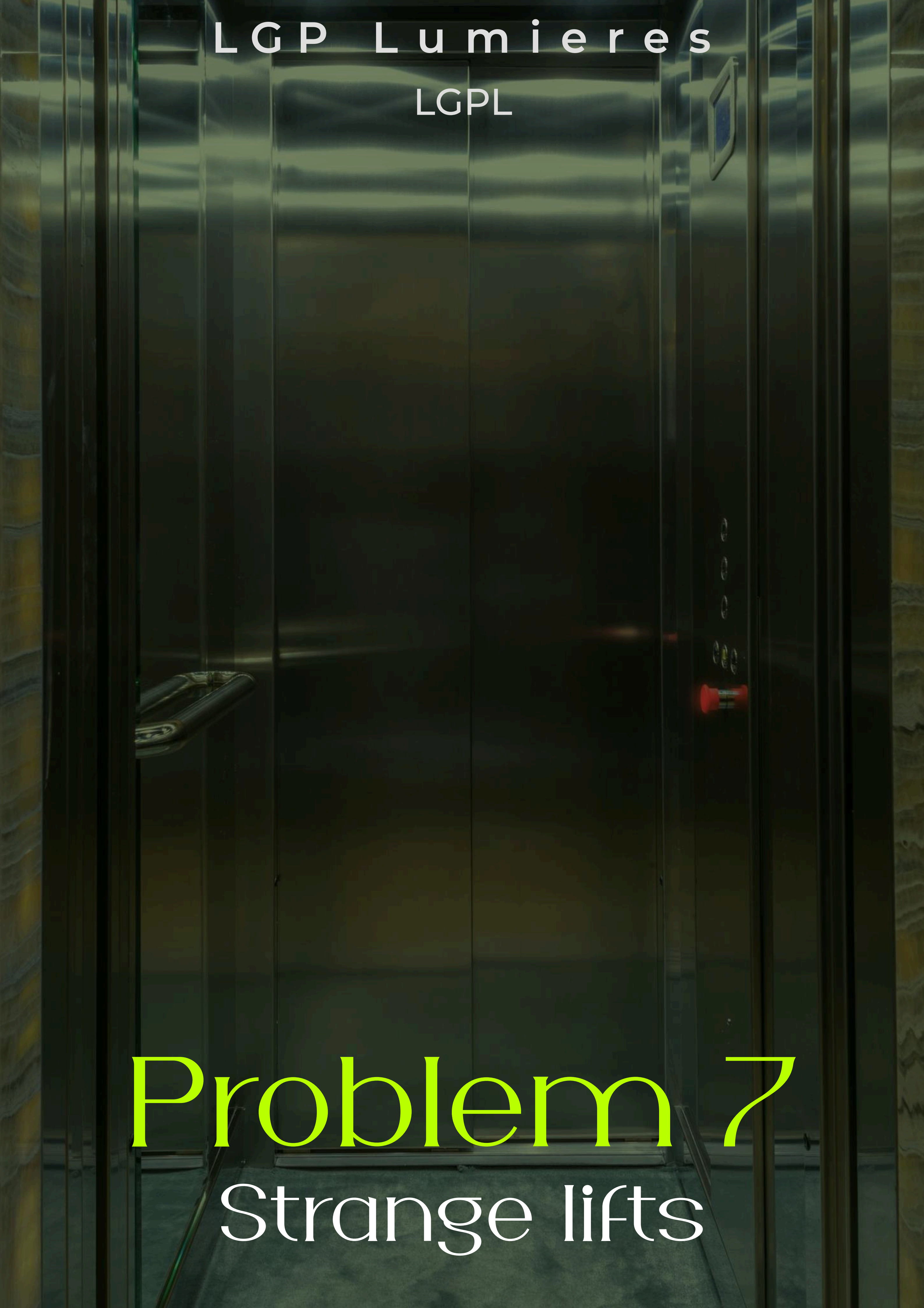

LGP Lumieres

LGPL

Problem 7

Strange lifts

PROBLEM 7

Strange Lifts

Summary

This problem is set in an infinitely tall building, with our protagonists Alice and Bob on the ground floor and the Nth floor, respectively. Alice can board a lift and go up a_i floors, where $a_i \in A$. The problem revolves around approximating the function $d_A(N)$, which measures the minimum number of steps Alice must take to reach Bob. We find global bounds on $d_A(N)$, approximate it for finite A, determine when it is unbounded, investigate $d_{A_t}(N)$, consider A as the set of Fibonacci numbers, and suggest and explore research directions.

Question	Result
1	solved
2	solved
3	solved
4	partially solved
5	partially solved
6	solved
7	solved

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Problem Statement: Imagine a building with infinitely many storeys. The storeys in this building are numbered by the non-negative integers in increasing order. Alice is standing on the zero-th storey of the building and she wants to meet with Bob, who lives on the $N-$ th storey for some $N \in \mathbb{N}$. Alice wants to use the building's lift, however there is one rule for using it. There is a set of numbers $A \subset \mathbb{N}$ with the only condition that $1 \in A$. Alice is allowed to go on the lift only from the $k-$ th storey to the $(k+a)-$ th storey for any $a \in A$.

First, let us rephrase $d_A(N)$ in the following way. Given $A = \{a_1, a_2, \dots a_n\}$, $1 = a_1 < a_2 < \cdots < a_n$, $N \in \mathbb{Z}^+$ such that

$$\sum_{i=1}^{n} a_i x_i = N$$

$d_A(N)$ is the minimum possible value of $x_1 + \cdots + x_n$

Definition 1): Throughout this document we will be using the term "greedy algorithm", which refers to a technique where you take the largest possible a_i at each step. If we determine that a set uses the greedy algorithm, we are done, since we have a formula for $d_A(N)$. Let $b_n = \left\lfloor \frac{N}{a_n} \right\rfloor$, and

$$b_i = \left\lfloor \frac{N - \sum_{k=i+1}^{n} a_k b_k}{a_i} \right\rfloor$$

Then, $d_A(N) = \sum\limits_{k=0}^{n} b_k$. (Here, a_n is the largest $a_i \leq N$.)

1 Bounds on $d_A(N)$

Question 1: Alice wants to estimate how long she will travel from storey to storey to get to Bob's apartment. She denotes the number of lift uses required to get to Bob's apartment as $d_A(N)$.

a) Can Alice always reach Bob's apartment? When it is possible to reach Bob's apartment, give a bound on $d_A(N)$ independent of A.

b) Decide whether $d_A(M+N)$ is always less than or equal to $d_A(M) + d_A(N)$.

(a) Upper bound independent of A

Since $1 \in A$, Alice can reach Bob by going up N steps one at a time. Clearly, $d_A(N) \leq N$ since we can always go N steps up one at a time. This bound is achieved when $a_2 > N$.

(b) Sub-additivity

Assume for the sake of contradiction $d_A(M+N) > d_A(M) + d_A(N)$. We can always go M floors up in $d_A(M)$ steps and N floors up in $d_A(N)$ steps. In this manner, we can go $M+N$ floors up in $d_A(M) + d_A(N)$ steps. This contradicts the minimality of $d_A(M+N)$.

Thus, $d_A(M+N) \leq d_A(M) + d_A(N)$

2 Approximating $d_A(N)$

Question 2: Alice understands that deriving a formula for $d_A(N)$ can be hard in many cases. Thus, in general she is interested in approximations for d_A as a function of N. She says that two functions $f, g : \mathbb{N} \to \mathbb{N}$ are approximately equal (or that f is approximately g) and writes it as $f \approx g$ if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\frac{1}{C}g(n) - C \leq f(n) \leq Cg(n) + c$$

Try to calculate d_A precisely or give an approximation in the following cases:

a) $A = \{1, a, b\}$ for $a, b \in \mathbb{N}$ arbitrary.

b) A is an arbitrary finite set.

c) $A = \{1\} \cup \{P(n)\}_{n \in \mathbb{Z}_{\geq 0}}$ where $P \in \mathbb{Z}[x]$ is a polynomial with non-negative integer coefficients.

(a) For $\{1, a, b\}$

Without loss of generality $a < b$. We claim that if $A = \{1, a, b\}$, $d_A(N) \approx \lfloor \frac{N}{b} \rfloor$ with $C = b$.

$$\left\lfloor \frac{N}{b} \right\rfloor \cdot \frac{1}{b} - b \leq \left\lfloor \frac{N}{b} \right\rfloor \leq d_A(N)$$

$$d_A(N) \leq N < b\left(\frac{N}{b}\right) + 1 \leq b\left\lfloor \frac{N}{b} \right\rfloor + b$$

So,

$$d_A(N) \approx \left\lfloor \frac{N}{b} \right\rfloor$$

.

Note

We can also have the better approximation $g(N) = \left\lfloor \frac{N}{b} \right\rfloor + \left\lfloor \frac{N - b\left\lfloor \frac{N}{b} \right\rfloor}{a} \right\rfloor$.

Proof: Notice that

$$\frac{1}{b}g(N) - b \leq g(N) \leq d_A(N) \leq N \leq b\left\lfloor \frac{N}{b} \right\rfloor + b \leq bg(N) + b$$

as desired. □

(b) A is finite

We claim $d_A(N) \approx \left\lfloor \frac{N}{a_n} \right\rfloor$ with $C = a_n$

$$\left\lfloor \frac{N}{a_n} \right\rfloor \cdot \frac{1}{a_n} - a_n \leq \left\lfloor \frac{N}{a_n} \right\rfloor \leq d_A(N)$$

$$d_A(N) \leq N \leq a_n \left\lfloor \frac{N}{a_n} \right\rfloor + a_n$$

So,

$$d_A(N) \approx \left\lfloor \frac{N}{a_n} \right\rfloor$$

Note

We can also have the better approximation. Let $b_n = \left\lfloor \frac{N}{a_n} \right\rfloor$, and

$$b_i = \left\lfloor \frac{N - \sum_{k=i+1}^{n} a_k b_k}{a_i} \right\rfloor$$

Then, $d_A(N) \approx \sum\limits_{k=1}^{n} b_k = g(N)$.

Proof: Notice that

$$\frac{1}{a_n} g(N) - a_n \leq g(N) \leq d_A(N) \leq N \leq b \left\lfloor \frac{N}{b} \right\rfloor + b \leq bg(N) + b$$

as desired. □

(c) $A = \{1\} \cup \{P(n) \mid n \geq 0\}$

We claim that $d_A(N) \approx 0$, or in other words, $d_A(N)$ is bounded above.

Assume for the sake of contradiction there exists P such that $d_A(N)$ for $A = \{1\} \cup \{P(n) \mid n \geq 0\}$ is unbounded.

Eric Kamke proved in 1921[1] that if $P(n)$ is a polynomial with no fixed divisor $d > 1$, then $\exists\, s$ such that

$$P(x_1) + P(x_2) + \cdots + P(x_t) = n$$

has a solution for large enough n and $t < s$.

Say n is large enough is $n \geq c$ for some constant c. For $n < c$, we have $d_A(n) \leq c - 1$ so $d_A(n)$ is bounded above.

Now, assume $n \geq c$.

Case 1: $d = 1$

Then, by the above theorem n can be represented in at most s steps, which is a contradiction.

Case 2: $d = a > 1$.

Let $Q(n) = \frac{P(n)}{a}$. Then, there a number s' such that every $n > c'$ can be represented in $\leq s'$ steps of size $Q(k)$. So, every multiple of a can be represented in s' moves at most.

Additionally, after going to the largest multiple of a less than n, we can reach n in at most $a - 1$ steps, contradicting our hypothesis.

Thus, $d_A(N) \approx 0$ for $A = \{1\} \cup \{P(n) \mid n \geq 0\}$.

3 $d_A(N) \not\approx 0$

Question 3: Alice wants to know for which sets A the value of $d_A(N)$ is bounded from above for all N.

a) Does there exist a function $h : \mathbb{N} \to \mathbb{N}$ such that if $A = \{1\} \cup \{x_1, x_2, \dots\}$ with $x_i \geq h(i)$ for all $i \in \mathbb{N}$, then $d_A \not\approx 0$?

b) Does there exist $p > 1$ such that the function d_A for $A = \{1\} \cup \{\lfloor p^n \rfloor | n \geq 1\}$ is not approximately 0?

c) Try to characterize all sets A for which $d_A(N)$ is bounded from above for all N.

(a) d_A is unbounded

Notice that $f(n) \approx 0 \implies -C \leq f(n) \leq C$

Note

Claim: $h(n) = n!$ satisfies this property.

Proof: Clearly, $d_A(N) \geq \left\lfloor \frac{N}{a_n} \right\rfloor$, where $a_n \leq N < a_{n+1}$

Notice that $a_{n+1} - 1$ lies in that range. So, if $\left\lfloor \frac{a_{n+1}-1}{a_n} \right\rfloor$ is unbounded, then $d_A(N)$ is also unbounded.

$\left\lfloor \frac{a_{n+1}-1}{a_n} \right\rfloor \leq c \implies \frac{a_{n+1}}{a_n} < c + 2 = C$

We know $\frac{a_2}{a_1} < C, \frac{a_3}{a_2} < C, \dots, \frac{a_n}{a_{n-1}} < C$.

Multiplying gives $\frac{a_n}{a_1} < C^{n-1}$, so $\left(\frac{a_n}{a_1}\right)^{\frac{1}{n-1}}$ is bounded above.

Note

Claim: $\lim\limits_{n \to \infty} \left(\frac{n!}{a_1}\right)^{\frac{1}{n-1}} = \infty$

Proof:

It is well known that $n! \geq \left(\frac{n}{e}\right)^n$. Thus,

$$\lim_{n\to\infty}\left(\frac{n!}{a_1}\right)^{\frac{1}{n-1}} \geq \lim_{n\to\infty}\left(\frac{n^n}{e^n \cdot a_n}\right)^{\frac{1}{n-1}} = \lim_{n\to\infty}\left(\frac{1}{a_n}\right)^{\frac{1}{n-1}} \cdot \left(\frac{n}{e}\right)^{1+\frac{1}{n-1}} = \lim_{x\to\infty}\left(\frac{n}{e}\right)\lim_{x\to\infty}\left(\frac{n}{e \cdot a_n}\right)^{\frac{1}{n-1}} = \infty$$

Now, $\left(\frac{a_n}{a_1}\right)^{\frac{1}{n-1}} \geq \left(\frac{n!}{a_1}\right)^{\frac{1}{n-1}}$ which is unbounded by our claim. However, we know $\left(\frac{a_n}{a_1}\right)^{\frac{1}{n-1}}$ is bounded above.

$\Rightarrow\Leftarrow$

So, $d_A(N)$ is unbounded as desired. $\square$

Note

Notice that any function of the form $h(1) = c \geq 1$, $h(n) = f(n)h(n-1) + k$ for $k > 1$ works in analogous fashion, where f is a linear polynomial of the form $an + b$, $a \geq 1$.

(b) $d_{A_t} \not\approx 0$

Definition 2): $A_t = \{\lfloor t^n \rfloor \mid n \geq 0\}$

We claim that there are infinitely many p that satisfy this property. In fact, all $p : p \geq 2, p \in \mathbb{N}$ satisfy this property.

As proved in $4a$), every natural number has a unique base p representation for $p \in \mathbb{N}$.

So, consider the number $p^k - 1 = (p-1)p^{k-1} + (p-1)p^{k-2} + \cdots + (p-1)$.

$d_A(N) = k(p-1) \to \infty$ as $k \to \infty$. So, d_{A_p} is unbounded as desired.

(c) When is $d_A(N)$ bounded above

We can rephrase this problem as finding all additive bases of finite magnitude with element 1. By Schnirelmann's theorem[3], this is only possible when $\sigma A = \inf\limits_{n\geq 1} \dfrac{A(n)}{n} > 0$.

More specifically, a set with Schnirelmann density ε is an additive basis of order at most $\left\lceil \frac{1}{\varepsilon} \right\rceil$.

4 Exploring d_{A_t}

Question 4: For a real number $t \geq 1$ denote $A_t = \{1\} \cup \{\lfloor t^n \rfloor \mid n \geq 1\}$. Alice finds sets A_t intriguing so she wants to understand more about different speeds of approaching Bob for them. Try to calculate d_A precisely or give an approximation in the following cases:

a) $A = A_2$ and $A = A_3$.
b) $A = A_t$ for $t \in \mathbb{N}$
c) $A = A_t$ for any $t \in \mathbb{R} \geq 1$.

We start by proving an important theorem.

Note

Claim: Every integer has a unique base k representation, for $k \in \mathbb{Z}^+$.

Proof: Assume for the sake of contradiction there is a number that can be represented as both

$$N = \sum_{i=0}^{n} a_i b^i = \sum_{i=0}^{m} c_i b^i$$

.

For the base case, note $a_0 = c_0$ since they are just the remainder when N is divided by b.

Now, say $c_k = a_k$, $c_{k+1} = a_{k+1}$, $\dots$, $c_0 = a_0$.

$$\sum_{i=k+1}^{n} a_i b^i = \sum_{i=k+1}^{m} a_i b^i$$

.

$$\sum_{i=0}^{n-k-1} a_{k+i+1} b^i = \sum_{i=1}^{m-k-1} a_{k+i+1} b^i = M$$

.

So, a_{k+1} is just the remainder when M is divided by b.

$\Rightarrow\Leftarrow$

(a) d_{A_2} and d_{A_3}

For $A = \{1, 2, 4, \dots\}$, we can always put 2^{i+1} instead of $2 \cdot 2^i$. So, we can have atmost 1 of each power of 2.

This is just the sum of the digits of the base 2 representation of a number, which is

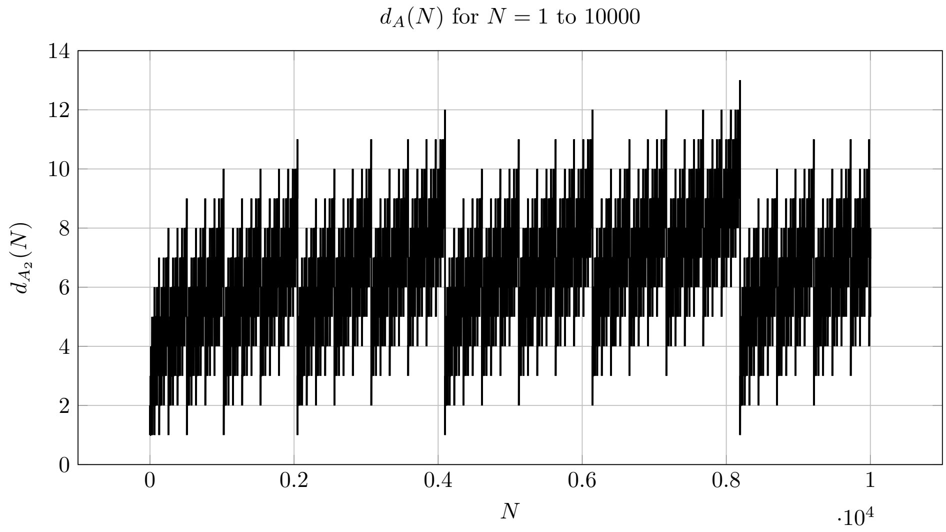
$$N - \sum_{i=1}^{\infty} \left\lfloor \frac{N}{2^i} \right\rfloor$$

For $A = \{1, 3, 9, \dots\}$, we can always put 3^{i+1} instead of $3 \cdot 3^i$. So, we can have atmost 2 of each power of 3.

This is just the sum of the digits of the base 3 representation of a number.

(b) d_{A_t} for $t \in \mathbb{N}$

For $A = \{1, t, t^2, \dots\}$, we can always put t^{i+1} instead of $t \cdot t^i$. So, we can have atmost $t - 1$ of each power of t.



This is just the sum of the digits of the base t representation of a number.

(c) Approximation for d_{A_t}

5 Comparing d_{A_t}

Question 5: Can you find an infinite set $T \subseteq \mathbb{R} \geq 1$ such that for any $t, t' \in T$ with $t \neq t'$ we have $d_{A_t} \neq d_{A_{t'}}$. Can you find such an uncountable set T?

Definition 3): Two numbers are said to be multiplicatively independent if their only common integer power is 1.

We claim that any set of multiplicatively independant numbers satisfies this property.

To prove this, we just need that if $p \neq q$ (WLOG $p > q$), $d_{A_p} \not\approx d_{A_q}$

For the sake of contradiction assume that $\exists\, C$ such that

$$\frac{1}{C} d_{A_p}(N) - C \leq d_{A_q}(N) \leq C d_{A_p}(N) + C$$

Then,

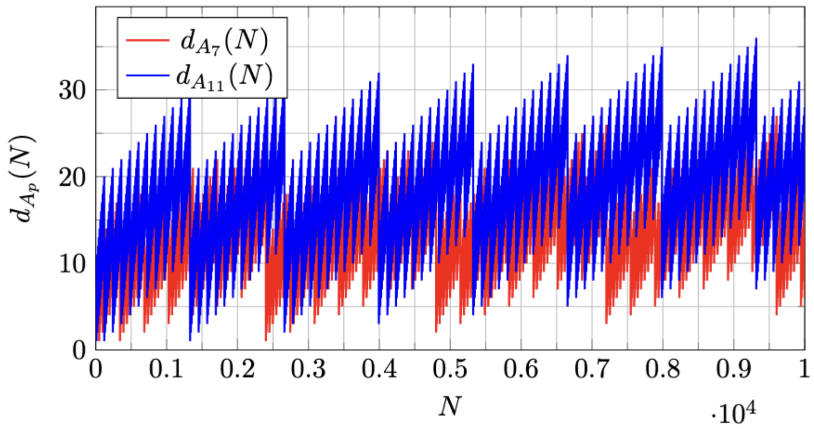
$$\frac{1}{C} - \frac{C}{d_{A_p}(N)} \leq \frac{d_{A_q}(N)}{d_{A_p}(N)} \leq C + \frac{C}{d_{A_p}(N)}$$

Let $N = p^n$

Thus, $d_{A_q}(p^n) \leq 2C \ \forall\, n$

By the Senge-Strauss Theorem [2], the number of integers the sum of whose digits in each of the bases p and q lies below a fixed bound is finite if and only if p and q are multiplicatively independent.

So, $d_{A_q}(p^n)+d_{A_p}(p^n)=d_{A_q}(p^n)+1\leq 2C+1$ for finitely many n. However, there are infinite choices for n, contradiction.



6 Set of Fibonacci numbers

Question 6: Can you calculate or approximate d_A for some other sets A? For example, for the set of Fibonacci numbers $A=\{1,2,3,5,8,13,\dots\}$.

Notice that we can replace $3f_n$ with $f_{n-2}+f_{n+2}$, $3f_1$ with 3 and $3f_2$ with f_1+f_4, so we cannot have more than 2 of each Fibonacci number.

Say $N=c_nf_n+c_{n-1}f_{n-1}+\cdots+c_1f_1$, where $c_i\in\{0,1,2\}$. If $2f_n$ comes in the decomposition for $n\geq 2$, we can replace it with $f_{n-2}+f_{n+1}$.

Lastly, we can replace f_n+f_{n-1} with f_{n+1}. All of these operations either decrease or preserve the number of times a lift is taken.

So, we can rewrite this as the number of ways to write N with no consecutive Fibonacci numbers and no number repeated.

Theorem (Zeckendorf, with a few edits to use the greedy algorithm): Every integer has a unique representation as a sum of non-consecutive distinct Fibonacci numbers, which can be obtained using the greedy algorithm. (Here, f_1 and f_2 are not considered distinct.)

Proof: We divide this into 4 parts.

Claim 1: $f_3+\cdots+f_{2n-1}=f_{2n}-1$

Proof: $f_3=2=f_4-1$.

Now, say $f_3 + \cdots + f_{2k-1} = f_{2k} - 1$

$f_3 + \cdots + f_{2k-1} + f_{2k+1} = f_{2k} + f_{2k+1} - 1 = f_{2k+2} - 1.$

Claim 2: $f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$

Proof: $f_2 = 1 = f_3 - 1.$

Now, say $f_2 + f_4 + \cdots + f_{2k} = f_{2k+1} - 1$

$f_2 + f_4 + \cdots + f_{2k} + f_{2k+2} = f_{2k+1} + f_{2k+2} - 1 = f_{2k+3} - 1.$

Claim 3: Every number can be represented as a sum of non-consecutive Fibonacci numbers using the greedy algorithm.

Proof: We proceed by induction. Clearly, $1 = f_1$ can be represented using the greedy algorithm. Now, say $1, 2, \ldots, k$ can be represented.

If $k + 1 = f_n$, then there is nothing to prove.

Otherwise, $\exists\ n$ such that $f_n < k + 1 < f_{n+1}$. Then, $k + 1 = f_n + (k + 1 - f_n)$, and $k + 1 - f_n < f_{n+1} - f_n = f_{n-1}$.

So, since $k + 1 - f_n$ has a unique representation using the greedy algorithm without involving f_n, we are done.

Claim 4: This representation is unique.

Say $\sum\limits_{x \in X} f_x = \sum\limits_{y \in Y} f_y$

Let $X - Y = A$, $Y - X = B$. Then,

$$\sum_{x \in A} f_x = \sum_{y \in B} f_y$$

However, say $\max A = v$ and $\max B = u$. Then, if WLOG $v > u$ we get a contradiction by Claim 1 and Claim 2 since we can't reach f_v with the sum of non consecutive Fibonacci numbers less that f_v.

Combining our claims, we get that every integer has a unique representation as the sum of non consecutive Fibonacci numbers using the greedy algorithm as desired. □

Definition 1): $h_0 = 0$, $h_1 = 1$ and $h_i = kh_{i-1} + h_{i-2}$

Now, let us consider the set $\{1, h_2, h_3, \ldots, h_n\}$. Notice that if we have kh_n and h_{n-1}, we can replace it with h_{n+1}. So, if the coefficient of h_i is ever k, the coefficient of h_{i-1} is 0.

Additionally, we can have at most $(k-1)h_1$ since $kh_1 = h_2$.

V.E Hogatt Junior wrote a paper describing such sequences. According to his work, every integer can be represented uniquely in this manner. In an analogous fashion to the proof in the Fibonacci sequence question, we get that this too uses the greedy algorithm, so we are done.

7 Research directions

Question 7: Suggest and study other research directions.

(a) Inserting a down button

One interesting research direction is the study of $d_A(N)$ when we are also allowed to go down a_i steps. Call this new function $d'_A(N)$

Notice that if we go a_i floors up and a_i floors down, then the total displacement is 0. So, either we go up a_i stories or down.

For a finite set, we claim $d'_A(N) \approx \frac{N}{a_n}$, where $a_n = \max(A)$.

This follows since

$$\frac{1}{a_n}\left\lfloor\frac{N}{a_n}\right\rfloor - a_n \leq \left\lfloor\frac{N}{a_n}\right\rfloor \leq d'_A(N) \leq N \leq a_n\left\lfloor\frac{N}{a_n}\right\rfloor + a_n$$

Now, let's consider $A = \{b^k \mid k \geq 0\}$ for some $b \in \mathbb{N}$

Notice that if we have $k \cdot b^n$ in our representation for $k > \left\lfloor\frac{b}{2}\right\rfloor$, we can replace it with $b^{n+1} - (b-k)\cdot b^n$. So, we can rephrase the problem with $-\left\lfloor\frac{b-1}{2}\right\rfloor \leq x_i \leq \left\lfloor\frac{b}{2}\right\rfloor$

This is known as the balanced representation of a number in base b.

Note

Claim: Every number has a unique balanced base b representation for all $b \in \mathbb{N}$

Proof: Assume for the sake of contradiction that there is an

$$N = \sum_{i=0}^{n} a_i b^i = \sum_{i=0}^{n} c_i b^i$$

where $-\left\lfloor\frac{b-1}{2}\right\rfloor \leq a_i, c_i \leq \left\lfloor\frac{b}{2}\right\rfloor$

Taking $\pmod b$ on both sides gives $a_0 \equiv c_0 \pmod b \implies a_0 = c_0$.

Now, assume $a_0 = c_0,\ a_1 = c_1,\ \ldots,\ a_k = c_k$.

$$N = \sum_{i=k+1}^{n} a_i b^i = \sum_{i=k+1}^{n} c_i b^i$$

$$N = \sum_{i=0}^{n-k-1} a_{i+k+1} b^i = \sum_{i=k+1}^{n-k-1} c_{i+k+1} b^i$$

Taking $\pmod b$ on both sides gives $a_{k+1} = c_{k+1}$, completing our proof.

Thus, for $A = A_b$, $d'_A(N)$ is just the sum of the digits of the balanced base b representation of N.

Additionally, If $a_i \geq i! \ \forall\ i$ then $d_A(N)$ is unbounded. The proof is the same as $3a$).

(b) Unexplored research directions

While researching this topic, I came across several interesting research directions that I couldn't explore. Here are 2 of them:

1 Given a function $f(n)$, when can you find a set A such that $d_A \approx f$.

2 Given a set A, consider all subsets B such that $1 \in B$. Find or approximate the average of $d_A(N)$ over all such subsets B.

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