Vedaant Tiwari
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EE16A HW3

1A $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ $\det = (1 \cdot 0) - (2 \cdot 1)$

$= -2 \neq 0$ invertible

1B $\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ $\det = (3 \cdot -1) - (2 \cdot 1)$

$-3 - 2 = -5 \neq 0$ invertible

1C $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ $\det = \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$

$\frac{1}{4} - \frac{3}{4} = \frac{-1}{2} \neq 0$ invertible

1D $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\det = 4 \neq 0$ invertible.

1E $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\det = (-1 + 0 + 0) - (0 + 1 + 0) = 1 - 1 = 0$

not invertible

1F $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 4 \end{bmatrix}$ $\det = (8 + 0 + 0) - (0 + 8 + 0)$ $8 - 8 = 0$

not invertible

1G $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $\det = 1 \det \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + 0 + 0$

$\det \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1 - 1 = 0$ $1 \cdot 0 = 0 \neq 0$

not invertible

1H $\begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$ $\det = \left(-1 + 0 + -\frac{1}{2}\right) - \left(0 + \frac{1}{2} + 1\right) = -2 \neq 0$

invertible

1J. $\begin{array}{c|c} 1 & \\ -1 & \\ 1 & \\ -1 & \\ & \end{array}\begin{array}{c} \begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix} \\ 1 \quad -1 \quad 1 \quad -1 \end{array}$ $\det = -1 \begin{vmatrix} 0 & -2 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 3 & 0 & -2 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{vmatrix}$

$$(-1)\begin{vmatrix} 0 & -2 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix} = \begin{matrix}(0 - 6 + 0) \\ -(1 + 0 - 16)\end{matrix} \quad -(-6-(-15)) \Rightarrow -9$$

$$(1)\begin{vmatrix} 3 & 0 & -2 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{vmatrix} = \begin{matrix}(0 + 0 + 0) \\ -(-12 + 3 + 0)\end{matrix} = 9$$

$\det = -9 + 9 = 0$ not invertible

2A $\begin{array}{cc} & \begin{matrix} 1 & \;2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$ = corresponding transition matrix

$$\vec{x}[n+1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}[n]$$

2B $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ $\quad \vec{x}[1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Universe 1

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$ $\quad \vec{x}[1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Alt. Universe

2C Using the data from timestep 1, we are unable to determine the water levels at timestep 0 because there are many previous timestep or initial states that when multiplied together with the corresponding transition matrix.

For Example in case 1:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

alt

$$x = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

As these two answers with different starting points show, it is impossible to determine the previous timestep, as this information is not preserved.

2D $A \cdot \vec{x}_1 = \vec{x}_2$ in order to determine the previous timestep $\vec{x}_1$, A must be invertible.

Since $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible, we are unable to recover the information in the previous matrix

In $\cancel{A}\ \cancel{A^{-1}} \cdot x_1 = \vec{x}_2\, A^{-1}$

$\vec{x}_1 = \vec{x}_2 \cdot A^{-1}$

Matrix A must be invertible for this to be true.

2E This statement implies that A is invertible, and by definition of invertibility, it is linearly independent, one to one, and onto.

$A \cdot \vec{x}_1 = \vec{x}_2$

in order to get $\vec{x}_1$ (previous timestep)

$\cancel{A^{-1}} \cancel{A} \cdot \vec{x}_1 = \vec{x}_2 A^{-1}$ A must be invertible

$I_n \vec{x}_1 = \vec{x}_2 A^{-1}$

2F If the entries of each column vector sums to one, the total water in the system is constant. No water is added or removed from this system

2G $\begin{bmatrix} 0 & 0 & 0 \\ 0.4 & 0.5 & 0.2 \\ 0 & 0.6 & 0.35 \end{bmatrix}$ The A matrix physically implies that there is an external source water so the total amount of water in the system is changing and is NOT constant. We know this because the rows and columns do not add up to 1.

2H $\begin{bmatrix} a_{1} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix}$ $[a_1 \cdots a_n] = 1$

$[xa_1 \ldots xa_{1n}] = x[a_{11} \cdots a_{1n}] = x$

Because the sum of the rows is one, multiplying by a uniform vector essentially becomes the uniform vector, regardless of how many columns / rows the two matrices have, as long as they are multipliable

3A The state transition matrix A is like as follows:

$$\begin{bmatrix} a & f & d \\ b & e & d \\ c & e & f \end{bmatrix} = A \qquad \begin{aligned} a+f+d&=1 \\ b+e+d&=1 \\ c+e+f&=1 \end{aligned}$$

$$0 < a,b,c,d,e,f < 1$$

$\vec{s}[n+1] = A\vec{s}[n]$ ← keeps track of the water distribution among the three reservoirs

3B $\begin{bmatrix} a & f & d \\ b & e & d \\ c & e & f \end{bmatrix}$ where $\begin{aligned} a+f+d&=1 \\ b+e+d&=1 \\ c+e+f&=1 \end{aligned}$

Because we know that the reservoir system is conservative, we know that the state transition matrix above rows sum to one which means that applying this system to a uniform vector will return the same uniform vector as proven in prop 2H.

Therefore $\vec{s}[n+1] = \vec{s}[n] = \vec{s}$ for any value of n.

The rows in this matrix add up to one, so we know that the total water stays the same and we can use the property with uniform vectors.

3C $A = \begin{bmatrix} 1/5 & 2/5 & 2/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix} = S$

In order to determine $\vec{s}[n]$ from the subsequent state $\vec{s}[n+1]$, A must be an invertible matrix

$A \cdot \vec{s}[n] = \vec{s}[n+1]$

$A^{-1} A \cdot \vec{s}[n] = A^{-1}\vec{s}[n+1]$

$I_n\, s[n] = A^{-1}\vec{s}[n+1]$ so A must be invertible in order to determine whether it is possible to determ $\vec{s}[n]$ from $\vec{s}[n+1]$

$$\det \begin{vmatrix} 1/5 & 2/5 & 2/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & 2/5 & 1/5 \end{vmatrix} = \left(\frac{1}{5^3} + \frac{8}{5^3} + \frac{8}{5^3}\right) = \frac{17}{5^3}$$
$$-\left(\frac{4}{5^3} + \frac{4}{5^3} + \frac{4}{5^3}\right) = \frac{12}{5^3}$$

$\frac{17}{5^3} - \frac{12}{5^3} = \frac{5}{5^3} = \boxed{\frac{1}{25}}$ $\quad \frac{1}{25} \neq 0$ so A is invertible, so it is possible to determine $\vec{s}[n]$ from $\vec{s}[n+1]$

1

$$\vec{x} \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix} \begin{matrix} \leftarrow \text{position} \\ \leftarrow \text{velocity} \\ \leftarrow \text{angle} \\ \leftarrow \text{angular velocity} \end{matrix}$$

$$\vec{x}[n+1] = \underset{\in \mathbb{R}^{4\times 4}}{A}\vec{x}[n] + \underset{\in \mathbb{R}^{4\times 1}}{\vec{b}}u[n]$$

4A $\vec{x}[1]$ when $n = 0$

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}u[0]$$

4B
$$\begin{aligned} \vec{x}[2] &= A\vec{x}[1] + \vec{b}u[1] \\ &= A[A\vec{x}[0] + \vec{b}u[0]] + \vec{b}u[1] \\ &= A^2x[0] + A\vec{b}u[0] + \vec{b}u[1] \end{aligned}$$

$$\begin{aligned} \vec{x}[3] &= Ax[2] + \vec{b}u[2] \\ &= A[A^2x[0] + A\vec{b}u[0] + \vec{b}u[1]] + \vec{b}u[2] \\ &= A^3x[0] + A^2\vec{b}u[0] + A\vec{b}u[1] + \vec{b}u[2] \end{aligned}$$

$$\begin{aligned} \vec{x}[4] &= Ax[3] + \vec{b}u[3] \\ &= A[A^3x[0] + A^2\vec{b}u[0] + A\vec{b}u[1] + \vec{b}u[2]] + \vec{b}u[3] \\ &= A^4x[0] + A^3\vec{b}u[0] + A^2\vec{b}u[1] + A\vec{b}u[2] + \vec{b}u[3] \end{aligned}$$

4C
$$\vec{x}[n] = A^n x[0] + \vec{b}\sum_{i=0}^{n-1} A^{n-1-i} \cdot u[i]$$

$$A = \begin{bmatrix} 1 & 0.05 & -0.01 & 0 \\ 0 & 0.22 & -0.17 & -0.01 \\ 0 & 0.10 & 1.14 & 0.10 \\ 0 & 1.6[illegible] & [illegible]5 & 1.14 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0.01 \\ 0.21 \\ -0.03 \\ -0.44 \end{bmatrix}$$

4 (cont)

$$\vec{x}[0] = \begin{bmatrix} -0.3853493 \\ 6.1032227 \\ 0.8120005 \\ -14 \end{bmatrix}$$

$$\vec{x}[t] = A^{t} \cdot \vec{x}[0] + \vec{b} \sum_{i=0}^{t-1} A^{t-1-i} \cdot u[i]$$

4D Two time steps

$$0 = \cancel{\vec{x}[2]} = A^2 \cdot \vec{x}[0] + A\vec{b}\,u[0] + \vec{b}\,u[1]$$

$$-A^2 x[0] = A\vec{b}\,u[0] + \vec{b}\,u[1]$$

$$Ax = b$$

$$\begin{bmatrix} Ab & \vec{b} & 0 & 0 \\ | & | & 0 & 0 \\ | & | & 0 & 0 \\ & & 0 & 0 \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix} = \begin{bmatrix} | \\ A^2 x[0] \\ | \end{bmatrix}$$

ipython, not possible in 3 time steps

4E Three time steps

$$0 = \cancel{\vec{x}[3]} = A^3 \vec{x}[0] + A^2 b\,u[0] + Ab\,u[1] + b\,u[2]$$

$Ax = b$, not poss

$$\begin{bmatrix} A^2 b & Ab & b & 0 \\ | & | & | & 0 \\ | & | & | & 0 \\ & & & 0 \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix} = \begin{bmatrix} | \\ A^3 x[0] \\ | \end{bmatrix}$$

4F Four Time step

0 $\cancel{\vec{x}[3]} = A^3\vec{x}[0] + A^2\vec{b}u[0] + A\vec{b}u[1] + \vec{b}u[2]$

$A^3x[0] = \vec{x}[3] - A^2\vec{b}u[0] + A\vec{b}u[1] + \vec{b}u[0]$

$Ax = b$

$$\begin{bmatrix} A^3b & A^2b & Ab & b \\ | & | & | & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix} = A^4x[0]$$

yes 4 step is enough

4G Ipython

4H A must be invertible and be an $N \times N$ matrix

4G A must be invertible and be an $N \times N$ matrix

5 Check Invertibility

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \det = 4-6 = 2 \neq 0, \text{ invertible}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \det = 1-0 \neq 0 \quad \text{invertible}$$

$$\begin{bmatrix} 6 & 4 & 3 \\ 5 & 2 & 1 \\ 12 & 8 & 6 \end{bmatrix} \quad \det = (72 + 120 + 48) - (72 + 48 + 120) = 0$$

not invertible

6 Worked on this HW alone. Finished what was assigned if I'm being honest. # 4 [illegible] took a day, 24 is as long as [illegible] it took