Binary search trees

1. Show how to implement the lower_bound and upper_bound functions for sets represented by binary search trees, in $O(h)$ time, where h is the height of the tree. By storing some additional information in the nodes, some other operations on sets can be performed in $O(h)$ time. Show how to do this for the following operations. Note that to ensure height is $O(\log n)$, it should be possible to update the extra information after every insert, erase or rotate operation.

(a) Find the kth smallest element in the set.

(b) Find the number of elements in the set with values in a given interval $[t_1, t_2]$.

(c) Find the closest pair in a set of integers, that is $\min |t_1 - t_2|$ for distinct pair of values t_1, t_2.

2. Suppose you have to maintain a map M from a finite subset of integers to integers. $M[x]$ is assumed to be 0 for integers not in the subset. The operations to be performed are:

(a) the subscript operator []: This can be used to read the value of $M[x]$ or assign a value $M[x] = y$. If x is not in the set, the assignment operation inserts it in the set, with y as the value of the map.

(b) Erase the element x from the set. It is now mapped to 0.

(c) $Add(x_1, y_1, c)$: Add a constant c to $M[x]$ for all elements x in the set such that $x_1 \leq x \leq y_1$.

(d) Find the element x in the set with maximum value of $M[x]$, if more than one, find the one with smallest magnitude.

3. Given a binary search tree containing numbers $1, 2, \ldots, n$, write an algorithm to find the number of permutations of n, such that inserting the numbers (without any balancing) in an empty tree, in the order given by the permutation, will result in the given binary search tree. Find a simple closed form expression for this in terms of some properties of the tree. For which tree is this number maximized, and how many such trees are there for which the maximum is achieved.

4. Suppose you are given a sequence of pairs of integers (a_i, b_i), which may be positive or negative, and a number S. It is required to find a substring of the sequence such that sum of the $a_i's$ in the substring is at least S and the sum of the $b_i's$ is minimum possible, subject to this constraint. Show how this can be done in $O(n \log n)$ time. If all the a_i are non-negative, show how this can be done in $O(n)$ time.