A Lecture Notes on

Discrete Mathematics:

Recurrence Relation

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In

CE/IT

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Recurrence Relation

Outline

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- Linear Recurrence Relation with constant coefficients
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- Homogeneous solution
- Non-Homogeneous Linear Recurrence Relation
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❖ Introduction

Consider the following instruction for generating a sequence

1. Start with 2
2. Give any term, add 5 to get the next term

If we list the terms of the sequence, we obtain

$2,7,12,17,22, \dots \dots \qquad ----(1)$

The first term is 2 from instruction 1. The second term is 7 as instruction 2 says to add 5 to get next term. The third term is 12 as instruction 2 says to add 5 to get next term.

By following instructions 1 and 2, we can compute any term in sequence.

If we denote the sequence (1) as $a_1, \ a_2, \dots ..$, we may rephrase instruction 1 as

$a_1 = 2 \qquad ----(2)$

and we may rephrase instruction 2 as

$\boldsymbol{a_n = a_{n-1} + 5 \quad ; n \geq 2} \qquad ----(3)$

Taking $n = 2$ in equation (3),

$a_2 = 2 + 5 = 7$

Taking $n = 3$ in equation (3),

$a_3 = a_2 + 5 = 7 + 5 = 12$

Equations (2) and (3) are equivalent to instruction 1 and 2.

Equation (3) is an example of recurrence relation.

❖ Definition

A recurrence relation for the sequence $a_0, \ a_1, \dots ..$ is an equation that relates a_n to certain of its predecessors $a_0, \ a_1, \dots .., a_{n-1}$.

This relation is also called difference equation.

Example:

Fibonacci numbers

The Fibonacci numbers are defined using the recurrence relation.

$F_n = F_{n-1} + F_{n-2}$

With values $F_0 = 0 \quad and \quad F_1 = 1$

Explicitly, recurrence yields the equations

$F_2 = F_1 + F_0$

$F_3 = F_2 + F_1$

$F_4 = F_3 + F_2$ etc.

We obtain the sequence of Fibonacci numbers which begins

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots \dots$

❖ Linear Recurrence Relation with constant coefficients

- The recurrence relation is called linear Recurrence relation if its degree is one.
- The general form of a linear recurrence relation of order k with constant coefficient is

  $C_0a_r + C_1a_{r-1} + C_2a_{r-2} \dots \dots + C_ka_{r-k} = f(r) \qquad ; r \geq k \qquad ----(1)$

  Where $C_0, C_1, \dots \dots, C_k$ are constants is called a linear recurrence relation with constant coefficients of order k.
- The order of recurrence relation is difference between greatest suffix and least suffix.

  For example,

  1. $a_r - 2a_{r-1} = 2r$ is a linear recurrence relation of order 1.
  2. $a_r + 2a_{r-3} = r^2$ is a linear recurrence relation of order 3.
  3. $a_r^2 + a_{r-1} = 5$ is **not** a linear recurrence relation as its degree is 2.
  4. $a_r + a_{r+1} + 2a_{r+2} = r^3$ is a linear recurrence relation of order 2.

❖ Homogeneous Linear Recurrence Relation

- The linear recurrence relation (1) is called a linear homogeneous recurrence relation of order k if $f(r) = 0$.

  For example,

  $a_r + 2a_{r-3} = 0$ is a linear homogeneous recurrence relation of order 3.

❖ Solution of Homogeneous Linear Recurrence Relation

Consider a linear homogeneous recurrence relation equation is

$C_0a_r + C_1a_{r-1} + C_2a_{r-2} \dots \dots + C_ka_0 = 0 \qquad ----(1)$

Suppose the solution of (1) is of the form

$a_r = C\alpha^r \qquad ----(2)$

Where α is the characteristic root of the equation (1) and C is the constant.

Substituting (2) in (1), we get,

$C_0(C\alpha^r) + C_1(C\alpha^{r-1}) + C_2(C\alpha^{r-2}) \dots \dots + C_k(C\alpha^0) = 0$

$C_0\alpha^r + C_1\alpha^{r-1} + C_2\alpha^{r-2} \dots \dots + C_k = 0 \qquad ----(3)$

Equation (3) is called the characteristic equation of the recurrence relation (1) and α is called the characteristic root.

- Equation (3) is a polynomial of degree k in terms of α. Thus equation (3) has k characteristic roots exists, say $\alpha_1, \alpha_2, \dots, \alpha_n$.
- Depending on the nature of the characteristic roots, a homogeneous solution can be written in different forms.
- The following two cases will exist.

Case – 1 When all the roots are real and distinct

$$a_r = A_1\alpha_1^r + A_2\alpha_2^r + \cdots A_r\alpha_r^r$$

Case – 2 When the roots are real and repeated

$$a_r = (A_1 + A_2 r)\alpha_1^r + A_3\alpha_2^r + \cdots A_r\alpha_r^r$$

Example – 1 (QB - 201)

Solve the recurrence relation $a_n = 2a_{n-1},\ a_0 = 1.$

Solution:

Rewrite the equation $a_n - 2a_{n-1} = 0$

The Characteristic equation is given by

$\alpha - 2 = 0$

$\therefore \alpha = 2$

The solution is

$a_n = A_1(2)^n$

Now, given that $a_0 = 1$

$a_0 = A_1(2)^0$

$\therefore A_1 = 1$

Hence, the required solution is

$a_n = (2)^n$

Example – 2 (QB - 203)

Solve the recurrence relation $a_n - 3a_{n-1} + 2a_{n-2} = 0.$

Solution:

The Characteristic equation is given by

$\alpha^2 - 3\alpha + 2 = 0$

$\therefore (\alpha - 1)(\alpha - 2) = 0$

$\therefore \alpha = 1, 2$

The solution is

$a_n = A_1(1)^n + A_2(2)^n = A_1 + A_2(2)^n$

Example – 3 (QB - 204)

Solve the recurrence relation $a_k - 5a_{k-1} + 6a_{k-2} = 0,\ a_0 = 2\ and\ a_1 = 5.$

Solution:

The Characteristic equation is given by

$\alpha^2 - 5\alpha + 6 = 0$

$\therefore (\alpha - 2)(\alpha - 3) = 0$

$\therefore \alpha = 2, 3$

The solution is

$a_k = A_1(2)^k + A_2(3)^k$

Now, given that $a_0 = 2 \; and \; a_1 = 5$

$a_0 = A_1(2)^0 + A_2(3)^0 = A_1 + A_2$

$\therefore A_1 + A_2 = 2 \qquad ----(1)$

Now,

$a_1 = A_1(2)^1 + A_2(3)^1 = 2A_1 + 3A_2$

$\therefore 2A_1 + 3A_2 = 5 \qquad ----(2)$

Solving (1) and (2),

$$\begin{array}{l} 2A_1 + 2A_2 = 4 \\ 2A_1 + 3A_2 = 5 \\ - \quad\quad - \quad\quad - \\ \hline -A_2 = -1 \end{array}$$

$\therefore \; A_2 = 1$

$\therefore A_1 = 1$

Hence, the required solution is

$a_k = (2)^k + (3)^k$

Example – 4 (QB - 206)

Solve the recurrence relation $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 0$.

Solution:

The Characteristic equation is given by

$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$

-2	1	6	12	8
	0	-2	-8	-8
	1	4	4	0

$\therefore (\alpha + 2)(\alpha^2 + 4\alpha + 4) = 0$

$\therefore (\alpha + 2)(\alpha + 2)(\alpha + 2) = 0$

$\therefore \alpha = -2, -2, -2$

The required solution is

$a_n = (A_1 + A_2 n + A_2 n^2)(-2)^n$

Example – 5 (QB - 207)

Solve the recurrence relation $S_k = S_{k-1} + S_{k-2},\; k \geq 2;\; S_0 = 1, S_1 = 1.$

Solution:

We have $S_k - S_{k-1} - S_{k-2} = 0$

The Characteristic equation is given by

$\alpha^2 - \alpha - 1 = 0$

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \alpha = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\therefore \alpha = \frac{1+\sqrt{5}}{2}, \qquad \frac{1-\sqrt{5}}{2}$$

The solution is

$$S_k = A_1 \left(\frac{1+\sqrt{5}}{2}\right)^k + A_2 \left(\frac{1-\sqrt{5}}{2}\right)^k$$

Given that $S_0 = 1, S_1 = 1$.

$S_0 = 1$

$\Rightarrow S_0 = A_1 + A_2$

$\Rightarrow A_1 + A_2 = 1 \qquad ----(1)$

$S_1 = 1$

$$\Rightarrow S_1 = A_1 \left(\frac{1+\sqrt{5}}{2}\right) + A_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$\Rightarrow A_1 \left(\frac{1+\sqrt{5}}{2}\right) + A_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 \qquad ----(2)$$

Solving (1) and (2),

$$A_1 \left(\frac{1+\sqrt{5}}{2}\right) + A_2 \left(\frac{1+\sqrt{5}}{2}\right) = \left(\frac{1+\sqrt{5}}{2}\right)$$

$$A_1 \left(\frac{1+\sqrt{5}}{2}\right) + A_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$-\qquad\qquad -\qquad\qquad -$

$$\sqrt{5}\, A_2 = \left(\frac{1+\sqrt{5}}{2}\right) - 1$$

$$\therefore A_2 = -\frac{1-\sqrt{5}}{2\sqrt{5}}$$

$$\therefore A_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

The required solution is

$$S_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}$$

Example – 6 (QB - 208)

Solve the recurrence relation $a_r - 7a_{r-1} + 16a_{r-2} - 12a_{r-3} = 0;\ a_0 = 1, a_1 = 4, a_2 = 8.$

Solution:

The Characteristic equation is given by

$\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0$

2	1	-7	16	-12
	0	2	-10	12
	1	-5	6	0

$\therefore (\alpha - 2)(\alpha^2 - 5\alpha + 6) = 0$

$\therefore \alpha = 2, 2, 3$

The solution is

$a_r = (A_1 + A_2 r)(2)^r + A_3(3)^r$

Given that $a_0 = 1, a_1 = 4, a_2 = 8$

$a_0 = 1$

$\Rightarrow A_1 + A_3 = 1 \qquad ----(1)$

$a_1 = 4$

$\Rightarrow 2(A_1 + A_2) + 3A_3 = 4$

$\Rightarrow 2A_1 + 2A_2 + 3A_3 = 4 \qquad ----(2)$

$a_2 = 8$

$\Rightarrow 4(A_1 + 2A_2) + 9A_3 = 8$

$\Rightarrow 4A_1 + 8A_2 + 9A_3 = 8 \qquad ----(3)$

Solving (1), (2) and (3), we get.

$A_1 = 5, \quad A_2 = 3, \quad A_3 = -4$

The required solution is

$a_r = (5 + 3r)(2)^r - 4(3)^r$

Example – 7 (QB - 222)

$a_r - 10a_{r-1} + 9a_{r-2} = 0\ with\ \ a_0 = 3\ and\ a_1 = 11$. Find homogeneous solution.

Solution:

The Characteristic equation is given by

$\alpha^2 - 10\alpha + 9 = 0$

$\therefore (\alpha - 1)(\alpha - 9) = 0$

$\therefore \alpha = 1, 9$

The solution is

$a_r = A_1(1)^r + A_2(9)^r$

Given that $a_0 = 3\ and\ a_1 = 11$

$a_0 = 3$

$\Rightarrow A_1 + A_2 = 3 \qquad ----(1)$

$a_1 = 11$

$\Rightarrow A_1 + 9A_2 = 11 \qquad ----(2)$

Solving (1) and (2), we get

$A_1 = 2, \qquad A_2 = 1$

Hence, the Homogeneous solution is

$a_r = 2(1)^r + (9)^r$

Example – 8 (QB - 223)

Consider $a_r - 8a_{r-1} + 16a_{r-2} = 0 \ where\ a_2 = 16\ and\ a_3 = 80$, Find solution.

Solution:

The Characteristic equation is given by

$\alpha^2 - 8\alpha + 16 = 0$

$\therefore (\alpha - 4)^2 = 0$

$\therefore \alpha = 4, 4$

The solution is

$a_r = (A_1 + A_2 r)(4)^r$

Given that $a_2 = 16\ and\ a_3 = 80$

$a_2 = 16$

$\Rightarrow 16A_1 + 32A_2 = 16$

$\Rightarrow A_1 + 2A_2 = 1 \qquad ----(1)$

$a_3 = 80$

$\Rightarrow 64A_1 + 192A_2 = 80$

$\Rightarrow 4A_1 + 12A_2 = 5 \qquad ----(2)$

Solving (1) and (2), we get

$A_1 = \frac{1}{2}, \qquad A_2 = \frac{1}{4}$

Hence, the Homogeneous solution is

$a_r = \left(\frac{1}{2} + \frac{1}{4}r\right)(4)^r$

Example – 9 (QB - 224)

Solve the recurrence relation: $d_n = 4(d_{n-1} - d_{n-2})$. Subject to initial conditions $d_0 = 1 = d_1$.

Solution:

$d_n - 4d_{n-1} + 4d_{n-2} = 0$

The Characteristic equation is given by

$\alpha^2 - 4\alpha + 4 = 0$

$\therefore (\alpha - 2)^2 = 0$

$\therefore \alpha = 2, 2$

The solution is

$d_n = (A_1 + A_2 n)(2)^n$

Given that $d_0 = 1 = d_1$

$d_0 = 1$

$\Rightarrow A_1 = 1 \qquad ----(1)$

$d_1 = 1$

$\Rightarrow 2A_1 + 2A_2 = 1 \qquad ----(2)$

Solving (1) and (2), we get

$A_1 = 1\,, \qquad A_2 = -\frac{1}{2}$

Hence, the Homogeneous solution is

$d_n = \left(1 - \frac{1}{2}n\right)(2)^n$

Example – 10 (QB - 225)

Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$, with initial conditions $a_0 = 1, a_1 = -2 \; and \; a_2 = -1.$

Solution:

$a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$

The Characteristic equation is given by

$\alpha^3 + 3\alpha^2 + 3\alpha + 1 = 0$

$\therefore (\alpha + 1)^3 = 0$

$\therefore \alpha = -1, -1, -1$

The solution is

$a_n = (A_1 + A_2 n + A_3 n^2)(-1)^n$

Given that $a_0 = 1, a_1 = -2 \; and \; a_2 = -1$

$a_0 = 1$

$\Rightarrow A_1 = 1 \qquad ----(1)$

$a_1 = -2$

$\Rightarrow -A_1 - A_2 - A_3 = -2$

$\Rightarrow A_1 + A_2 + A_3 = 2 \qquad ----(2)$

$a_2 = -1$

$\Rightarrow A_1 + 2A_2 + 4A_3 = -1 \qquad ----(3)$

Solving (1), (2) and (3), we get.

$A_1 = 1, \qquad A_2 = 3, \qquad A_3 = -2$

The required solution is

$a_n = (1 - 2n + 3n^2)(-1)^n$

❖ Non – Homogeneous Linear Recurrence Relation

The general form of a linear recurrence relation of order k with constant coefficient is

$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} \ldots\ldots + C_k a_{r-k} = f(r) \qquad ; r \geq k \qquad ----(1)$

If $f(r) \neq 0$ then it is called as linear non-homogeneous recurrence relation.

❖ Solution of Non – Homogeneous Linear Recurrence Relation

The general solution of (1) is

$a_r = a_r^{(h)} + a_r^{(p)}$

Where $a_r^{(h)}$ is a homogeneous solution and $a_r^{(p)}$ is a particular solution of (1).

❖ Method to find Particular solution

$\boldsymbol{f(r)}$ ***or RHS***	**Form of Particular solution**
A constant, d	A constant, P
A Linear function $d_0 + d_1 r$	A linear function $P_0 + P_1 r$
An n^{th} degree polynomial $d_0 + d_1 r + d_2 r^2 + \cdots + d_n r^n$	An n^{th} degree polynomial $P_0 + P_1 r + P_2 r^2 + \cdots + P_n r^n$
d^r	$P \cdot d^r$
An exponential function $d \cdot b^r$; provided b is not the characteristic root.	$P \cdot b^r$
An exponential function $d \cdot b^r$; provided b is the characteristic root with multiplicity m	$P \cdot r^m \, b^r$
$b^r \cdot P(r)$ if b is a root of characteristic equation of multiplicity m	$r^m \, (P_0 + P_1 r + P_2 r^2 + \cdots + P_n r^n) \, b^r$
$b^r \cdot P(r)$ if b is not a root of characteristic equation	$b^r \, (P_0 + P_1 r + P_2 r^2 + \cdots + P_n r^n)$
$\sin d_r$ or $\cos d_r$	$A \sin d_r + B \cos d_r$

Example – 1 (QB - 213)

Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 9;\ a_0 = 0, a_1 = 1$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha^2 - 5\alpha + 6 = 0$

$\therefore (\alpha - 2)(\alpha - 3) = 0$

$\therefore\ \alpha = 2, 3$

$a_n^{(h)} = A_1(2)^n + A_2(3)^n$

For Particular solution,

We have 9 in RHS, which is a constant.

Let $a_n^{(p)} = P$

$\therefore\ a_{n-1} = P = a_{n-2}$

Substituting above values in given recurrence relation

$P - 5P + 6P = 9$

$\therefore 2P = 9$

$\therefore \boldsymbol{P = \frac{9}{2}}$

$$a_n^{(p)} = \frac{9}{2}$$

Thus, the general solution is

$a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = A_1(2)^n + A_2(3)^n + \frac{9}{2}$$

Given that $a_0 = 0, a_1 = 1$.

$a_0 = 0$

$$\Rightarrow A_1 + A_2 + \frac{9}{2} = 0$$

$$\Rightarrow A_1 + A_2 = -\frac{9}{2} \qquad ----(1)$$

$a_1 = 1$

$$\Rightarrow A_1(2)^1 + A_2(3)^1 + \frac{9}{2} = 1$$

$$\Rightarrow 2A_1 + 3A_2 = -\frac{7}{2} \qquad ----(2)$$

Solving (1) and (2),

$A_1 = -10, \qquad A_2 = \frac{11}{2}$

Hence,

$a_n = -10(2)^n + \frac{11}{2}(3)^n + \frac{9}{2}$

Example – 2 (QB - 209)

Find the particular solution of the recurrence relation $a_{n+2} - 3a_{n+1} + 2a_n = 5^n$.

Solution:

Let $a_n^{(p)} = P \cdot 5^n$

$\therefore a_{n+1} = P \cdot 5^{n+1}$

$\therefore a_{n+2} = P \cdot 5^{n+2}$

Substituting above values in given recurrence relation

$P \cdot 5^{n+2} - 3\, P \cdot 5^{n+1} + 2\, P \cdot 5^n = 5^n$

$\therefore 25\, P\, 5^n - 15\, P\, 5^n + 2\, P\, 5^n = 5^n$

$\therefore 12\, P\, 5^n = 5^n$

$\therefore 12\, P = 1$

$\therefore \boldsymbol{P = \frac{1}{12}}$

$a_n^{(p)} = \frac{1}{12} \cdot 5^n$

Example – 3 (QB - 211)

Solve the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha^2 + 5\alpha + 6 = 0$

$\therefore (\alpha + 2)(\alpha + 3) = 0$

$\therefore \ \alpha = -2, -3$

$a_n^{(h)} = A_1(-2)^n + A_2(-3)^n$

For Particular solution,

Here $f(n) = 3\, n^2$

Let $a_n^{(p)} = P_0 + P_1 n + P_2 n^2$

$a_{n-1} = P_0 + P_1(n-1) + P_2(n-1)^2$

$a_{n-2} = P_0 + P_1(n-2) + P_2(n-2)^2$

Substituting above values in given recurrence relation

$P_0 + P_1 n + P_2 n^2 + 5P_0 + 5P_1(n-1) + 5P_2(n-1)^2 + 6P_0 + 6P_1(n-2) + 6P_2(n-2)^2 = 3n^2$

$\therefore P_0 + P_1 n + P_2 n^2 + 5P_0 + 5P_1 n - 5P_1 + 5P_2 n^2 - 10P_2 n + 5P_2 + 6P_0 + 6P_1 n - 12P_1$

$+6P_2 n^2 - 24P_2 n + 24P_2 = 3n^2$

$\therefore 12P_0 + 12P_1 n + 12P_2 n^2 - 17P_1 - 34P_2 n + 29P_2 = 3n^2$

$\therefore (12P_0 - 17P_1 + 29P_2) + (12P_1 - 34P_2)n + 12P_2 n^2 = 3n^2$

Comparing on both the sides, we get,

$12P_0 - 17P_1 + 29P_2 = 0$

$12P_1 - 34P_2 = 0$

$12P_2 = 3 \Rightarrow \boldsymbol{P_2} = \frac{\mathbf{1}}{\mathbf{4}}$

$\therefore 12P_1 = \frac{34}{4} \Rightarrow \boldsymbol{P_1} = \frac{\mathbf{17}}{\mathbf{24}}$

$\therefore 12P_0 = \frac{289}{24} - \frac{29}{4} = \frac{1156 - 696}{96} = \frac{460}{96} = \frac{115}{24} \Rightarrow \boldsymbol{P_0} = \frac{\mathbf{115}}{\mathbf{288}}$

$a_n^{(p)} = \frac{115}{288} + \frac{17}{24}n + \frac{1}{4}n^2$

Thus, the general solution is

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = A_1(-2)^n + A_2(-3)^n + \frac{115}{288} + \frac{17}{24}n + \frac{1}{4}n^2$

Example – 4 (QB - 210)

Solve the recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = n + 3^n$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha^2 - 4\alpha + 4 = 0$

$\therefore (\alpha - 2)(\alpha - 2) = 0$

$\therefore \ \alpha = 2, 2$

$a_n^{(h)} = (A_1 + A_2 n)(2)^n$

For Particular solution,

Here, $f(n) = n + 3^n$

Let $a_n^{(p)} = (P_0 + P_1 n) + P_2 \cdot 3^n$

$a_{n-1} = (P_0 + P_1(n-1)) + P_2 \cdot 3^{n-1}$

$a_{n-2} = (P_0 + P_1(n-2)) + P_2 \cdot 3^{n-2}$

Substituting above values in given recurrence relation

$$(P_0 + P_1 n) + P_2 \cdot 3^n - 4\big(P_0 + P_1(n-1)\big) - 4P_2 \cdot 3^{n-1} + 4(P_0 + P_1(n-2)) + 4P_2 \cdot 3^{n-2}$$
$$= n + 3^n$$

$$\therefore P_0 + P_1 n + P_2 \cdot 3^n - 4P_0 - 4P_1 n + 4P_1 - \frac{4}{3} P_2\, 3^n + 4P_0 + 4P_1 n - 8P_1 + \frac{4}{9}\, P_2\, 3^n = n + 3^n$$

$$\therefore P_0 + P_1 n + \frac{1}{9}\, P_2 \cdot 3^n - 4P_1 = n + 3^n$$

$$\therefore (P_0 - 4P_1) + P_1 n + \frac{1}{9}\, P_2 \cdot 3^n = n + 3^n$$

Comparing on both the sides, we get,

$(P_0 - 4P_1) = 0$

$\boldsymbol{P_1 = 1}$

$\frac{1}{9}\, P_2 = 1 \Rightarrow \boldsymbol{P_2 = 9}$

$\boldsymbol{P_0 = 4}$

$a_n^{(p)} = (4 + n) + 9 \cdot 3^n$

Thus, the general solution is

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = (A_1 + A_2 n)(2)^n + (4 + n) + 9 \cdot 3^n$

Example – 5 (QB - 215)

Solve the recurrence relation $a_r - 7a_{r-1} + 12a_{r-2} = r.4^r$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha^2 - 7\alpha + 12 = 0$

$\therefore (\alpha - 4)(\alpha - 3) = 0$

$\therefore\ \alpha = 3, 4$

$a_r^{(h)} = A_1(3)^r + A_2(4)^r$

For Particular solution,

Here, $f(r) = r.4^r$ and 4 is a root of characteristic equation of multiplicity 1.

Let $a_r^{(p)} = r\,(P_0 + P_1 r)\, 4^r = (P_0 r + P_1 r^2)\, 4^r$

$a_{r-1} = [P_0(r-1) + P_1(r-1)^2]\, 4^{r-1}$

$a_{r-2} = [P_0(r-2) + P_1(r-2)^2]\, 4^{r-2}$

Substituting above values in given recurrence relation

$$(P_0r + P_1r^2)\ 4^r - 7[P_0(r-1) + P_1(r-1)^2]4^{r-1} + 12[P_0(r-2) + P_1(r-2)^2]\ 4^{r-2} = r.4^r$$

$$\therefore P_0r\ 4^r + P_1r^2\ 4^r - 7P_0r4^{r-1} + 7P_04^{r-1} - 7P_1r^24^{r-1} + 14P_1r4^{r-1} - 7P_14^{r-1} + 12P_0r4^{r-2}$$
$$-24P_04^{r-2} + 12P_1r^24^{r-2} - 48P_1r4^{r-2} + 48P_14^{r-2} = r.4^r$$

$$\therefore P_0r\ 4^r + P_1r^2\ 4^r - \frac{7}{4}P_0r4^r + \frac{7}{4}P_04^r - \frac{7}{4}P_1r^24^r + \frac{14}{4}P_1r4^r - \frac{7}{4}P_14^r + \frac{12}{16}P_0r4^r$$
$$-\frac{24}{16}P_04^r + \frac{12}{16}P_1r^24^r - \frac{48}{16}P_1r4^r + \frac{48}{16}P_14^r = r.4^r$$

$$\therefore \frac{1}{4}P_04^r + \frac{1}{2}P_1r4^r + \frac{5}{4}P_14^r = r.4^r$$

$$\therefore \left(\frac{1}{4}P_0 + \frac{5}{4}P_1\right)4^r + \frac{1}{2}P_1r4^r = r.4^r$$

Comparing on both the sides, we get,

$$\frac{1}{4}P_0 + \frac{5}{4}P_1 = 0$$

$$\frac{1}{2}P_1 = 1 \Rightarrow \boldsymbol{P_1 = 2}$$

$$\therefore \boldsymbol{P_0 = -10}$$

$$a_r^{(p)} = (P_0r + P_1r^2)\ 4^r = (-10r + 2r^2)\ 4^r$$

Thus, the general solution is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = A_1(3)^r + A_2(4)^r + (-10r + 2r^2)\ 4^r$$

Example – 6 (QB - 214)

Solve the recurrence relation $a_n - 6a_{n-1} + 9a_{n-2} = (n+1)3^n$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$$\alpha^2 - 6\alpha + 9 = 0$$

$$\therefore (\alpha - 3)(\alpha - 3) = 0$$

$$\therefore\ \alpha = 3, 3$$

$$a_n^{(h)} = (A_1 + A_2n)(3)^n$$

For Particular solution,

Here, $f(n) = (n+1)3^n$ and 3 is a double root of characteristic equation of multiplicity 2.

Let $a_n^{(p)} = n^2\ (P_0 + P_1n)\ 3^n = (P_0n^2 + P_1n^3)\ 3^n$

$$a_{n-1} = [P_0(n-1)^2 + P_1(n-1)^3]\ 3^{n-1}$$

$$a_{n-2} = [P_0(n-2)^2 + P_1(n-2)^3]\ 3^{n-2}$$

Substituting above values in given recurrence relation

$$(P_0n^2 + P_1n^3)\,3^n - 6[P_0(n-1)^2 + P_1(n-1)^3]3^{n-1} + 9[P_0(n-2)^2 + P_1(n-2)^3]\,3^{n-2} = (n+1)3^n$$

$\therefore P_0n^23^n + P_1n^33^n - 2P_0n^23^n + 4P_0n3^n - 2P_03^n - 2P_1n^33^n + 2P_13^n + 6P_1n^23^n - 6P_1n3^n$
$\quad + P_0n^23^n - 4P_0n3^n + 4P_03^n + P_1n^33^n - 8P_13^n - 6P_1n^23^n + 12P_1n3^n = n3^n + 3^n$

$\therefore\ 2P_03^n + 6P_1n3^n - 6P_13^n = n3^n + 3^n$

$\therefore (6P_1)n3^n + (2P_0 - 6P_1)3^n = n3^n + 3^n$

Comparing on both the sides, we get,

$6P_1 = 1 \qquad \Rightarrow \boldsymbol{P_1 = \frac{1}{6}}$

$2P_0 - 6P_1 = 1 \qquad \Rightarrow \boldsymbol{P_0 = 1}$

$$a_n^{(p)} = \left(n^2 + \frac{1}{6}\,n^3\right)\,3^n$$

Thus, the general solution is

$a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = (A_1 + A_2n)(3)^n + \left(n^2 + \frac{1}{6}\,n^3\right)\,3^n$$

Example – 7 (QB – 227)

Find total solution of $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1.$

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha^2 + 5\alpha + 6 = 0$

$\therefore (\alpha + 2)(\alpha + 3) = 0$

$\therefore\ \alpha = -2, -3$

$a_r^{(h)} = A_1(-2)^r + A_2(-3)^r$

For Particular solution,

Here $f(r) = 3r^2 - 2r + 1$

Let $a_r^{(p)} = P_0 + P_1r + P_2r^2$

$a_{r-1} = P_0 + P_1(r-1) + P_2(r-1)^2$

$a_{r-2} = P_0 + P_1(r-2) + P_2(r-2)^2$

Substituting above values in given recurrence relation

$$P_0 + P_1r + P_2r^2 + 5P_0 + 5P_1(r-1) + 5P_2(r-1)^2 + 6P_0 + 6P_1(r-2) + 6P_2(r-2)^2 = 3r^2 - 2r + 1$$

$$\therefore P_0 + P_1 r + P_2 r^2 + 5P_0 + 5P_1 r - 5P_1 + 5P_2 r^2 - 10P_2 r + 5P_2 + 6P_0 + 6P_1 r - 12P_1$$
$$+6P_2 r^2 - 24P_2 r + 24P_2 = 3r^2 - 2r + 1$$

$$\therefore 12P_0 + 12P_1 r + 12P_2 r^2 - 17P_1 - 34P_2 r + 29P_2 = 3r^2 - 2r + 1$$

$$\therefore (12P_0 - 17P_1 + 29P_2) + (12P_1 - 34P_2)r + 12P_2 r^2 = 3r^2 - 2r + 1$$

Comparing on both the sides, we get,

$$12P_0 - 17P_1 + 29P_2 = 1$$

$$12P_1 - 34P_2 = -2$$

$$12P_2 = 3 \qquad \Rightarrow \boldsymbol{P_2 = \frac{1}{4}}$$

$$\therefore 12P_1 = -2 + \frac{34}{4} = \frac{13}{2} \qquad \Rightarrow \boldsymbol{P_1 = \frac{13}{24}}$$

$$\therefore 12P_0 = 1 + \frac{221}{24} - \frac{29}{4} = \frac{71}{24} \qquad \Rightarrow \boldsymbol{P_0 = \frac{71}{288}}$$

$$a_r^{(p)} = P_0 + P_1 r + P_2 r^2$$

$$a_r^{(p)} = \frac{71}{288} + \frac{13}{24} r + \frac{1}{4} r^2$$

Thus, the general solution is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = A_1(-2)^r + A_2(-3)^r + \frac{71}{288} + \frac{13}{24} r + \frac{1}{4} r^2$$

Example – 8

Solve $a_r - 2a_{r-1} + a_{r-2} = 7$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$$\alpha^2 - 2\alpha + 1 = 0$$

$$\therefore (\alpha - 1)(\alpha - 1) = 0$$

$$\therefore \ \alpha = 1, 1$$

$$a_r^{(h)} = (A_1 + A_2 r)(1)^r$$

For Particular solution,

Let $a_r^{(p)} = P\, r^2$

The question arises that why $P\, r^2$ has been chosen corresponding to 7 and not P or Pr. The reason is simple. Since 1 is a double root of characteristic equation.

$a_{r-1} = P\ (r-1)^2$

$a_{r-2} = P\ (r-2)^2$

Substituting above values in given recurrence relation

$P\ r^2 - 2P\ (r-1)^2 + P\ (r-2)^2 = 7$

$\therefore\ P\ r^2 - 2\ P\ r^2 + 4P\ r - 2P + P\ r^2 - 4P\ r + 4P = 7$

$\therefore 2P = 7$

$\therefore \boldsymbol{P = \frac{7}{2}}$

$a_r^{(p)} = P\ r^2 = \frac{7}{2}\ r^2$

Thus, the general solution is

$a_r = a_r^{(h)} + a_r^{(p)}$

$a_r = (A_1 + A_2 r)(1)^r + \frac{7}{2}\ r^2$

Example – 9 (QB - 232)

Solve the equation $a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r,\ r \geq 2,$

with given boundary conditions $a_0 = 1,\ a_1 = 1.$

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha^2 - 5\alpha + 6 = 0$

$\therefore (\alpha - 2)(\alpha - 3) = 0$

$\therefore\ \alpha = 2, 3$

$a_r^{(h)} = A_1(2)^r + A_2(3)^r$

For Particular solution,

Here $f(r) = 2^r + r$

Let $a_r^{(p)} = (P_0\ r\ 2^r) + (P_1 + P_2 r)$

$a_{r-1} = [P_0\ (r-1)2^{r-1}] + P_1 + P_2(r-1)$

$a_{r-2} = [P_0\ (r-2)2^{r-2}] + P_1 + P_2(r-2)$

Substituting above values in given recurrence relation

$$(P_0\ r\ 2^r) + (P_1 + P_2 r) - 5[P_0\ (r-1)2^{r-1}] - 5P_1 - 5P_2(r-1) + 6[P_0\ (r-2)2^{r-2}] + 6P_1 + 6P_2(r-2) = 2^r + r$$

$$\therefore P_0\, r\, 2^r + P_1 + P_2 r - \frac{5}{2} P_0 r 2^r + \frac{5}{2} P_0\, 2^r - 5P_1 - 5P_2 r + 5P_2 + \frac{3}{2}\, P_0\, r\, 2^r - 3P_0\, 2^r + 6P_1$$

$$+6P_2\, r - 12P_2 = 2^r + r$$

$$\therefore 2P_1 + 2P_2\, r - \frac{1}{2}\, P_0\, 2^r - 7P_2 = 2^r + r$$

$$\therefore (2P_1 - 7P_2) + (2P_2)r + \left(-\tfrac{1}{2}\, P_0\right)\, 2^r = 2^r + r$$

Comparing on both the sides, we get,

$$2P_1 - 7P_2 = 0$$

$$2P_2 = 1 \qquad \Rightarrow \boldsymbol{P_2} = \frac{\mathbf{1}}{\mathbf{2}}$$

$$-\frac{1}{2}\, P_0 = 1 \qquad \Rightarrow \boldsymbol{P_0} = \mathbf{-2}$$

$$\therefore \boldsymbol{P_1} = \frac{\mathbf{7}}{\mathbf{4}}$$

$$a_r^{(p)} = (P_0\, r\, 2^r) + (P_1 + P_2 r)$$

$$a_r^{(p)} = (-2\, r\, 2^r) + \left(\frac{7}{4} + \frac{1}{2} r\right)$$

Thus, the general solution is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = A_1 (2)^r + A_2 (3)^r + (-2\, r\, 2^r) + \left(\frac{7}{4} + \frac{1}{2} r\right) \qquad ----(1)$$

Given that $a_0 = 1,\ a_1 = 1.$

$$a_0 = 1$$

$$\Rightarrow A_1 + A_2 + \frac{7}{4} = 1$$

$$\Rightarrow A_1 + A_2 = -\frac{3}{4} \qquad ----(2)$$

$$a_1 = 1$$

$$\Rightarrow 2A_1 + 3A_2 - 4 + \frac{9}{4} = 1$$

$$\Rightarrow 2A_1 + 3A_2 = \frac{11}{4} \qquad ----(3)$$

Solving (2) and (3),

$$A_1 = -5, \qquad A_2 = \frac{17}{4}$$

From (1),

$$a_r = -5(2)^r + \frac{17}{4}(3)^r + (-2\, r\, 2^r) + \left(\frac{7}{4} + \frac{1}{2} r\right)$$

Example – 10 (QB - 229)

Find the general solution of $a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha^2 + 5\alpha + 6 = 0$

$\therefore (\alpha + 2)(\alpha + 3) = 0$

$\therefore \alpha = -2, -3$

$a_r^{(h)} = A_1(-2)^r + A_2(-3)^r$

For Particular solution,

Here $f(r) = 42 \cdot 4^r$

Let $a_r^{(p)} = P\ 4^r$

$a_{r-1} = P\ 4^{r-1}$

$a_{r-2} = P\ 4^{r-2}$

Substituting above values in given recurrence relation

$P\ 4^r + 5\ P\ 4^{r-1} + 6\ P\ 4^{r-2} = 42 \cdot 4^r$

$\therefore P\ 4^r + \frac{5}{4}\ P\ 4^r + \frac{3}{8}\ P\ 4^r = 42 \cdot 4^r$

$\therefore \left(P + \frac{5}{4}\ P + \frac{3}{8}\ P\right) 4^r = 42 \cdot 4^r$

Comparing on both the sides, we get,

$P + \frac{5}{4}\ P + \frac{3}{8}\ P = 42$

$\therefore \frac{21}{8}\ P = 42$

$\therefore \boldsymbol{P = 16}$

$a_r^{(p)} = P\ 4^r = 16 \cdot 4^r$

Thus, the general solution is

$a_r = a_r^{(h)} + a_r^{(p)}$

$a_r = A_1(-2)^r + A_2(-3)^r + 16 \cdot 4^r$

Example – 11 (QB - 219)

Solve the recurrence relation $a_n - 3a_{n-1} + 2a_{n-2} = 2^n$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha^2 - 3\alpha + 2 = 0$

$\therefore (\alpha - 2)(\alpha - 1) = 0$

$\therefore \ \alpha = 1, 2$

$a_n^{(h)} = A_1(1)^n + A_2(2)^n$

For Particular solution,

Here, $f(n) = 2^n$ and 2 is a root of characteristic equation.

Let $a_n^{(p)} = P\, n\, 2^n$

$a_{n-1} = P\ (n-1)\ 2^{n-1}$

$a_{n-2} = P\ (n-2)\ 2^{n-2}$

Substituting above values in given recurrence relation

$P\, n\, 2^n - 3P\ (n-1)2^{n-1} + 2P\ (n-2)2^{n-2} = 2^n$

$\therefore P\, n\, 2^n - \frac{3}{2} P\, n\, 2^n + \frac{3}{2}\ P\, 2^n + \frac{1}{2}\ P\, n\, 2^n - P\, 2^n = 2^n$

$\therefore \frac{1}{2}\ P\, 2^n = 2^n$

Comparing on both the sides, we get,

$\frac{1}{2}\ P = 1$

$\therefore \boldsymbol{P = 2}$

$a_n^{(p)} = 2\, n\, 2^n = n\, 2^{n+1}$

Thus, the general solution is

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = A_1(1)^n + A_2(2)^n + n\, 2^{n+1}$

Example – 12 (QB - 221)

Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = n^2\ 2^n$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha^2 - 2\alpha + 1 = 0$

$\therefore (\alpha - 1)(\alpha - 1) = 0$

$\therefore \ \alpha = 1, 1$

$a_n^{(h)} = (A_1 + A_2 n)(1)^n$

For Particular solution,

Here, $f(n) = n^2\ 2^n$

Let $a_n^{(p)} = (P_0 + P_1 n + P_2 n^2)\ 2^n$

$a_{n+1} = [P_0 + P_1(n+1) + P_2(n+1)^2]\ 2^{n+1}$

$a_{n+2} = [P_0 + P_1(n+2) + P_2(n+2)^2]\ 2^{n+2}$

Substituting above values in given recurrence relation

$$[P_0 + P_1(n+2) + P_2(n+2)^2]2^{n+2} - 2[P_0 + P_1(n+1) + P_2(n+1)^2]2^{n+1}$$
$$+ (P_0 + P_1 n + P_2 n^2)\ 2^n = n^2\ 2^n$$

$$\therefore [P_0 + P_1 n + 2P_1 + P_2 n^2 + 4nP_2 + 4P_2]\ 2^{n+2} - 2[P_0 + P_1 n + P_1 + P_2 n^2 + 2nP_2 + P_2]\ 2^{n+1}$$
$$+(P_0 + P_1 n + P_2 n^2)\ 2^n = n^2\ 2^n$$

$$\therefore 4P_0\ 2^n + 4P_1 n\ 2^n + 8P_1\ 2^n + 4P_2 n^2\ 2^n + 16P_2\ n\ 2^n + 16P_2\ 2^n$$
$$-4P_0\ 2^n - 4P_1\ n\ 2^n - 4\ P_1 2^n - 4P_2\ n^2 2^n - 8P_2\ n\ 2^n - 4P_2 2^n$$
$$+P_0\ 2^n + P_1\ n\ 2^n + P_2\ n^2 2^n = n^2\ 2^n$$

$$\therefore\ P_0\ 2^n + P_1\ n\ 2^n + 4P_1\ 2^n + P_2\ n^2 2^n + 8P_2\ n\ 2^n + 12P_2 2^n = n^2\ 2^n$$

$$\therefore (P_0 + 12P_2 + 4P_1)2^n + (P_1 + 8P_2)n\ 2^n + (P_2)n^2\ 2^n = n^2\ 2^n$$

Comparing on both the sides, we get,

$P_0 + 12P_2 + 4P_1 = 0$

$P_1 + 8P_2 = 0$

$\boldsymbol{P_2 = 1}$

$\therefore \boldsymbol{P_1 = -8}$

$\therefore \boldsymbol{P_0 = 20}$

$a_n^{(p)} = (P_0 + P_1 n + P_2 n^2)\ 2^n$

$a_n^{(p)} = (20 - 8n + n^2)\ 2^n$

Thus, the general solution is

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = (A_1 + A_2 n) + (20 - 8n + n^2)\ 2^n$

Example – 13 (QB - 226)

Find total solution of $a_{r+2} + 2a_{r+1} - 3a_r = 4$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha^2 + 2\alpha - 3 = 0$

$\therefore (\alpha - 1)(\alpha + 3) = 0$

$\therefore\ \alpha = 1, -3$

$a_r^{(h)} = A_1(1)^r + A_2(-3)^r$

For Particular solution,

Here, $f(r) = 4$

Let $a_r^{(p)} = P$

$a_{r+1} = P$

$a_{r+2} = P$

Substituting above values in given recurrence relation

$P + 2P - 3P = 4$

$\therefore 0 = 4$ which is not possible

Let $a_r^{(p)} = P \cdot r$

$a_{r+1} = P \cdot (r + 1)$

$a_{r+2} = P \cdot (r + 2)$

Substituting above values in given recurrence relation

$P \cdot (r + 2) + 2P \cdot (r + 1) - 3P \cdot r = 4$

$\therefore 4P = 4$

$\therefore \boldsymbol{P = 1}$

$a_r^{(p)} = r$

Thus, the general solution is

$a_r = a_r^{(h)} + a_r^{(p)}$

$a_r = A_1(1)^r + A_2(-3)^r + r$

Example – 14 (QB - 212)

Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 3n + 5$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha^2 - 2\alpha + 1 = 0$

$\therefore (\alpha - 1)(\alpha - 1) = 0$

$\therefore\ \alpha = 1, 1$

$a_n^{(h)} = (A_1 + A_2 n)(1)^n = A_1 + A_2 n$

For Particular solution,

Here, $f(n) = 3n + 5$

Guess $a_n^{(p)} = P_0 + P_1 n$

But the terms in $a_n^{(p)}$ (i.e. 1 and n) are matching with the terms in $a_n^{(h)}$. Thus we need to multiply $a_n^{(p)}$ by suitable power of n so as no terms can match. Thus, we multiply it by n^2.

$\therefore a_n^{(p)} = P_0 n^2 + P_1 n^3$

$a_{n+1} = P_0(n+1)^2 + P_1(n+1)^3$

$a_{n+2} = P_0(n+2)^2 + P_1(n+2)^3$

Substituting above values in given recurrence relation

$P_0(n+2)^2 + P_1(n+2)^3 - 2P_0(n+1)^2 - 2P_1(n+1)^3 + P_0 n^2 + P_1 n^3 = 3n + 5$

$\therefore\ P_0(n^2 + 4n + 4) + P_1(n^3 + 8 + 6n^2 + 12n) - 2\,P_0(n^2 + 2n + 1) - 2P_1(n^3 + 1 + 3n^2 + 3n)$

$+P_0 n^2 + P_1 n^3 = 3n + 5$

$\therefore P_0 n^2 + 4P_0 n + 4P_0 + P_1 n^3 + 8P_1 + 6P_1 n^2 + 12P_1 n - 2\,P_0 n^2 - 4\,P_0 n - 2P_0 - 2P_1 n^3 - 2P_1$

$- 6P_1 n^2 - 6P_1 n + P_0 n^2 + P_1 n^3 = 3n + 5$

$\therefore -2P_0 n + 2P_0 + 6P_1 + 6P_1 n = 3n + 5$

$\therefore (6P_1)n + (2P_0 + 6P_1) = 3n + 5$

Comparing on both the sides, we get,

$6P_1 = 3 \qquad \Rightarrow \boldsymbol{P_1 = \frac{1}{2}}$

$2P_0 + 6P_1 = 5 \qquad \Rightarrow \boldsymbol{P_0 = 1}$

$a_n^{(p)} = P_0 n^2 + P_1 n^3$

$a_n^{(p)} = n^2 + \frac{1}{2} n^3$

Thus, the general solution is

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = A_1 + A_2 n + n^2 + \frac{1}{2} n^3$

Example – 15 (QB - 218)

Solve the recurrence relation $a_{n+1} - a_n = n^2$.

Solution:

For Homogeneous solution,

The Characteristic equation is given by

$\alpha - 1 = 0$

$\therefore\ \alpha = 1$

$a_n^{(h)} = A_1$

For Particular solution,

Here, $f(n) = n^2$

Guess $a_n^{(p)} = P_0 + P_1 n + P_2 n^2$

But the terms in $a_n^{(p)}$ (i.e. 1) is matching with the terms in $a_n^{(h)}$. Thus we need to multiply $a_n^{(p)}$ by suitable power of n so as no terms can match. Thus, we multiply it by n.

$\therefore a_n^{(p)} = (P_0 + P_1 n + P_2 n^2)n = P_0 n + P_1 n^2 + P_2 n^3$

$a_{n+1} = P_0(n+1) + P_1(n+1)^2 + P_2(n+1)^3$

$\quad = P_0 n + P_0 + P_1 n^2 + 2P_1 n + P_1 + P_2 n^3 + 3P_2 n^2 + 3P_2 n + P_2$

Substituting above values in given recurrence relation

$P_0 n + P_0 + P_1 n^2 + 2P_1 n + P_1 + P_2 n^3 + 3P_2 n^2 + 3P_2 n + P_2$

$\quad -P_0 n - P_1 n^2 - P_2 n^3 = n^2$

$\therefore (P_0 + P_1 + P_2) + (2P_1 + 3P_2)n + (3P_2)n^2 = n^2$

Comparing on both the sides, we get,

$P_0 + P_1 + P_2 = 0$

$2P_1 + 3P_2 = 0$

$3P_2 = 1 \qquad \Rightarrow \boldsymbol{P_2 = \frac{1}{3}}$

$\therefore 2P_1 = -1 \qquad \Rightarrow \boldsymbol{P_1 = -\frac{1}{2}}$

$\therefore \boldsymbol{P_0 = \frac{1}{6}}$

$a_n^{(p)} = P_0 n + P_1 n^2 + P_2 n^3$

$a_n^{(p)} = \frac{1}{6}n - \frac{1}{2}n^2 + \frac{1}{3}n^3$

Thus, the general solution is

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = A_1 + \frac{1}{6}n - \frac{1}{2}n^2 + \frac{1}{3}n^3$