A Lecture Notes on

Discrete Mathematics:

Propositional and Predicate Logic

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Propositional and Predicate Logic

Outline

- Normal Forms
- Disjunctive Normal Forms
- Conjunctive Normal Forms
- Arguments
- Valid Argument and Fallacy Arguments
- Predicates, Universe of Discourse
- Quantifiers: Universal Quantifiers and Existential Quantifiers

❖ Propositional Logic

❖ Normal Forms

- In logic, with the help of truth table we can compare if two statements are equivalent. But when more statements or propositions are involved, then this method is not practical.
- [As if n propositions are involved, its truth values will be 2^n]
  Hence, it is necessary to apply another method.
- One method is to transform S_1 and S_2 to some standard form $S_1{}'$ and $S_2{}'$. Such that a simple comparison of S_1 and S_2 should establish whether $S_1{}' \equiv S_2{}'$.
- The standard forms are called normal forms or canonical forms.

➢ Disjunctive Normal Forms (DNF)

- Disjunctive normal form is a disjunction $(\vee)$ of **fundamental conjunctions** $(\wedge)$.
- Now fundamental conjunctions $(\wedge)$ are conjunction of simple statements [i.e. joining two statements by $' \wedge '$].
- i.e. $p \wedge q,\ \sim p \wedge q,\ \sim p \wedge \sim q,\ p \wedge \sim p,\ q \wedge \sim q,\ p \wedge \sim q,\ \sim p$ are fundamental conjunctions.
- Hence disjunction of fundamental conjunctions is joining fundamental conjunction by $' \vee '$.

For example,

1. $(p \wedge q) \vee p \vee (q \wedge \sim p)$
2. $(p \wedge q \wedge r) \vee (p \wedge q' \wedge r) \vee (p' \wedge q \wedge r)$
3. $(p \wedge q) \vee r$

Remark:

- DNF of a given formula is not unique but all different forms are equivalent.
- $q \wedge \sim q, p \wedge \sim p$ are always false. Hence, if a fundamental conjunction contains at least one pair of $(p\ and\ \sim p)$ or $(q\ and\ \sim q)$ etc., it will be false.

❖ Conjunctive Normal Forms (CNF)

- Conjunctive normal form is a conjunction $(\wedge)$ of **fundamental disjunctions** $(\vee)$.
- Now fundamental disjunctions $(\vee)$ are disjunction of simple statements [i.e. joining two statements by $' \vee '$].
- i.e. $p,\ p \vee q,\ \sim p \vee q,\ \sim p \vee \sim q,\ p \vee \sim p,\ \sim p, \sim q$ are fundamental disjunctions.
- Hence conjunction of fundamental disjunctions is joining fundamental disjunction by $' \wedge '$.

For example,

1. $(p \vee q) \wedge (q \vee r) \wedge (\sim p \vee \sim r)$
2. $(p \vee q \vee \sim r) \wedge (p \vee q) \wedge (\sim p \vee q \vee \sim r)$

3. $p \wedge (\sim q \vee \sim r)$

Remark:

- CNF of a given formula is not unique but all different forms are equivalent.

❖ Equivalent Formulae

$p \vee \boldsymbol{F} \Leftrightarrow p$	$p \wedge \boldsymbol{T} \Leftrightarrow p$
$p \vee \boldsymbol{T} \Leftrightarrow \boldsymbol{T}$	$p \wedge \boldsymbol{F} \Leftrightarrow \boldsymbol{F}$
$\mathrm{p} \vee \sim p \Leftrightarrow \boldsymbol{T}$	$\mathrm{p} \wedge \sim p \Leftrightarrow \boldsymbol{F}$

Example – 1 (QB - 160)

Find conjunctive normal form and disjunctive normal form for the following without using truth table. $(p \rightarrow q) \wedge (q \rightarrow p)$

Solution:

$(p \rightarrow q) \wedge (q \rightarrow p)$

$\equiv (\sim p \vee \mathrm{q}) \wedge (\sim q \vee \mathrm{p})$ **CNF** [As $p \rightarrow q \equiv (\sim p \vee \mathrm{q})$]

$\equiv [(\sim p) \wedge (\sim q \vee \mathrm{p})] \vee [\mathrm{q} \wedge (\sim q \vee \mathrm{p})]$ [Using Distributive law]

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge \mathrm{p}) \vee (\mathrm{q} \wedge \sim q) \vee (\mathrm{q} \wedge \mathrm{p})$

$\equiv (\sim p \wedge \sim q) \vee \mathrm{F} \vee \mathrm{F} \vee (\mathrm{q} \wedge \mathrm{p})$

$\equiv (\sim p \wedge \sim q) \vee (\mathrm{q} \wedge \mathrm{p})$ **DNF**

Example – 2 (QB - 159)

Obtain conjunctive normal form of

(1) $(\sim p \rightarrow q) \wedge (p \rightarrow q)$

(2) $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

Solution:

(1) $(\sim p \rightarrow q) \wedge (p \rightarrow q)$

$\equiv (\sim(\sim p) \vee q) \wedge (\sim p \vee \mathrm{q})$ [As $p \rightarrow q \equiv (\sim p \vee \mathrm{q})$]

$\equiv (p \vee q) \wedge (\sim p \vee \mathrm{q})$ **CNF**

(2) $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

$\equiv [p \vee (\sim p \wedge q \wedge r)] \wedge [q \vee (\sim p \wedge q \wedge r)]$

$\equiv [(p \vee \sim p) \wedge (p \vee q) \wedge (p \vee r)] \wedge [(q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r)]$

$\equiv [T \wedge (p \vee q) \wedge (p \vee r)] \wedge [(q \vee \sim p) \wedge (q) \wedge (q \vee r)]$

$\equiv (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge (q) \wedge (q \vee r)$ **CNF**

Example – 3 (QB - 161)

Obtain disjunctive normal form of

1. $(p \to q) \wedge (\sim p \wedge q)$
2. $\big(p \wedge (p \to q)\big) \to q$

Solution:

1. $(p \to q) \wedge (\sim p \wedge q)$

$\equiv (\sim p \vee \mathrm{q}) \wedge (\sim p \wedge q)$ [As $p \to q \equiv (\sim p \vee \mathrm{q})$]

$\equiv [\sim p \wedge (\sim p \wedge q)] \vee [\mathrm{q} \wedge (\sim p \wedge q)]$ [Using Distributive law]

$\equiv [(\sim p \wedge \sim p) \wedge q] \vee [(\mathrm{q} \wedge q) \wedge \sim p]$ [By associative and commutative law]

$\equiv (\sim p \wedge q) \vee (q \wedge \sim p)$ **DNF** [By idempotent law]

2. $\big(p \wedge (p \to q)\big) \to q$

$\equiv \big(p \wedge (\sim p \vee \mathrm{q})\big) \to q$

$\equiv \sim[p \wedge (\sim p \vee \mathrm{q})] \vee \mathrm{q}$

$\equiv \sim p \vee (\sim(\sim p \vee \mathrm{q})) \vee \mathrm{q}$

$\equiv \sim p \vee (\mathrm{p} \wedge \sim q) \vee \mathrm{q}$ **DNF**

Example – 4 (QB - 162)

Obtain the conjunction normal form of each of the following:

1. $p \wedge (p \to q)$
2. $\sim(p \vee \mathrm{q}) \leftrightarrow (p \wedge q)$

Solution:

1. $p \wedge (p \to q)$

$\equiv p \wedge (\sim p \vee \mathrm{q})$ **CNF**

2. $\sim(p \vee \mathrm{q}) \leftrightarrow (p \wedge q)$

$\equiv [\sim(p \vee \mathrm{q}) \to (p \wedge q)] \wedge [(p \wedge q) \to \sim(p \vee \mathrm{q})]$ [$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$]

$\equiv [(p \vee \mathrm{q}) \vee (p \wedge q)] \wedge [(\sim p \vee \sim q) \vee (\sim \mathrm{p} \wedge \sim q)]$

$\equiv [(p \vee \mathrm{q}) \vee \mathrm{p}] \wedge [(p \vee \mathrm{q}) \vee \mathrm{q}] \wedge [(\sim p \vee \sim q) \vee (\sim \mathrm{p})] \wedge [(\sim p \vee \sim q) \vee (\sim q)]$

$\equiv (p \vee \mathrm{q}) \wedge (p \vee \mathrm{q}) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q)$

$\equiv (p \vee \mathrm{q}) \wedge (\sim p \vee \sim q)$ **CNF**

Example – 5

Show that $\big(\sim p \wedge (\sim q \wedge r)\big) \vee (q \wedge r) \vee (p \wedge r) \equiv r$.

Solution:

$\big(\sim p \wedge (\sim q \wedge r)\big) \vee (q \wedge r) \vee (p \wedge r)$

$\equiv \big(\sim p \wedge (\sim q \wedge r)\big) \vee (q \wedge r) \vee (p \wedge r)$

$\equiv \big(\sim p \wedge (\sim q \wedge r)\big) \vee (r \wedge (q \vee p))$

$\equiv \big((\sim p \wedge \sim q) \wedge r\big) \vee (r \wedge (q \vee p))$

$\equiv \big((\sim p \wedge \sim q) \wedge r\big) \vee ((q \vee p) \wedge r)$

$\equiv \big((\sim p \wedge \sim q) \vee (q \vee p)\big) \wedge r$

$\equiv \big(\sim(p \vee q) \vee (p \vee q)\big) \wedge r$

$\equiv \boldsymbol{T} \wedge r$

$\equiv r$

Example – 6 (QB - 166)

Obtain DNF for the following: $p \vee [\sim p \rightarrow (q \vee (q \rightarrow \sim r))]$.

Solution:

$p \vee [\sim p \rightarrow (q \vee (q \rightarrow \sim r))]$

$\equiv p \vee [\sim p \rightarrow (q \vee (\sim q \vee \sim r))]$

$\equiv p \vee [\sim p \rightarrow (q \vee \sim q \vee \sim r)]$

$\equiv p \vee [\sim p \rightarrow (T \vee \sim r)]$

$\equiv p \vee [p \vee (T \vee \sim r)]$

$\equiv p \vee [p \vee T]$

$\equiv p \vee [T]$

$\equiv T$

Example – 7

Use the laws of logic to show that $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ is a tautology.

Solution:

$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$

$\equiv \sim[(p \rightarrow q) \wedge \sim q] \vee \sim p$ [Implication law]

$\equiv \sim[(\sim p \vee q) \wedge \sim q] \vee \sim p$

$\equiv \sim[\sim q \wedge (\sim p \vee q)] \vee \sim p$ [Commutative law]

$\equiv \sim[(\sim q \wedge \sim p) \vee (\sim q \wedge q)] \vee \sim p$ [Distributive law]

$\equiv \sim[(\sim q \wedge \sim p) \vee F] \vee \sim p$

$\equiv \sim[(\sim q \wedge \sim p)] \vee \sim p$

$\equiv (q \vee p) \vee \sim p$

$\equiv q \vee (p \vee \sim p)$

$\equiv q \vee T$

$\equiv T$

❖ Truth Table Method to find DNF

Let p be a proposition containing n variables $p_1, p_2, \dots \dots, p_i, \dots, p_k, \dots p_n$.

To find its DNF from the truth table.

Step – 1: Consider the true values (T) from p.

Step – 2: Form the conjunction $('\wedge')$,

$$(p_1 \wedge \ p_2 \wedge \dots \dots \wedge p_i \wedge \dots \wedge p_k \wedge \dots \wedge p_n)$$

Where if p_i is true consider p_i and

if p_k is false consider $\sim p_k$

Such term is called minterm.

Step – 3: The disjunction of minterms is the DNF of the given form.

Example – 1 (QB - 164)

Eliminating conditional and biconditional, find DNF which is logically equivalent to given form. $[p \leftrightarrow (q \vee r)] \rightarrow \sim p$ Using truth table.

Solution:

p	q	r	$\sim p$	$q \vee r$	$p \leftrightarrow (q \vee r)$	$[p \leftrightarrow (q \vee r)] \rightarrow \sim p$
T	T	T	F	T	T	F
T	T	F	F	T	T	F
T	F	T	F	T	T	F
T	F	F	F	F	F	T
F	T	T	T	T	F	T
F	T	F	T	T	F	T
F	F	T	T	T	F	T
F	F	F	T	F	T	T

Consider only (T) from last column and choose corresponding values (T) from p, q, r.

e.g., for marked row $(\rightarrow)$, corresponding p is true, q and r are false, so consider $(p \wedge q' \wedge r')$ *or* $(p \wedge \sim q \wedge \sim r)$.

Hence, the logically equivalent form of

$$[p \leftrightarrow (q \vee r)] \rightarrow \sim p \equiv (p \wedge q' \wedge r') \vee (p' \wedge q \wedge r) \vee (p' \wedge q \wedge r') \vee (p' \wedge q' \wedge r) \vee (p' \wedge q' \wedge r')$$

Example – 2

Find DNF of $[(p \rightarrow q) \rightarrow q] \rightarrow p$ by truth table method.

Solution:

p	q	$p \to q$	$(p \to q) \to q$	$[(p \to q) \to q] \to p$
T	T	T	T	T
T	F	F	T	T
F	T	T	T	F
F	F	T	F	T

Hence, the logically equivalent form of

$[(p \to q) \to q] \to p \equiv (p \wedge q) \vee (p \wedge q') \vee (p' \wedge q')$

Example – 3 (QB - 163)

Find dnf of $[p \to (q \wedge r)] \wedge [\sim p \to (\sim p \wedge \sim r)]$ by truth table method.

Solution:

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$\sim p$	$\sim r$	$\sim p \wedge \sim r$	$\sim p \to (\sim p \wedge \sim r)$	$[p \to (q \wedge r)] \wedge$ $[\sim p \to (\sim p \wedge \sim r)]$
T	T	T	T	T	F	F	F	T	T
T	T	F	F	F	F	T	F	T	F
T	F	T	F	F	F	F	F	T	F
T	F	F	F	F	F	T	F	T	F
F	T	T	T	T	T	F	F	F	F
F	T	F	F	T	T	T	T	T	T
F	F	T	F	T	T	F	F	F	F
F	F	F	F	T	T	T	T	T	T

Hence, the DNF is

$[p \to (q \wedge r)] \wedge [\sim p \to (\sim p \wedge \sim r)] \equiv (p \wedge q \wedge r) \vee (p' \wedge q \wedge r') \vee (p' \wedge q' \wedge r')$

The CNF is

$[p \to (q \wedge r)] \wedge [\sim p \to (\sim p \wedge \sim r)]$

$$\equiv (p' \vee q' \vee r) \wedge (p' \vee q \vee r') \wedge (p' \vee q \vee r) \wedge (p \vee q' \vee r') \wedge (p \vee q \vee r')$$

Example – 4 (QB - 168)

Obtain CNF and DNF for the following using truth table: $(p \to q) \wedge (q \vee (p \wedge r))$

Solution:

p	q	r	$p \to q$	$p \wedge r$	$q \vee (p \wedge r)$	$(p \to q) \wedge (q \vee (p \wedge r))$
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	F	T	T	F
T	F	F	F	F	F	F

F	T	T	T	F	T	T
F	T	F	T	F	T	T
F	F	T	T	F	F	F
F	F	F	T	F	F	F

Hence, CNF is

$(p \to q) \wedge (q \vee (p \wedge r)) \equiv (p' \vee q \vee r') \wedge (p' \vee q \vee r) \wedge (p \vee q \vee r') \wedge (p \vee q \vee r)$

DNF is

$(p \to q) \wedge (q \vee (p \wedge r)) \equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge r') \vee (p' \wedge q \wedge r) \vee (p' \wedge q \wedge r')$

- **Arguments**

  Argument $P_1, P_2, \dots \dots, P_n \vdash Q$ is said to be valid if Q is true whenever all the premises are true.

- **Theorem**

  Argument $P_1, P_2, \dots \dots, P_n \vdash Q$ is said to be valid if and only if the proposition

  $P_1 \wedge P_2 \wedge \dots \dots \wedge P_n \rightarrow Q$ is a tautology.

- **Note**

  An argument which is not valid is said to be fallacy.

Example – 1

Check the validity of the following: $p \rightarrow \sim q,\ r \rightarrow q,\ r \vdash \sim p.$

Solution:

Here, the premises are

$P_1 : p \rightarrow \sim q$

$P_2 : r \rightarrow q$

$P_3 : r$

$Q : \sim p$

p	q	r	$\sim q$	$p \rightarrow \sim q$	$r \rightarrow q$	$\sim p$
T	T	T	F	F	T	F
T	T	F	F	F	T	F
T	F	T	T	T	F	F
T	F	F	T	T	T	F
F	T	T	F	T	T	T
F	T	F	F	T	T	T
F	F	T	T	T	F	T
F	F	F	T	T	T	T

In all premises only 5^{th} entry will have all T $(true)$ and corresponding $\sim p$ $(conclusion)$ is also T $(true)$.

Hence, argument is valid.

Another Method:

p	q	r	$\sim q$	$p \to \sim q$	$r \to q$	$\sim p$	$(p \to \sim q) \wedge (r \to q) \wedge r$	$(p \to \sim q) \wedge (r \to q) \wedge r \to \sim p$
T	T	T	F	F	T	F	F	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	F	F	T
T	F	F	T	T	T	F	F	T
F	T	T	F	T	T	T	T	T
F	T	F	F	T	T	T	F	T
F	F	T	T	T	F	T	F	T
F	F	F	T	T	T	T	F	T

Hence, $(p \to \sim q) \wedge (r \to q) \wedge r \to \sim p$ is a tautology.

So, argument is valid.

Example – 2 (QB - 173)

Check the validity of the following:

If 7 is less than 4, then 7 is not prime number.

7 is not less than 4.

---

7 is prime number.

Solution:

Let

p: 7 is less than 4.

q: 7 is prime number.

Argument: $p \to \sim q,\ \sim p \vdash q$

p	q	$\sim q$	$p \to \sim q$	$\sim p$
T	T	F	F	F
T	F	T	T	F
F	T	F	T	T
F	F	T	T	T

In all premises last two entries are true and corresponding conclusion value is T and F.

So, argument is not valid.

Example – 3 (QB - 175)

Check the validity of the following:

If I study, then I will not fail in DM.

If I do not play cricket, then I will study.

But I failed in DM.

Therefore, I must have played cricket.

Solution:

Let

p: I study.

q: I will fail in DM

r: I play cricket.

Argument: $p \to \sim q,\ \sim r \to p,\ q \vdash r$

p	q	r	$\sim q$	$\sim r$	$p \to \sim q$	$\sim r \to p$
T	T	T	F	F	F	T
T	T	F	F	T	F	T
T	F	T	T	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	T	T
F	T	F	F	T	T	F
F	F	T	T	F	T	F
F	F	F	T	T	T	F

In all premises only 5^{th} entry will have all $T\ (true)$ and corresponding $r\ (conclusion)$ is also $T\ (true)$.

Hence, argument is valid.

Example – 4 (QB - 171)

Check the validity of the following:

If I like mathematics then I will study.

Either I will study or I will fail.

If I fail then I do not like mathematics.

Solution:

Let

p: I like mathematics.

q: I will study.

r: I will fail.

Argument: $p \to q,\ q \vee r \vdash r \to \sim p$

p	q	r	$p \to q$	$q \vee r$	$\sim p$	$r \to \sim p$
T	T	T	T	T	F	F
T	T	F	T	T	F	T
T	F	T	F	T	F	F
T	F	F	F	F	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	T	F	T	T

In all premises 1^{st} entry will have all T $(true)$ and corresponding $r \to \sim p$ $(conclusion)$ is F.

Hence, argument is not valid.

Example – 5 (QB - 172)

Determine the validity of argument given:

S1: If a man is a bachelor, he is unhappy.

S2: If a man is unhappy, he dies young.

S: Bachelors die young.

Solution:

Let

p: A man is bachelor.

q: A man is unhappy.

r: A man dies young.

Argument: $p \to q,\ q \to r\ \vdash p \to r$

p	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

In all premises entry will have all T $(true)$ and corresponding $p \to r$ $(conclusion)$ is T.

Hence, argument is valid.

Example – 6 (QB - 174)

Determine the validity of argument given:

If two sides of a triangle are equal, then the opposite angles are equal.

Two sides of a triangle are not equal.

The opposite angles are not equal.

Solution:

Let

p: two sides of a triangle are equal.

q: opposite angles are equal.

Argument: $p \to q,\ {\sim}p\ \vdash {\sim}q$

p	q	$p \to q$	${\sim}p$	${\sim}q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

In all premises 3^{rd} entry will have all T $(true)$ and corresponding ${\sim}q$ $(conclusion)$ is F.

Hence, argument is not valid.

❖ Predicate Logic

❖ Introduction

Propositional logic is not enough to express the meaning of all statements in mathematics and natural language.

Eg. (1) Is "$x > 1$" true or false? – which is true for $3 > 1$ and false for $-4 > 1$.

(2) Is "x is great tennis player" True or false? - If suppose A is a great tennis player then it is true but B is a common man then it is false.

These types of sentence we cannot express as in terms of propositional logic. For these we use the concept of **predicate logic.**

Such sentences are known as **propositional functions** or **predicates.**

❖ Predicate

- A predicate $p(x)$ is a sentence that contains a finite number of variables and becomes a proposition, when specific values are substituted for the variables.

  Where $p(x)$ is a propositional function and x is a predicate variable.

- A predicate $p(x)$ contains n variables $x_1, x_2, \dots \dots, x_n$ is called an n-place predicate.

  1. $'x > 9'$ is a one place predicate and is denoted by $p(x)$.
  2. $'x + y \leq 8'$ is two place predicate and is denoted by $p(x, y)$.
  3. $'x + y + z = 9'$ is three place predicate and is denoted by $p(x, y, z)$.

- The set of all possible values that may be substituted in place of variables is called the **universe of discourse.**

  For example,

  1. "x is a tennis player"

     Here x is set of all human names which is universe of discourse.

  2. $p(x): x - 7 = 4$

     Here, x is the set of natural numbers which is universe of discourse.

Remark

1. A predicate is usually not a proposition but every proposition is a predicate (i.e. propositional function).
2. When specific values are given to the variables appearing in predicate, variables are bound. If all the variables are bound in a predicate the predicate becomes a proposition.

❖ Quantifier

Quantifiers are words that refers to quantities such as “some” or “all” and indicate how frequently a certain statement is true.

There are two types:

1. Universal Quantifier
2. Existential Quantifier

❖ Universal Quantifier

- The phrase ‘for all’ denoted by ‘$\forall$’ is called the universal quantifier.

  Eg. Let ‘All students are smart’.

  Let $p(x)$ denote ‘x is smart’

  then the above sentence can be written as $\forall \boldsymbol{x}\, \boldsymbol{p}(\boldsymbol{x})$.

- "$For\ all\ x,\ p(x)$", which is interpreted as $'For\ all\ values\ of\ x, p(x)\ is\ true'$ is a proposition in which variable x is said to be **universally quantified** and $'\forall'$ is known as **Universal quantifier**.

❖ Existential Quantifiers

- The phrase ‘there exists’ denoted by ‘$\exists$’ is called the existential quantifier.

  Eg. Let ‘There exists x such that $x^2 = 9$’

  Let $p(x)$ denote ‘$x^2 = 9$’

  Then the above sentence can be written as $\exists \boldsymbol{x}\, \boldsymbol{p}(\boldsymbol{x})$.

- $\exists x\, p(x)$ means $'$there exists a value of x in the universe of discourse for which $p(x)$ is true$'$.
- $\exists x$ can be read as other ways also,

  Eg. There exists an x

  There is an x

  For some x

  There is at least one x

❖ Negation of quantified statements

Negation of $\forall\ x\ p(x)$ is $\forall\ x\ p(x)$ is not true or it is not the case that for all $x, p(x)$ is true. This means atleast for some $x\ p(x)$ is not true, in other words there exists an x such that $\sim p(x)$ is true.

Hence, $\sim[\forall x\, p(x)] \equiv \exists x\, [\sim p(x)]$

And $\sim[\exists x\, p(x)] \equiv \forall x\, [\sim p(x)]$

Example – 1

Over the universe of book defined propositions

B(x): x has blue cover.

M(x): x is maths book.

I(x): x published in India.

Translate the following.

(1) $\forall\, x\big(M(x) \wedge I(x)\big) \rightarrow B(x)$

(2) There are maths books published outside India.

Solution:

(1) All mathematics book published in India has blue cover.

(2) $\exists\, x\ (M(x) \wedge \sim I(x))$

Example – 2 (QB - 182)

Rewrite the following statements using quantifiers and predicate symbols:

1. All birds can fly
2. Not all birds can fly
3. Some men are genius
4. Some numbers are not rational
5. There is a student who likes mathematics but not Geography.
6. Each integer is either even or odd

Solution:

1. $p(x)$: x is a bird

   $q(x)$: x can fly.

   Then statement can be written as,

   $\forall x\ [p(x)\ \rightarrow\ q(x)]$

2. $\sim\big[\forall x\ [p(x) \rightarrow\ q(x)]\big] \equiv \exists x\ [\sim[p(x) \rightarrow\ q(x)]]$

   $\equiv \exists x\ [\sim[\ \sim p(x) \vee q(x)]]$

   $\equiv \exists x\ [p(x) \wedge \sim q(x)]$

3. $A(x)$: x is a man.

   $B(x)$: x is a genious.

   Then the statement can be written as,

   $\exists x\ [A(x) \wedge B(x)]$

4. $N(x)$: x is a number.

   $Q(x)$: x is rational.

Then the statement can be written as,

$\exists x\ [N(x) \wedge \sim Q(x)]$

5. $S(x)$: x is a student.

   $M(x)$: x likes mathematics.

   $G(x)$: x likes Geography.

   Then the statement in symbolic form is

   $\exists x\ [S(x) \wedge M(x) \wedge \sim G(x)]$

6. $I(x)$: x is an integer.

   $E(x)$: x is even.

   $O(x)$: x is odd.

   Then the statement in symbolic form is

   $\forall\ x\ [I(x)\ \rightarrow \{E(x) \vee O(x)\}]$

Example – 3 (QB - 186)

Negate each of the statement.

1. $\forall x, |x| = x$
2. $\exists x, x^2 = x$
3. If there is a riot, then someone is killed.
4. It is day light and all the people are arisen.

Solution:

1. $\exists x,\ |x| \neq\ x.$
2. $\forall x,\ x^2 \neq x$
3. p: There is a riot.

   q: Someone is killed.

   Given statement is $p \rightarrow q$

   Hence, $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q) \equiv p \wedge \sim q \equiv$ There is a riot and someone is not killed.

4. p: It is a day light.

   q: All the people are arisen.

   Given statement is $p \wedge q$.

   Hence $\sim(p \wedge q) \equiv\ \sim p \vee \sim q$.

   Hence either it is not a day light or all the people are not arisen.

Example – 4 (QB - 181)

Write the following statements in symbolic form, using quantifiers.

1. All students have taken a course in communication skills.
2. There is a girl student in the class who is also a sports person.

3. Some students are intelligent, but not hard working.

Solution:

1. $p(x)$: x is a student

   $q(x)$: x has taken a course in communication skills.

   Then the statement in symbolic form is

   $\forall x\ (p(x)\ \rightarrow\ q(x))$

2. $S(x)$: x is a student

   $G(x)$: x is a girl

   $SP(x)$: x is a sports person

   Then the statement in symbolic form is

   $\exists x\ (S(x) \wedge G(x) \wedge SP(x))$

3. $S(x)$: x is a student

   I(x): x is intelligent

   H(x): x is hardworking

   Then the statement in symbolic form is

   $\exists x\ (S(x) \wedge I(x) \wedge \sim H(x))$

Example – 5 (QB - 185)

Write the following two propositions in symbols.

1. for every number x there is a number y such that $y = x + 1$.
2. There is a number y such that, for every number $x, y = x + 1$.

Solution:

Let $p(x, y)$ denote the predicate $y = x + 1$

1. $\forall x\ \exists y\ p(x, y)$
2. $\exists y\ \forall x\ p(x, y)$

Example – 6

Negate each of the statements:

1. $\exists x\ P(x)\ \vee\ \forall y\ Q(y)$
2. $\forall x\ P(x)\ \wedge\ \exists y\ Q(y)$

Solution:

1. $\sim[\exists x\ P(x)\ \vee\ \forall y\ Q(y)] \equiv \forall x\ \left(\sim P(x)\right)\ \wedge\ \exists y\ (\sim Q(y))$
2. $\sim[\forall x\ P(x)\ \wedge\ \exists y\ Q(y)] \equiv \exists x\ \left(\sim P(x)\right) \vee\ \forall y\ (\sim Q(y))$

Example – 7

Write the following statement into symbols

1. A number x, is less than 7 and greater than 5.
2. For a given number x there is a greater number y.
3. For two given numbers x and y, there is a number z such that the difference of x and y is less than the product of x^2 and z.
4. The numbers x, y, z are such that $x + y$ is greater than xz.

Solution:

1. $p(x)$: x is less than 7

   $q(x)$: x is greater than 5

   Then the statement in symbolic form is

   $\exists x\ [p(x) \wedge q(x)]$
2. $p(x, y)$: y is greater than x.

   Then the statement in symbolic form is

   $\forall x\ \exists y\ p(x, y)$
3. $p(x, y)$: The difference of x and y

   $q(x, z)$: The product of x^2 and z

   $r(x, y, z)$: The difference of x and y is less than the product of x^2 and z

   Then the statement in symbolic form is

   $\forall x\ \forall y\ \exists z\ r(x, y, z)$
4. $p(x, y, z)$: $x + y$ is greater than xz

   Then the statement in symbolic form is

   $\exists x\ \exists y\ \exists z\ p(x, y, z)$