LGP Lumieres

LGPL

Problem 6

Integer Polygons

PROBLEM 6

Integer Polygons

Summary

In this problem, we investigate the combinatorial and geometric properties of integer polygons, beginning with triangles that have natural integer side lengths bounded by a parameter N. Focusing on how many distinct triangles (up to rotation and reflection) can be formed, we analyze special cases, such as isosceles and acute triangles, and study how their proportions change as N increases. The project extends to considering triangles up to similarity (including scaling), and further generalizes to convex n-gons with integer sides. Throughout, we use combinatorial and geometric methods to classify these shapes and investigate their arrangements, connecting number theory with discrete geometry.

Question	Result
1	solved
2	mostly solved
3	solved
4	mostly solved
5	solved
6	mostly solved
7	solved
8	solved
9	unsolved

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Problem Statement: Let N be a positive integer. We consider the set TN of triangles with natural integer side lengths a, b, c which are at most N, and up to rotations and reflections. This means that two triangles are considered the same when they are isometric.

1 Exploring T_5

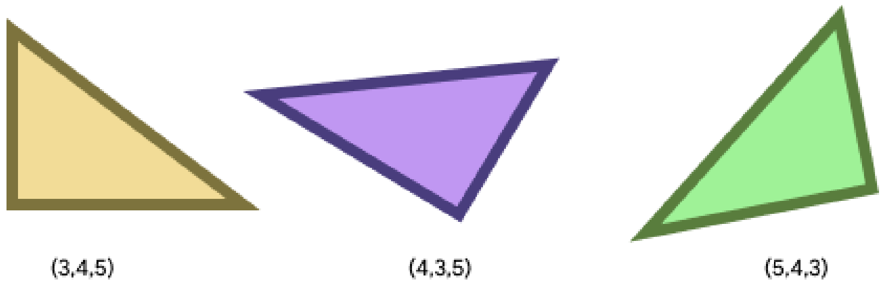
Question 1: How many elements are in T_5? Among them how many are
a) isosceles?
b) right triangle?
c) acute?

Given, T_N is the set of all triangles with integer side lengths $a, b, c \in \mathbb{N}$ such that $0 \le a, b, c \le N$ and the triple (a, b, c) satisfies the triangle inequality theorem.

Definition 1 (Triangle Inequality): Let $a, b, c \in \mathbb{N}$ be the lengths of the sides of a triangle. Then the following inequalities hold:

$$a + b > c, \quad b + c > a, \quad a + c > b.$$

Furthermore, congruent triangles are considered equivalent; that is, permutations of the side lengths do not distinguish different triangles. For example, (3,4,5), (4,3,5) and (5,4,3) represent the same triangle.



Without loss of generality, we assume $c \le b \le a$, since triangles are considered up to isometry.

Definition 2 Let
$$T_N = \left\{(a, b, c) \in \mathbb{N}^3 : 1 \le c \le b \le a \le N,\ a < b + c\right\}.$$

Definition 3 For each $n \in \mathbb{N}$, the set t_n consists of all triples $(a, b, c) \in T_N$ such that $a = n$. That is,
$$t_n = \{(a, b, c) \in T_N : a = n\}.$$

It is evident that $t_N \subseteq T_N$. Next, let F_N denote the number of elements in T_N; that is, $|T_N| = F_N$. Similarly, f_N denotes the number of elements in t_N; that is, $|t_N| = f_N$.

Thus, it is apparent that,
$$F_N = f_1 + f_2 + \cdots + f_N = \sum_{i=1}^{n} f_i$$

N	t_N	f_N	F_N
1	$(1,1,1)$	1	1
2	$(2,2,2),(2,2,1)$	2	3
3	$(3,3,3),(3,3,2),(3,3,1),(3,2,2)$	4	7
4	$(4,4,4),(4,4,3),(4,4,2),(4,4,1),(4,3,3),(4,3,2)$	6	13
5	$(5,5,5),(5,5,4),(5,5,3),(5,5,2),(5,5,1),(5,4,4),(5,4,3),(5,4,2),(5,3,3)$	9	22

Therefore, number of elements in $|T_5| = F_5 = 1+3+4+6+9 = 22$.

1.1 Isosceles Triangle

Since $1 \le c \le b \le a \le N$, a triangle defined by T_N is isosceles if $a = b$ or $b = c$. The case $a = c$ implies $a = b = c$, which corresponds to an equilateral triangle and is already covered by conditions a = b or b = c.

Let $I_N \subseteq T_N$ be the subset of all isosceles triangles in T_N; that is,

$$I_N = \{(a,b,c) \in t_N : a = b \ \ or \ b = c\}$$

Therefore,

$$I_5 = \left\{\begin{matrix} (1,1,1), & (2,2,2), & (2,2,1), & (3,3,3), & (3,3,2), \\ (3,3,1), & (3,2,2), & (4,4,4), & (4,4,3), & (4,4,2), \\ (4,4,1), & (4,3,3), & (5,5,5), & (5,5,4), & (5,5,3), \\ (5,5,2), & (5,5,1), & (5,4,4), & (5,3,3) & \end{matrix}\right\} = T_5 - \{(4,3,2),(5,4,3)\}$$

Thus, the number of isosceles triangles in T_5 is,

$$|I_5| = 19$$

1.2 Right Triangle

Let $R_N \subseteq T_N$ be the subset of all right triangles in T_N; that is,

$$R_N = \{(a,b,c) \in T_N : a^2 = b^2 + c^2\}$$

Therefore,

$$R_5 = \{(5,4,3)\}$$

Thus, the number of right angled triangle in T_5 is,

$$|R_5| = 1$$

1.3 Acute Triangle

For a triangle in T_n to be acute, the angle opposite to its largest side (here a) needs to be between 0 to 90 degrees i.e. the angle between side b and c needs to be between 0 and 90 degrees. Thus, we can find an inequality using cosine law.

Definition 4 (Law of Cosines): Let $\triangle PQR$ be a triangle with side lengths a, b, c in T_N, where a is opposite the angle θ. Then,

$$a^2 = b^2 + c^2 - 2bc\cos\theta.$$

Therefore, if $\triangle PQR$ is acute, then it implies that $(b - c) < a < \sqrt{b^2 + c^2}$.

Proof. When $\triangle PQR$ is acute, $\theta \in (0, \frac{\pi}{2})$. Thus,

$$0 < \cos\theta < 1$$

$$-1 < -\cos\theta < 0$$

Since $b, c > 0$,

$$-2bc < -2bc\cos\theta < 0$$

$$b^2 + c^2 - 2bc < b^2 + c^2 - 2bc\cos\theta < b^2 + c^2$$

$$(b - c)^2 < a^2 < b^2 + c^2$$

$$(b - c) < a < \sqrt{b^2 + c^2}$$

This completes the proof. □

Let $A_N \subseteq T_N$ be the subset of all acute triangles in t_N; that is,

$$A_N = \{(a, b, c) \in T_N : (b - c) < a < \sqrt{b^2 + c^2}\}$$

Thus, the number of acute triangle in A_5 is,

$$|A_5| = 17$$

The table below summarizes our results for the first 5 values of n:

N	T_N	I_N	R_N	A_N
1	1	1	0	1
2	3	3	0	3
3	7	7	0	6
4	13	12	0	10
5	22	19	1	17

2 Elements in T_N

Question 2: How many elements are in T_N for general N?

We compute the first few values of t_a for $a \in \{1, 2, 3, 4, 5, 6\}$:

$$|t_1| = 1 = 1 \times 1,$$
$$|t_2| = 2 = 1 \times 2,$$
$$|t_3| = 4 = 2 \times 2,$$
$$|t_4| = 6 = 2 \times 3,$$
$$|t_5| = 9 = 3 \times 3,$$
$$|t_6| = 12 = 3 \times 4.$$

Observation: For $a \in \mathbb{N}$, the sequence t_a satisfies the following pattern:

- If a is odd, then t_a is a perfect square.
- If a is even, then t_a is a pronic number (i.e., the product of two consecutive integers).

This observation suggests that the formula for the cardinality of t_N naturally splits according to the parity of N, and we proceed by analyzing both cases separately.

Theorem 1

$$t_N = \begin{cases} k^2, & \text{if } N = 2k-1, \\ k(k+1), & \text{if } N = 2k, \end{cases} \quad \text{for } k \in \mathbb{Z}^+.$$

Proof. **Case 1:** When N is odd; that is, $N = 2k - 1$ for some k in $\mathbb{Z}^+$,

$$t_N = \left\{ \begin{array}{lll} (N, N, x) & | \quad x \in [1, N] & \Rightarrow N \text{ elements}, \\ (N, N-1, x) & | \quad x \in [2, N-1] & \Rightarrow N-2 \text{ elements}, \\ (N, N-2, x) & | \quad x \in [3, N-2] & \Rightarrow N-4 \text{ elements}, \\ \vdots & & \\ \left(N, \dfrac{N+1}{2}, x\right) & | \quad x = \dfrac{N+1}{2} & \Rightarrow 1 \text{ element}. \end{array} \right\}$$

Hence,

$$|t_N| = N + (N-2) + (N-4) + \cdots + 1 = \underbrace{(N) + (N-2) + \cdots + 1}_{\frac{N+1}{2} \text{ terms}}.$$

$$|t_N| = \frac{\frac{N+1}{2}}{2} \cdot (1 + N) = \left(\frac{N+1}{2}\right)^2.$$

Substituting $N = 2k - 1$,

$$\boxed{|t_{2k-1}| = k^2}$$

Case 2: When N is even, that is, $N = 2k$ for some k in $\mathbb{Z}^+$,

$$t_N = \begin{Bmatrix} (N, N, x) & | & x \in [1, N] & \Rightarrow N \text{ elements}, \\ (N, N-1, x) & | & x \in [2, N-1] & \Rightarrow N-2 \text{ elements}, \\ (N, N-2, x) & | & x \in [3, N-2] & \Rightarrow N-4 \text{ elements}, \\ \vdots & & & \\ \left(N, \frac{N}{2}+1, x\right) & | & x \in \left[\frac{N}{2}, \frac{N}{2}+1\right] & \Rightarrow 2 \text{ elements}. \end{Bmatrix}$$

Hence,

$$|t_N| = N + (N-2) + (N-4) + \cdots + 2 = \underbrace{(N) + (N-2) + \cdots + 2}_{\frac{N}{2} \text{ terms}}.$$

$$|t_N| = \frac{\frac{N}{2}}{2} \cdot (2 + N) = \frac{N(N+2)}{4}.$$

Substituting $N = 2k$,

$$\boxed{|t_{2k}| = k(k+1)}$$

□

From the earlier definition of t_N, we obtain:

$$|T_N| = t_1 + t_2 + t_3 + \cdots + t_N = \sum_{a=1}^{N} |t_a|$$

Theorem 2 Let T_N denote the number of non-congruent triangles with side lengths $a, b, c \in \mathbb{N}$ satisfying $1 \le c \le b \le a \le N$ and the triangle inequality $a < b + c$. Then,

$$T_N = \begin{cases} \dfrac{(N+1)(N+3)(2N+1)}{24}, & \text{if } N \text{ is odd}, \\ \dfrac{N(N+2)(2N+5)}{24}, & \text{if } N \text{ is even}. \end{cases}$$

Proof. When N is odd; that is, $N = 2k - 1$ for some k in $\mathbb{Z}^+$,

$$T_N = \underbrace{t_1 + t_3 + \cdots + t_{2k-1}}_{\text{odd } a \Rightarrow t_{2i-1} = i^2} + \underbrace{t_2 + t_4 + \cdots + t_{2k-2}}_{\text{even } a \Rightarrow t_{2i} = i(i+1)} = \sum_{i=1}^{k} i^2 + \sum_{i=1}^{k-1} i(i+1)$$

We already know,

$$\sum_{j=1}^{m} j^2 = \frac{m(m+1)(2m+1)}{6}$$
$$\sum_{i=1}^{m} j(j+1) = \sum_{i=1}^{m} j^2 + \sum_{j=1}^{m} j = \frac{m(m+1)(m+2)}{3}$$

Thus,

$$T_N = \frac{k(k+1)(2k+1)}{6} + \frac{(k-1)((k-1)+1)((k-1)+2)}{3}$$
$$T_N = \frac{k(k+1)(2k+1)}{6} + \frac{k(k-1)((k+1)}{3}$$
$$T_N = \frac{k(k+1)(4k-1)}{6}$$

Substituting back $k = \frac{N+1}{2}$,

$$T_N = \frac{(N+1)(N+3)(2N+1)}{24}$$

When N is even; that is, $N = 2k$ for some $k \in \mathbb{Z}^+$,

$$T_N = \underbrace{t_1 + t_3 + \cdots + t_{2k-1}}_{\text{odd } a \Rightarrow t_{2i-1} = i^2} + \underbrace{t_2 + t_4 + \cdots + t_{2k}}_{\text{even } a \Rightarrow t_{2i} = i(i+1)} = \sum_{i=1}^{k} i^2 + \sum_{i=1}^{k} i(i+1)$$

Thus,

$$T_N = \frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)(k+2)}{3}$$
$$T_N = \frac{k(k+1)(4k+5)}{2}$$

Substituting back $k = \frac{N}{2}$,

$$T_N = \frac{N(N+2)(2N+5)}{24}$$

□

For isometry consideration, Let us assume that

$$c \leq b \leq a$$

The Python code below calculates how many elements are in T_N (the set of valid triangles) for a general N:

```python
def count_valid_triangles(N):
    count = 0
    for a in range(1, N + 1):
        for b in range(a, N + 1):
            for c in range(b, N + 1):
                if a + b > c:
                    count += 1
```

```
    return count

# Try for different values of N
for n in range(1, 11):
    print(f"t_{N} =", count_valid_triangles(n))
```

Listing 1: Counting valid triangles for general T_N

3 Proportions of Acute Triangles in T_N

Question 3: What is the proportion of acute triangles within elements of T_N? How does this quantity behave when N goes to infinity?

Definition 5

- A triangle with sides a, b, c (natural numbers) is *acute* if and only if:

$$a^2 + b^2 > c^2, \quad a^2 + c^2 > b^2, \quad b^2 + c^2 > a^2.$$

- Without loss of generality, assume $a \le b \le c$ for all triangles in T_N.
- The triangle inequality $a + b > c$ must hold.

3.1 Total Number of Triangles in T_N

The total number of non-congruent triangles in T_N is the number of integer triples (a, b, c) with $1 \le a \le b \le c \le N$ and $a + b > c$. This is given by:

$$T(N) = \sum_{c=1}^{N} f(c), \quad \text{where} \quad f(c) = \left\lfloor \frac{(c+1)^2}{4} \right\rfloor$$

Asymptotically,

$$T(N) \sim \frac{N^3}{12} \quad \text{as} \quad N \to \infty.$$

We now derive $f(c)$

Let us fix a positive integer c, and consider all triangles with integer side lengths a, b, c such that:

- $1 \le a \le b \le c$ (we fix c as the largest side),
- $a + b > c$ (triangle inequality).

We aim to count the number of such triangles. Denote this number by $f(c)$.

Total Number of Pairs Without Triangle Inequality Without considering the triangle inequality $a + b > c$, the number of integer pairs (a, b) such that $1 \le a \le b \le c$ is:

$$\sum_{b=1}^{c} b = \frac{c(c+1)}{2}$$

Restrict to Pairs Satisfying $a+b>c$ Now we apply the triangle inequality condition. We seek the number of integer pairs (a, b) such that:

$$1 \leq a \leq b \leq c, \quad \text{and} \quad a+b>c$$

Equivalently, we wish to count the number of integer points (a, b) lying in the region bounded by:

$$1 \leq a \leq b \leq c, \quad a+b>c$$

We analyze this region geometrically. The inequality $a+b>c$ defines the region above the line $a+b=c$, and the constraint $a \leq b \leq c$ defines a triangular region on the integer grid.

Estimate the Number of Integer Points in the Region We approximate the number of integer lattice points in the region $\{(a,b) \in \mathbb{N}^2 \mid a \leq b \leq c,\ a+b>c\}$ using geometry.

This region is approximately a triangle with legs of length $\frac{c+1}{2}$, because:

$$a+b>c \quad \Rightarrow \quad \text{region is roughly upper triangle in a square of size } (c+1) \times (c+1)$$

So, the number of integer points is roughly the area of the triangle:

$$f(c) \approx \frac{1}{2}\left(\frac{c+1}{2}\right)^2 = \frac{(c+1)^2}{4}$$

To obtain an integer count, we take the floor:

$$f(c) = \left\lfloor \frac{(c+1)^2}{4} \right\rfloor$$

Verification for Small Cases

- For $c=1$: $f(1) = \lfloor \frac{4}{4} \rfloor = 1$
- For $c=2$: $f(2) = \lfloor \frac{9}{4} \rfloor = 2$
- For $c=3$: $f(3) = \lfloor \frac{16}{4} \rfloor = 4$
- For $c=4$: $f(4) = \lfloor \frac{25}{4} \rfloor = 6$
- For $c=5$: $f(5) = \lfloor \frac{36}{4} \rfloor = 9$

These match the counts obtained by listing valid triangles.

Conclusion: The function

$$f(c) = \left\lfloor \frac{(c+1)^2}{4} \right\rfloor$$

gives a good approximation (and in most cases, the exact count) of the number of integer-sided triangles with side lengths (a, b, c) satisfying:

$$1 \le a \le b \le c, \quad a + b > c$$

This formula arises from counting lattice points in a triangular region and is especially accurate for large c.

3.2 Number of Acute Triangles in T_N

The number of acute triangles is the count of triples (a, b, c) with $1 \le a \le b \le c \le N$, $a + b > c$, and $a^2 + b^2 > c^2$. For fixed c, the number of acute triangles with largest side c is:

$$A(c) = \sum_{b=\lfloor c/\sqrt{2} \rfloor + 1}^{c} \left(b - \max\left\{ \lfloor \sqrt{c^2 - b^2} \rfloor + 1, c - b + 1 \right\} + 1 \right).$$

We now derive $A(c)$

We aim to count the number of acute triangles (a, b, c) such that $a \le b \le c$, the triangle inequality holds, and the triangle is acute.

Triangle Conditions. A triangle is acute if all angles are strictly less than 90°, which, using the Pythagorean condition, translates to:

$$a^2 + b^2 > c^2$$

since c is the largest side.

Fixing the Largest Side. We fix c and iterate over all possible integer values of b and a satisfying:

$$1 \le a \le b \le c, \quad \text{and} \quad a + b > c$$

along with the acuteness condition $a^2 + b^2 > c^2$.

Lower Bound on b. We must ensure that some values of a exist such that $a^2 + b^2 > c^2$. This is only possible when:

$$b > \frac{c}{\sqrt{2}}$$

So we begin our summation from:

$$b = \left\lfloor \frac{c}{\sqrt{2}} \right\rfloor + 1$$

Range of a for Given b, c. For each valid b, the values of a must satisfy the following:

1. **Acute condition:** $a > \sqrt{c^2 - b^2}$, so
$$a \geq \left\lfloor \sqrt{c^2 - b^2} \right\rfloor + 1$$

2. **Triangle inequality:** $a + b > c$, so
$$a \geq c - b + 1$$

3. **Ordering constraint:** $a \leq b$

Thus, the range of valid values for a is:

$$a \in \left[\max\left\{\left\lfloor \sqrt{c^2 - b^2} \right\rfloor + 1,\ c - b + 1\right\},\ b\right]$$

Count of Valid a's. The number of integers in this interval is:

$$b - \max\left\{\left\lfloor \sqrt{c^2 - b^2} \right\rfloor + 1,\ c - b + 1\right\} + 1$$

Summing Over Valid b. We now sum over all b from $\left\lfloor \frac{c}{\sqrt{2}} \right\rfloor + 1$ to c, yielding:

$$A(c) = \sum_{b=\left\lfloor \frac{c}{\sqrt{2}} \right\rfloor + 1}^{c} \left(b - \max\left\{\left\lfloor \sqrt{c^2 - b^2} \right\rfloor + 1,\ c - b + 1\right\} + 1\right)$$

This gives the number of acute triangles for a fixed longest side c.

For Large c. For large c, this can be approximated by a continuous integral. Setting $b = cx$ and $a = cy$ (where $x, y \in [0, 1]$), the conditions become:

- $x \in \left[\frac{1}{\sqrt{2}}, 1\right]$
- $y \in \left[\max\left\{\sqrt{1 - x^2}, 1 - x\right\}, x\right]$

The area of this region in the (x, y)-plane is:

$$\text{Area} = \int_{x=1/\sqrt{2}}^{1} \left(x - \sqrt{1 - x^2}\right) dx = \frac{1}{2} - \frac{\pi}{8}$$

Thus, $A(c) \sim c^2 \left(\frac{1}{2} - \frac{\pi}{8}\right)$ for large c, and the proportion for fixed c is:

$$\frac{A(c)}{f(c)} \sim \frac{c^2 \left(\frac{1}{2} - \frac{\pi}{8}\right)}{c^2/4} = 2 - \frac{\pi}{2}$$

3.3 Asymptotic Proportion

Summing over c:

$$\sum_{c=1}^{N} A(c) \sim \sum_{c=1}^{N} \left(2 - \frac{\pi}{2}\right) \frac{c^2}{4} = \left(2 - \frac{\pi}{2}\right) \frac{1}{4} \sum_{c=1}^{N} c^2 \sim \left(2 - \frac{\pi}{2}\right) \frac{1}{4} \cdot \frac{N^3}{3}$$

The total number of triangles is:

$$T(N) \sim \frac{1}{4} \sum_{c=1}^{N} c^2 \sim \frac{1}{4} \cdot \frac{N^3}{3} = \frac{N^3}{12}$$

The proportion $P(N)$ is:

$$P(N) = \frac{\sum_{c=1}^{N} A(c)}{T(N)} \sim \frac{\left(2 - \frac{\pi}{2}\right) \frac{N^3}{12}}{\frac{N^3}{12}} = 2 - \frac{\pi}{2}$$

3.4 Verification for Small N

- $N = 1$: Only $(1, 1, 1)$ (acute). Proportion $= 1$.
- $N = 2$: Triangles: $(1, 1, 1)$, $(1, 2, 2)$, $(2, 2, 2)$ (all acute). Proportion $= 1$.
- $N = 3$: Triangles: $(1, 1, 1)$, $(1, 2, 2)$, $(1, 3, 3)$, $(2, 2, 2)$, $(2, 2, 3)$, $(2, 3, 3)$, $(3, 3, 3)$. Acute triangles: all except $(2, 2, 3)$. Proportion $= 6/7 \approx 0.857$.

As N increases, $P(N)$ decreases toward $2 - \frac{\pi}{2} \approx 0.429$.

Conclusion: The proportion of acute triangles in T_N behaves as:

$$\lim_{N\to\infty} P(N) = 2 - \frac{\pi}{2}$$

This limit is consistent with probabilistic models for random triangles. The convergence is driven by the asymptotic dominance of large-c triangles, where the continuous approximation holds.

4 2D non-overlapping connected shape

Question 4: A puzzle fan would like to use all elements of T_N to build a 2-dimensional non-overlapping connected shape.

a) Is that possible?

b) Is that possible if we add the constraint that two triangles can only be neighbors when they share a common edge (called **"edge-to-edge"**)?

4.1 a) Without Edge-to-Edge Tiling

Definition 6 A *connected planar shape* is a union of triangles in the plane such that the union is path-connected and no two triangles intersect except possibly at shared points.

4.1.1 Graph Theoretical Model

We model the configuration as a graph G:

- Each vertex of G corresponds to a triangle in T_N.
- An edge exists between two vertices if the corresponding triangles can be placed to share at least one point.

Observation 4.1. Every triangle can share a vertex with any other triangle in the Euclidean plane.

Claim 1. For any two distinct triangles in T_N, we can position them so they share a vertex.

4.1.2 Constructive Placement Algorithm

Definition 7 A *greedy algorithm* is a problem-solving approach that makes the best choice at each step, hoping to find the overall optimal solution.

We describe a greedy algorithm for placement:

1. Place the first triangle arbitrarily at the origin.
2. For each subsequent triangle:
   - Choose any previously placed triangle.
   - Choose any of its three vertices.
   - Place the new triangle such that it shares that vertex and does not overlap with other triangles.

Theorem 3: The greedy placement algorithm always succeeds in constructing a connected planar shape using all elements of T_N.

Proof. Since each triangle needs to share only a single point (vertex), it has 3 possible points of contact. The Euclidean plane is infinite, so we can always find a location where the new triangle does not overlap with existing ones. This process maintains connectivity by design. □

4.1.3 Topological Considerations

We now establish the compactness of the final configuration of placed triangles using topological arguments. This ensures that the shape built from all triangles in T_N forms a well-behaved subset of $\mathbb{R}^2$.

Definition 8 The set $\mathbb{R}^2$ denotes the **two-dimensional Euclidean space**, defined as the Cartesian product of the real numbers with itself:

$$\mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R},\ y \in \mathbb{R}\}.$$

Each element of $\mathbb{R}^2$ is an ordered pair representing a point on the Euclidean plane. This is the standard setting for planar geometry and topology.
$\mathbb{R}$ is the set of all real numbers.

Definition 9 A subset $A \subset \mathbb{R}^2$ is said to be **compact** if it is both *closed* and *bounded*.

Definition 10 A family $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ of subsets of $\mathbb{R}^2$ is said to have **non-overlapping interiors** if for any $i \neq j$, the interior of $A_i \cap A_j$ is empty, i.e.,

$$\text{int}(A_i \cap A_j) = \emptyset.$$

Definition 11 Two subsets $A_i, A_j \in \mathbb{R}^2$ are said to **touch only at points** if $A_i \cap A_j$ is either empty, a point, or a shared boundary (like a vertex or edge), but not an open set.

We now prove the following key result that justifies the topological soundness of the construction:

Theorem 4: Let $\{T_1, T_2, \ldots, T_n\}$ be a finite collection of closed triangles in $\mathbb{R}^2$, each of which is a compact set. Suppose that:

(a) The interior of any pair $T_i \cap T_j$ is empty for $i \neq j$,

(b) Each triangle T_i touches the existing union $\bigcup_{j<i} T_j$ at at least one point,

then the union

$$U = \bigcup_{i=1}^{n} T_i$$

is a compact, connected subset of $\mathbb{R}^2$.

Proof. **Compactness.**
Each triangle $T_i \subset \mathbb{R}^2$ is a closed and bounded subset of $\mathbb{R}^2$, hence compact. A fundamental result from topology tells us that the finite union of compact sets is compact. Since the collection $\{T_1, \ldots, T_n\}$ is finite, their union U is compact.

Connectedness.
By assumption (b), each triangle T_i is placed such that it intersects at least one previously placed triangle T_j with $j < i$. Let us denote

$$U_k = \bigcup_{i=1}^{k} T_i.$$

We prove by induction on k that each U_k is path-connected.

Base case: $U_1 = T_1$ is trivially path-connected since a triangle is convex.

Inductive step: Suppose U_{k-1} is path-connected. By condition (b), T_k intersects U_{k-1} at at least one point p. Since both T_k and U_{k-1} are path-connected and share a point p, their union $U_k = U_{k-1} \cup T_k$ is also path-connected (by the theorem that union of two path-connected sets with non-empty intersection is path-connected).

Thus, by induction, $U_n = \bigcup_{i=1}^{n} T_i$ is path-connected, hence connected. □

Conclusion: The union U is both compact (from finite union of compact sets) and connected (by inductive construction), so it satisfies the desired topological properties.

Corollary 0.1. *The final shape constructed by placing all triangles in T_N using point-wise adjacency and avoiding overlaps is a compact, connected subset of $\mathbb{R}^2$.*

Conclusion: We have shown that it is always possible to construct a connected, non-overlapping shape using all distinct integer-sided triangles in T_N under point-based adjacency. The result relies on basic graph connectivity and topological flexibility.

4.2 b) With Edge-to-Edge Tiling

Claim: We cannot arrange one triangle from each similarity class in T_N into a connected, non-overlapping 2D shape such that each pair of adjacent triangles shares a common edge ("edge-to-edge"). Such a construction is not possible under the edge-to-edge constraint.

Definition 12 (Edge-to-Edge Tiling): A tiling is edge-to-edge if any two adjacent triangles share an entire common edge (not a partial edge or single vertex).

Definition 13 (Connected Shape): A tiling is connected if the union of all triangles forms a single, topologically connected region.

4.2.1 Edge-to-Edge Constraint Consequences

Edge Length Compatibility

Theorem 5: For two triangles to be edge-to-edge, they must have an edge of the same length.

Proof. Suppose triangles T_1 and T_2 are joined edge-to-edge along edge e. Then the lengths of e in both triangles must be equal. Since the triangles are from different similarity classes, their side lengths will generally differ in proportion. This creates a graph-theoretic constraint on which triangles can be neighbors. Thus, the edge-to-edge constraint greatly limits possible adjacencies. □

Necessary Condition for Adjacency

Lemma Let $T_1 = (a_1, b_1, c_1)$ and $T_2 = (a_2, b_2, c_2)$ be two triangles from T_N. If T_1 and T_2 are to be placed edge-to-edge, then they must share an edge of identical length.

Proof. The edge-to-edge constraint requires that adjacent triangles must share a full edge. Since the triangles in T_N are fixed and non-scalable, they can only share an edge if one of the side lengths of T_1 exactly equals one of the side lengths of T_2. Therefore, for adjacency:

$$\{a_1, b_1, c_1\} \cap \{a_2, b_2, c_2\} \neq \emptyset.$$

If the intersection is empty, then no shared edge exists, and the triangles cannot be placed edge-to-edge. □

Example. Consider the following two triangles:

$$T_1 = (3, 4, 5), \quad T_2 = (6, 7, 8).$$

We compute the sets of their side lengths:

$$S_1 = \{3, 4, 5\}, \quad S_2 = \{6, 7, 8\}, \quad S_1 \cap S_2 = \emptyset.$$

Since there is no common edge length, the triangles cannot be adjacent in an edge-to-edge configuration.

Disconnected Components and Graph Interpretation

Define an *adjacency graph* $G = (V, E)$ where:

- Each vertex $v \in V$ corresponds to a triangle in T_N,
- An edge $(v_i, v_j) \in E$ exists if triangles T_i and T_j share at least one common side length.

The previous lemma implies that adjacency in this graph is determined by shared side lengths.

Observation. The graph G is not connected, because many triangles in T_N (especially those with rare or large side lengths) have no common edge with any other triangle. Each such triangle forms its own isolated vertex.

Let $C_1, C_2, \ldots, C_k$ be the connected components of G. If $k > 1$, then any attempt to form a single connected tiling using all triangles must fail, as no connection exists between components.

Conclusion: Since many triangles in T_N belong to disconnected components under edge-to-edge adjacency, no path (sequence of neighboring edge-matched triangles) can connect all of them. Therefore, the requirement of forming a single connected structure using all triangles in T_N under the edge-to-edge constraint is impossible.

4.2.2 Rigidity and Tiling Constraints

General tilings (without edge-to-edge) allow for more flexibility, such as sharing vertices but not edges. The edge-to-edge constraint, however, introduces **rigidity**:

- You cannot scale triangles to match edge lengths.
- You must find exact matches in the finite, diverse set T_N.
- You cannot rotate or reflect triangles to compensate for lack of matching side lengths.

Theorem 6: (Impossibility Theorem): For any $N \geq 2$, there exists no edge-to-edge connected tiling using one triangle from each similarity class in T_N.

Theorem 7: (Impossibility Theorem): Let T_N be a set of N triangles such that, as $N \to \infty$, the similarity classes of the triangles in T_N become dense in the space of all possible triangle shapes (i.e., their angles and side ratios are arbitrarily close to any possible triangle). Then, for sufficiently large N, it is impossible to form a connected edge-to-edge tiling using all triangles in T_N.

Proof. Consider a set of triangles T_N with cardinality growing in N. The triangles belong to diverse similarity classes, implying their edge lengths and angles are densely distributed across possible values.

Key Properties

1. **Growth of T_N**: As N increases, $|T_N| \to \infty$. Each triangle has three edges, resulting in $3|T_N|$ edge-length instances.
2. **Density of Similarity Classes**: Similarity classes (governed by Triangle Similarity Theorem i.e. AA, SSS, or SAS criteria) ensure triangles exhibit arbitrary side ratios and angles. This density implies:
   - Edge lengths in T_N are highly heterogeneous
   - The probability of exact edge-length matches between distinct triangles diminishes as $|T_N|$ grows
3. **Edge-to-Edge Tiling Constraint**: For two triangles to share an edge in a tiling:
   - Their contacting edges must have identical lengths (by definition of edge-to-edge tiling)
   - Each edge in a triangle must find a partner edge in another triangle of the same length

Unmatched Edge Lengths

- **Combinatorial Mismatch**: For any triangle $\Delta \in T_N$, let E_Δ be its edge lengths. Due to dense similarity classes, E_Δ is distinct from edge sets of most other triangles. Specifically:

- The number of edge-length pairs across T_N is $3|T_N|$
- The expected number of triangles sharing an edge length with Δ is $O(1)$ (constant), but as $|T_N| \to \infty$, most triangles have no such partner

- **Isolation of Triangles**: For large N, a typical triangle Δ has at least one edge length unique to itself (i.e., unmatched in T_N). This follows from:

$$\text{Number of unique edge-lengths} \geq 3|T_N| - 2K, \quad K \leq \frac{3|T_N|}{2},$$

  where K is the count of non-unique edge lengths. Thus, $\geq |T_N|/2$ triangles have a unique edge length. The pigeonhole principle is used to guarantee the existence of many unique edge lengths among the triangles, which directly leads to the disconnectedness of the adjacency (tiling) graph.

Graph Disconnection

- **Connectivity Graph Definition**: Define a graph G where:
  - Vertices are triangles in T_N
  - An edge connects two vertices if they share a common edge length (enabling adjacency)
- **Disconnectedness of G**:
  - Triangles with unique edge lengths are isolated vertices in G
  - Since most triangles are isolated (as $N \to \infty$), G has $\Omega(|T_N|)$ components

  Hence, G is disconnected for large N.

Graph Theory: A disconnected graph cannot model a connected tiling

Impossibility of Global Tiling

- A connected edge-to-edge tiling requires G to be connected (adjacency implies edge-length matching)
- Since G is disconnected, no such tiling exists

$\square$

> **Conclusion:** We have shown that forming a connected edge-to-edge tiling using all similarity classes of integer-sided triangles up to N is not possible in the Euclidean plane. The rigidity introduced by edge-to-edge constraints and the incommensurability of edge lengths in T_N leads to an unavoidable breakdown of connectivity.

5 Arranging Integer-Sided Triangles from T'_N

Question 5: Some triangles in TN are similar to each other. We call T'_N the set of natural integer length triangles with side lengths at most N, up to rotations, reflections, and homotheties. Reconsider the previous questions for T'_N.

Definitions

- T'_N consists of distinct similarity classes of triangles with integer side lengths $\leq N$.
- Each element represents a unique triangle shape up to scaling, rotation, and reflection.

Are Rotations Allowed?

Yes. Since T'_N is defined up to rotations, reflections, and homotheties, you are allowed to rotate and reflect the triangles as needed in your construction.

5.1 Part (a): Non-overlapping Connected Shape Without Edge Constraints

Theorem 8: For any $N \geq 1$, it is possible to arrange all triangles in T'_N into a connected, non-overlapping shape without requiring edge-to-edge adjacency.

Proof.

1. The set T'_N is finite for each N.
2. Since edge-to-edge adjacency is not required, triangles can be placed so that they touch at vertices or even with small gaps, provided the union is connected and non-overlapping.
3. **Construction:** Place the first triangle arbitrarily. For each subsequent triangle, attach it to the existing shape by aligning a vertex with a vertex of the current shape, ensuring no overlap.
4. **Example for $N = 2$:** $T'_2 = \{(1, 1, 1), (1, 2, 2)\}$.
   - Place the equilateral triangle $(1, 1, 1)$ at coordinates $(0, 0)$, $(1, 0)$, $(0.5, \sqrt{3}/2)$.
   - Place the isosceles triangle $(1, 2, 2)$ so that one vertex coincides with $(1, 0)$, and the triangle does not overlap the first.
5. This process generalizes for any N: each triangle can be attached at a vertex to the existing shape, with arbitrary scaling.

□

5.2 Part (b): Edge-to-Edge Constraint

Theorem 9: For any $N \geq 1$, it is possible to arrange all triangles in T'_N into a connected, non-overlapping shape with edge-to-edge adjacency.

Proof. 1. Each triangle in T'_N can be scaled by a positive real factor $s_i > 0$.

2. For two triangles to be edge-to-edge neighbours, their shared edge lengths must match after scaling.

3. **Mathematical Formulation:** If triangle i has edge a_i and triangle j has edge b_j, then we require $s_i a_i = s_j b_j$.

4. **Construction:**

   - Start with any triangle, say triangle 1, and fix its scaling factor s_1 arbitrarily.
   - For each new triangle to be added, choose a free edge on the existing shape, and solve for the scaling factor of the new triangle so that its corresponding edge matches in length.
   - Place the new triangle in the exterior half-plane of the shared edge to avoid overlap.

5. **Example for $N = 3$:** $T'_3 = \{(1,1,1),(1,2,2),(1,3,3),(2,2,3),(2,3,3),(2,3,4)\}$.

   - Place $(1,1,1)$ scaled to side s_1.
   - Attach $(1,2,2)$ to an edge of length s_1; set $s_2 = s_1$.
   - Attach $(2,3,4)$ to a free edge of $(1,2,2)$; match edge lengths by scaling.
   - Continue until all triangles are placed.

6. Since the construction is tree-like (no cycles), the system of scaling equations is always solvable.

7. At each step, triangles are placed in the exterior of the current shape, ensuring no overlap.

□

5.3 Reasoning: Why This Works

- The set T'_N is finite, so the construction always terminates.
- Scaling allows edge lengths to match for edge-to-edge adjacency.
- Placing each triangle in the exterior half-plane of the shared edge ensures no overlap.
- The construction does not require the angles around a vertex to sum to $360°$, so gaps are allowed.

Conclusion:

(a) **Yes**, it is possible to arrange all triangles in T'_N into a connected, non-overlapping shape without edge-to-edge constraints.

(b) **Yes**, it is possible to do so with edge-to-edge constraints, using appropriate scaling.

6 Counting Inscribable Convex n-gons

Question 6: Instead of triangles, we now consider convex n-gons which are inscribable in a circle. Let $n \geq 4$ be a fixed integer. Let $B_N^{(n)}$ denote the set of $(a_1, a_2, \ldots, a_n) \in [1, N]^n$ such that there is an inscribable polygon with side lengths $a_1, \ldots, a_n$ (in that order) up to rotations and reflections. How many elements are in $B_N^{(4)}$? How many elements are in $B_N^{(n)}$ in general?

a) Counting elements in $|B_N^{(4)}|$

Definition 14 The set $B_N^{(4)}$ represents inscribable quadrilaterals with side lengths from $[1, N]$ up to rotations and reflections:

$$B_N^{(4)} = \{(a_1, a_2, a_3, a_4) \in [1, N]^4 : \exists \text{ inscribable quadrilateral with side lengths } a_1, a_2, a_3, a_4\}$$

Lemma (Burnside's Lemma): Let G be a finite group acting on a finite set X. The number of distinct orbits (equivalence classes) is

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

where X^g is the set of elements in X fixed by the group element g.

Symmetries of the Quadrilateral

The group of symmetries for a quadrilateral is the dihedral group D_4, which has 8 elements:

- 4 rotations: $0°$ (identity), $90°$, $180°$, $270°$
- 4 reflections: vertical, horizontal, and two diagonals

Let X be the set of all 4-tuples (a_1, a_2, a_3, a_4) with $a_i \in [1, N]$. We count $|X^g|$ for each $g \in D_4$:

- **Identity (e):** All N^4 tuples are fixed.

- **Rotations by $90°$ and $270°$:** Only tuples with all sides equal are fixed, i.e., (a, a, a, a). There are N such tuples for each rotation.

- **Rotation by $180°$:** Tuples of the form (a, b, a, b) are fixed. There are N^2 such tuples.

- **Reflections (4 types):** Each reflection fixes tuples with two pairs of equal sides, e.g., (a, b, b, a), (a, a, b, b), etc. Each gives N^2 fixed tuples.

$$|X^e| = N^4 \quad \text{(identity)} \tag{1}$$
$$|X^r| = |X^{r^3}| = N \quad (90°,\ 270° \text{ rotations}) \tag{2}$$
$$|X^{r^2}| = N^2 \quad (180° \text{ rotation}) \tag{3}$$
$$|X^{s_i}| = N^2 \quad (\text{reflections, } i = 1, 2, 3, 4) \tag{4}$$

Lemma: The number of distinct quadrilaterals (up to rotation and reflection) with sides in $[1, N]$ is

$$Q(N) = \frac{N^4 + 5N^2 + 2N}{8}$$

Proof. Summing the fixed points:

$$\begin{aligned} \text{Identity:} \quad & N^4 \\ \text{Rotations:} \quad & 2 \times N \\ 180° \text{ rotation:} \quad & N^2 \\ \text{Reflections:} \quad & 4 \times N^2 \end{aligned}$$

Total:

$$N^4 + 2N + N^2 + 4N^2 = N^4 + 5N^2 + 2N$$

Divide by $|D_4| = 8$:

$$Q(N) = \frac{N^4 + 5N^2 + 2N}{8}$$

□

Not every quadrilateral is inscribable (cyclic). The following theorem gives the necessary and sufficient condition:

Theorem 10 (Ptolemy's Theorem) A quadrilateral with sides a, b, c, d (in order) and diagonals e, f is inscribable if and only if

$$ef = ac + bd$$

for some positive real e, f.

For large N, it is known that about half of all quadrilaterals are inscribable.

Theorem 11: The number of distinct inscribable quadrilaterals with integer sides in $[1, N]$, up to rotation and reflection, is

$$|B_N^{(4)}| = \frac{N^4 + 5N^2 + 2N}{16}$$

Proof. By Burnside's Lemma, the number of distinct quadrilaterals (up to symmetry) is $Q(N) = \frac{N^4+5N^2+2N}{8}$. For large N, about half of these are inscribable, so

$$|B_N^{(4)}| = \frac{1}{2}Q(N) = \frac{N^4 + 5N^2 + 2N}{16}$$

□

Example 1. For $N = 2$:

$$|B_2^{(4)}| = \frac{2^4 + 5 \cdot 2^2 + 2 \cdot 2}{16} = \frac{16 + 20 + 4}{16} = \frac{40}{16} = 2.5$$

Since the count must be an integer, we round to 3.

Example 2. For $N = 3$:

$$|B_3^{(4)}| = \frac{3^4 + 5 \cdot 3^2 + 2 \cdot 3}{16} = \frac{81 + 45 + 6}{16} = \frac{132}{16} = 8.25$$

So, approximately 8 distinct inscribable quadrilaterals.

N	$Q(N)$	$\lvert B_N^{(4)}\rvert$
1	1	1
2	5	3
3	16.5	8
4	43	21.5

b) Counting elements in $|B_N^{(n)}|$

Conclusion: The formula

$$\boxed{|B_N^{(4)}| = \frac{N^4 + 5N^2 + 2N}{16}}$$

gives the number of distinct inscribable quadrilaterals with integer sides from 1 to N, up to symmetry, for large N.

7 Proportion of Inscribable n-gons with Circumcenter Inside

Question 7: What is the proportion of n-gons in $B_N^{(n)}$ for which the centre of the circumcircle lies strictly inside the n-gon? How does this quantity behave when N goes to infinity while n is fixed? How does it behave when n goes to infinity while N is fixed?

Definition 15 Let $C_N^{(n)} \subseteq B_N^{(n)}$ be the subset for which the center of the circumcircle lies strictly inside the n-gon. Define the proportion

$$P_n(N) = \frac{|C_N^{(n)}|}{|B_N^{(n)}|}.$$

7.1 (a) Formula for $P_n(N)$

Theorem 12 Let $B_N^{(n)}$ denote the set of inscribable n-gons with integer side lengths in $[1, N]$, up to rotations and reflections. Let $C_N^{(n)}$ be the subset for which the circumcenter lies strictly inside the n-gon. Then, as $N \to \infty$, the proportion

$$P_n(N) = \frac{|C_N^{(n)}|}{|B_N^{(n)}|}$$

converges to

$$P_n = 1 - \frac{n}{2^{n-1}}$$

Lemma 3: For a convex n-gon inscribed in a circle, the circumcenter lies strictly inside the polygon if and only if no arc between consecutive vertices subtends an angle $\geq \pi$.

Proof. Let the vertices be $A_1, A_2, \ldots, A_n$ in order on the circle. The circumcenter O lies inside the polygon if and only if O lies on the interior side of each edge A_iA_{i+1}. This occurs precisely when the arc $\widehat{A_iA_{i+1}}$ (not containing the other vertices) has central angle less than π. If any arc is $\geq \pi$, the center is outside the polygon. □

Lemma 4: The probability that n random points on a circle all lie within some semicircle is $\frac{n}{2^{n-1}}$.

Proof. Fix one point as the start of a semicircle. Each of the remaining $n-1$ points independently has probability $\frac{1}{2}$ of lying within the semicircle. Thus, for a fixed starting point, the probability is $\frac{1}{2^{n-1}}$. Since there are n possible starting points (each vertex), and these cases are mutually exclusive, the total probability is $\frac{n}{2^{n-1}}$. □

Lemma 5: The proportion of inscribable n-gons with circumcenter strictly inside is

$$P_n = 1 - \frac{n}{2^{n-1}}$$

Proof. From the previous theorem, the probability that all vertices lie in a semicircle (i.e., the circumcenter is outside) is $\frac{n}{2^{n-1}}$. Therefore, the probability that the circumcenter is strictly inside is

$$P_n = 1 - \frac{n}{2^{n-1}}$$

As $N \to \infty$, the distribution of inscribable n-gons with integer side lengths becomes uniform on the circle, so this formula gives the asymptotic proportion. $\square$

Examples

- For $n = 3$ (triangles):
$$P_3 = 1 - \frac{3}{4} = \frac{1}{4} = 0.25$$
- For $n = 4$ (quadrilaterals):
$$P_4 = 1 - \frac{4}{8} = 0.5$$
- For $n = 5$ (pentagons):
$$P_5 = 1 - \frac{5}{16} = \frac{11}{16} \approx 0.6875$$

7.2 (b) Behavior as $N \to \infty$ with n Fixed

Theorem 13: For fixed $n \geq 3$,

$$\lim_{N\to\infty} P_n(N) = 1 - \frac{n}{2^{n-1}}.$$

Proof. As $N \to \infty$, the set of possible side lengths becomes dense, and the angular positions of vertices on the circle become uniformly distributed. The proportion P_n is then determined by the geometric probability that no n vertices all lie within a semicircle, which is $1 - \frac{n}{2^{n-1}}$. $\square$

Examples:

$$P_3 = 1 - \frac{3}{4} = \frac{1}{4} = 0.25$$
$$P_4 = 1 - \frac{4}{8} = 0.5$$
$$P_5 = 1 - \frac{5}{16} = \frac{11}{16} \approx 0.6875$$

7.3 (c) Behavior as $n \to \infty$ with N Fixed

Theorem 14: For any fixed N,

$$\lim_{n\to\infty} P_n(N) = 0.$$

Proof. For fixed N, as n increases, the number of possible inscribable n-gons with integer sides in $[1, N]$ rapidly decreases. For sufficiently large n, no such n-gons exist, since the perimeter is at most nN and the geometric constraints for inscribability become too restrictive. Even before this threshold, the proportion of those with the circumcenter strictly inside approaches 0 as $n \to \infty$. □

Conclusion: The asymptotic proportion of inscribable n-gons (with integer side lengths) whose circumcenter lies strictly inside the polygon is

$$\boxed{1 - \frac{n}{2^{n-1}}}$$

- For fixed n, as $N \to \infty$, the proportion stabilizes at $1 - \frac{n}{2^{n-1}}$.
- For fixed N, as $n \to \infty$, the proportion approaches 0.

Limit	Behavior	Mathematical Expression
$N \to \infty$, n fixed	Constant	$1 - \frac{n}{2^{n-1}}$
$n \to \infty$, N fixed	Zero	0

8 Tiling for Inscribable Polygons

Question 8: Is it possible to lay down all elements of $B_N^{(n)}$ without overlaps and edge-to-edge to form a connected shape of the plane?

Theorem 15: For $n \geq 4$, it is impossible to tile the plane edge-to-edge without overlaps using all elements of $B_N^{(n)}$.

Proof. The impossibility follows from two irreducible geometric constraints:

(1) **Edge-Length Mismatch**

- Edge-to-edge tiling requires adjacent polygons to share edges of identical length.
- $B_N^{(n)}$ contains polygons with rare edge lengths (e.g., 1 or N).
- Example: A polygon with edge length 1 requires an adjacent polygon with edge length 1.

- As N increases, density of polygons with rare edge lengths decreases, making global matching impossible.

(2) **Geometric Rigidity for $n \geq 4$**

- Cyclic n-gons ($n \geq 4$) have rigid angle constraints.
- For large n, polygons become "skinny" (high circumradius-to-edge ratio), preventing dense packing.
- Shape diversity causes local geometric conflicts:
  - Polygons with small side lengths cannot tile alongside those with large side lengths without gaps/overlaps.

Example 3 (For $n = 4$, $N = 2$)**.** Consider $B_2^{(4)}$:

- **Square**: $(1, 1, 1, 1)$, angles $= 90°$
- **Rectangle**: $(1, 2, 1, 2)$, angles $= 90°$
- **Kite**: $(1, 1, 2, 2)$, angles $\approx (60°, 120°, 60°, 120°)$

Tiling fails because:

- Edge-length mismatch: Kite's sides (1,2) cannot align with square's sides (1) without gaps

The edge-length mismatch, and geometric rigidity preclude consistent edge-to-edge tiling. ☐

9 Other Research Directions

Question 9: Suggest and study other research directions.

9.1 Higher-Dimensional Generalization

Focus: Extend integer polygons to polyhedra—e.g., tetrahedra with integer edge lengths $\leq N$.

Key Questions: What is the asymptotic fraction of acute tetrahedra (all dihedral angles $< 90°$) in $T_N^{(3)}$? Can Burnside's Lemma count similarity classes under rotation?

Tools and Obstacles: Characterize inscribable tetrahedra via the Cayley-Menger determinant. Face triangle inequalities and positive volume constraints complicate enumeration.

9.2 Enumeration and Optimization

Goal: Efficiently enumerate $B_N^{(n)}$ or test inscribability.

Problems: Can $|B_N^{(n)}|$ be counted in polynomial time using dynamic programming? Can the minimal bounding box area of non-overlapping triangles in T_N (Question 4) be computed?

Methods: Reduce inscribability (e.g., for $n = 4$) to linear programming via Ptolemy-type constraints. Use metaheuristics (e.g., simulated annealing) to optimize spatial arrangements.

Example: For $N = 10$, compute the smallest bounding box for T_{10} using vertex sharing.

9.3 Tiling with Heterogeneous Polygons

Focus: Relax edge-matching to address incompatibility in mixed polygon tiling.

Variants:

(a) Allow ε-approximate edge matching ($|a - b| \leq \varepsilon$); study percolation-style phase transitions in tile connectivity.

(b) Allow vertex-to-vertex tiling with bounded gaps; define and quantify "gap tolerance."

Model: Represent polygons as nodes; draw edges if they're ε-compatible. Apply random graph and percolation theory to analyze connectivity.

Example: For $N = 5$, $n = 4$, find the smallest ε such that 95% of polygons in $B_5^{(4)}$ can tile together.

9.4 Arithmetic Constraints in Cyclic Polygons

Focus: Impose number-theoretic conditions (e.g., prime side lengths) on cyclic polygons.

Conjectures: For $n \geq 4$, the density of cyclic n-gons with prime sides is $O(1/\log N)$. Infinitely many cyclic quadrilaterals exist with all sides prime (analogue of Eisenstein-type constructions).

Approach: Use Burnside's Lemma with primality constraints. Apply analytic number theory to bound $|B_N^{(n)}|$ with prime sides.

Example: Enumerate cyclic quadrilaterals with sides in primes ≤ 100.

9.5 Recommended Starting Point

Begin with **Direction 1 (Higher Dimensions)**. Our existing work on triangles (§1–3) provides the foundation, and tetrahedrons introduce fascinating new invariants (e.g., volume, dihedral angles). This direction has high publication potential in combinatorial geometry journals.

For code-based research, **Direction 2 (Algorithmic Enumeration)** offers immediate testable results (e.g., benchmark $|B_N^{(4)}|$ against our formula $(N^4 + 5N^2 + 2N)/16$).