LGP Lumieres

LGPL

Problem 8

Wandering Bacteria

PROBLEM 8

Wandering Bacteria

Summary

This problem investigates the stochastic evolution of a one-dimensional bacterial colony, *E.tim*, which grows and shrinks exclusively at its endpoints with discrete-time dynamics. Each end of the colony independently undergoes unit-length changes: with probability λ, it grows by one unit; with probability μ, it shrinks by one unit; and with remaining probability, it remains unchanged. The primary focus is on determining the expected colony length L_n at time n, conditioned on the initial length $L_0 > 0$, and understanding its dependence on the growth and shrinkage probabilities. Two cases are analyzed in detail: pure growth ($\mu = 0$) and combined growth and shrinkage ($\mu > 0$). In the pure growth case, the expected length evolves linearly as $\mathbb{E}[L_n] = L_0 + 2\lambda n$. When shrinkage is permitted, the expected length still follows a linear trend but with adjusted rate, given by $\mathbb{E}[L_n] = L_0 + 2(\lambda - \mu)n$. These results model the dynamics of fluctuating biological systems under probabilistic growth mechanisms and lay the groundwork for further exploration into extinction times and steady-state behavior.

Question	Result
1	solved
2	not solved
3	not solved
4	not solved
5	not solved
6	not solved

Contents

Problem Statement: The bacteria *E.tim* are a kind of bacteria that live in a very dynamic pluricellular mode. The bacteria live stuck together in a 1D structure as a 'stick'. This colony can grow and shrink over time. This evolution only happens at the ends of that colony, that is, over time a colony can grow or shrink by its ends with some probability rate. We denote by $0 < \lambda \leq 1$ the probability of growth at each end of the colony. We denote by $0 < \mu \leq 1$ the probability of shrinking of the colony at those said ends. In our model, we consider a discrete evolution in time. More precisely, this means that, at each time $n \in \mathbb{N}$, and at each end of the colony, there is a probability λ that the length increases by 1, and a probability μ that it decreases by 1. (If $\mu \neq 0$, it may happen that the colony grows and shrinks simultaneously, either both at the same end, or each at a different end, in which case it keeps the same length. Also, the colony may grow, or shrink, at both its ends at the same time, in which case the length would increase or decrease by 2.) We denote by L_n the length at time $n \in \mathbb{N}$, and we always assume that $L_0 > 0$. If for some n, $L_n = 0$, the colony dies and the process stops.

1 Length of the Colony

Question 1: We first wonder about the length of the colony as time elapses.

a) Assume that the colony can only grow with probability λ, that is, $\mu = 0$. How does the length of the colony evolve over time, depending on L_0? What is the expected length at time n?

b) Assume that $\mu \neq 0$. What is the expected length of the colony at time n, depending on L_0?

a) Pure Growth

Theorem 1 (Linear Growth of Colony Length when $\mu = 0$): Let $(L_n)_{n \geq 0}$ be the discrete-time process describing the length of an *E.tim* bacterial colony at time $n \in \mathbb{N}$, with initial length $L_0 > 0$. Denote by $\lambda \in (0, 1]$ the probability that at any given time step and at a given end of the colony the length increases by one unit, independently at each end and across time steps, and assume no shrinking occurs (i.e. $\mu = 0$). Then for every $n \geq 0$ one can write

$$L_n = L_0 + \sum_{k=1}^{n} (X_k + Y_k),$$

where for each $k \geq 1$, the random variables X_k and Y_k each take the value 1 with probability λ and 0 with probability $1 - \lambda$, and all these $2n$ random variables are mutually independent. It follows that the expected length at time n satisfies

$$\mathbb{E}[L_n] = L_0 + 2\lambda\, n.$$

Proof. We begin by constructing a suitable probability space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ carry random variables $X_k, Y_k : \Omega \to \{0, 1\}$ for $k = 1, 2, \ldots$ such that $\mathbb{P}(X_k = 1) = \mathbb{P}(Y_k = 1) = \lambda$, $\mathbb{P}(X_k = 0) = \mathbb{P}(Y_k = 0) = 1 - \lambda$, and all X_k, Y_k are independent. Here $X_k = 1$ indicates a unit growth at the first end during the step from time $k - 1$ to k, while $Y_k = 1$ indicates growth at the second end in the same step.

By the model's rule, during step k the total change in length is $X_k + Y_k$, so we have the recursion $L_k = L_{k-1} + X_k + Y_k$ for each $k \geq 1$. To derive an explicit formula, observe that for $n = 0$ this gives

$L_0 = L_0$, and assuming $L_{n-1} = L_0 + \sum_{k=1}^{n-1}(X_k + Y_k)$ one obtains

$$L_n = L_{n-1} + X_n + Y_n = L_0 + \sum_{k=1}^{n-1}(X_k + Y_k) + (X_n + Y_n) = L_0 + \sum_{k=1}^{n}(X_k + Y_k).$$

Thus by induction the claimed summation formula holds for all $n \geq 0$.

Having expressed L_n as a sum of independent 0–1 random variables plus the constant L_0, we compute its expectation by linearity: since each X_k and Y_k equals 1 with probability λ and 0 otherwise, we have $\mathbb{E}[X_k] = \lambda$ and $\mathbb{E}[Y_k] = \lambda$. Therefore,

$$\mathbb{E}[L_n] = \mathbb{E}\Big[L_0 + \sum_{k=1}^{n}(X_k + Y_k)\Big] = L_0 + \sum_{k=1}^{n}(\mathbb{E}[X_k] + \mathbb{E}[Y_k]) = L_0 + \sum_{k=1}^{n}(\lambda + \lambda) = L_0 + 2\lambda\, n.$$

This completes the proof. □

b) Growth with Shrinking

Theorem 2 (Expected Length of Colony when $\mu \neq 0$): Let $(L_n)_{n\geq 0}$ be the discrete-time process describing the length of an *E.tim* bacterial colony at time $n \in \mathbb{N}$, with initial length $L_0 > 0$. Suppose that at each time step and at each end of the colony, the length increases by $+1$ with probability λ, decreases by -1 with probability μ, and stays the same with probability $1 - \lambda - \mu$, independently across ends and time steps (with $0 \leq \mu \leq 1$, $0 < \lambda \leq 1$, and $\lambda + \mu \leq 1$). Then for every $n \geq 0$ one has

$$L_n = L_0 \; + \; \sum_{k=1}^{n}(X_k + Y_k),$$

where for each k, X_k, Y_k are independent random variables taking values in $\{-1, 0, 1\}$ according to

$$\mathbb{P}(X_k = 1) = \lambda, \quad \mathbb{P}(X_k = -1) = \mu, \quad \mathbb{P}(X_k = 0) = 1 - \lambda - \mu,$$

$$\mathbb{P}(Y_k = 1) = \lambda, \quad \mathbb{P}(Y_k = -1) = \mu, \quad \mathbb{P}(Y_k = 0) = 1 - \lambda - \mu.$$

Moreover, the expected length at time n is

$$\mathbb{E}[L_n] \; = \; L_0 \; + \; 2\,(\lambda - \mu)\, n.$$

Proof. Construct a probability space carrying independent variables $X_k, Y_k : \Omega \to \{-1, 0, 1\}$ with the stated distributions. By the model, each step k changes the length by $X_k + Y_k$, so

$$L_k = L_{k-1} + X_k + Y_k \quad (k \geq 1).$$

An easy induction shows

$$L_n = L_0 + \sum_{k=1}^{n}(X_k + Y_k) \quad \text{for all } n \geq 0.$$

Taking expectations and using linearity,

$$\mathbb{E}[L_n] = L_0 + \sum_{k=1}^{n} \big(\mathbb{E}[X_k] + \mathbb{E}[Y_k]\big).$$

Since each X_k satisfies

$$\mathbb{E}[X_k] = 1 \cdot \lambda + (-1) \cdot \mu + 0 \cdot (1 - \lambda - \mu) = \lambda - \mu,$$

and similarly $\mathbb{E}[Y_k] = \lambda - \mu$, it follows that

$$\mathbb{E}[L_n] = L_0 + \sum_{k=1}^{n} 2(\lambda - \mu) = L_0 + 2(\lambda - \mu)\, n.$$

This completes the proof. $\square$