LGP Lumieres

LGPL

Problem 9

A Game with Light

PROBLEM 9

A Game with Light

Summary

In this problem, we investigate the conditions under which it is possible to reach an all-green configuration on an $n \times n$ grid of lights, where each light cycles through the colors red, blue and green. Pressing a light changes its own color as well as the colors of its adjacent (neighboring) lights, advancing each cyclically through the three colors. We begin by systematically labeling the lights and introducing notation to model the effect of pressing a light. Initially, we use modular arithmetic to describe the color transitions and then proceed to a more general and rigorous treatment using linear algebra over finite fields.

Question	Result
1	solved
2	solved
3	solved
4	solved
5	solved
6	mostly solved
7	partially solved
8	solved

For the case $n = 2$, we show that the order of taps is irrelevant and that any initial configuration can be transformed into the all green configuration. We extend this result to arbitrary n, and derive a function to count the number of unsolvable initial configurations. Finally, we explore generalizations to scenarios with m colors.

Contents

Problem Statement: Guillaume is playing with a special cube where each of its six faces contains an $n \times n$ grid of lights. Each light can be in one of three states: **red, blue,** or **green.** When Guillaume presses a light, its state changes cyclically as follows:

$$\text{Red} \to \text{Blue} \to \text{Green} \to \text{Red}.$$

However, pressing a light does not only affect itself! It also changes the state of its adjacent lights on the same face (not diagonals), as well as the lights directly connected across the edges of the cube. Guillaume's goal is to turn all the lights **green**, but he quickly realizes that this is not an easy task! He wonders if there is a strategy to solve the puzzle efficiently.

Notations

$\mathbb{Z}/3\mathbb{Z}$ The integers modulo 3

x The initial/current light state vector

e_i The vector representing the effect of pressing light A_i

p Press vector indicating number of times each light is pressed

M Matrix describing light interaction

Preliminaries

We position the cube in a 3-dimensional space so that its center lies at the origin, and its edges are aligned with the Cartesian axes x, y, z. Each face is identified according to the outward normal direction of the axis it faces:

Face F_t	**Orientation**
F_1	$+z$ (front)
F_2	$-z$ (back)
F_3	$+y$ (top)
F_4	$-y$ (bottom)
F_5	$+x$ (right)
F_6	$-x$ (left)

Definition 1 For each face F_t, where $t \in \{1, 2, \ldots, 6\}$, the cardinality is defined as

$$|F_t| = t.$$

Each face F_t, for $t \in \{1, 2, \ldots, 6\}$, is treated as a distinct 2 dimensional grid. We assign to each light on F_t a unique coordinate $(r, c) \in \{1, 2, \ldots, n\}^2$, where:

- r (row index) increases from bottom to top (i.e., row 1 is the bottommost, row n is the topmost),
- c (column index) increases from left to right (i.e., column 1 is the leftmost, column n is the rightmost).

Definition 2 Each light is denoted by A_{index}, where the index is given by

$$\text{index} = n^2(t-1) + n(r-1) + c,$$

with $t \in \{1, 2, \ldots, 6\}$ indicating the face number, and $r, c \in \{1, 2, \ldots, n\}$ representing the row and column of the light on that face, respectively.

This way, each light gets a unique number based on where it is on the cube. The index always lies in the range:

$$\text{index} \in \{1, 2, \ldots, 6n^2\}.$$

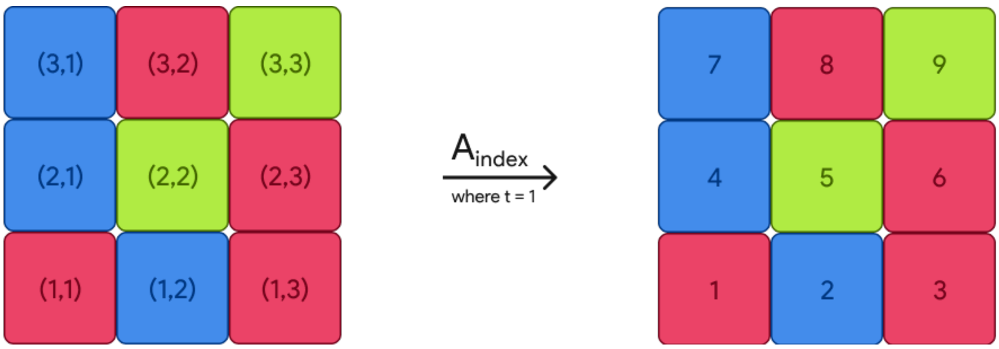


Figure 1: A_{index} on face F_1 for $n = 3$.

It is given that whenever a light, say A_i, is tapped, its colour changes cyclically from red to blue to green, and then back to red.
Suppose we encode the colours numerically as follows:

- Red $\mapsto 0$
- Blue $\mapsto 1$
- Green $\mapsto 2$

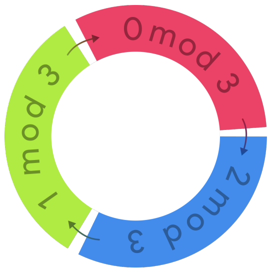
Tapping a light then corresponds to incrementing its value by 1 (modulo 3), i.e., $x \mapsto x + 1 \mod 3$. This gives us the following equivalence classes under modulo 3 arithmetic:

$$\text{Red} = \{0, 3, 6, 9, \ldots\} \equiv 0 \mod 3$$

$$\text{Blue} = \{1, 4, 7, 10, \ldots\} \equiv 1 \mod 3$$

$$\text{Green} = \{2, 5, 8, 11, \ldots\} \equiv 2 \mod 3$$

Thus, each light's colour state can be viewed as an element of the finite field $\mathbb{Z}_3$, with successive taps moving its value forward in the cycle.



Definition 3 The vector

$$x \in (\mathbb{Z}/3\mathbb{Z})^{6n^2}$$

is defined to represent the colour configuration of all lights in the cube.

Thus,

$$x = (x_1, x_2, \dots, x_{6n^2}) \in (\mathbb{Z}/3\mathbb{Z})^{6n^2}$$

where each component $x_i \in \{0, 1, 2\}$ represents the colour state of the light A_i, encoded as:

$$0 \equiv \text{Red}, \quad 1 \equiv \text{Blue}, \quad 2 \equiv \text{Green}.$$

Definition 4 For any light A_i, its colour in the configuration x is denoted by

$$x(A_i) := x_i \in \mathbb{Z}_3.$$

We are given that the action of tapping a light not only affects the light itself but also the lights that share an edge with it.

Definition 5 The neighbor set of a light A_i is defined as

$$N_i = \{j \mid A_j \text{ shares an edge with } A_i\},$$

that is, the set of index of lights that are adjacent to A_i.

Since each light is represented as a square unit on an $n \times n$ grid, it has exactly four adjacent lights along its edges (excluding diagonals). Thus, it is evident that $|N_i| = 4$.
Pressing (tapping) a light A_i increments the colour of A_i itself and all lights A_j such that $j \in N_i$.
To mathematically represent this action, we introduce a vector

$$e_i \in (\mathbb{Z}/3\mathbb{Z})^{6n^2}.$$

Definition 6 For each light A_i, the vector

$$e_i \in (\mathbb{Z}/3\mathbb{Z})^{6n^2}$$

is defined by

$$(e_i)_k = \begin{cases} 1 & \text{if } k = i \text{ or } k \in N_i, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Therefore, tapping A_i updates the colour state x as:

$$x \mapsto x + e_i \pmod 3$$

Which means:

$$x_k \mapsto \begin{cases} x_k + 1 \pmod 3 & \text{if } k = i \text{ or } k \in N_i \\ x_k & \text{otherwise} \end{cases} \tag{2}$$

Definition 7 For each light A_i, the function

$$f_i : (\mathbb{Z}/3\mathbb{Z})^{6n^2} \to (\mathbb{Z}/3\mathbb{Z})^{6n^2}$$

is defined by

$$f_i(x) \coloneqq x + e_i.$$

1 In the grand algebraic scheme, does order matter?

Question 1: Guillaume wonders whether the **order** in which he presses the lights affects the final result. If he presses the same set of lights but in a different sequence, will he always obtain the same final state?

We already know the action of pressing a light A_i and A_j, where $i, j \in \{1, 2, \ldots, 6n^2\}$, can be represented using the function f:

$$f_i(x) = x + e_i \quad \text{and} \quad f_j(x) = x + e_j.$$

Then, the action of pressing in a particular sequence can be written as:

$$\text{Pressing } A_i \text{ then } A_j : \quad f_j \circ f_i(x),$$

$$\text{Pressing } A_j \text{ then } A_i : \quad f_i \circ f_j(x).$$

Theorem 1: The final state of the cube is independent of the order in which a fixed set of lights is pressed. That is, for any lights A_i and A_j, the corresponding functions commute:

$$f_j \circ f_i(x) = f_i \circ f_j(x).$$

Proof. We are well aware that $(\mathbb{Z}/3\mathbb{Z})^{6n^2}$ denotes the set of all vectors of length $6n^2$, where each component is an element of $\mathbb{Z}_3 = \{0, 1, 2\}$, and vector addition is performed componentwise modulo 3.

Let $a, b \in (\mathbb{Z}/3\mathbb{Z})^{6n^2}$ where

$$a = (a_1, a_2, \dots, a_{6n^2}) \quad \text{and} \quad b = (b_1, b_2, \dots, b_{6n^2}).$$

In this space, vector addition is defined component-wise:

$$a + b = (a_1,\, a_2, \dots,\, a_n) + (b_1,\, b_2, \dots,\, b_n) = (a_1 + b_1,\, a_2 + b_2,\, \dots,\, a_{6n^2} + b_{6n^2}) \pmod 3.$$

That is, for each $k \in \{1, 2, \dots, 6n^2\}$,

$$(a + b)_k = (a_k + b_k) \pmod 3.$$

Similarly,

$$(b + a)_k = (b_k + a_k) \pmod 3.$$

Since addition in $\mathbb{Z}_3$ is commutative,

$$a_k + b_k = b_k + a_k \pmod 3.$$

Therefore,

$$(a + b)_k = (b + a)_k \pmod 3,$$

which implies

$$a + b = b + a.$$

Now,

$$f_j \circ f_i(x) = f_j(f_i(x)) = f_j(x + e_i) = x + e_i + e_j,$$

and

$$f_i \circ f_j(x) = f_i(f_j(x)) = f_i(x + e_j) = x + e_j + e_i.$$

Thus,

$$f_j \circ f_i(x) - x = e_i + e_j \quad \text{and} \quad f_i \circ f_j(x) - x = e_j + e_i.$$

Replacing a with e_i and b with e_j, we get

$$e_i + e_j = e_j + e_i.$$

Therefore,

$$f_j \circ f_i(x) - x = f_i \circ f_j(x) - x,$$

which implies

$$f_j \circ f_i(x) = f_i \circ f_j(x).$$

This completes the proof. □

Using similar reasoning, we can show that for any finite sequence of taps, the order does not matter.

Theorem 2: Let $\{f_{i_1}, f_{i_2}, \ldots, f_{i_m}\}$ be a set of tap functions. Then, for any permutation σ of $\{1, 2, \ldots, m\}$,

$$f_{i_{\sigma(1)}} \circ f_{i_{\sigma(2)}} \circ \cdots \circ f_{i_{\sigma(m)}}(x) = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}(x).$$

This follows because addition in $(\mathbb{Z}/3\mathbb{Z})^{6n^2}$ is commutative; that is, $x + e_{i_1} + e_{i_2} + \cdots + e_{i_m}$ is independent of the order in which the e_{i_k} are added.

This conclusion can also be understood by modeling the cumulative effect of pressing a collection of lights via matrix-vector multiplication. That is, there exists a fixed matrix

$$M \in \mathbb{Z}_3^{6n^2 \times 6n^2}$$

such that pressing a collection of lights, encoded by a vector $p \in (\mathbb{Z}/3\mathbb{Z})^{6n^2}$ (where p_i indicates how many times light A_i is pressed, modulo 3), results in the configuration

$$x' = x + Mp \pmod 3.$$

A natural question is whether the *order* in which the lights are pressed (that is, the sequence in which the entries of p are accumulated) has any effect on the final outcome. The answer is no, and this follows from the fundamental algebraic structure of the system:

Theorem 3 (Order Independence of Presses)**:** Let $x \in (\mathbb{Z}/3\mathbb{Z})^{6n^2}$ be the initial configuration and let $p \in (\mathbb{Z}/3\mathbb{Z})^{6n^2}$ be any press-vector. Then the final configuration

$$x' = x + Mp \pmod 3$$

is independent of the order in which the individual light presses are applied. That is, any two pressing sequences that yield the same net vector p will result in the same final state x'.

Proof. Each individual press of a light A_j corresponds to adding the jth column $M_{\cdot j}$ of the matrix M to the configuration vector x. Pressing several lights in sequence adds the corresponding columns:

$$x' = x + M_{\cdot j_1} + M_{\cdot j_2} + \cdots + M_{\cdot j_k} \pmod 3.$$

But vector addition over $\mathbb{Z}_3$ is both associative and commutative, so the order in which the columns $M_{\cdot j_i}$ are added is irrelevant:

$$M_{\cdot j_1} + M_{\cdot j_2} + \cdots + M_{\cdot j_k} = M_{\cdot j_{\sigma(1)}} + M_{\cdot j_{\sigma(2)}} + \cdots + M_{\cdot j_{\sigma(k)}}$$

for any permutation $\sigma \in S_k$. Therefore, all pressing sequences with the same multiset of tapped lights produce the same final vector x'. $\square$

Conclusion: The system is linear and additive over the field $\mathbb{Z}_3$, and the light presses commute. Thus, Guillaume may press the lights in any order he likes. The final configuration depends only on how many times each light is pressed (modulo 3), not on the sequence.

2 Red light green light

Question 2: In this question, Guillaume assumes that $n = 2$. If all lights are red at the beginning, is it possible to turn all lights green?

When $n = 2$, we have *index* $\in [1, 24]$; that is, there are 24 lights (4 lights per face).

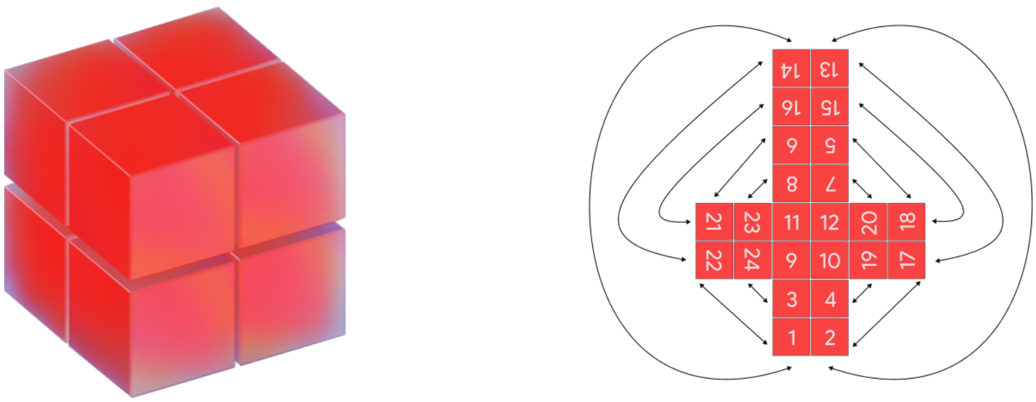


Figure 2

Given that all 24 lights are red at the beginning, the initial configuration is

$$x = (0, 0, \ldots, 0) \in (\mathbb{Z}/3\mathbb{Z})^{24}.$$

For this question, Guillaume wants to reach the final configuration x', where all 24 lights are green:

$$x' = (2, 2, \ldots, 2) \in (\mathbb{Z}/3\mathbb{Z})^{24}.$$

If it is possible to reach the final state x' from the initial state x in a finite number of steps, this is equivalent to

$$x' = f_{i_m} \circ \ldots \circ f_{i_1}(x).$$

Using the vector $p \in (\mathbb{Z}/3\mathbb{Z})^{24}$ from part (1), this can be written as

$$x' = x + p_1e_1 + p_2e_2 + \ldots + p_{24}e_{24} \pmod 3$$

$$x' = x + \sum_{i=1}^{24} p_ie_i \pmod{3}.$$

Thus, for each $k \in [1, 24]$,

$$x'_k = x_k + \left(\sum_{i=1}^{24} p_i(e_i)\right)_k \pmod 3$$

Since we have established that $x'_k = 2$ and $x_k = 0$, it follows that

$$x'_k = 2 = \left(\sum_{i=1}^{24} p_i(e_i)\right)_k \pmod{3}.$$

Let us first look at

$$\left(\sum_{i=1}^{24} e_i\right)_k.$$

Observation: We already know that $(e_i)_k = 1$ when $i = k$ or $i \in N_k$, and 0 otherwise. Thus,

$$\left(\sum_{i=1}^{24} e_i\right)_k = \underbrace{1}_{\text{self}} + \underbrace{4}_{\text{neighbors}} = 5 \equiv 2 \pmod{3}.$$

This observation implies

$$\sum_{i=1}^{24} e_i \equiv (2, 2, \ldots, 2) \pmod{3}.$$

Also,

$$x'_k = \left(\sum_{i=1}^{24} p_ie_i\right)_k = \left((1)\, p_k + (1) \sum_{i \in N_k} p_i + (0) \sum_{i \notin N_k \cup \{k\}} p_i\right) \pmod{3}.$$

$$x'_k = p_k + \sum_{i \in N_k} p_i \pmod{3}.$$

Let us assume $N_k = \{k_1, k_2, k_3, k_4\}$. Then,

$$x'_k = (p_k + p_{k_1} + p_{k_2} + p_{k_3} + p_{k_4}) \pmod{3}.$$

$$2 = (p_k + p_{k_1} + p_{k_2} + p_{k_3} + p_{k_4}) \pmod{3}.$$

The uniform solution that satisfies the condition for all $p_i \in \mathbb{Z}_3$ is

$$p_i = 1 \quad \text{for all } i \in [1, 24].$$

Thus, for each k,

$$x'_k = 2 = (1+1+1+1+1) \pmod 3 = 5 \pmod 3 = 2.$$

Note that when $p_i = 1$ for all i,

$$\sum_{i=1}^{24} p_i e_i = \sum_{i=1}^{24} e_i.$$

Therefore,

$$x' = x + \sum_{i=1}^{24} e_i.$$

where

$$\sum_{i=1}^{24} e_i = (5, 5, \dots, 5) \equiv (2, 2, \dots, 2) \pmod 3.$$

Observation: Other possible choices of $\{p_i\}$ would lead to imbalance because:

- Any deviation from $p_i = 1$ for all i would result in some lights receiving fewer or more contributions due to the overlapping of neighbor sets.
- Since each light appears as a neighbor in the neighbor sets of multiple other lights, unequal p_i would create discrepancies to satisfy the condition $x'_k = 2 \pmod 3$ uniformly for all k.

A_i	N_i				A_i	N_i			
1	14	22	3	2	13	14	15	17	2
2	1	4	17	13	14	13	16	22	1
3	24	9	4	1	15	5	16	18	13
4	3	2	10	19	16	21	6	14	15
5	6	7	18	15	17	2	13	18	19
6	5	8	16	21	18	15	5	20	17
7	12	8	5	20	19	10	20	17	4
8	7	9	23	11	20	12	7	19	18
9	10	11	3	24	21	22	23	6	16
10	4	9	12	19	22	1	14	24	21
11	9	12	8	23	23	8	11	24	21
12	10	11	7	20	24	22	23	9	3

Table 1: N_i for each A_i when $n = 2$

Claim: The only way to transition from an all-red configuration to an all-green configuration is by pressing each light exactly once (or, more generally, $3y+1$ times for some integer y), since pressing a light three times has no net effect modulo 3.

To provide a concrete mathematical foundation for this claim and to rigorously prove both the **existence** and **uniqueness** of this solution, we now turn to a linear algebraic approach.

Theorem 4: For $n = 2$ and $x = 0 \cdot \vec{\mathbf{1}}$, the press-vector $p = (1, 1, \ldots, 1)^\top$ satisfies

$$M\,p \;\equiv\; 2\,\vec{\mathbf{1}} \pmod 3,$$

and hence pressing each light once turns the all-red cube into an all-green cube.

Proof. **Existence:** Fix any light A_i. On a 2×2 face, A_i has exactly:

$$\underbrace{1}_{\text{itself}} + \underbrace{2}_{\substack{\text{its two orthogonal}\\ \text{neighbors on the same face}}} + \underbrace{2}_{\substack{\text{its two edge-}\\ \text{neighbors on adjacent faces}}} = 5$$

press-contributions when *every* light is pressed once. Since $5 \equiv 2 \pmod 3$, it follows that the ith entry of $M\,p$ is 2. As i was arbitrary, we conclude

$$M\,p = (2, 2, \ldots, 2)^\top = 2\,\vec{\mathbf{1}} \pmod{3}.$$

Therefore

$$x' \;=\; x + M\,p \;\equiv\; 0 + 2\,\vec{\mathbf{1}} \;=\; 2\,\vec{\mathbf{1}} \pmod 3,$$

so all 24 lights become green. This completes the proof of existence of an all-green solution for $n = 2$.

$$M = \begin{pmatrix}
1&1&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\
1&1&0&1&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
1&0&1&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&1&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1
\end{pmatrix}$$

Figure 3: The matrix M for $n = 2$.

Uniqueness: Suppose $q \in (\mathbb{Z}/3\mathbb{Z})^{24}$ is another solution:

$$M\,q \equiv 2\,\vec{\mathbf{1}} \pmod 3.$$

Define $d = q - \vec{\mathbf{1}}$. Then

$$M\,d = M\,q - M\,\vec{\mathbf{1}} \equiv 2\,\vec{\mathbf{1}} - 2\,\vec{\mathbf{1}} = 0 \pmod 3,$$

so d lies in the null space of M over $\mathbb{Z}/3\mathbb{Z}$.

Now observe: by cube symmetry, every light has the same structural role. Hence, any vector $d \in \ker M$ must be constant across all positions (since rotating the cube permutes coordinates and preserves M). That is,

$$d = (k, k, \ldots, k)^\top \quad \text{for some } k \in \mathbb{Z}/3\mathbb{Z}.$$

Let us compute $M\,d$ for such a constant vector:

$$M\,d = M\,(k\,\vec{\mathbf{1}}) = k\,M\,\vec{\mathbf{1}}.$$

As shown earlier, $M\,\vec{\mathbf{1}} \equiv 2\,\vec{\mathbf{1}} \pmod 3$, so

$$M\,d \equiv k \cdot 2\,\vec{\mathbf{1}} \equiv 2k\,\vec{\mathbf{1}} \pmod 3.$$

For this to vanish modulo 3, we require $2k \equiv 0 \pmod 3$, which implies $k \equiv 0 \pmod 3$. Hence $d = 0$ and $q = \vec{\mathbf{1}}$. Therefore, the only solution is $p = (1, 1, \ldots, 1)^\top$. $\square$

3 Universal Solvability for $n = 2$

Question 3: In this question, Guillaume assumes that $n = 2$. No matter the original state of the cube, is it possible to turn all lights green?

Theorem 5: Let $M \in \mathbb{Z}_3^{24\times 24}$ be the pressing-matrix for the 2×2 cube of lights, and let $x \in (\mathbb{Z}/3\mathbb{Z})^{24}$ be any initial colouring. Then there exists a unique press-vector $p \in (\mathbb{Z}/3\mathbb{Z})^{24}$ such that

$$x + M\,p \equiv 2\,\vec{\mathbf{1}} \pmod 3,$$

i.e. every initial configuration can be turned to all green.

Proof. First, note that in the matrix M every diagonal entry is 1, and all off-diagonal entries lie in $\{0, 1\} \subset \mathbb{Z}_3$. We perform Gaussian elimination over $\mathbb{Z}_3$ and exhibit 24 nonzero pivots as follows.

First we use the equation corresponding to row 1,

$$p_1 + p_{i_2} + p_{i_3} + p_{i_4} + p_{i_5} \;\equiv\; 0 \pmod 3,$$

where $\{i_2, i_3, i_4, i_5\}$ are the four neighbors of light A_1. Since the coefficient of p_1 is 1, we solve

$$p_1 \;=\; -\big(p_{i_2} + p_{i_3} + p_{i_4} + p_{i_5}\big) \pmod 3$$

and then add suitable multiples of this row to rows i_2, i_3, i_4, i_5 to eliminate p_1 from them. This uses the first pivot, which is nonzero in $\mathbb{Z}_3$.

Next, assume we have eliminated $p_1, \ldots, p_{k-1}$ using $k - 1$ nonzero pivots. Among the remaining rows and variables, pick the smallest index k whose diagonal entry is still 1 (such an index always exists because no row-addition operation can turn a diagonal 1 into 0). Write that row as

$$p_k \;+\; \sum_{j>k} c_{kj}\, p_j \;\equiv\; 0 \pmod 3,$$

solve for

$$p_k \;=\; -\sum_{j>k} c_{kj}\, p_j \pmod 3,$$

and then add appropriate multiples of this row to all remaining rows to eliminate p_k. This provides the kth pivot, again nonzero.

Continuing in this manner for $k = 1, 2, \ldots, 24$, we obtain 24 nonzero pivots in $\mathbb{Z}_3$. Hence

$$\det M \;\not\equiv\; 0 \pmod 3,$$

so M is invertible over $\mathbb{Z}_3$. Therefore for any right-hand side $b = 2\vec{1} - x$ there is a unique solution

$$p = M^{-1} b,$$

and consequently

$$x + M\,p \;\equiv\; 2\,\vec{1} \pmod 3.$$

This completes the proof. □

Conclusion: For $n = 2$, every initial configuration $x \in (\mathbb{Z}/3\mathbb{Z})^{24}$ can be transformed into the all-green configuration $x' \equiv 2 \pmod 3$.

Conclusion: Since M is invertible over $\mathbb{Z}/3\mathbb{Z}$, it follows that

$$\operatorname{span}\{e_1, \ldots, e_{24}\} = (\mathbb{Z}/3\mathbb{Z})^{24}.$$

Therefore, every initial configuration can be expressed as a linear combination $\sum p_i e_i$. Thus, for $n = 2$, any initial configuration can be transformed into any desired final configuration.

4 Minimum Number of Moves in the Worst Case for $n = 2$

Question 4: What is the minimum number of moves that Guillaume needs to solve the worst-case scenario which has a solution on a 2×2 cube face?

Theorem 6: On the 2×2 cube of lights, define a *move* to be a single tap of any one light (so tapping the same light twice counts as two moves). Then for every initial configuration there is a unique solution press-vector $p \in (\mathbb{Z}/3\mathbb{Z})^{24}$ and the maximal number of moves required over all configurations is

$$\max_{x\in(\mathbb{Z}/3\mathbb{Z})^{24}} \sum_{i=1}^{24} p_i \;=\; 48.$$

In other words, the worst-case configuration can be solved in exactly 48 moves, and no configuration requires more.

Proof. We recall that the pressing-matrix

$$M \;\in\; \mathbb{Z}_3^{24\times 24}$$

is invertible (as shown in the universal-solvability proof), so for every initial colouring $x \in (\mathbb{Z}/3\mathbb{Z})^{24}$ there is a unique press-vector

$$p \;=\; M^{-1}\big(2\vec{\mathbf{1}} - x\big) \in (\mathbb{Z}/3\mathbb{Z})^{24},$$

and executing p_i taps at light A_i (mod 3) turns x into the all-green configuration $2\vec{\mathbf{1}}$. We now count moves.

Encode each entry $p_i \in \{0, 1, 2\}$ as the number of moves at light A_i, so the total moves is

$$\|p\| \;=\; \sum_{i=1}^{24} p_i,$$

where $p_i = 2$ contributes two taps. Since each $p_i \le 2$, we have

$$\|p\| \;\le\; 2\cdot 24 \;=\; 48$$

for every $p \in (\mathbb{Z}/3\mathbb{Z})^{24}$. Thus no solution ever requires more than 48 moves.

It remains to show that 48 moves are actually needed for some configuration. Consider the press-vector

$$p^* = (2, 2, 2, \ldots, 2) \in (\mathbb{Z}/3\mathbb{Z})^{24},$$

which indeed has $\|p^*\| = 48$. Its action produces the right-hand side

$$M\,p^* \;=\; 2\,M\,\vec{\mathbf{1}} \;\equiv\; 2\cdot\big(2\,\vec{\mathbf{1}}\big) \;=\; 4\,\vec{\mathbf{1}} \;\equiv\; \vec{\mathbf{1}} \pmod 3,$$

because $M\,\vec{\mathbf{1}} = (2, 2, \ldots, 2)^\top$ on the 2×2 cube. Therefore if the initial colouring x^* satisfies

$$2\,\vec{\mathbf{1}} \;-\; x^* \;=\; M\,p^* \;\equiv\; \vec{\mathbf{1}} \pmod 3,$$

we get $x^* \equiv 2\,\vec{\mathbf{1}} - \vec{\mathbf{1}} = \vec{\mathbf{1}}$. That is, $x^* = (1, 1, \ldots, 1)$ is the *all-blue* configuration. Its unique solution vector is p^*, requiring exactly 48 moves.

Combining these observations, the maximum over all x of the minimal number of moves is $\|p^*\| = 48$. No configuration demands more, and the all-blue configuration demands exactly 48. This completes the proof. $\square$

5 Worst-Case Number of Moves for Arbitrary n

Question 5: Reconsider the previous question for arbitrary n?

Theorem 7: Let $n \geq 2$ and let

$$M \in \mathbb{Z}_3^{6n^2 \times 6n^2}$$

be the pressing-matrix for the $n \times n$ cube of lights, and suppose every initial configuration admits at least one solution to

$$x + M\,p \equiv 2\,\vec{\mathbf{1}} \pmod{3}.$$

Then the maximal number of moves required (where each tap counts as one move) is exactly

$$\max_{x} \min_{p\,:\,x+Mp=2\vec{1}} \sum_{i=1}^{6n^2} p_i = 12\,n^2.$$

Proof. Since each entry $p_i \in \{0,1,2\}$, the total number of taps $\sum_i p_i$ never exceeds $2 \cdot 6n^2 = 12n^2$.

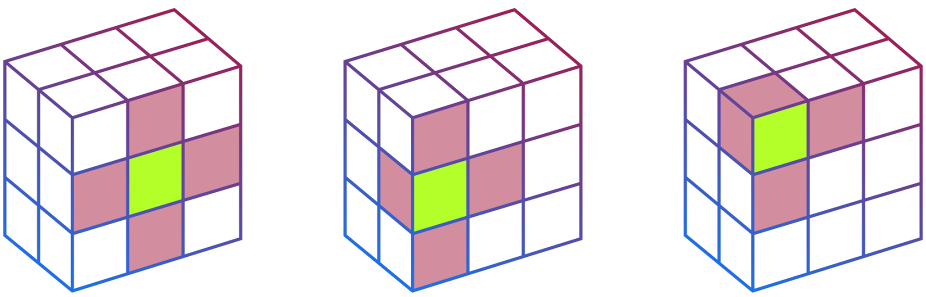
To show this bound is attained, let

$$p^* = (2,2,\ldots,2)^\top \in \mathbb{Z}_3^{6n^2},$$

so $\|p^*\| = \sum_i 2 = 12n^2$. We claim p^* is the unique solution for the all-blue configuration $x^* = (1,1,\ldots,1)^\top$.

First observe that for each light A_i on the cube:

- if A_i is an *interior* light on its face, it has 4 face-neighbors and 0 neighbors across cube edges;
- if A_i is on a *face-edge* but not a corner, it has 3 face-neighbors and 1 across-edge neighbor;
- if A_i is a *corner* light, it has 2 face-neighbors and 2 across-edge neighbors.



In each case the total number of presses that affect A_i when every entry of the vector $\vec{\mathbf{1}} = (1,1,\ldots,1)$ is pressed once is

$$1\text{ (itself)} + \{\text{face-neighbors}\} + \{\text{edge-neighbors}\} = 1+4+0 = 1+3+1 = 1+2+2 = 5.$$

Hence

$$M\,(1,1,\ldots,1)^\top \;=\; 5\,\vec{\mathbf{1}} \;\equiv\; 2\,\vec{\mathbf{1}} \pmod 3,$$

and therefore

$$M\,p^* \;=\; 2\,M\,(1,1,\ldots,1)^\top \;\equiv\; 2\cdot 2\,\vec{\mathbf{1}} \;=\; 4\,\vec{\mathbf{1}} \;\equiv\; \vec{\mathbf{1}} \pmod 3.$$

For the all-blue x^* we have $2\,\vec{\mathbf{1}} - x^* = \vec{\mathbf{1}}$, so indeed

$$M\,p^* \equiv 2\,\vec{\mathbf{1}} - x^*,$$

showing p^* solves $x^* + Mp = 2\vec{\mathbf{1}}$. By the invertibility of M, this solution is unique and uses $12n^2$ moves. Thus, the worst-case number of required moves is exactly $12n^2$. □

6 Number of Unsolvable Configurations

Question 6: How many unsolvable initial configurations are there as a function of n?

Theorem 8: Let $n \geq 2$ and let

$$M \;\in\; \mathbb{Z}_3^{6n^2 \times 6n^2}$$

be the pressing-matrix of the $n \times n$ cube of lights, and let

$$r(n) \;=\; \operatorname{rank}_{\mathbb{Z}_3}(M), \quad d(n) \;=\; 6n^2 - r(n)$$

be the rank and nullity of M. Then the number of initial colourings that *cannot* be solved to all green is

$$\#\{\, x \in \mathbb{Z}_3^{6n^2} \;:\; x + Mp \not\equiv 2\,\vec{\mathbf{1}} \text{ has no solution } p\} \;=\; 3^{6n^2} \;-\; 3^{r(n)} \;=\; 3^{6n^2}\,\bigl(1 - 3^{-\,d(n)}\bigr)$$

Proof. Every initial configuration $x \in \mathbb{Z}_3^{6n^2}$ is solvable precisely when the equation

$$x + M\,p \;\equiv\; 2\,\vec{\mathbf{1}} \pmod 3$$

has a solution $p \in \mathbb{Z}_3^{6n^2}$. Rewrite this as

$$M\,p \;\equiv\; 2\,\vec{\mathbf{1}} - x.$$

Since M has rank $r(n)$, its image $\operatorname{Im}(M) \subseteq \mathbb{Z}_3^{6n^2}$ is an $\mathbb{F}_3$–subspace of dimension $r(n)$, hence

$$\#\bigl(\operatorname{Im}(M)\bigr) \;=\; 3^{r(n)}.$$

Thus there are exactly $3^{r(n)}$ choices of right–hand sides $b = 2\vec{\mathbf{1}} - x$ for which $M\,p = b$ is solvable. Since the map $x \mapsto 2\vec{\mathbf{1}} - x$ is a bijection on $\mathbb{Z}_3^{6n^2}$, there are likewise $3^{r(n)}$ solvable initial configurations x.

The total number of all initial colourings is 3^{6n^2}, so the number of *unsolvable* ones is

$$3^{6n^2} - 3^{r(n)}.$$

Finally, since $d(n) = 6n^2 - r(n)$, we may factor

$$3^{6n^2} - 3^{r(n)} = 3^{6n^2}\left(1 - 3^{-d(n)}\right),$$

completing the proof. $\square$

7 Cellular Automaton Rule

Question 7: Now, instead of manually pressing the lights, the system evolves automatically according to a cellular automaton rule: a light remains unchanged unless it is influenced by at least four adjacent lights of the same colour.

- If a light is surrounded by at least four adjacent lights of the same colour, it advances in the cycle
$$\text{Red} \to \text{Blue} \to \text{Green} \to \text{Red}.$$
- If a light has exactly one red and one blue neighbour, it flips to the opposite state in the cycle.

a) For which initial conditions does the system evolve to a stationary configuration?

b) What is the longest time for the system to evolve to a periodic configuration?

a) Convergence to Stationary Configurations under the Cellular Automaton Rule

Theorem 9: Let the cube C_n have $6n^2$ lights, each taking a value in $\mathbb{Z}_3 = \{0, 1, 2\}$ (Red $= 0$, Blue $= 1$, Green $= 2$). Each light has exactly four face-neighbors and two edge-neighbors. Define the synchronous update

$$F\colon \mathbb{Z}_3^{6n^2} \longrightarrow \mathbb{Z}_3^{6n^2}$$

by

$$(F(x))_i = \begin{cases} x_i + 1 \pmod 3, & \text{if among the four face-neighbors of } i \text{ either} \\ & \quad \text{they are all equal, or exactly one is 0 and one is 1,} \\ x_i, & \text{otherwise.} \end{cases}$$

Then for every initial configuration $x^{(0)}$ there is a time $T \le 6n^2$ such that

$$x^{(T+1)} = x^{(T)}.$$

Moreover, the fixed points of F are exactly those configurations in which no light sees

1. 4 identical face-neighbors, and

2. exactly 1 red and 1 blue among its face-neighbors.

Proof. Define

$$A(x) \ = \ \big|\{\, i : (F(x))_i \neq x_i \}\big|.$$

Clearly $0 \le A(x) \le 6n^2$, and $A(x) = 0$ if and only if x is a fixed point. We show that

$$A\big(F(x)\big) \ < \ A(x) \quad \text{whenever } A(x) > 0,$$

which forces convergence in at most $6n^2$ steps.

Let

$$S \ = \ \{\, i : (F(x))_i \neq x_i \}, \quad S' \ = \ \{\, i : (F \circ F)(x)_i \neq (F(x))_i \}.$$

We will prove $S' \subseteq \{1, \dots, 6n^2\} \setminus S$, whence

$$|S'| \ \le \ 6n^2 - |S| \ < \ |S| \implies A\big(F(x)\big) < A(x).$$

(i) No $i \in S$ can lie in S'. Once i has flipped in $F(x)$, its four face-neighbors no longer satisfy either trigger condition.

(ii) No $j \notin S$ can enter S'. An inactive j (i.e. $(F(x))_j = x_j$) must have face-neighbor colours forming one of the following *allowed* multisets of counts $(\#0, \#1, \#2)$:

$$(3,1,0), \quad (2,2,0), \quad (2,1,1).$$

We check each pattern:

Case A: $(3,1,0)$ WLOG the colours are $(0,0,0,1)$. Label them (a,b,c,d). Let k be the number of zeros that flip $(0 \to 1,\ 1 \to 2)$. One checks:

- $k = 0,1,2,4$ produce multisets not of the form "all equal" nor "one 0 and one 1."
- $k = 3$ could yield $(1,1,1,1)$ only if the original 1 remains unflipped; but that 1 could not have been inactive in $(0,0,0,1)$, a contradiction.

Case B: $(2,2,0)$ WLOG the colours are $(0,0,1,1)$. Let m be the number of 0's that flip.

- $m = 0,1,3,4$ clearly avoid forbidden patterns.
- $m = 2$ could give $(1,1,1,1)$ only if both 1's remain unflipped, which contradicts their inactivity.

Case C: $(2,1,1)$ WLOG the colours are $(0,0,1,2)$ with flips $0 \to 1,\ 1 \to 2,\ 2 \to 0$. A similar enumeration excludes both "all equal" and "one 0 and one 1" without violating inactivity of any neighbor.

In all cases no inactive j can become active in $F(x)$. Thus $A(F(x)) < A(x)$, proving convergence in $\leq 6n^2$ steps. Finally, $A(x) = 0$ exactly when no cell meets a flip-trigger, i.e. precisely the fixed-point condition claimed. □

b) Maximum Transient Length before Periodicity

Theorem 10: Under the same hypotheses and notation as before, the synchronous automaton F on $\mathbb{Z}_3^{6n^2}$ reaches a periodic configuration (in fact a fixed point) in at most $6n^2$ steps. Equivalently, every orbit under F has transient length at most $6n^2$.

Proof. Recall the activity count

$$A(x) = \#\{\, i : (F(x))_i \neq x_i\},$$

which we showed strictly decreases as long as $A(x) > 0$. Since $0 \leq A(x) \leq 6n^2$ for all x, it follows that after at most

$$6n^2$$

applications of F one must have $A(x^{(T)}) = 0$. But $A(x^{(T)}) = 0$ precisely means $F(x^{(T)}) = x^{(T)}$, i.e. $x^{(T)}$ is a fixed point. Fixed points are periodic of period one, and no other periodic cycles occur. Hence the longest possible transient before entering a periodic orbit is exactly $6n^2$. □

8 Further Exploration

Question 8: Suggest and study other research directions.

a) Generalisation to m colours

i) Commutativity of Pressing for m colours

Theorem 11: Let $m \geq 2$ be an integer, and let C_n be the cube with $6n^2$ lights, each light's colour encoded by an element of the ring $\mathbb{Z}/m$. Number the lights $1, 2, \ldots, 6n^2$. For each light i, define the *press-operator* $T_i \colon (\mathbb{Z}/m)^{6n^2} \to (\mathbb{Z}/m)^{6n^2}$ by

$$T_i(x)_k = \begin{cases} x_k + 1 \pmod{m}, & \text{if } k = i \text{ or } k \sim i, \\ x_k, & \text{otherwise,} \end{cases}$$

where "$k \sim i$" means k is one of the four face-neighbors or two edge-neighbors of i. Then for any two presses i and j we have

$$T_i \circ T_j \;=\; T_j \circ T_i,$$

and more generally, if $p \in (\mathbb{Z}/m)^{6n^2}$ records how many times each light is pressed (modulo m), then the *net effect* of pressing each light i exactly p_i times (in any order) is the linear map

$$x \longmapsto x + M\,p \pmod{m},$$

where $M \in \{0,1\}^{6n^2\times 6n^2}$ is the adjacency matrix with $M_{k,i} = 1$ if and only if $k = i$ or $k \sim i$. In particular, pressing the same multiset of lights in any sequence yields the same final configuration.

Proof. We work in the free $\mathbb{Z}/m$-module $(\mathbb{Z}/m)^{6n^2}$. Let e_i denote the standard basis vector with a 1 in coordinate i and zeros elsewhere. Observe that the column $M_{\cdot,i}$ of M is exactly

$$M_{\cdot,i} = e_i \;+\; \sum_{j\sim i} e_j \quad \in (\mathbb{Z}/m)^{6n^2}.$$

By definition, applying the press operator T_i to a configuration x produces

$$T_i(x) \;=\; x \;+\; M_{\cdot,i} \pmod{m}.$$

Hence for any two distinct lights i and j,

$$T_i\big(T_j(x)\big) = T_i\big(x + M_{\cdot,j}\big) = x + M_{\cdot,j} + M_{\cdot,i} \pmod{m},$$

while

$$T_j\big(T_i(x)\big) = T_j\big(x + M_{\cdot,i}\big) = x + M_{\cdot,i} + M_{\cdot,j} \pmod{m}.$$

Since addition in $\mathbb{Z}/m$ is commutative,

$$x + M_{\cdot,j} + M_{\cdot,i} \;=\; x + M_{\cdot,i} + M_{\cdot,j},$$

we conclude

$$T_i \circ T_j \;=\; T_j \circ T_i.$$

In particular, regardless of the order in which one applies T_i and T_j, the final result is the same.

More generally, let $p = (p_1, \ldots, p_{6n^2}) \in (\mathbb{Z}/m)^{6n^2}$ record that we press light i a total of p_i times (where all arithmetic is modulo m). Writing a sequence of presses as $T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_k}$, one sees by repeated application that

$$T_{i_k} \circ \cdots \circ T_{i_2} \circ T_{i_1}(x) = x \;+\; \sum_{r=1}^{k} M_{\cdot,i_r} = x \;+\; \sum_{i=1}^{6n^2} p_i\, M_{\cdot,i} = x + M\,p,$$

where p_i counts how many times index i appears among $i_1, \ldots, i_k$. Commutativity of each pair T_i, T_j guarantees that any reordering of the same multiset of presses produces the same sum $\sum p_i M_{\cdot,i}$. Thus the *net effect* depends only on the total press-vector p and not on the order of presses.

Finally, the map

$$p \longmapsto x + M\,p$$

is plainly a $\mathbb{Z}/m$-linear map in the variable p. This establishes both the commutativity of individual presses and the overall linearity of the pressing process. Consequently, pressing the same set of lights in any sequence yields the same final state. $\square$

ii) Number of Unsolvable Initial Conditions with m colours

Theorem 12: Let $m \geq 2$ be an integer with prime factorization

$$m = \prod_{i=1}^{t} p_i^{e_i},$$

and let C_n be the cube with $6n^2$ lights, each coloured by an element of $\mathbb{Z}/m$. Fix a target colour $j \in \mathbb{Z}/m$. Let

$$M \in \{0,1\}^{6n^2 \times 6n^2}$$

be the adjacency matrix (with $M_{k,i} = 1$ iff light k is i itself or a neighbor of i). Then an initial colouring $x \in (\mathbb{Z}/m)^{6n^2}$ is solvable (i.e. there exists $p \in (\mathbb{Z}/m)^{6n^2}$ with $x + Mp \equiv j\vec{\mathbf{1}} \pmod{m}$) if and only if for each prime power $p_i^{e_i}$ the reduction $M \bmod p_i \in \mathbb{F}_{p_i}^{6n^2 \times 6n^2}$ has rank r_i. Consequently, the total number of *unsolvable* initial configurations is

$$\#\{\text{unsolvable}\} = m^{6n^2} - \prod_{i=1}^{t} p_i^{e_i r_i}.$$

Proof. An initial colouring $x \in (\mathbb{Z}/m)^{6n^2}$ is solvable exactly when the linear congruence

$$M\,p \equiv j\,\vec{\mathbf{1}} - x \pmod{m}$$

admits a solution $p \in (\mathbb{Z}/m)^{6n^2}$. By the Chinese Remainder Theorem,

$$(\mathbb{Z}/m)^{6n^2} \cong \prod_{i=1}^{t} (\mathbb{Z}/p_i^{e_i})^{6n^2},$$

and a vector $b \in (\mathbb{Z}/m)^{6n^2}$ lies in the image of M modulo m if and only if its reduction modulo each $p_i^{e_i}$ lies in the image of M over $\mathbb{Z}/p_i^{e_i}$. Moreover, reduction further to $\mathbb{Z}/p_i$ shows that the image modulo $p_i^{e_i}$ has size $p_i^{e_i r_i}$, where

$$r_i = \operatorname{rank}_{\mathbb{F}_{p_i}}\big(M \bmod p_i\big).$$

Hence the number of solvable right–hand-sides $j\vec{\mathbf{1}} - x$ is

$$\prod_{i=1}^{t} p_i^{e_i r_i},$$

and since $x \mapsto j\vec{\mathbf{1}} - x$ is a bijection of $(\mathbb{Z}/m)^{6n^2}$, the number of solvable initial x is the same. Therefore the number of unsolvable x is

$$m^{6n^2} - \prod_{i=1}^{t} p_i^{e_i r_i},$$

as claimed. □

Appendix. Calculate d(n).

The following code can be used to find nullity for arbitrary n in Question 6:

```
from sage.all import GF, Matrix
import matplotlib.pyplot as plt

# --- Define adjacency and helper functions ---
face_adj = {
    1: {'up':5,'down':6,'left':4,'right':2},
    2: {'up':5,'down':6,'left':1,'right':3},
    3: {'up':5,'down':6,'left':2,'right':4},
    4: {'up':5,'down':6,'left':3,'right':1},
    5: {'up':3,'down':1,'left':4,'right':2},
    6: {'up':1,'down':3,'left':4,'right':2},
}

def make_positions(n):
    return [(f, i, j)
            for f in range(1,7)
            for i in range(1, n+1)
            for j in range(1, n+1)]

def same_face_neighbors(pos, n):
    f, i, j = pos
    for di, dj in [(1,0),(-1,0),(0,1),(0,-1)]:
        ii, jj = i+di, j+dj
        if 1 <= ii <= n and 1 <= jj <= n:
            yield (f, ii, jj)

def edge_neighbor(pos, direction, n):
    f, i, j = pos
    f2 = face_adj[f][direction]
    if direction == 'up':    return (f2, n, j)
    if direction == 'down':  return (f2, 1, j)
    if direction == 'left':  return (f2, i, n)
    if direction == 'right': return (f2, i, 1)

def all_neighbors(pos, n):
    nbrs = set(same_face_neighbors(pos, n))
    for d in ['up','down','left','right']:
        nbrs.add(edge_neighbor(pos, d, n))
    return nbrs

def compute_stats(n):
    positions = make_positions(n)
    idx = {pos: k for k, pos in enumerate(positions)}
```

```
    dim = len(positions)
    M = Matrix(GF(3), dim, dim, sparse=True)
    for k, pos in enumerate(positions):
        M[k, k] += 1
        for q in all_neighbors(pos, n):
            M[idx[q], k] += 1
    rank = M.rank()
    nullity = dim - rank
    return nullity

# --- Compute and print for a range of n ---
results = []
for n in range(1, 201):
    null = compute_stats(n)
    results.append((n, null))
    print(f"n={n:2d}  nullity={null:3d}")

# --- List all n with zero nullity ---
zero_null_ns = [n for n, null in results if null == 0]
print("\nValues of n with zero nullity:", zero_null_ns)

ns    = [n for n, null in results]
nulls = [null for n, null in results]

plt.figure(figsize=(10, 7))
plt.plot(ns, nulls, marker='o')
plt.xlabel("Grid size n")
plt.ylabel("Nullity")
plt.title("Nullity vs. n")
plt.grid(True)
plt.tight_layout()
plt.show()
```