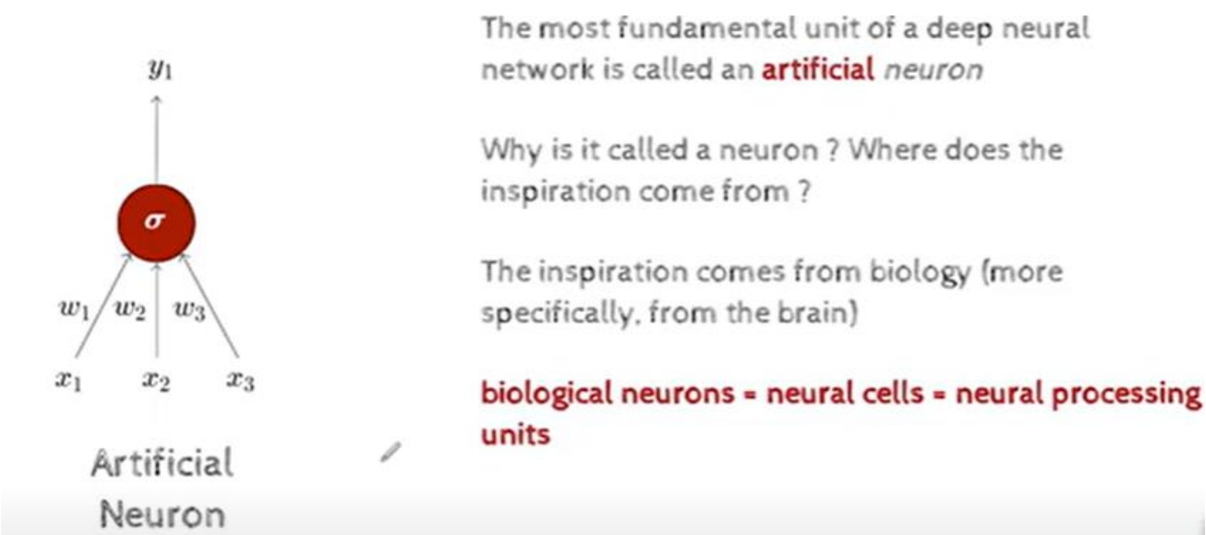
McCulloch Pitts (MP) neurons

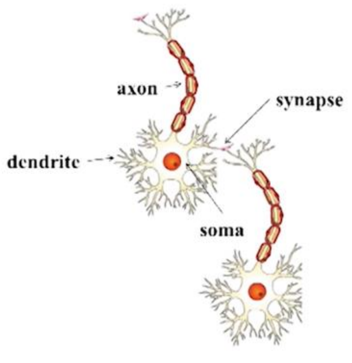
Courtesy: Prof. M. Khapra



Biological neurons

Biological neurons in detail here.

In the 1890s,
the term neuron was coined
for the neural processing units or the cells in our brain.

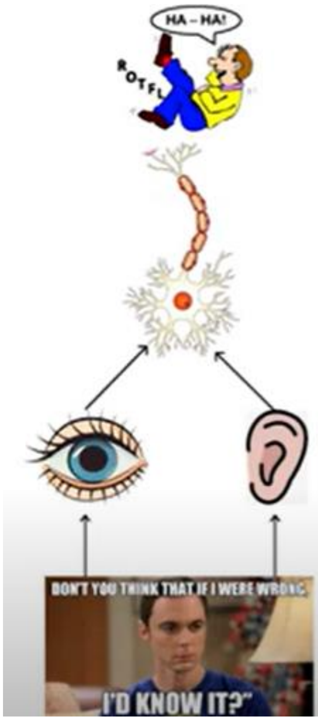


Biological Neurons*

- **dendrite:** receives signals from other neurons
- **synapse:** point of connection to other neurons
- **soma:** processes the information
- **axon:** transmits the output of this neuron

*Image adapted from
`https://cdn.vectorstock.com/i/composite/12,25/neuron-cell-vector-81225.jpg`

An extremely simplified version of how it works:

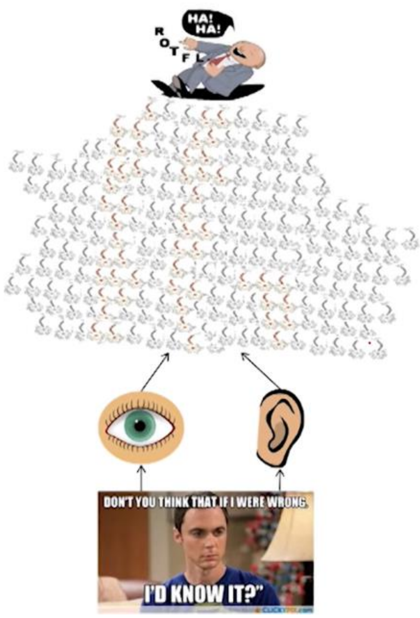


Let us see a very cartoonish illustration of how a neuron works

Our sense organs interact with the outside world

They relay information to the neurons

The neurons (may) get activated and produces a response (laughter in this case)

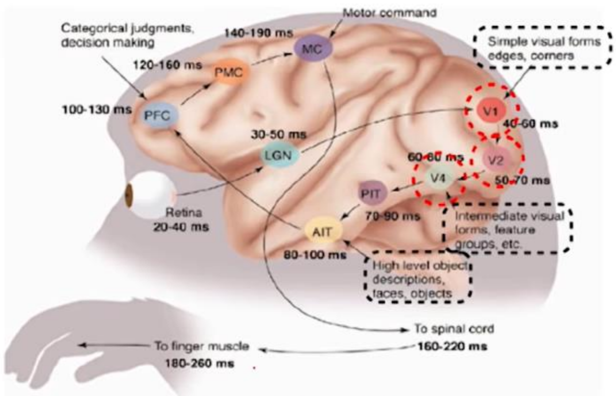


- Of course, in reality, it is not just a single neuron which does all this
- There is a massively parallel interconnected network of neurons
- The sense organs relay information to the lowest layer of neurons
- Some of these neurons may fire (in red) in response to this information and in turn relay information to other neurons they are connected to
- These neurons may also fire (again, in red) and the process continues eventually resulting in a response (laughter in this case)

An average human brain has 100 billion neurons!

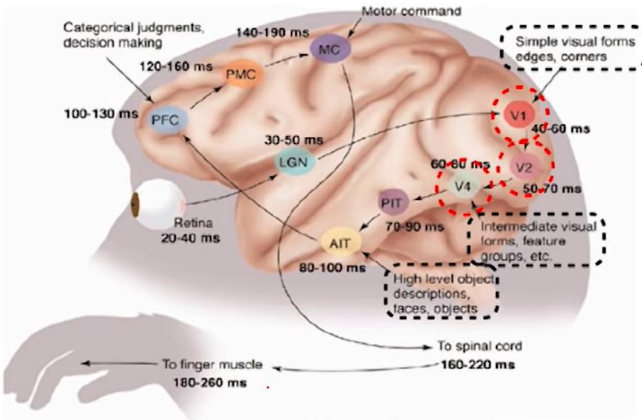
A simplified illustration

- This massively parallel network also ensures that there is division of work
- Each neuron may perform a certain role or respond to a certain stimulus

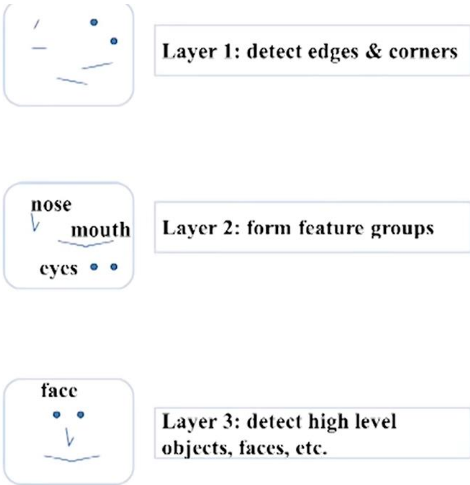


[picture from Simon Thorpe]

- The neurons in the brain are arranged in a hierarchy
- We illustrate this with the help of visual cortex (part of the brain) which deals with processing visual information
- Starting from the retina, the information is relayed to several layers (follow the arrows)
- We observe that the layers $V1$, $V2$ to AIT form a hierarchy (from identifying simple visual forms to high level objects)



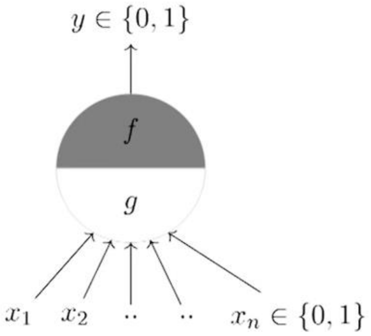
[picture from Simon Thorpe]



Sample illustration of hierarchical processing*

McCulloch Pitts neurons

1943
McCulloch and Pitts proposed
a highly simplified computational model of the brain.

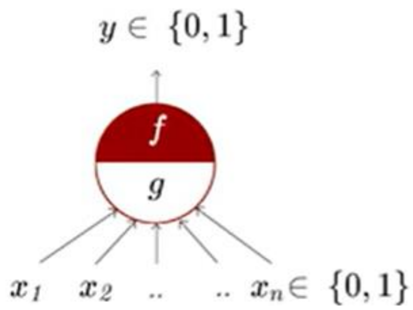


Motivation?

Our brain is capable of
very complex processing,
taking a lot of inputs from various sources and then,
help us take various decisions and actions.

What if you want a computer to do this?

Want a model which is very similar to how the brain works or
at least how we think the brain works.



McCulloch (neuroscientist) and Pitts (logician) proposed a highly simplified computational model of the neuron (1943)

g aggregates the inputs and the function f takes a decision based on this aggregation

It will take a lot of inputs and these inputs are all binary.
The output is Boolean.
If it's 0, then the neuron does not fire.
If it's 1, the neuron fires.
Eg.
Say, I am trying to make a decision
whether I should watch a movie or not.
x_1 could be is the genre of the movie:
thriller.
Another variable x_n the actor
Matt Damon.

The director
Christopher Nolan,
the music director and so on.
You want this neuron to help us make that decision.

- g aggregates the inputs and the function f takes a decision based on this aggregation
- The inputs can be excitatory or inhibitory

Inhibitory input

$y = 0$ if any x_i is inhibitory, else

$$g(x_1, x_2, ..., x_n) = g(x) = \sum_{i=1}^{n} x_i$$

$$y = f(g(x)) = 1 \quad \text{if} \quad \underline{g(x) \geq \theta}$$

θ is called the thresholding parameter

Say, you are taking input from many sources and
one of them is "am I ill today?"
If so, irrespective of who the actor, director or whatever is,
I am not going to watch the movie
because I just cannot leave my bed.
(i.e.) if this input is on, the output is always 0.
Means, the neuron is never going to fire.

Excitatory input
will NOT cause the neuron to fire on its OWN,
but combined with all the other inputs
could cause the neuron to fire.
How?
Take the sum (aggregation) of all the input to the neuron.

$$g(x1, x2, x3, \ldots, xn) = g(X) = \sum_{i=1}^{n} xi$$

This count gives the number of inputs which are on,
the number of inputs which have value 1.

$$y = f(g(x)) = 1 \ \ if \ \ g(X) \geq \theta$$

$$y = f(g(x)) = 0 \ \ if \ \ g(X) < \theta$$

Function y takes this g as the input.
Means, it takes this sum as the input and
if the sum is greater than a certain threshold, then it fires.
Else, not fire.
(i.e.) depending on the actor, director, and genre and so on, I decide.
At least two of these three conditions are satisfied.

At least I am happy with the actor and the director
even though the genre is not something that I care about.

Instead, if I cared about all the factors, I'd set the threshold higher.

A very simplified model.
θ is called the thresholding parameter--
the value which decides whether the neuron is going to fire or not.
The overall thing is "thresholding logic".

This is McCulloch Pitts neuron.

Let us implement some boolean functions using this McCulloch Pitts (MP) neuron

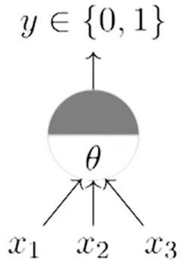
Why Boolean functions?

Because we have overly simplified the way we take decisions:
take many Boolean inputs (x_1 to x_n) and
based on that we produce a Boolean output.

That is a Boolean function that we are trying to learn from x to y.

Can cast many decision problems in this framework.
Eg.
whether to come for the class today or not.

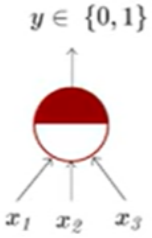
A concise representation of the McCulloch Pitts neuron:



A McCulloch Pitts unit

It takes a few Boolean inputs and
it has certain threshold.
If the sum of these inputs crosses this threshold,
then the neuron will fire and otherwise, not.

Say, we are trying to learn the AND function.



AND function

$f(g(x))=1 \quad g(x) \geq \theta$

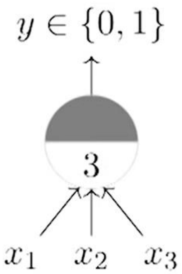
What should Θ be?

When would the AND function fire?

When all the inputs are on.

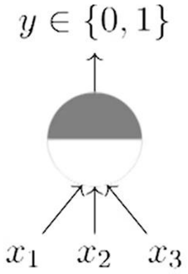
So, what should the threshold be?

3.



AND function

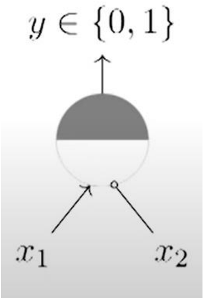
The OR function.



OR function

The threshold should be 1.

A few more.
x1 and !x2



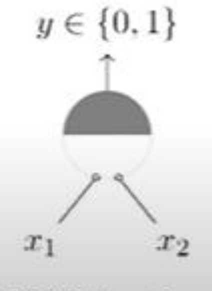
x_1 AND $!x_2$*

The circle, instead of the arrow head,
means that this input is an inhibitory input.
If that is on, then the neuron is not going to fire.

What should the threshold be for this?

If x2 is on, it is not going to fire.
So, two rows are ruled out.
0 0
0 1
1 0
1 1
Out of the remaining two, when do we want it to fire?
What should be the threshold?
1

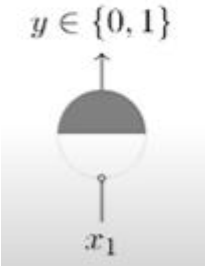
Now, what about NOR?



NOR function

The threshold should be 0.

What about this?



NOT function

0.

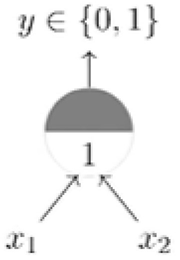
So, if we have a certain number of input variables and
the function that we are trying to model
(the decision that we are trying to make)
is a boolean function,
then we could represent using these MP neurons.

Whether all boolean functions can be represented in this way or not
is still not clear.
Some counterexamples later.

Can any boolean function be represented using a McCulloch Pitts unit ?

Let's first see the geometric interpretation of what the MP neuron is trying to do.

Geometric interpretation



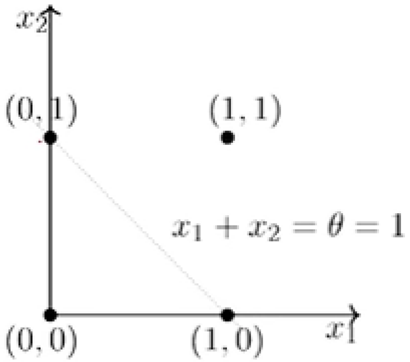
OR function

$x_1 + x_2 = \sum_{i=1}^{2} x_i \geq 1$

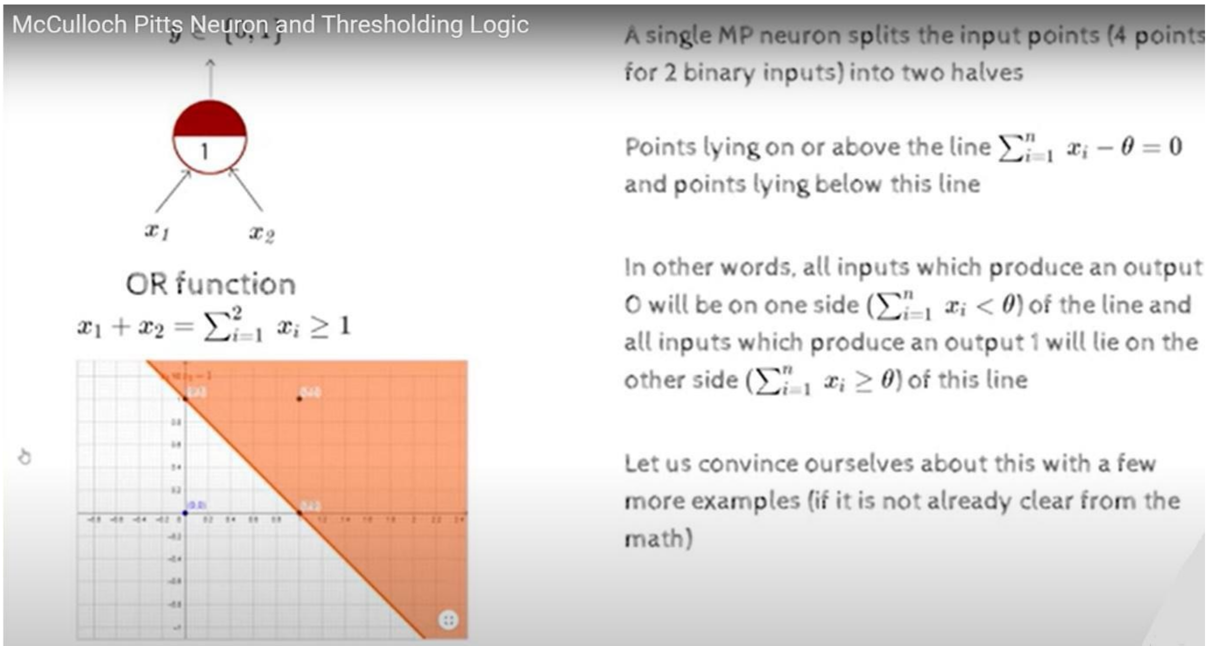
The neuron will fire if the sum is >= 1.
Ignoring the > part,
x1 + x2 = 1 is an equation of a line.

Here,
since we are dealing with boolean inputs and we have two axes x1 and x2,
how many input points can we have?
4:
0 0
0 1
1 0
1 1.
Only four inputs are valid here.
This is not a real numbered access.
Only boolean inputs possible here.

What does the line x1 + x2 = 1 (of the OR function) tell?



The line that passes through 1, 0 and 0, 1.



What do we want?

For all those inputs for which the output is actually 1,
they should lie on the line
or on the positive side on the line, and
for all those inputs for which the output is 0,
they should lie on the other side of the line.

Is that happening?

Yes.

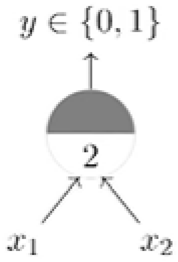
So, what is the MP neuron actually learning?

A linear decision boundary--
dividing the input points into two halves such that
all the points for which the output is 0 lie below this line and
all the points for which the output is 1
either lie on this line or above the line.

A few more examples.

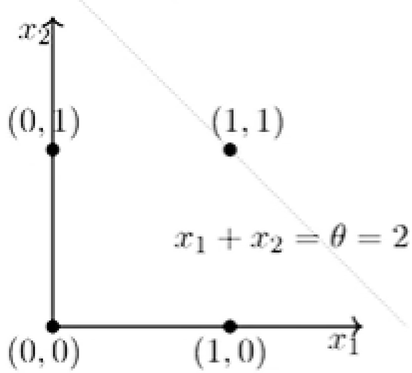
AND
Tautology (always ON)

For the AND function, what’s the decision boundary?

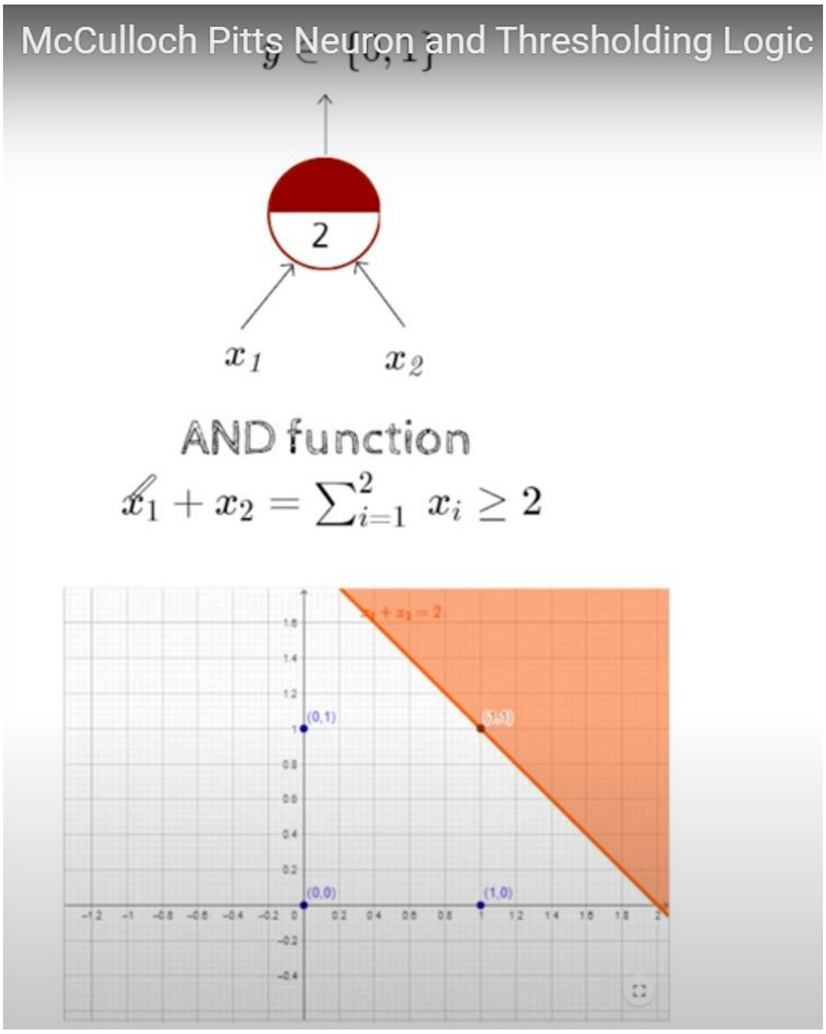


AND function
$x_1 + x_2 = \sum_{i=1}^{2} x_i \geq 2$

AND function
$x_1 + x_2 = \sum_{i=1}^{2} x_i \geq 2$



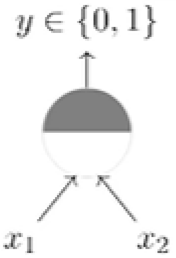
Passing through that 1, 1 and intercepting x2 around 2, 0 and x1 this around 0, 2.



Our condition is satisfied in that

all the inputs for which we want the output to be 2 are on or above the line and
all the inputs for which we want the output to be 0 are below the line.

Now, what about tautology (neuron always ON) function?



Tautology (always ON)

What is the threshold?

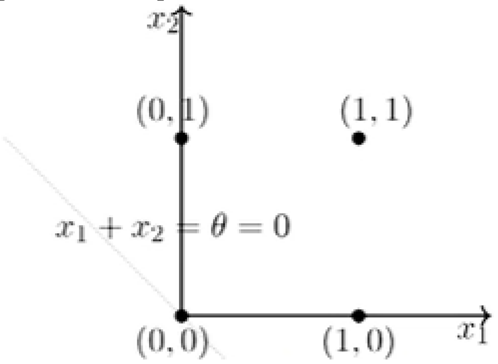
0

What would the line be?

x1 + x2 = 0

which passes through the origin.

Again, all the points are either on or above the line.



We are going to call the upper half as a positive half space and down one as the negative half space.

What if we have more than 2 inputs?

In a two dimensional case,

when we just had x1 and x2,

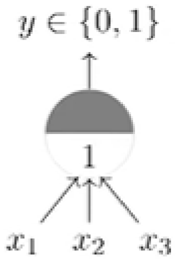
we are trying to find a separating line.

In 3-D?

Plane.

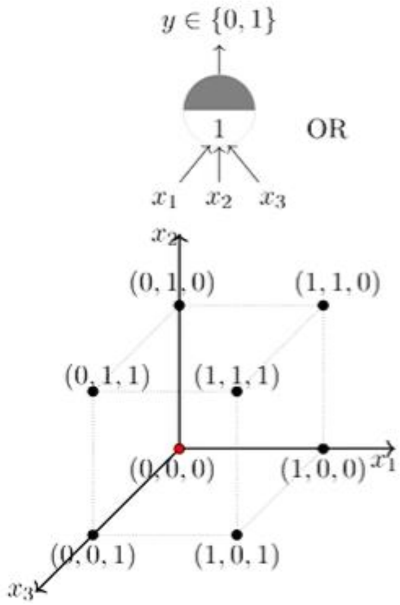
In higher dimensions?

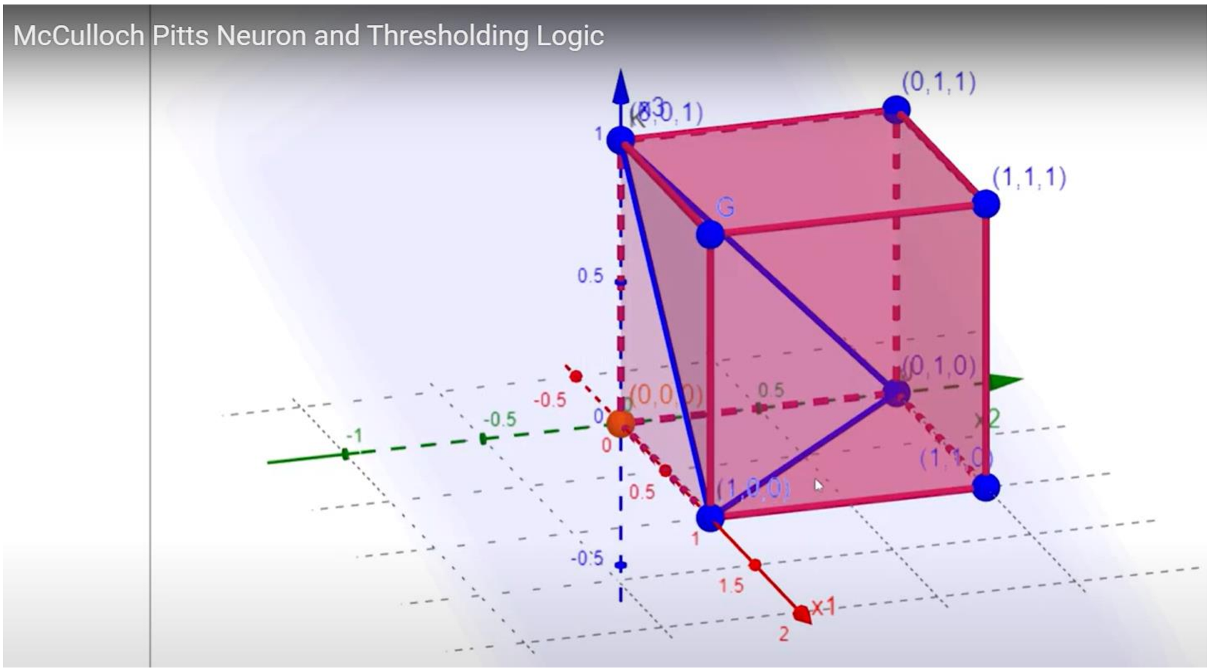
Hyper plane.

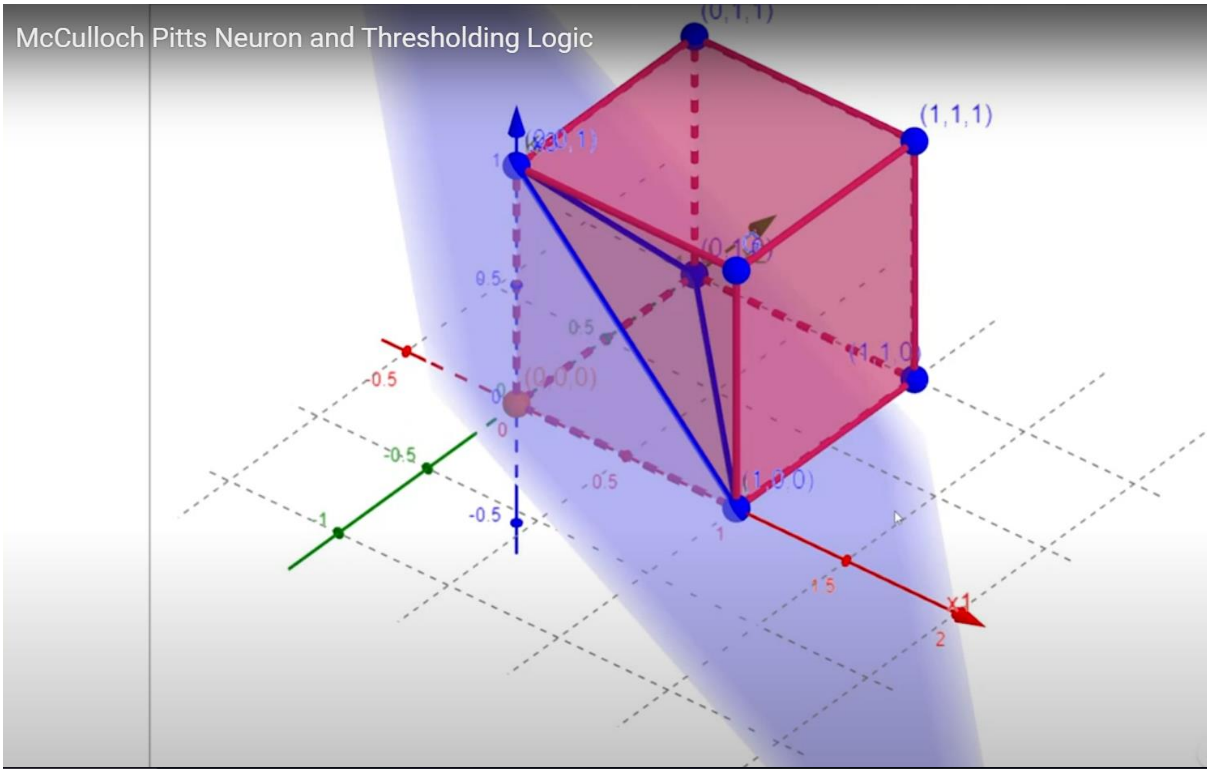


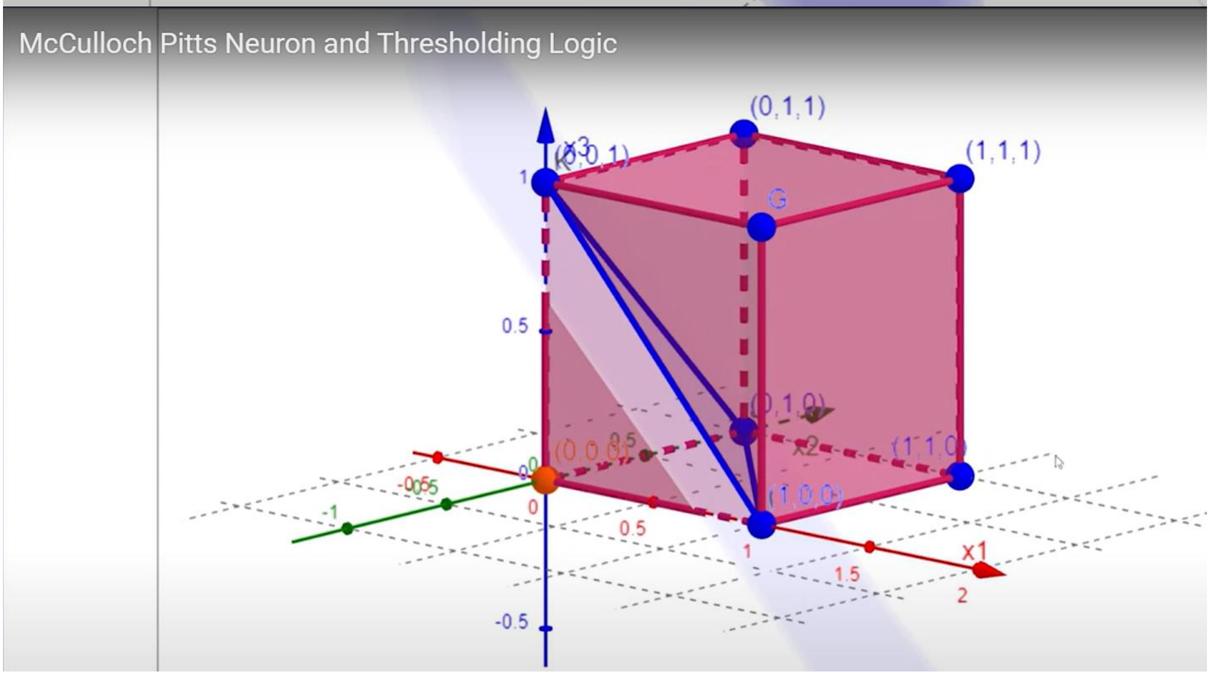
OR

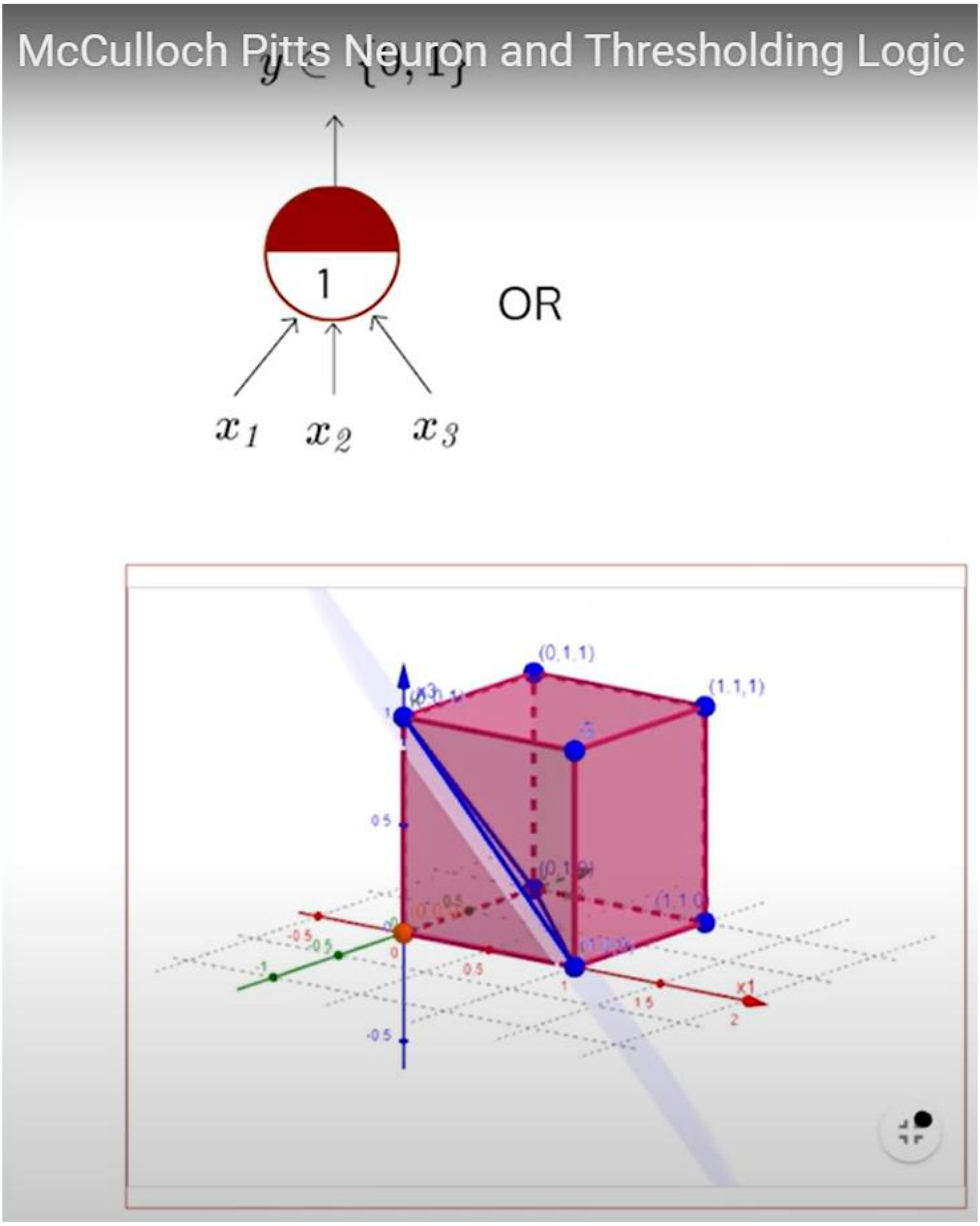
3-D case:


$y \in \{0, 1\}$
1
OR
x_1 x_2 x_3
x_2
(0, 1, 0)
(1, 1, 0)
(0, 1, 1)
(1, 1, 1)
(0, 0, 0)
(1, 0, 0)
x_1
(0, 0, 1)
(1, 0, 1)
x_3


McCulloch Pitts Neuron and Thresholding Logic
(0,1,1)
(1,1,1)
G
(0,1,0)
(0,0,0)
(1,0,0)
x2
x1

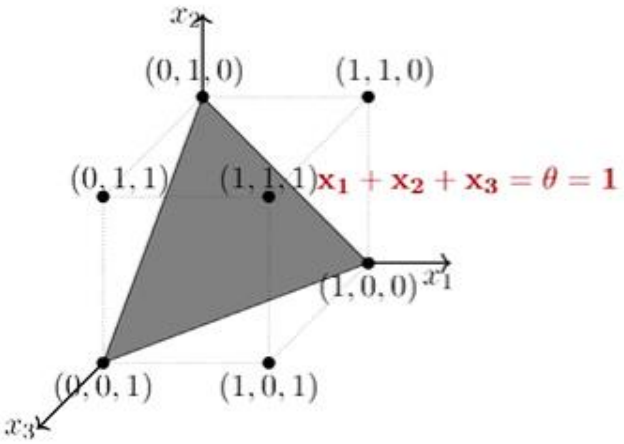

McCulloch Pitts Neuron and Thresholding Logic
(0,1,1)
(1,1,1)
(1,1,0)
(0,0,0)
(1,0,0)
x1


McCulloch Pitts Neuron and Thresholding Logic
(0,1,1)
(1,1,1)
x3
G
(0,1,0)
(1,1,0)
(0,0,0)
(1,0,0)
x2
x1



Three axes here.
But not all points are possible.
How many are possible?
8
Which is the function that we are trying to implement?
OR.
Out of these eight points, for how many is the output 1?
7 and

for 1, it is 0.
So, what is the kind of plane that we are looking for?
A plane such that 7 points lie on or above it and
1 point lies below it.
Which is that point?
0,0,0
What is the equation of that hyperplane?
x1+ x2 + x3=1.



All the seven points are visible,
but the points 0,0,0 is not because
it is on the other side of the plane.

So, this is doable in three dimensions and
in higher dimensions also (we could find a hyperplane).

Informally,

- A single McCulloch Pitts Neuron can be used to represent boolean functions which are linearly separable
- Linear separability (for boolean functions) : There exists a line (plane) such that all inputs which produce a 1 lie on one side of the line (plane) and all inputs which produce a 0 lie on other side of the line (plane)