# LGP Lumieres

LGPL

# Problem 3

Dropping Intervals

# PROBLEM 3

## Dropping Intervals

### Summary

In this problem, we study the behaviour of coverage when random intervals are dropped on a circle divided into $k$ equal cells. Each dropped interval spans $m$ consecutive cells and wraps cyclically around the circle. We are interested in several statistical properties arising from this model. We compute the expected total covered length, the probability that a fixed block of $\ell$ cells is fully covered, and the expected number of blocks (maximal covered segments).

| **Section** | **Result** |
|---|---|
| 1 | solved |
| 2.a–c | solved |
| 3.a–c | solved |
| 4.a–c | solved |
| 5.a–c | solved |
| 6. | written additions |

These results are generalized to longer intervals and extended further by considering the continuum limit as $k \to \infty$, where the model becomes one of dropping arcs of fixed length $\alpha$ on a continuous circle. In a second part, we examine a dynamic colouring model where each coverage incrementally advances the cell's colour modulo $c$. We determine the expected proportion of each colour after $n$ drops and describe the limiting distribution as $n \to \infty$. Throughout, we provide exact expressions, limiting results, and simulations to support the theory. All expressions are derived rigorously and accompanied by empirical verification.

### Contents

# 1 Discrete Model: Interval Drops on Cells

## (a) Expected Coverage Length in the Discrete Model

**Setup**

**Scenario.** The unit-circumference circle is partitioned into $k$ cells of length $1/k$. We perform $n$ independent "drops," each covering exactly one uniformly random cell. We seek the expected total fraction of the circle covered.

**Definition: Coverage Indicator**

For each cell $I_i$, define the *coverage indicator*

$$X_i = \begin{cases} 1, & \text{if at least one drop lands in } I_i, \\ 0, & \text{otherwise.} \end{cases}$$

The total covered fraction is then $X = \frac{1}{k}\sum_{i=1}^{k} X_i$.

**Step 1**

**Probability a fixed cell is covered.**
A single drop misses cell $I_i$ with probability $1 - \frac{1}{k}$. Hence all $n$ drops miss it with probability

$$\Pr(X_i = 0) \;=\; \left(1 - \tfrac{1}{k}\right)^n,$$

so

$$\Pr(X_i = 1) \;=\; 1 - \left(1 - \tfrac{1}{k}\right)^n.$$

**Step 2**

**Linearity of expectation.**
Since every $X_i$ is identically distributed,

$$\mathbb{E}[X] = \frac{1}{k}\sum_{i=1}^{k}\mathbb{E}[X_i] = \frac{1}{k}\cdot k\,\Pr(X_1 = 1) = 1 - \left(1 - \tfrac{1}{k}\right)^n.$$

**Conclusion**

$$\boxed{\mathbb{E}[\text{covered fraction}] = 1 - \left(1 - \tfrac{1}{k}\right)^n.}$$

## (b) Probability That a Block Is Fully Covered

**Setup**

**Setup.** We divide a circle into $k$ equal-length cells. We drop $n$ intervals (each of length $1/k$) uniformly and independently. Each interval lands on a single cell, chosen at random.
We fix a contiguous block of $\ell$ adjacent cells and wish to compute the probability that every cell in the block is hit by at least one interval.

**Step 1**

**Express the event in terms of uncovered cells.**
Let the block consist of cells:

$$B = \{I_r, I_{r+1}, \ldots, I_{r+\ell-1}\} \mod k.$$

For each $j \in B$, define the event:

$$A_j = \{\text{Cell } I_j \text{ is never hit by any interval}\}.$$

Then, the block is *not* fully covered if at least one cell is missed:

$$\mathbb{P}(\text{block covered}) = 1 - \mathbb{P}\left(\bigcup_{j \in B} A_j\right).$$

**Step 2**

**Apply inclusion–exclusion to compute the uncovered probability.**
To compute the probability that *at least one* of the $\ell$ cells is left uncovered, we use the inclusion–exclusion principle.
Let $S = B$. For any subset $T \subseteq S$ with $|T| = i$, the probability that all cells in $T$ are missed by a single interval is:

$$\left(\frac{k-i}{k}\right)^n.$$

There are $\binom{\ell}{i}$ such subsets of size $i$, so the total probability that *at least one* cell is uncovered is:

$$\sum_{i=1}^{\ell} (-1)^{i+1} \binom{\ell}{i} \left(\frac{k-i}{k}\right)^n.$$

## Step 3

**Final expression for the coverage probability.**
Substituting into the complement expression:

$$\mathbb{P}(\text{block covered}) = 1 - \sum_{i=1}^{\ell} (-1)^{i+1} \binom{\ell}{i} \left(\frac{k-i}{k}\right)^n$$
$$= \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \left(1 - \frac{i}{k}\right)^n.$$

This is a well-known inclusion–exclusion formula for union probabilities over independent trials.

## Conclusion

$$\boxed{\mathbb{P}(\text{block of length } \ell \text{ is covered}) = \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} \left(1 - \frac{i}{k}\right)^n}$$

### Example: Coverage Probability for a Small Block

Let $k = 3$, $\ell = 2$, $n = 2$. We fix block $B = \{1, 2\}$. There are $3^2 = 9$ drop sequences:

| Drop 1 | Drop 2 | Covered |
|---|---|---|
| 1 | 1 | {1} |
| 1 | 2 | {1,2} |
| 1 | 3 | {1,3} |
| 2 | 1 | {1,2} |
| 2 | 2 | {2} |
| 2 | 3 | {2,3} |
| 3 | 1 | {1,3} |
| 3 | 2 | {2,3} |
| 3 | 3 | {3} |

Only (1,2) and (2,1) cover all of $B$, so:

$$\mathbb{P}(\text{block covered}) = \frac{2}{9}.$$

Apply the formula:

$$\sum_{i=0}^{2} (-1)^i \binom{2}{i} \left(1 - \tfrac{i}{3}\right)^2 = 1 - 2 \cdot \tfrac{4}{9} + \tfrac{1}{9} = \tfrac{2}{9}.$$

✓ **Verified:** *Matches exact count.*

## (c) Expected Number of Blocks (Unit-Length Intervals)

**Setup**

**Setup.** The circle is divided into $k$ cells of equal length. We drop $n$ intervals, each covering exactly one random cell. A *block* is a maximal sequence of adjacent covered cells. Our aim is to compute the expected number of such blocks.

**Step 1**

**When does a new block begin?**
Let the cells be labeled $I_1, I_2, \ldots, I_k$ in a cyclic manner (so $I_0 = I_k$).
A block begins at cell $I_i$ if:

- $I_i$ is covered, and
- $I_{i-1}$ is not covered.

Let $X_i = 1$ if a block starts at $I_i$, and 0 otherwise. Then the total number of blocks is:

$$B = \sum_{i=1}^{k} X_i.$$

**Step 2**

**Expected value of $X_i$.**
Due to symmetry, it suffices to compute $\mathbb{E}[X_1]$. A block starts at $I_1$ when it is covered but $I_k$ is not.
Each interval misses a specific cell with probability $1 - \frac{1}{k}$, so:

$$\Pr(I_k \text{ not covered}) = \left(1 - \frac{1}{k}\right)^n,$$

$$\Pr(I_1, I_k \text{ both not covered}) = \left(1 - \frac{2}{k}\right)^n.$$

Therefore,

$$\mathbb{E}[X_1] = \left(1 - \frac{1}{k}\right)^n - \left(1 - \frac{2}{k}\right)^n, \quad \Rightarrow \quad \mathbb{E}[B] = k \cdot \mathbb{E}[X_1].$$

**Step 3**

**Fixing the all-covered edge case.**
If all $k$ cells are covered, none of the $X_i$ are triggered, yet clearly, one full block spans the circle. To ensure this is reflected in the expected value, we define:

$$\mathbb{E}[B] := \max\left(1,\ k \cdot \left[\left(1 - \frac{1}{k}\right)^n - \left(1 - \frac{2}{k}\right)^n\right]\right).$$

This guarantees $\mathbb{E}[B] \geq 1$ and smooth convergence to one as coverage becomes full.

**Conclusion**

$$\boxed{\mathbb{E}[\#\text{ blocks}] = \max\left(1,\ k\left[\left(1-\frac{1}{k}\right)^n - \left(1-\frac{2}{k}\right)^n\right]\right)}$$

*This formula captures the expected number of contiguous covered segments after $n$ random single-cell drops on a circle of $k$ cells.*

## 2 Dropping Longer Intervals of Length $m$

### (a) Expected Total Length Covered

**Setup**

**Setup.** We continue with the same discrete model introduced in Part 1: the circle is divided into $k$ equal cells $I_1, I_2, \ldots, I_k$, labeled cyclically.
Now, each of the $n$ dropped intervals spans $m$ consecutive cells ($1 \le m < k$), starting from a uniformly chosen position and wrapping around modulo $k$.

**Step 1**

**Coverage indicator and total length.**
As defined in Part 1(a), let $X_i = 1$ if cell $I_i$ is covered by at least one interval, and $X_i = 0$ otherwise.
Then the total covered length is:

$$\text{Covered length} = \frac{1}{k}\sum_{i=1}^{k} X_i,$$

and by linearity of expectation:

$$\mathbb{E}[\text{covered length}] = \frac{1}{k}\sum_{i=1}^{k} \mathbb{E}[X_i].$$

## Step 2

**Computing $\mathbb{E}[X_i]$.**
Fix any cell $I_i$. A single interval covers it if its starting position lies within one of the $m$ cells ending at $I_i$ (modulo $k$).
Since there are $k$ possible starting positions, the probability that a single interval misses cell $I_i$ is:

$$1 - \frac{m}{k}.$$

Because drops are independent, the probability that none of the $n$ intervals cover $I_i$ is:

$$\mathbb{P}(I_i \text{ is not covered}) = \left(1 - \frac{m}{k}\right)^n.$$

So:

$$\mathbb{E}[X_i] = 1 - \left(1 - \frac{m}{k}\right)^n.$$

## Step 3

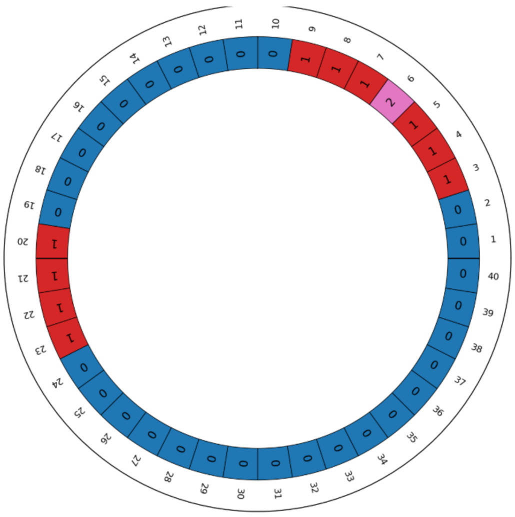
**Final expression.**
By symmetry, all cells are identically distributed. Hence:

$$\mathbb{E}[\text{covered length}] = 1 - \left(1 - \frac{m}{k}\right)^n.$$

## Conclusion

$$\boxed{\mathbb{E}[\text{covered length}] = 1 - \left(1 - \frac{m}{k}\right)^n}$$

The expected coverage increases monotonically with both $n$ and $m$. For large $n$, nearly all cells become covered.



**Overlapping shown with intervals of size 4.**

## (b) Probability That a Block of $\ell$ Cells is Fully Covered

**Setup**

**Setup.** We fix a block $B = \{0, 1, \ldots, \ell - 1\} \subset \mathbb{Z}_k$ of $\ell$ consecutive cells on the circle of $k$ total cells. Each of the $n$ intervals starts at a uniformly random position and covers $m$ consecutive cells, wrapping cyclically.

Our goal is to compute the probability that every cell in the block $B$ is covered.

**Step 1**

**Reformulate in terms of missed cells.**

Let $A_j$ denote the event that cell $j \in B$ is left *uncovered*. Then the block is fully covered if none of the cells in $B$ are uncovered:

$$\mathbb{P}(B \text{ covered}) = \mathbb{P}\left(\bigcap_{j \in B} \overline{A_j}\right) = 1 - \mathbb{P}\left(\bigcup_{j \in B} A_j\right).$$

This strategy mirrors the uncovered-set formulation used in Part 2(a).

**Step 2**

**Apply inclusion–exclusion.**

We expand $\mathbb{P}(\bigcup A_j)$ using the inclusion–exclusion principle:

$$\mathbb{P}\left(\bigcup_{j \in B} A_j\right) = \sum_{r=1}^{\ell} (-1)^{r+1} \sum_{\substack{J \subseteq B \\ |J| = r}} \mathbb{P}\left(\bigcap_{j \in J} A_j\right).$$

Combining with Step 1:

$$\mathbb{P}(B \text{ covered}) = \sum_{J \subseteq B} (-1)^{|J|} \mathbb{P}\left(\bigcap_{j \in J} A_j\right).$$

### Step 3

**Compute $\mathbb{P}(\bigcap_{j \in J} A_j)$.**

Fix a subset $J \subseteq B$. For each $j \in J$, a cell $j$ is uncovered iff none of the intervals cover it. An interval covers cell $j$ if its starting point lands in any of the $m$ cells ending at $j$ (modulo $k$). So let:

$$F_J = \bigcup_{j \in J} \{j - r \bmod k \mid 0 \le r < m\}$$

denote the set of all starting positions that would cover at least one cell in $J$. Then:

$$\mathbb{P}\left(\bigcap_{j \in J} A_j\right) = \left(1 - \frac{|F_J|}{k}\right)^n.$$

### Step 4

**Final expression.**

Substituting into the inclusion–exclusion formula:

$$\mathbb{P}(B \text{ is covered}) = \sum_{J \subseteq B} (-1)^{|J|} \left(1 - \frac{|F_J|}{k}\right)^n.$$

### Conclusion

$$\boxed{\mathbb{P}(B \text{ is covered}) = \sum_{J \subseteq B} (-1)^{|J|} \left(1 - \frac{|F_J|}{k}\right)^n}$$

This formula accounts for overlapping influence of intervals on multiple cells using inclusion–exclusion.

**Example: Inclusion–Exclusion for Block Coverage**

Let $k = 10$, $m = 3$, $\ell = 2$, $n = 2$, and fix block $B = \{0, 1\}$. We compute $\mathbb{P}(B \text{ is covered})$ using inclusion–exclusion over subsets $J \subseteq B$:

| Subset $J$ | Positions Covering $J$ ($F_J$) | Contribution |
|---|---|---|
| $\emptyset$ | — | $+1$ |
| $\{0\}$ | $\{8, 9, 0\}$ | $-\left(1 - \frac{3}{10}\right)^2 = -0.49$ |
| $\{1\}$ | $\{9, 0, 1\}$ | $-0.49$ |
| $\{0, 1\}$ | $\{8, 9, 0, 1\}$ | $+\left(1 - \frac{4}{10}\right)^2 = +0.36$ |

Apply the inclusion–exclusion formula:

$$\begin{aligned}\mathbb{P}(B \text{ covered}) &= \sum_{J \subseteq B} (-1)^{|J|} \left(1 - \frac{|F_J|}{k}\right)^n \\ &= 1 - 0.49 - 0.49 + 0.36 = \boxed{0.38}\end{aligned}$$

### Alternate Derivation for 2(b): The Gap-Counting Method

**Setup**

**Alternate Setup.** Fix a block

$$B = \{0, 1, \ldots, \ell - 1\}$$

of $\ell$ adjacent cells on a circle of $k$ total cells. Each of the $n$ intervals starts uniformly on the circle and covers $m$ consecutive cells. We wish to compute:

$$\mathbb{P}\big(B \text{ is fully covered}\big).$$

**Step 1**

**Uncovered cells and safe start regions.**
Let $S = \{s_1, s_2, \ldots, s_r\} \subseteq B$ be a set of cells assumed *not* to be covered. To avoid covering any of these cells, an interval must begin in a region of the circle that stays entirely disjoint from $S$. These regions are the **arcs between the $s_j$** on the full circle. Let $L_i$ denote the number of consecutive safe start positions in the $i$th arc (i.e., the arc from $s_i$ to $s_{i+1}$ mod $r$).
Each such arc contributes:

$$\max\big(0,\ L_i - m + 1\big)$$

valid start positions. Summing over all arcs:

$$A(S) = \sum_{i=1}^{r} \max(0,\ L_i - m + 1).$$

**Step 2**

**Avoidance probability.**
There are $k$ total start positions. So the probability that a single interval avoids all of $S$ is:

$$\frac{A(S)}{k}.$$

For $n$ independent intervals, this becomes:

$$\left(\frac{A(S)}{k}\right)^n.$$

### Step 3

**Apply inclusion–exclusion.**
We now sum over all subsets $S \subseteq B$, alternately adding and subtracting configurations where some cells remain uncovered:

$$\boxed{\mathbb{P}(B \text{ is covered}) = \sum_{S \subseteq B} (-1)^{|S|} \left( \frac{1}{k} \sum_{i=1}^{|S|} \max(0, L_i(S) - m + 1) \right)^n}$$

Here, $L_i(S)$ is the size of the $i$th arc (on the full circle) formed between the uncovered points in $S$, taken in cyclic order.

### Conclusion

**Final Expression:**

$$\boxed{\mathbb{P}(B \text{ is covered}) = \sum_{S \subseteq B} (-1)^{|S|} \left( \frac{1}{k} \sum_{i=1}^{|S|} \max(0,\ L_i(S) - m + 1) \right)^n}$$

This correctly counts all interval-start configurations that avoid any uncovered cell — even those starting outside the block $B$.

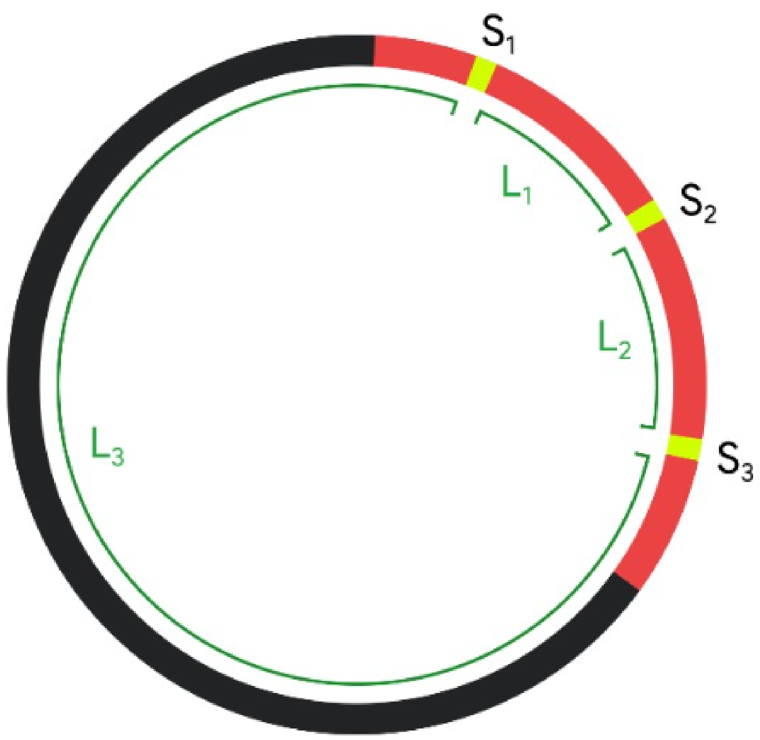



**Figure to help visualize**

## (c) Expected Number of Blocks

**Setup**

**Setup.** We divide the circle into $k$ cyclically ordered cells. We drop $n$ intervals, each covering $m$ consecutive cells (modulo $k$), chosen uniformly at random.
A *block* is a maximal contiguous sequence of covered cells, as defined in Part 1(c). We aim to compute the expected number of such blocks after all $n$ drops.

**Step 1**

**Using symmetry and indicator variables.**
As established in Part 1(c), a block begins at cell $I_i$ if it is covered while its preceding cell $I_{i-1}$ is not.
Let $X_i$ be the indicator variable that a block starts at cell $I_i$. Then the total number of blocks is:

$$B = \sum_{i=1}^{k} X_i.$$

By symmetry and linearity of expectation:

$$\mathbb{E}[B] = \sum_{i=1}^{k} \mathbb{E}[X_i] = k \cdot \mathbb{E}[X_1].$$

**Step 2**

**Computing $\mathbb{E}[X_1]$.**
A block starts at $I_1$ if $I_1$ is covered and $I_k$ is not.

- Probability that a specific cell is uncovered by a single drop is $1 - \frac{m}{k}$,
- So the probability that cell $I_k$ is not covered after all $n$ drops is:

$$\left(1 - \frac{m}{k}\right)^n.$$

- Probability that both $I_1$ and $I_k$ are not covered simultaneously is:

$$\left(1 - \frac{m+1}{k}\right)^n.$$

Thus, the probability that a block starts at $I_1$ is:

$$\mathbb{E}[X_1] = \left(1 - \frac{m}{k}\right)^n - \left(1 - \frac{m+1}{k}\right)^n.$$

**Step 3**

**Final expression and adjustment.**
Plugging back into the expectation:

$$\mathbb{E}[B] = k \cdot \left[\left(1 - \frac{m}{k}\right)^n - \left(1 - \frac{m+1}{k}\right)^n\right].$$

However, this expression can fall below 1 when coverage is nearly full — even though, as argued in Part 1(c), a fully covered circle still contains exactly one block.
To ensure this, we adjust the formula:

$$\mathbb{E}[B] = \max\left(1,\ k\left[\left(1 - \frac{m}{k}\right)^n - \left(1 - \frac{m+1}{k}\right)^n\right]\right).$$

**Conclusion**

$$\boxed{\mathbb{E}[\#\text{ blocks}] = \max\left(1,\ k\left[\left(1 - \frac{m}{k}\right)^n - \left(1 - \frac{m+1}{k}\right)^n\right]\right)}$$

This expression gives the expected number of contiguous covered segments (blocks) after randomly dropping $n$ length-$m$ intervals on a $k$-cell circle. The use of $\max(1, \cdot)$ ensures correct behavior even when the entire circle becomes covered.

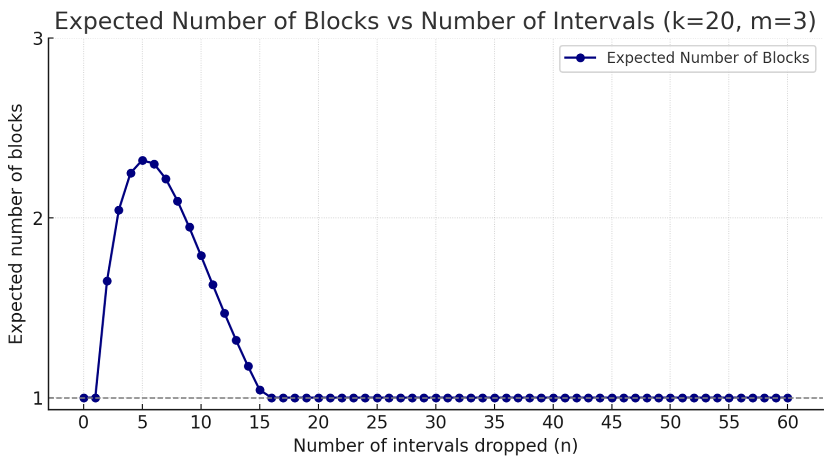



**Figure.** Expected number of blocks (for $k = 20, m = 3$) as a function of $n$, using the formula from above.

**Graph Interpretation**

The curve initially rises as isolated blocks form, reaching a peak where partial coverage produces the most fragmentation. As $n$ increases further, overlaps merge adjacent blocks, and eventually the entire circle is covered—yielding a single block. This explains why the graph has a clear maximum before dropping smoothly to 1.

## 3 The Continuum Limit

**Setup**

**Setup.** We now consider the continuous version of our model by letting the number of cells $k \to \infty$ and interval length $m \to \infty$, while keeping the ratio $\alpha = \frac{m}{k} \in (0, 1)$ fixed.
In this limit, each discrete interval becomes a continuous arc of length $\alpha$ on the unit circle. Our goal is to determine how much of the circle is expected to be covered after $n$ such arcs are dropped independently and uniformly at random.

### (a) Expected Fraction of the Circle Covered

**Step 1**

**Start from the discrete expectation.**
Previously, we showed that in the discrete case:

$$\mathbb{E}[\text{covered length}] = 1 - \left(1 - \frac{m}{k}\right)^n.$$

Define the fixed ratio $\alpha = \frac{m}{k}$. As $k \to \infty$ and $m \to \infty$, the unit circle becomes continuous, and this formula approaches:

$$\lim_{k\to\infty} \left(1 - \left(1 - \frac{m}{k}\right)^n\right) = 1 - (1 - \alpha)^n.$$

**Conclusion**

$$\boxed{\mathbb{E}[\text{fraction covered}] = 1 - (1 - \alpha)^n}$$

This gives the expected fraction of the circle covered by $n$ arcs of length $\alpha$, each dropped independently and uniformly at random.

**Interpretation**

Each arc of length $\alpha$ covers an $\alpha$-sized portion of the circle. Since drops are independent, the uncovered portion of a fixed point after $n$ arcs is $(1 - \alpha)^n$. Subtracting this from 1 gives the total expected coverage.

## (b) Probability That a Fixed Arc Is Fully Covered

**Setup**

**Setup.** On the unit circle $S^1$, fix the target arc

$$A = [0, \lambda], \quad \lambda \in (0, 1].$$

We drop $n$ random arcs $I_1, \dots, I_n$, each of (arc-)length $\alpha \in (0, 1)$, by choosing centers uniformly on $S^1$. Compute

$$\mathbb{P}\big(A \subseteq I_1 \cup \cdots \cup I_n\big).$$

**Step 1**

**Recast via uncovered points and inclusion–exclusion.**
The arc $A$ is fully covered if and only if no point $x \in A$ is left uncovered. Define

$$E_x = \{\text{no arc covers } x\}.$$

Then

$$\mathbb{P}(A \text{ covered}) = 1 - \mathbb{P}\left(\bigcup_{x \in A} E_x\right).$$

Using the inclusion–exclusion principle over finite subsets of bad points:

$$\mathbb{P}(A \text{ covered}) = \sum_{r=0}^{\infty} (-1)^r \int_{0 < x_1 < \cdots < x_r < \lambda} \mathbb{P}(E_{x_1} \cap \cdots \cap E_{x_r})\, dx_1 \cdots dx_r.$$

## Step 2

**Safe-center measure $\ell_r(x_1, \ldots, x_r)$.**
For fixed $x_1 < \cdots < x_r < \lambda$, an arc of length $\alpha$ avoids all $x_i$ iff its center lies in none of the $\alpha/2$ neighborhoods of any $x_i$.
That means centers must lie in the $r$ disjoint open gaps between the $x_i$, taken modulo 1:

$$\begin{aligned} g_1 &= x_2 - x_1, \\ &\vdots \\ g_{r-1} &= x_r - x_{r-1}, \\ g_r &= (1 - x_r) + x_1. \end{aligned}$$

A safe center in each gap must be $\alpha/2$ away from both endpoints, so the usable portion is length $\max(0, g_i - \alpha)$. Total safe length:

$$\ell_r = \sum_{i=1}^{r-1} \max(0, x_{i+1} - x_i - \alpha) + \max(0, 1 - x_r + x_1 - \alpha).$$

By independence over $n$ arcs:

$$\mathbb{P}(E_{x_1} \cap \cdots \cap E_{x_r}) = [\ell_r(x_1, \ldots, x_r)]^n .$$

## Step 3

**Final integral form.**
Substitute into the inclusion–exclusion formula:

$$\boxed{\mathbb{P}([0, \lambda] \subseteq \textstyle\bigcup_{j=1}^n I_j) = \sum_{r=0}^\infty (-1)^r \int_{0<x_1<\cdots<x_r<\lambda} [\ell_r(x_1, \ldots, x_r)]^n \, dx_1 \cdots dx_r,}$$

with

$$\ell_r = \sum_{i=1}^{r-1} \max(0, x_{i+1} - x_i - \alpha) + \max(0, 1 - x_r + x_1 - \alpha).$$

## Conclusion

$$\boxed{\mathbb{P}\left([0, \lambda] \subseteq \bigcup_{j=1}^n I_j\right) = \sum_{r=0}^\infty (-1)^r \int_{0<x_1<\cdots<x_r<\lambda} \left[\sum_{i=1}^r \max(0, g_i - \alpha)\right]^n dx_1 \cdots dx_r}$$

Here, the gaps $g_1, \ldots, g_{r+1}$ are the circular distances between consecutive uncovered points:

$$\begin{aligned} &g_1 = x_1, \quad g_i = x_i - x_{i-1} \text{ for } 2 \leq i \leq r-1, \\ &g_r = 1 - x_r + x_1 \text{ (wraparound)}. \end{aligned}$$

This formula gives the exact probability that the arc $[0, \lambda]$ is fully covered by $n$ arcs of length $\alpha$, dropped uniformly on the circle.

## (c) Expected Number of Blocks

**Setup**

**Setup.** We consider the continuous model, where $n$ arcs of fixed length $\alpha \in (0, 1)$ are dropped independently and uniformly at random on the unit circle $S^1$.
A *block* is defined as a maximal contiguous covered segment. A block begins at a point $x \in S^1$ if:

- $x$ lies inside some arc, and
- the point immediately before $x$ (in the circular sense) is not covered by any arc.

Our goal is to compute the expected number of such blocks formed after $n$ arcs are dropped.

**Step 1**

**Block initiation from an arc.**
Suppose one arc is dropped, centered at a random point $c \in S^1$. It covers the interval $[c - \alpha/2,\ c + \alpha/2)$ on the circle.
This arc starts a block if its left endpoint lies in a region that was not already covered by any of the other $n - 1$ arcs. Since arcs are placed independently, this probability depends only on the coverage of that point by the remaining arcs.

**Step 2**

**Probability that an arc starts a new block.**
For a single arc, the probability that a fixed point is not covered by one random arc is $1 - \alpha$. Hence, the probability that a point is not covered by any of the other $n - 1$ arcs is:

$$(1 - \alpha)^{n-1}.$$

Since arcs are independent, this is exactly the chance that the left endpoint of a given arc begins a new block.

**Step 3**

**Expected number of blocks.**
Each of the $n$ arcs has an independent probability of $(1 - \alpha)^{n-1}$ of starting a new block. So the expected number of such starting points (i.e., blocks) is:

$$\mathbb{E}[\#\text{ blocks}] = n \cdot (1 - \alpha)^{n-1}.$$

However, once the entire circle is covered, we should not return an expected value below 1 — there must be at least one block. We therefore take the maximum between this value and 1:

$$\mathbb{E}[\#\text{ blocks}] = \max\left(1,\ n \cdot (1 - \alpha)^{n-1}\right).$$

**Conclusion**

$$\boxed{\mathbb{E}[\#\text{ blocks}] = \max\left(1,\; n \cdot (1-\alpha)^{n-1}\right)}$$

This formula gives the expected number of contiguous covered segments formed by $n$ arcs of length $\alpha$ dropped independently on the unit circle. It correctly accounts for merging of segments as $n$ increases, and ensures that once the circle is fully covered, at least one block remains.

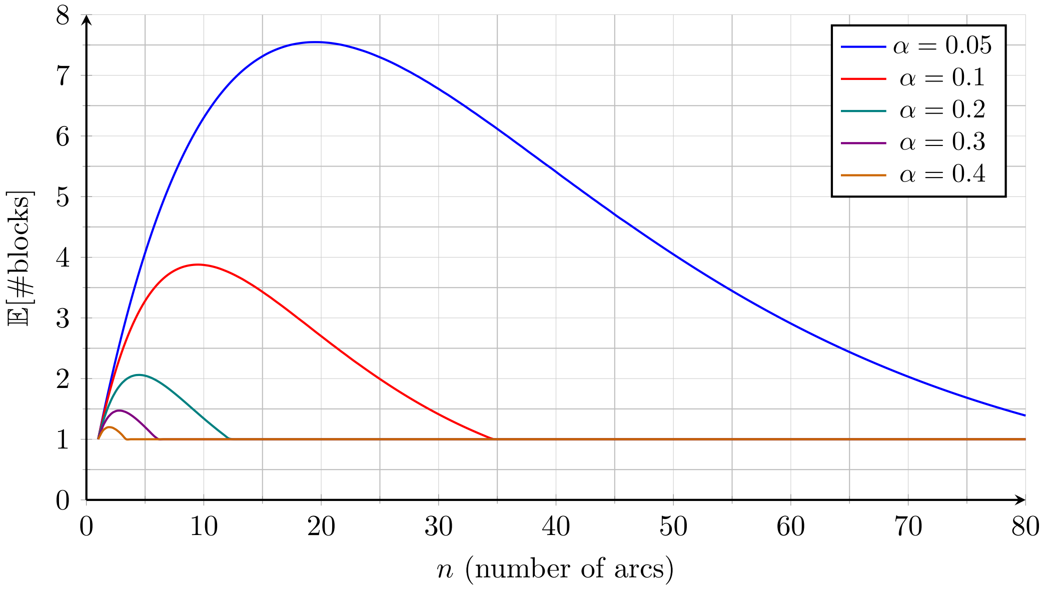



Figure 1: **Expected number of blocks vs. number of arcs** $n$, for various arc lengths $\alpha$. Smaller $\alpha$ values lead to more gradual coverage (slower decay toward 1), while larger arcs cause faster full coverage. All curves flatten to 1 once the full circle is likely covered, representing a single wrap-around block.

# 4 Colour Cycles and Limit Behavior

**Setup**

**Setup.** We divide the circle into $k$ equally spaced cells, each initially coloured $C_0$. Each of the $n$ random intervals (arcs) covers $m$ consecutive cells, corresponding to a coverage proportion $\alpha = \frac{m}{k}$. Each time a cell is hit by an interval, its colour increments cyclically:

$$C_0 \to C_1 \to \cdots \to C_{c-1} \to C_0 \to \cdots .$$

We aim to compute the expected total length of cells of each colour $C_j$, where $j = 0, 1, \ldots, c-1$.

## (a) Expected Total Length of Each Colour

**Step 1**

**Modelling a single cell.**
Fix a single cell on the circle. Each dropped interval independently covers it with probability $\alpha = \frac{m}{k}$. Let $T \sim \text{Binomial}(n, \alpha)$ be the number of times the cell is hit. After $T$ increments, its final colour is $C_{T \bmod c}$. Hence, the probability that the cell ends up with colour $C_j$ is:

$$\mathbb{P}(T \equiv j \mod c).$$

**Step 2**

**Extracting modulo-$c$ residues using Fourier filters.**
We want to isolate:

$$\sum_{\substack{r=0 \\ r \equiv j \mod c}}^{n} \binom{n}{r} \alpha^r (1-\alpha)^{n-r}.$$

To filter the terms where $r \equiv j \mod c$, we use the **roots-of-unity filter**:

$$\frac{1}{c} \sum_{s=0}^{c-1} \omega^{(r-j)s} = \begin{cases} 1, & \text{if } r \equiv j \mod c, \\ 0, & \text{otherwise,} \end{cases}$$

where $\omega = e^{2\pi i/c}$ is a primitive $c$th root of unity.
*(See: Hardy Wright, *An Introduction to the Theory of Numbers*, Theorem 91, 6th ed.)*
This identity enables us to extract binomial coefficients in a fixed congruence class modulo $c$ from the full binomial expansion.

**Step 3**

**Application to the binomial distribution.**
Apply the filter:

$$\mathbb{P}(T \equiv j \mod c) = \sum_{r=0}^{n} \binom{n}{r} \alpha^r (1-\alpha)^{n-r} \cdot \frac{1}{c} \sum_{s=0}^{c-1} \omega^{-js} \omega^{rs}.$$

Change the order of summation:

$$= \frac{1}{c} \sum_{s=0}^{c-1} \omega^{-js} \sum_{r=0}^{n} \binom{n}{r} (\alpha\omega^s)^r (1-\alpha)^{n-r}.$$

Recognize the inner sum as the binomial expansion of:

$$(1-\alpha+\alpha\omega^s)^n.$$

Therefore,

$$\mathbb{P}(T \equiv j \mod c) = \frac{1}{c} \sum_{s=0}^{c-1} \omega^{-js} (1-\alpha+\alpha\omega^s)^n.$$

**Step 4**

**Expected total length of colour $C_j$.**
Since each cell behaves identically and independently, and there are $k$ cells in total:

$$\mathbb{E}[Z_j] = k \cdot \mathbb{P}(T \equiv j \mod c), \quad \mathbb{E}[X_j] = \frac{1}{k} \cdot \mathbb{E}[Z_j] = \mathbb{P}(T \equiv j \mod c).$$

So the expected total *length* (fraction of the circle) in colour $C_j$ is simply the probability that a fixed cell ends in colour $C_j$.

**Conclusion**

$$\boxed{\mathbb{E}[X_j] = \frac{1}{c} \sum_{s=0}^{c-1} \omega^{-js} (1-\alpha+\alpha\omega^s)^n}$$

This formula expresses the expected proportion of the circle that ends in colour $C_j$, using a discrete Fourier transform (roots-of-unity filter) to select cells whose number of hits lands in a specific residue class modulo $c$.

## (b) Asymptotic Colour Proportions as $n \to \infty$

**Setup**

**Setup.** Recall from Part 4(a) that after $n$ drops, the expected fraction of the circle coloured $C_j$ is

$$\mathbb{E}[X_j(n)] = \frac{1}{c}\sum_{s=0}^{c-1} \omega^{-js}\big(1-\alpha+\alpha\,\omega^s\big)^n,$$

where $\alpha = \frac{m}{k}$ and $\omega = e^{2\pi i/c}$. We now let $n \to \infty$ (with $\alpha, c$ fixed) and determine $\lim_{n\to\infty} \mathbb{E}[X_j(n)]$.

**Step 1**

**Isolate the dominant term.**
Split the sum into the $s = 0$ term and the remainder:

$$\mathbb{E}[X_j(n)] = \frac{1}{c}\Big[\underbrace{\omega^{-j\cdot 0}(1-\alpha+\alpha\cdot 1)^n}_{=1} + \sum_{s=1}^{c-1} \omega^{-js}\,(1-\alpha+\alpha\,\omega^s)^n\Big].$$

Since $\omega^0 = 1$, the first term is exactly $1/c$.

**Step 2**

**Show all other terms vanish.**
For $1 \le s \le c-1$, $\omega^s$ is a non-trivial $c$th root of unity, so $\Re(\omega^s) < 1$. Hence

$$\big|1-\alpha+\alpha\,\omega^s\big| < 1.$$

Raising to the $n$th power gives $\big|1-\alpha+\alpha\,\omega^s\big|^n \to 0$ exponentially fast as $n \to \infty$. Therefore each summand with $s \ge 1$ tends to zero in the limit.

**Step 3**

**Take the limit.**
Combining Steps 1–2,

$$\lim_{n\to\infty} \mathbb{E}[X_j(n)] = \frac{1}{c}\Big[1+0\Big] = \frac{1}{c}.$$

**Conclusion**

$$\boxed{\lim_{n\to\infty} \mathbb{E}[X_j(n)] = \frac{1}{c}.}$$

In the limit of many drops, each colour $C_j$ occupies exactly one-$c$th of the circle in expectation. This follows rigorously from the roots-of-unity expansion.

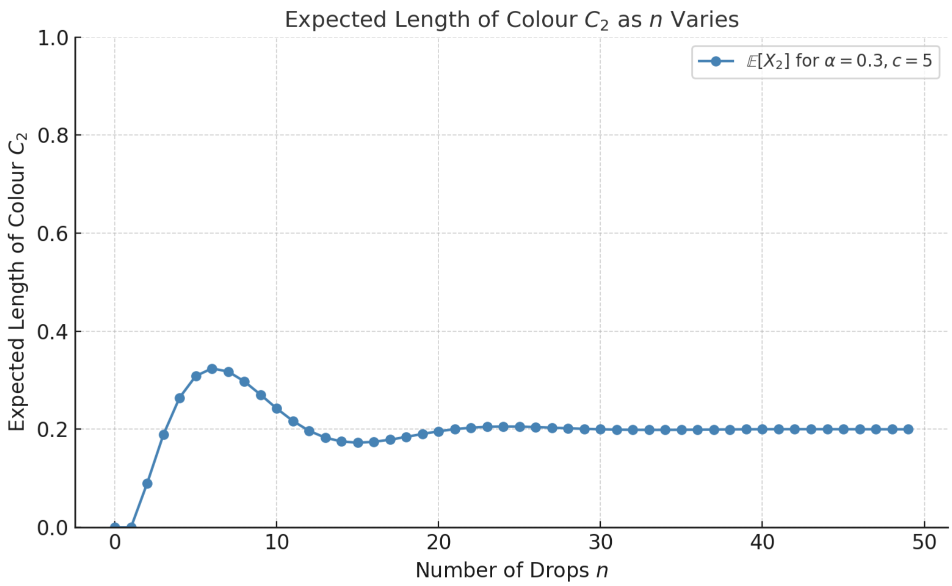
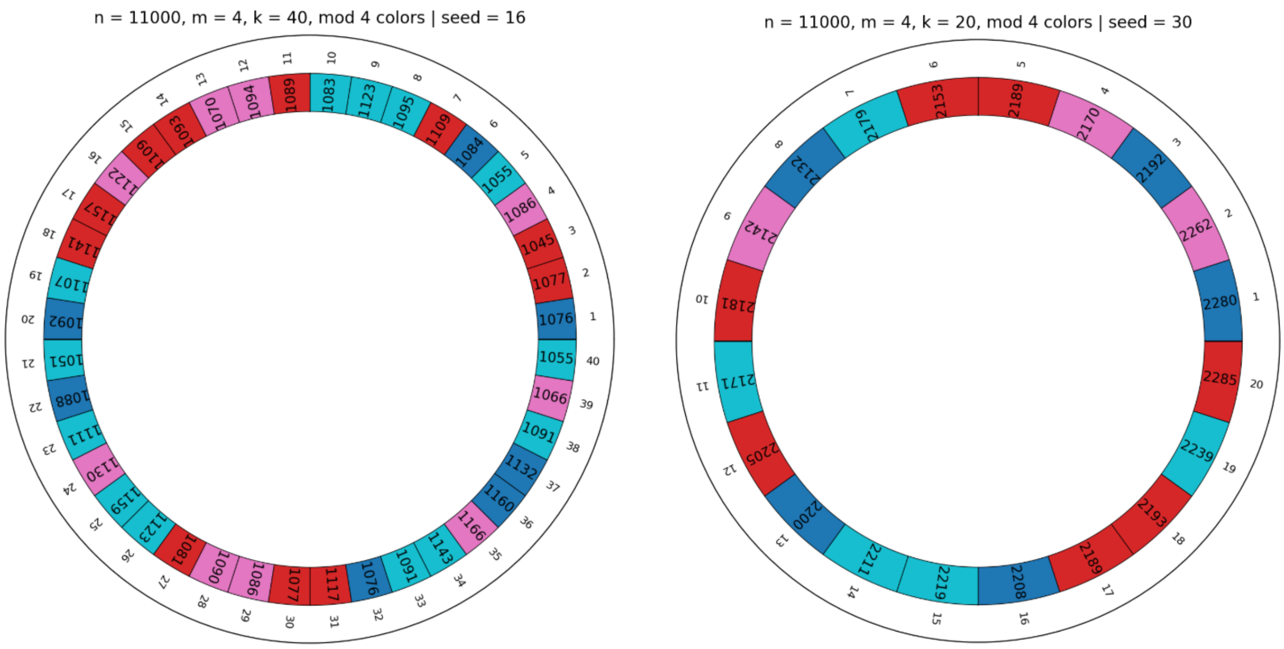



Figure 2: Evolution of Colour Proportions. The graph shows how the expected length of a fixed colour class $C_j$ evolves as the number of interval drops $n$ increases. Initially, the entire circle starts in colour $C_0$, so other colours appear gradually as overlaps accumulate. Due to the cyclic colour rule ($C_0 \to C_1 \to \cdots \to C_{c-1} \to C_0$), the distribution temporarily oscillates but eventually stabilizes: each colour occupies approximately $\frac{1}{c}$ of the circle.




(a) $k = 40$, $m = 4$, $c = 4$, $n = 11000$ (b) $k = 20$, $m = 4$, $c = 4$, $n = 11000$

Figure 3: **Colour evolution after many drops.** Each arc drop increments the colour of cells it hits cyclically mod $c$. These diagrams show the final configuration for different cell counts $k$, with the same total drop count $n = 11000$. For large $n$, the distribution of colours stabilizes as predicted by the roots-of-unity formula in Part 4a, and visually verifies the oscillatory convergence behavior described in Part 4b.

## 5 Continuous Coverage and Limit Behavior

**Setup**

**Setup.** We consider a unit circle $S^1$, and drop $n$ arcs of fixed length $\alpha \in (0,1)$, with centers chosen independently and uniformly at random over the circle.
This is the continuous analogue of the discrete setting in earlier sections. Our goal is to study three quantities:

1. The expected total length covered,
2. The probability that a fixed arc of length $\lambda$ is fully covered,
3. The expected number of contiguous covered regions (blocks).

### (a) Expected Total Length Covered

**Step 1**

**Probability that a point is covered.**
Fix any point $x \in S^1$. An arc of length $\alpha$ covers $x$ if its center lies within an interval of length $\alpha$ centered at $x$, i.e., the interval $[x - \alpha/2,\ x + \alpha/2]$ modulo 1.
Since arc centers are uniform over $S^1$, the probability that a single arc covers $x$ is:

$$\mathbb{P}(\text{arc covers } x) = \alpha.$$

Therefore, the probability that none of the $n$ arcs cover $x$ is:

$$(1-\alpha)^n,$$

and the probability that $x$ is covered by at least one arc is:

$$\mathbb{P}(x \text{ is covered}) = 1 - (1-\alpha)^n.$$

**Step 2**

**Expected total coverage.**
By symmetry of the circle and uniform randomness, every point $x \in S^1$ has the same probability of being covered.
Hence, the expected covered length is:

$$\mathbb{E}[\text{total length covered}] = \int_{S^1} \mathbb{P}(x \text{ is covered})\, dx = \int_{S^1} \left(1 - (1-\alpha)^n\right)\, dx.$$

But this integrand is constant over the circle, so the integral equals:

$$\left(1 - (1-\alpha)^n\right) \cdot \text{length}(S^1) = 1 - (1-\alpha)^n.$$

**Conclusion**

$$\boxed{\mathbb{E}[\text{total length covered}] = 1 - (1-\alpha)^n}$$

This formula gives the expected proportion of the unit circle covered by $n$ randomly placed arcs of length $\alpha$.

## (b) Probability That a Fixed Arc of Length $L$ Is Fully Covered

**Setup**

**Setup.** On the unit circle $S^1$ of circumference 1, we fix an arc

$$A = [0, L], \quad 0 < L \leq 1,$$

and then drop $n$ random arcs $I_1, \ldots, I_n$, each of length $\alpha \in (0, 1)$, with centers chosen uniformly on $S^1$. We want

$$\mathbb{P}\big(A \subseteq I_1 \cup \cdots \cup I_n\big).$$

**Step 1**

**Rephrase as "no uncovered point."**
Let for each $x \in A$,

$$E_x = \{\text{every arc misses } x\}.$$

Then $A$ is fully covered exactly when no $E_x$ occurs:

$$\{A \text{ covered}\} = \Big(\bigcup_{x \in A} E_x\Big)^c, \quad \mathbb{P}(A \text{ covered}) = 1 - \mathbb{P}\Big(\bigcup_{x \in A} E_x\Big).$$

**Step 2**

**Apply inclusion–exclusion.**
By the inclusion–exclusion principle (Bonferroni's inequalities),

$$\mathbb{P}\Big(\bigcup_{x \in A} E_x\Big) = \sum_{r=1}^{\infty} (-1)^{r+1} \sum_{0 < x_1 < \cdots < x_r < L} \mathbb{P}\big(E_{x_1} \cap \cdots \cap E_{x_r}\big).$$

Equivalently,

$$\mathbb{P}(A \text{ covered}) = \sum_{r=0}^{\infty} (-1)^r \sum_{0 < x_1 < \cdots < x_r < L} \mathbb{P}\big(E_{x_1} \cap \cdots \cap E_{x_r}\big),$$

where the $r = 0$ term is understood to be 1.

**Step 3**

**Probability all $r$ points are missed.**

Fix $0 < x_1 < \cdots < x_r < L$. A single arc of length $\alpha$ covers $x_i$ iff its center lies in $[x_i - \frac{\alpha}{2},\ x_i + \frac{\alpha}{2}]$ (mod 1). Let

$$U = \bigcup_{i=1}^{r} [x_i - \tfrac{\alpha}{2},\ x_i + \tfrac{\alpha}{2}] \pmod 1,$$

and write $|U|$ for its total length. Then one arc avoids all $x_i$ with probability $1 - |U|$, and by independence

$$\mathbb{P}\big(E_{x_1} \cap \cdots \cap E_{x_r}\big) = \big(1 - |U|\big)^n.$$

**Step 4**

**From sums to integrals.**

Summing over every ordered choice $0 < x_1 < \cdots < x_r < L$ is equivalent to integrating over that simplex. Hence

$$\sum_{0<x_1<\cdots<x_r<L} \big(1 - |U|\big)^n \longrightarrow \int_{0<x_1<\cdots<x_r<L} \big[\,1 - |U(x_1, \dots, x_r)|\,\big]^n \, dx_1 \cdots dx_r.$$

Putting it all together,

$$\boxed{\mathbb{P}\big(A \subseteq \bigcup_{j=1}^{n} I_j\big) = \sum_{r=0}^{\infty} (-1)^r \int_{0<x_1<\cdots<x_r<L} \big[\,1 - |U(x_1, \dots, x_r)|\,\big]^n \, dx_1 \cdots dx_r.}$$

**Conclusion**

**Remarks.**

- The union $U$ is taken *mod 1*, so any wrap-around of intervals is handled automatically.
- No separate "outside $A$" term is needed—the measure $|U|$ on $S^1$ already accounts for centers both inside and outside the target arc.
- For $r = 1$, one recovers the elementary term $-\int_0^L (1-\alpha)^n dx = L\,(1-\alpha)^n$.
- Absolute convergence and the use of inclusion–exclusion on this continuum are justified by the classical Bonferroni inequalities (see Feller[**Feller1971**], Ch.1).

## (c) Expected Number of Covered Blocks

**Setup**

**Setup.** On the unit circle $S^1$, we drop $n$ arcs of fixed length $\alpha \in (0, 1)$, choosing each center independently and uniformly. A *block* is a maximal connected covered segment (see Section 3.1.c). We seek

$$\mathbb{E}[\#\text{blocks}].$$

**Step 1**

**Block-start criterion.**
An arc centered at $c$ covers $\left[c-\frac{\alpha}{2},\ c+\frac{\alpha}{2}\right]$. Its left endpoint

$$x = c - \frac{\alpha}{2}$$

begins a new block precisely if none of the other $n-1$ arcs covers $x$.

**Step 2**

**Avoidance probability.**
Another arc covers $x$ iff its center falls in the length-$\alpha$ interval $[x, x+\alpha]$. Hence the chance it *avoids* $x$ is

$$1-\alpha,$$

and by independence all $n-1$ others avoid $x$ with probability

$$(1-\alpha)^{n-1}.$$

**Step 3**

**Linearity of expectation.**
Each of the $n$ arcs has the same probability $(1-\alpha)^{n-1}$ to start a block. Therefore

$$\mathbb{E}[\#\text{blocks}] = n \ \times \ (1-\alpha)^{n-1}.$$

**Conclusion**

$$\boxed{\mathbb{E}[\#\text{blocks}] \ = \ n\,(1-\alpha)^{n-1}.}$$

Since $\alpha \in (0,1)$ and $n \geq 1$, one checks easily $n(1-\alpha)^{n-1} \geq 1$, so this already enforces at least one block.

**Dropping intervals simulation**

**Open the simulation!**

Python based simulation using streamlit and matplot lib to simulate dropping random intervals on a circle

## 6 Research Directions

### Future Horizons

**1. Threshold for Certainty.** Investigate the minimal number of arcs $n^*(\alpha)$ so that full coverage becomes almost sure—seek sharp asymptotics as $\alpha \to 0$.

**2. Peak Block Phenomenon.** Characterize the drop–count $n_{\max}(k, m)$ at which the expected number of covered blocks attains its maximum, and describe its scaling in $k, m$.

**3. Variable-Length Ensembles.** Allow each arc's length to vary according to a random law; study how the length-distribution modulates coverage and fragmentation statistics.

**4. Continuum Limits on Curves.** Extend to non-circular geometries (ellipses, smooth loops) and derive the corresponding inclusion–exclusion integrals.

**5. Spatial Correlation Models.** Replace independent centers by mildly correlated patterns—probe how repulsion or clustering alters cover times and block counts.

**6. Adaptive Placement Rules.** Design dynamic schemes that choose each new center based on the current largest gap—compare random vs. gap-targeting in expectation.

**7. Rare-Event Analysis.** Quantify the probabilities of exceptionally slow or fast coverage via large-deviation estimates and identify phase-transition thresholds.

**8. Algorithmic Complexity.** Determine whether exact coverage probabilities admit efficient approximation algorithms when $k, n$ grow large.