel{\tfrac{1}{2}}(T_2 + T_3) = P_2\,2 \quad \rightarrow \quad -2P2 = -\cancel{T_2} - \cancel{T_3}$$
$$\cancel{2}\cancel{\tfrac{1}{2}}(T_3 + T_4) = P_3\,2 \qquad 2P_3 = \cancel{T_3} + \cancel{T_4}$$
$$\cancel{2}\cancel{\tfrac{1}{2}}(T_4 + T_5) = P_4\,2 \qquad -2P_4 = -\cancel{T_4} - \cancel{T_5}$$
$$\cancel{2}\cancel{\tfrac{1}{2}}(T_5 + T_6) = P_5\,2 \qquad 2P_5 = +\cancel{T_5} + \cancel{T_6}$$
$$2\tfrac{1}{2}(T_6 + T_1) = P_6\,2 \qquad -2P_6 = -\cancel{T_6} - \cancel{T_1}$$

$$2(P1 - P_2 + P_3 - P_4 + P_5 - P_6) = 0$$

No this is not possible
as it is impossible to extract the total tips in terms of a single person tip (due to there being an even amount of tippers here are two examples that lead to the exact same value of P_1 through P_6





These two unique assignments of T_1 through T_6 result in the exact same P_1 through P_6 results

5B

$$\cancel{2}\cancel{\tfrac{1}{2}}(T_1 + T_2) = 2 \cdot P_1 \qquad 2P_1 = T_1 + \cancel{T_2}$$
$$\cancel{2}\cancel{\tfrac{1}{2}}(T_2 + T_3) = 2 \cdot P_2 \qquad -2P_2 = -\cancel{T_2} - \cancel{T_3}$$
$$\cancel{2}\cancel{\tfrac{1}{2}}(T_3 + T_4) = 2 \cdot P_3 \qquad 2P_3 = \cancel{T_3} + \cancel{T_4}$$
$$\cancel{2}\cancel{\tfrac{1}{2}}(T_4 + T_5 = 2 \cdot P_4 \qquad -2P_4 = -\cancel{T_4} - \cancel{T_5}$$
$$2\tfrac{1}{2}(T_5 + T_1) = 2 \cdot P_5 \qquad 2P5 = \cancel{T_5} + T_1$$

$$T_1 = 2(P_1 - P_2 + P_3 - P_4 + P_5)$$

Since we can isolate T_1, we can use it to solve for T_2 … to T_5.

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5C If n is the total number of people sitting around a table, if n is odd, we can figure out everyone's tip, because it allows us to isolate at least 1 persons tip and work backwards to figure out everyone's tip. If n is even, it is not possible to isolate a single person's tip as everything zeros out.

6
$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \lambda \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{R(\theta)} \begin{bmatrix} P_x \\ P_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

A
$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} P_x \\ P_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} (P_x)(R_{xx}) + (P_y)(R_{xy}) \\ P(x)(R_{yx}) + (P_y)(R_{yy}) \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

$$q_x = P_x(R_{xx}) + P_y(R_{xy}) + T_x$$
$$q_y = P(x)(R_{yx}) + P_y(R_{yy}) + T_y$$

q and P are known, R and T are unknowns (6 unknowns)

Because there are 6 unknowns, we will need 6 equations and need at least 3 pairs of common points $\vec{p}$ and $\vec{q}$ to solve for the unknowns

6B
$$(q_{1x}) = P_{1x}(R_{xx}) + P_{1y}(R_{xy}) + T_x$$
$$(q_{1y}) = P_{1x}(R_{yx}) + P_{1y}(R_{yy}) + T_y$$
$$(q_{2x}) = P_{2x}(R_{xx}) + P_{2y}(R_{xy}) + T_x$$
$$(q_{2y}) = P_{2x}(R_{yx}) + P_{2y}(R_{yy}) + T_y$$
$$(q_{3x}) = P_{3x}(R_{xx}) + P_{3y}(R_{xy} + T_x$$
$$(q_{3y}) = P_{3x}(R_{yx}) + P_{3y}(R_{yy}) + T_y$$

6C
$$\begin{array}{cccccc} R_{xx} & R_{xy} & R_{yx} & R_{yy} & T_x & T_y \\ \end{array}$$
$$\begin{bmatrix} P_{1x} & P_{1y} & 0 & 0 & 1 & 0 \\ 0 & 0 & P_{1x} & P_{1y} & 0 & 1 \\ P_{2x} & P_{2y} & 0 & 0 & 1 & 0 \\ 0 & 0 & P_{2x} & P_{2y} & 0 & 1 \\ P_{3x} & P_{3y} & 0 & 0 & 1 & 0 \\ 0 & 0 & P_{3x} & P_{3y} & 0 & 1 \end{bmatrix}$$

$p_1 = \begin{bmatrix} 200 \\ 700 \end{bmatrix}$ $q_1 \begin{bmatrix} 162.2576 \\ 563.5662 \end{bmatrix}$

$p_2 \begin{bmatrix} 310 \\ 620 \end{bmatrix}$ $q_2 \begin{bmatrix} 255.4281 \\ 456.7469 \end{bmatrix}$

$p_3 \begin{bmatrix} 390 \\ 660 \end{bmatrix}$ $q_3 \begin{bmatrix} 355.2[illegible] \\ 49[illegible] \end{bmatrix}$

6D If $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ are collinear $(\vec{p}_2 - \vec{p}_1) = k(\vec{p}_3 - \vec{p}_1)$ for some $k \in \mathbb{R}$

If three points are collinear, then $p_1 p_2$ and p_3 will only map to themselves, and there will be only 1 unique point of reference for stitching the two images together. Because the system is linearly dependent, it will not be possible to stitch these images together with just one "real" reference point or line defined by these 3 collinear points.

6E If the three points are not collinear, meaning that the system is indeed linearly independent, then the system is fully determined just like in problem 6B/6C. In this example, the points are NOT collinear, and we are able to define the whole system because we can isolate variables when the points are NOT collinear.

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6E cont.

In the case that the points are not collinear, then using the definition of colinearity and linear dependency, we can conclude that the system is linearly independent, and this allows us to proceed with the math in 6B/6C to solve the equation and merge the two images

6F

$$\begin{matrix} q_x \\ q_y \end{matrix} = \lambda \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

$$a = \lambda \cos\theta$$
$$b = \lambda \sin\theta$$

$$\begin{matrix} q_x = \\ q_y = \end{matrix} \begin{bmatrix} a & -b \\ b & +a \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

$$\begin{array}{cccc} a & b & T_x & T_y \end{array}$$
$$\begin{bmatrix} p_x & -p_y & 1 & 0 \\ p_y & p_x & 0 & 1 \\ p_x' & -p_y' & 1 & 0 \\ p_y' & p_x' & 0 & 1 \end{bmatrix}$$

This equation allows us to recover the transformation with only two distinct points. generating the matrix above. We can then use this matrix to get an equation in matrix/vector form to solve this merging of images!

7 Find the inverse of $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$$\left[\begin{matrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}\right| \left.\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right]$$

R1 − R3
R2 − R4