DA-IICT

MC 315 Independent Project

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Independent Project Report

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Contents

Introduction

Patterns that repeat themselves no matter how much you zoom in reveal complexity, the closer we look into them. Fractals break our intuition about shapes, dimensions, and structure. Fractals are the centre of understanding patterns in nature, modelling irregular phenomena and exploring the connection between mathematics and complexity. One of the most intriguing properties of fractals is their scaling behaviour, where the structures look similar at different magnification levels and yet they defy the general definitions of dimension.

This report presents an exploration into the scaling properties of fractals and how they arise in various systems. The aim is to understand how these objects scale, how their dimensions can be measured using various mathematical methods, and how scaling differs across different fractals. The following section will introduce the concept of fractals in more detail, including key examples and classifications.

1 Countable and Uncountable sets

This section overviews some parts of set theory where we will talk about parts of infinity that are useful in understanding the fractals.

Are some infinities larger than others? The answer is yes. In the late 19th century, mathematician Georg Cantor introduced a way to compare infinite sets. He stated that two sets X and Y have the same size—what we call cardinality—if there's a way to pair each element of Y, and vice versa. Such a mapping is called a **one-to-one correspondence**. It can be said that each element of one set, after some transformation or with a relation, has exactly one element in the other set that it can be paired with.

An infinite set more commonly known to us is the set of natural numbers N 1,2,3,4,.... This set provides a basis for comparison—if another set X can be put into one-to-one correspondence with the natural numbers, then X is said to be countable. Otherwise, X is uncountable. These sets that are countable but head till infinity we coin a term called infinitely countable.

Is the set of the even numbers(E) countable
Solution
Let $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ be the set of natural numbers and let $E = \{2, 4, 6, 8, \dots\}$ be the set of even numbers.
We define a function:

$$f : \mathbb{N} \to E \quad \text{by} \quad f(n) = 2n$$

Injectivity:
Assume $f(n_1) = f(n_2)$. Then:

$$2n_1 = 2n_2 \Rightarrow n_1 = n_2$$

Therefore, f is injective (one-to-one).
Surjectivity:
Let $e \in E$. Then there exists some $n \in \mathbb{N}$ such that $e = 2n$. Thus:

$$f(n) = e$$

So every even number has a pre-image in $\mathbb{N}$, making f surjective (onto).

Is the set of the even numbers(E) countable
Solution
Let $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ be the set of natural numbers and let $O = \{1, 3, 5, 7, \dots\}$ be the set of odd numbers.
We define a function:

$$f : \mathbb{N} \to O \quad \text{by} \quad f(n) = 2n - 1$$

Injectivity:
Assume $f(n_1) = f(n_2)$. Then:

$$2n_1 - 1 = 2n_2 - 1 \Rightarrow 2n_1 = 2n_2 \Rightarrow n_1 = n_2$$

Therefore, f is injective (one-to-one).

Surjectivity:
Let $o \in O$. Then there exists some $n \in \mathbb{N}$ such that $o = 2n - 1$. Thus:

$$f(n) = o$$

So every odd number has a pre-image in $\mathbb{N}$, making f surjective (onto).

2 What are Fractals

Let us start the study of fractals by first knowing the history of how it was identified and how it became one of the most observed concepts in almost every field.

2.1 History of fractals

Benoit Mandelbrot is often characterised as the **father of fractal geometry**. However, it can be said that the notion of fractals and similar structures existed much more before Mandelbrot coined these terms.

The origins of fractal geometry can be traced back to the 17th and 18th centuries, when mathematicians started experimenting with odd, seemingly strange curves and patterns. Many of the fractals and their descriptions go to the mathematicians of the past, like Georg Cantor (1883), Giuseppe Peano (1890), David Hilbert (1891), Helge von Koch (1904), Waclaw Sierpinski (1916), Gaston Julia (1918), or Felix Hausdorff (1919), to name just a few. Karl Weierstrass's(1872) study of nowhere-differentiable curves revealed mathematical irregularities that defied classical geometry. These objects were continuous everywhere yet had no specified tangent at any point, a condition that puzzled mathematicians for decades.

In the last part of the 19th century, Felix Klein and Henri Poincaré introduced the term "self-inverse" fractals. One of the next milestones occurred in 1904, when Helge von Koch, expanding Poincaré's ideas and dissatisfied with Weierstrass's abstract and analytic formulation, provided a more geometric definition that included hand-drawn drawings of a comparable function, now known as the Koch snowflake. A decade later, in 1915, Wacław Sierpinski built his famous triangle, followed by his carpet the next year. By 1918, two French mathematicians, Pierre Fatou and Gaston Julia, working independently, arrived simultaneously at the results describing what is now seen as fractal behaviour associated with mapping complex numbers and iterative functions and leading to further ideas about attractors and repellors. Very shortly after that work, Felix Hausdorff expanded on the notion of "dimension" to allow for sets to have non-integer dimensions.

In 1975, Mandelbrot solidified hundreds of years of thought and mathematical development in coining the word "Fractal", from the Latin word fractus, which means "broken" or "fragmented." Many mathematicians played a key role in Mandelbrot's concept of a new geometry. However, they did not view their creations as conceptual steps towards a new perception or a new geometry of nature. Instead, they were regarded as exceptional objects, counterexamples, mathematical monsters.'One of the main reason for this was the lack of visualising tools and the only reliance on the hand-drawn images and tools, which restricted them to visualise and create new types of fractal sets to analyse them further.

2.2 Fractals Definations

Fractals are complex structures that exhibit self-similarity, that is their patterns repeat at different scales. However, there is no single universal definition of a fractal; different types of fractals and different perspectives of fractals emphasise on different characteristics. Benoit Mandelbrot coined the definition of fractal as :

"A fractal is by definition a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension"

This mathematical view focuses on the idea of fractional dimensions as a key property of fractals, highlighting their non-Euclidean nature.

In practice, however, not all fractals fit neatly into this definition. Some are better defined through their recursive construction (like geometric fractals such as the Sierpinski triangle, Sierpinski's Gasket), while others arise from iterative dynamical processes (like the Mandelbrot and Julia sets). In more general terms, a fractal is any structure that has great detail at arbitrarily small sizes, is too irregular to be easily explained using traditional study of geometry, and frequently exhibits self-similarity or statistical similarity.

2.3 Examples

Let us look at the various types of fractals and where are the different field of the nature they are found in

2.3.1 In Nature

1. **In Plants**

(a) Romanesco Broccoli : Source

(b) Branches and root of tree Source

Figure 1: Repetition nature in plants

(a) Veins of Leaves : Source

(b) Fern : Source

Figure 2: Repetition nature in leaves

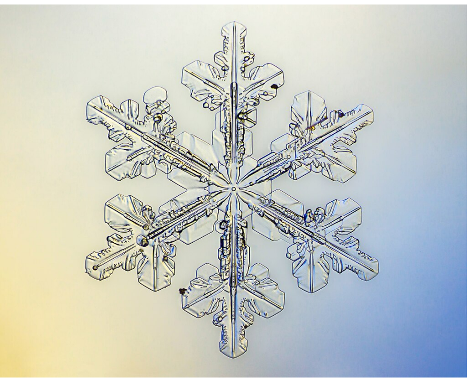
2. **In Lightnings and Thunder**

(a) High-voltage breakdown within a 4in (100mm) block of acrylic glass Source

(b) Real lightning showing self-similarity. Source

Figure 3: Lightning and thunder showing self-similarity in both artificial and natural environments.

3. **In Snowflakes**



(a) Snowflake Source

(b) Frost crystals occurring naturally on cold glass. Source

Figure 4: Different Snowflakes showing self-similar nature

4. **In Geography**

(a) River Tributaries : Source

(b) Ganges tributaries form above. Source

Figure 5: River Tributaries

2.3.2 In Geometric/Iterative Fractals

1. **Cantor Set**

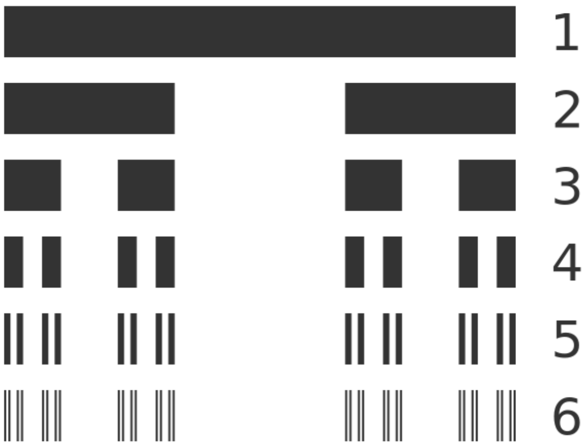


Figure 6: This is a Cantor set till 6 iterations : Source

2. **Fractal tree**

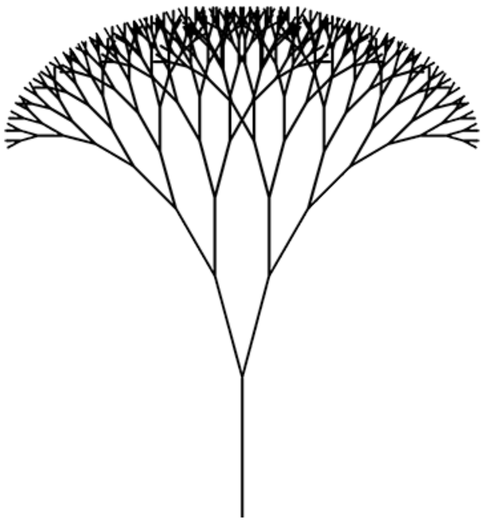


Figure 7: Fractal tree: Source

3. **Sierpinski's Triangle**

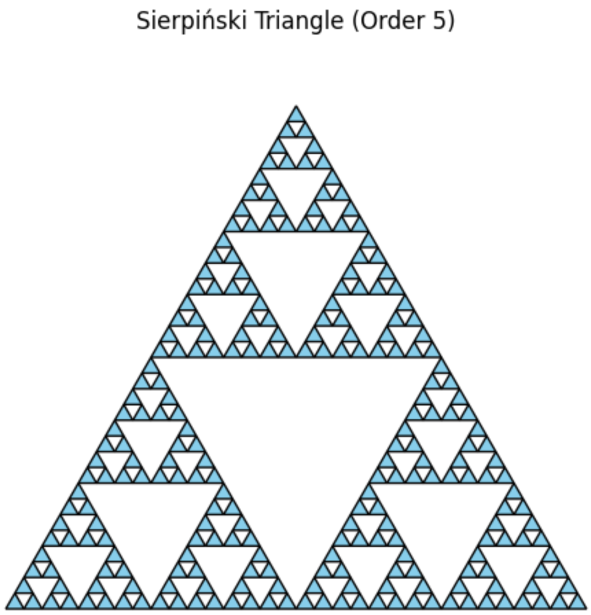


Figure 8: Sierpinski at 5^{th} iteration

4. **Koch's Snowflake**

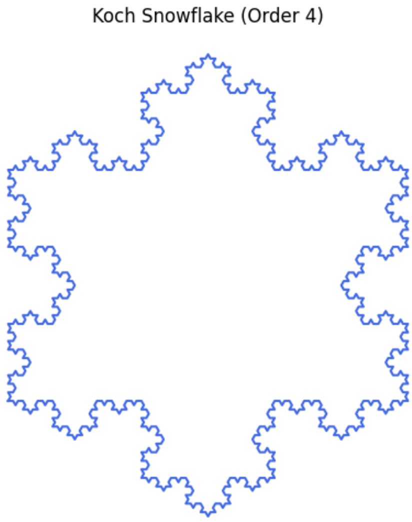


Figure 9: Koch's Snowflake at 4^{th} iteration

5. **Mandelbrot Set**

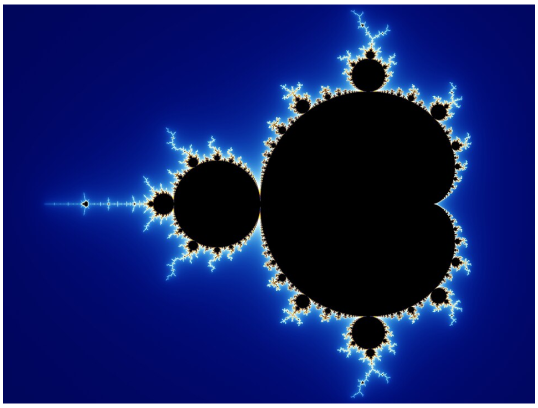


Figure 10: Mandelbrot Set : Source

Apart from these, any other geometrical structures show fractal behaviours such as the Fibonacci sequence(Golden Spiral), etc. Though there might not be named structures but with simple repetition of a function on a geometric shape can give us a fractal.

2.3.3 In Dynamical Systems

1. **Logistic Map** The logistic map equation is given by:

$$x_{n+1} = rx_n(1 - x_n)$$

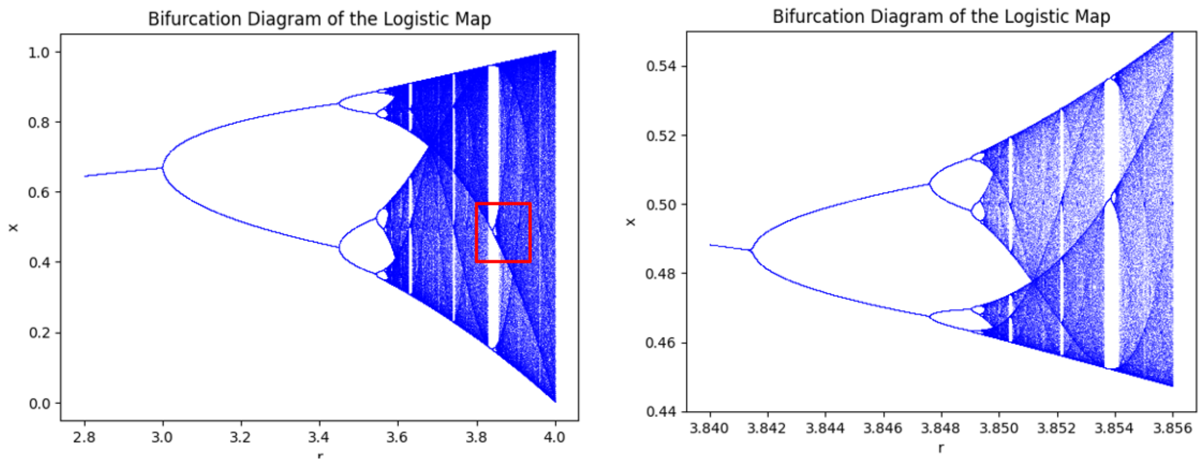


Figure 11: This is a logistic Map. The right side figure is the zoomed-in version to see the self-similarity.

2. **Lorenz Attractor** The following equations are the Lorenz Equations

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(r - z) - y$$

$$\frac{dz}{dt} = xy - bz$$

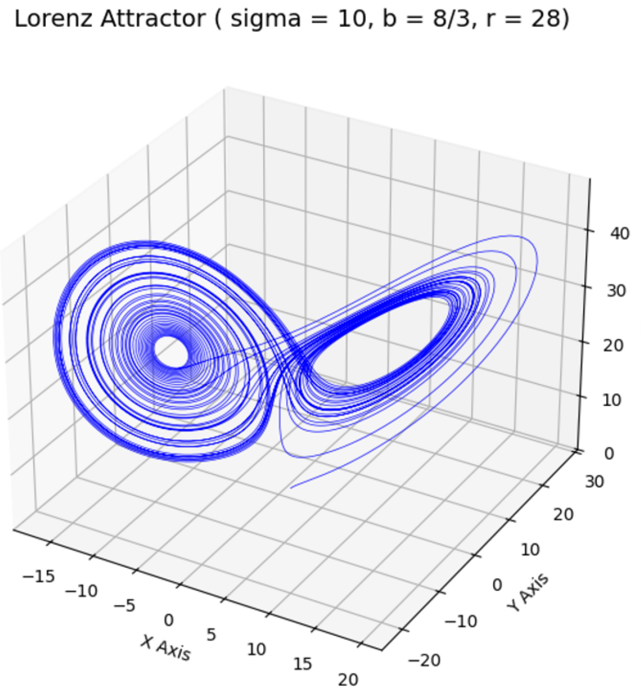


Figure 12: This is the Lorenz Attractor.

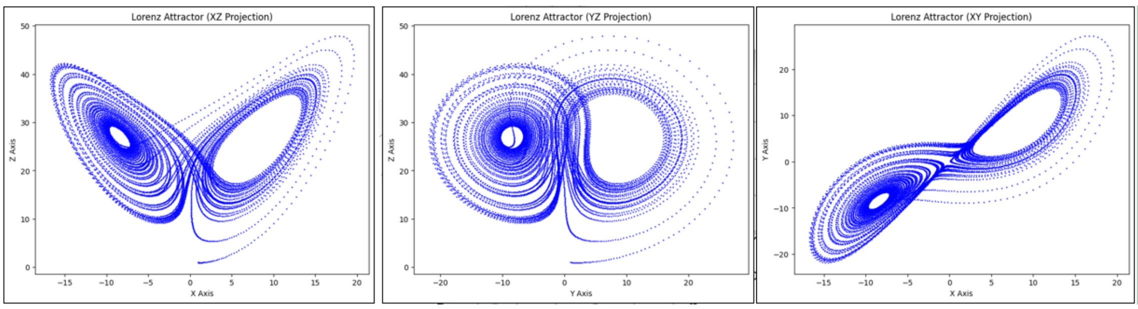


Figure 13: This is the Lorenz Attractor in 2-D version

2.3.4 In Biology

Fractals appear extensively in the human body, where many complex structures must fit within limited space while maximising efficiency. One of the most prominent examples is the network of blood vessels, which branches repeatedly into smaller and smaller vessels, forming a self-similar structure that allows efficient transport of blood throughout the body.

Similarly, the bronchial tree of the lungs has a fractal design, which allows for optimum surface area for gas exchange in a small space. Neurons also have fractal properties, with dendrites branching repeatedly to build massive, complicated networks optimised for connectivity and signal transmission.

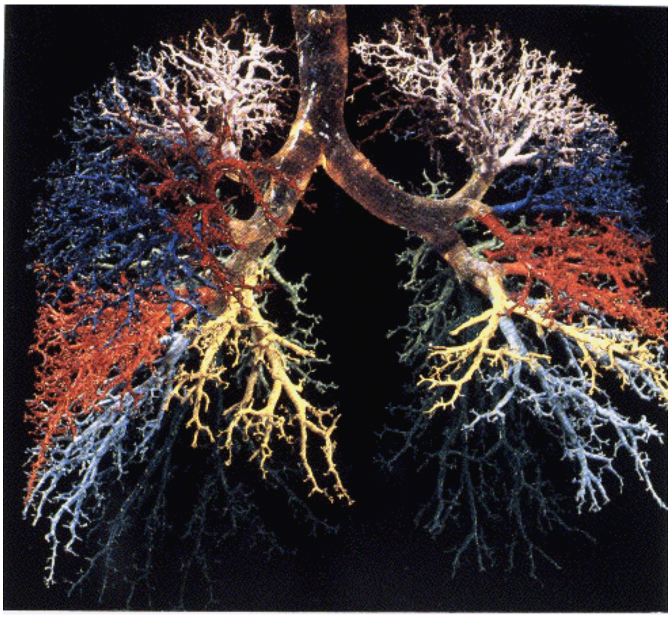


Figure 14: This is a picture of a resin-filled lung with its bronchis being visible, showing a self-similar structure like a fractal treeUniversity of Washington

3 Scaling and Self-Similarity

Fractals are intricate structures defined by their repeating patterns across scales, commonly referred to as self-similarity. In mathematics, a self-similar entity is one whose pieces resemble the whole, either exactly or roughly. In strict self-similarity, the object may be broken down into components that are precise reproductions of the original.

The concept of scaling is a fundamental feature of many natural and complex systems. An object or process exhibits scaling if it remains essentially the same when its size is multiplied by a factor. This idea leads to the identification of scaling laws, such as power laws.

Power Law
A mathematical relationship where one variable (y) is proportional to another variable (x) raised to a constant power,

$$y = ax^n$$

A special form of self-similarity is **scale invariance**, where the object looks the same at any level of magnification and fractals follow power laws because of this property. This recurring pattern results in mathematical relationships wherein, instead of changing linearly with size, values such as length, area, or frequency vary according to a power of that size. Because of this lack of a characteristic size, measurements and distributions naturally obey power laws. Many complex systems exhibit these scaling characteristics.

The Koch snowflake is a classic example: no matter how closely one zooms in on one of its sides, the jagged pattern repeats perfectly at every level. This lack of a characteristic scale—where no single size dominates, is a defining feature of many fractals and complex systems. However, not all repeating patterns are fractal. For instance, a straight line may resemble itself at every segment but lacks the detailed complexity and structure on smaller scales that fractals possess.

- A ***scaling relation*** is present if there are continuous and real-valued functions f and g, such that upon a scale transformation, $x \to \lambda x$, the following holds:
$$f(\lambda x) = g(\lambda) f(x)$$
where λ is a real positive number and $g(\lambda) = \lambda^d$.

- Therefore, it can be written as
$$f(\lambda x) = \lambda^d f(x)$$
where λ is the scaling factor, g is the scaling function, and d is the dimension of the function.

To rigorously define fractals, mathematicians use methods like box-counting and Hausdorff dimension, which help quantify how detail increases with scale. Since perfect fractals cannot be physically constructed since they require infinite magnification, for analysing them practically, one focuses on finite approximations of them.

So now let us look at how to find the dimension of these fractals and what do we exactly mean when we say the dimension of fractals.

4 Fractal Dimensions

4.1 Why do fractals defy the normal norms of dimension?

Fractals pose a challenge to traditional geometric knowledge because they do not conform to standard, integer-based dimensions. Traditional shapes like lines, squares, and cubes exist neatly in one, two, or three dimensions, respectively. But no matter how much one zooms in, a new structure keeps appearing because fractals have limitless complexity and self-similarity at every size. They cannot be measured using traditional methods like length or area.

For example, a Koch snowflake has a topological dimension of 1, yet its length is infinite, and it occupies more space than a line but less than a plane. This in-between, fragmented behaviour cannot be described adequately by whole-number dimensions, highlighting the need for a new way to define and understand their structure.

They are said to have filled their space qualitatively and quantitatively differently from how an ordinary geometrical set does. This is not like density; it measures the complexity of the curve/set/fractal.

4.2 Why We Need to Find the Dimension of Fractals?

To quantify the complexity of fractals and compare how they fill space, we use the concept of fractal dimension, capturing how detail increases with scale. This allows us to distinguish between fractals with similar shapes but varying degrees of complexity, and to make sense of patterns that traditional geometry cannot describe. For example, a curve with a fractal dimension of 1.9 behaves very differently from one with a dimension of 1.1, although both are 'curves' in a topological sense. It gives us a powerful tool to mathematically describe and analyse forms that are otherwise too complex for classical geometry.

5 Scaling / Similarity Dimension

One of the simplest ways to define the dimension of a fractal is through self-similarity. Self-similar fractals are composed of smaller, scaled-down copies of themselves, continuing recursively to arbitrarily small scales. To understand how this relates to dimension, we can begin with classical examples of self-similar shapes, such as line segments, squares, or cubes.

Consider a square divided into smaller, equally sized squares. If the square is divided into 4 smaller squares by halving its length in each direction (scale factor $r = 2$), it takes $m = 4$ small squares to reconstruct the original. If the same square is divided into 9 smaller squares by scaling down by $r = 3$, it takes $m = 3$ parts to cover the whole. This demonstrates that in two dimensions, the number of self-similar pieces grows proportionally to r^2, where the exponent (2 in this case) reflects the space's dimension.

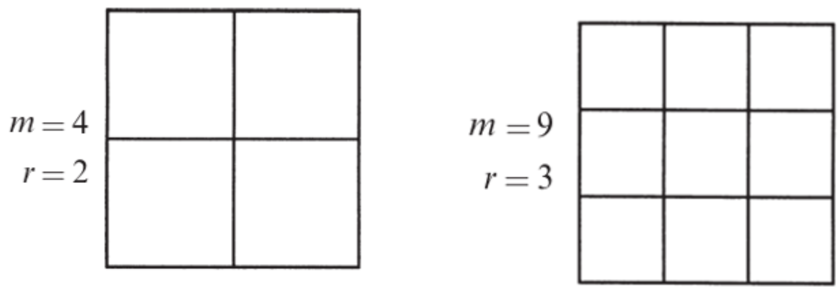


Figure 15: These both are squares of length 1 unit, where the first figure is divided into squares of length $\frac{1}{2}$ and the second square is divides into squares of length $\frac{1}{3}$. Here, r is the scale factor and m is the number of boxes.

This reasoning extends naturally to three dimensions. For example, if a cube is scaled down by a factor of r, then r^3 smaller cubes are required to fill the original volume, illustrating the three-dimensional nature of the shape. These relationships form the basis of the similarity dimension, defined for any self-similar object.

Suppose that a self-similar set is composed of m copies of itself scaled down by a factor of r. Then the similarity dimension d is the exponent defined by $m = \frac{1}{r^d}$ or equivalently, (taking log both sides)

$$d = \frac{\ln m}{\ln(\frac{1}{r})}$$

Here it is $\frac{1}{r}$ because each time it is reduced by the factor $\frac{1}{r}$

5.1 Example: Cantor Set

The Cantor Set is a well-known example of a self-similar fractal constructed by iteratively removing the middle third of a line segment. At each iteration, the number of segments increases while their individual lengths decrease.

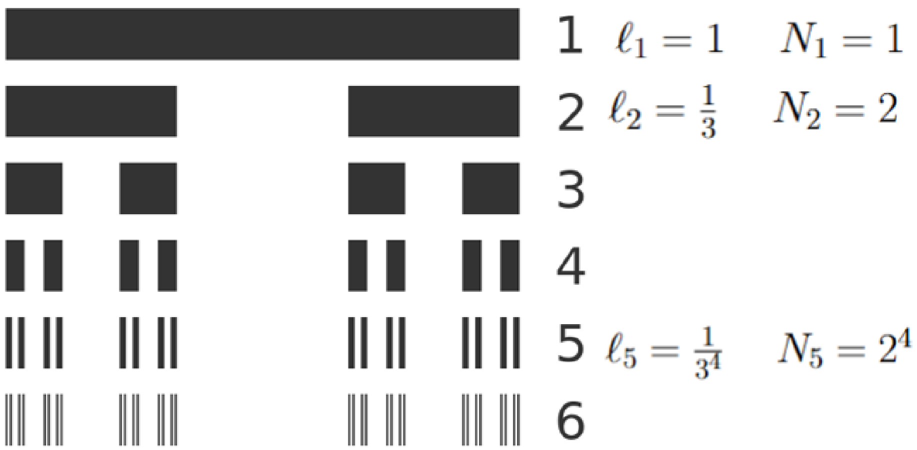


Figure 16: Cantor Set with 6 iterations

Let ℓ_n be the length of a segment at the n-th iteration, and N_n the number of such segments. For the Cantor set, these values follow:

$$\ell_n = \frac{1}{3^{n-1}}, \quad N_n = 2^{n-1}$$

The similarity dimension d of the fractal can be calculated using the formula:

$$d = \frac{\ln(N_n)}{\ln(1/\ell_n)}$$

Substituting the expressions for N_n and ℓ_n, we obtain:

$$d = \frac{(n-1)\ln 2}{(n-1)\ln 3} = \frac{\ln 2}{\ln 3}$$

Thus, the similarity dimension of the Cantor set is:

$$d \approx 0.631$$

This non-integer value reflects that the Cantor Set occupies more space than a collection of discrete points (dimension 0), but less than a full line (dimension 1).

5.2 Example: Sierpinski's Triangle

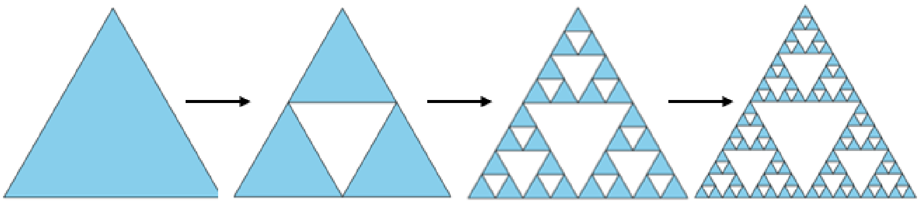


Figure 17: Seirpinski triangle for 4 iterations

The Sierpinski triangle is created by recursively removing the central (equilateral) triangle from an equilateral triangle. At each iteration, the number of remaining triangles triples, while the side length of each triangle is halved. Let $\ell_n = \frac{1}{2^n}$ denote the side length of a triangle in the n-th iteration, and $N_n = 3^n$ be the total number of triangles at that stage.

The similarity dimension d can be calculated using the relation:

$$d = \frac{\ln(N_n)}{\ln(1/\ell_n)}$$

Substituting the expressions for N_n and ℓ_n, we get:

$$d = \frac{n \ln 3}{n \ln 2} = \frac{\ln 3}{\ln 2}$$

Thus, the fractal (similarity) dimension of the Sierpinski triangle is:

$$d \approx 1.585$$

This fractional value indicates that the Sierpinski triangle is more space-filling than a 1D object but less than a 2D solid.

Table 1: Iteration Properties of the Sierpiński Triangle

Iteration (n)	**No. of Triangles** (N_n)	**Side Length** (ℓ_n)	**Total Perimeter**	**Total Area** (A_n)
0	$3^0 = 1$	$\frac{1}{2^0} = 1$	3	$A_0 = \frac{\sqrt{3}}{4}$
1	3^1	$\frac{1}{2^1}$	$\frac{3^2}{2^1}$	$\frac{3}{4}A_0$
2	3^2	$\frac{1}{2^2}$	$\frac{3^3}{2^2}$	$\left(\frac{3}{4}\right)^2 A_0$
3	3^3	$\frac{1}{2^3}$	$\frac{3^4}{2^3}$	$\left(\frac{3}{4}\right)^3 A_0$
n	3^n	$\frac{1}{2^n}$	$\frac{3^{n+1}}{2^n}$	$\left(\frac{3}{4}\right)^n A_0$

5.3 Example: Koch's Snowflake

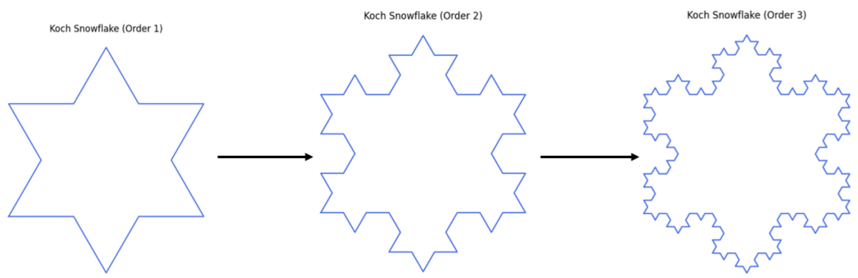


Figure 18: Koch's Snowflake for 3 iterations

The Koch curve is a well-known fractal generated by replacing each line segment with four new segments, each one-third the length of the original. Let $\ell_n = \frac{1}{3^n}$ be the length of a segment at iteration n, and $N_n = 4^n$ be the total number of segments at that stage. The similarity dimension d is given by:

$$d = \frac{\ln(N_n)}{\ln(1/\ell_n)} = \frac{n \ln 4}{n \ln 3} = \frac{\ln 4}{\ln 3} \approx 1.2619$$

This dimension shows that the Koch curve fills more space than a simple line but less than a full surface.

Table 2: Iteration Properties of the Koch Curve

Iteration (n)	**No. of Segments (N_n)**	**Segment Length (ℓ_n)**	**Total Length**
0	$4^0 = 1$	1	1
1	4^1	$\frac{1}{3}$	$\frac{4}{3}$
2	4^2	$\frac{1}{3^2}$	$\left(\frac{4}{3}\right)^2$
3	4^3	$\frac{1}{3^3}$	$\left(\frac{4}{3}\right)^3$
n	4^n	$\frac{1}{3^n}$	$\left(\frac{4}{3}\right)^n$

6 Box-Dimension

The box-counting method is a standard technique used to estimate the fractal dimension of sets that may not exhibit exact self-similarity. The approach involves covering the fractal set with boxes (or cubes) of a fixed size ε, and counting the number of boxes $N(\varepsilon)$ required to fully cover the structure.

For classical geometric shapes, this method aligns with intuitive notions of dimension. For instance, for a smooth curve of length L, we find that $N(\varepsilon) \propto \frac{L}{\varepsilon}$, and for a 2D region of area A, $N(\varepsilon) \propto \frac{A}{\varepsilon^2}$.

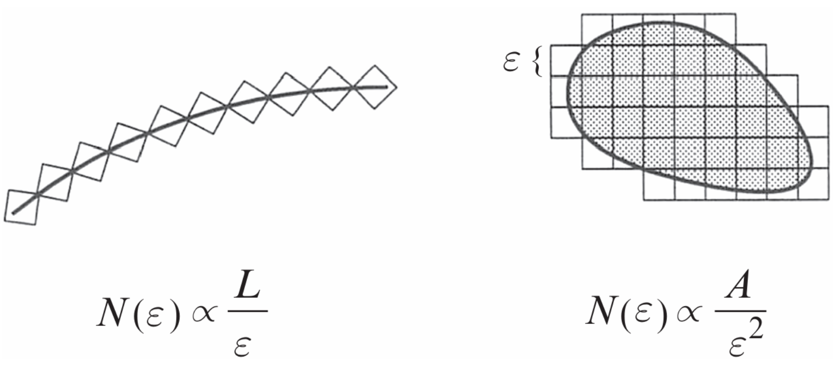


Figure 19

These follow the general power-law form:

$$N(\varepsilon) \propto \varepsilon^{-d}$$

The exponent d can be interpreted as the dimension of the set. For fractals, this exponent typically yields a non-integer, indicating the object's complex scaling behaviour. The box-counting dimension is also called the capacity dimension. Therefore, the box-counting dimension can be given as:

$$d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}$$

6.1 Example: Coast of Great Britain with Ireland

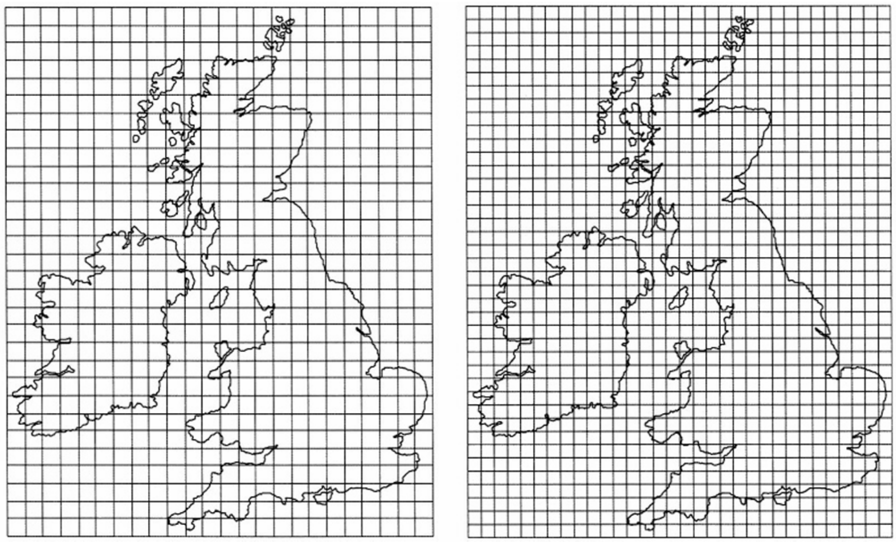


Figure 20: The coastline of Great Britain with Ireland covered in boxes of different ϵ (Chaos and Fractals)

As an example, let us start to understand it form the classic example, the coastline of Great Britain. The figure shows an outline of the coast with two underlying grids. Having normalised the width of the entire grid to 1 unit, the mesh sizes are $\frac{1}{24}$ and $\frac{1}{32}$. The box count yields 194 and 283 boxes that intersect the coastline in the corresponding grids. From these data it is now easy to derive the box-counting dimension. When entering the data into a log/log diagram, the slope of the line that connects the two points is

$$d = \frac{\log 283 - \log 194}{\log 32 - \log 24} \approx \frac{2.45 - 2.29}{1.51 - 1.38} \approx 1.31.$$

6.2 Example: Cantor Set

To compute the box dimension of the Cantor set, consider that at each step the end points of the cantor set are taken as set i.e at S_1 the set is 0,1 at S_2 we have two sets $\{0,\frac{1}{3}\}$ and $\{\frac{2}{3},1\}$ and S_3 will have sets $\{0,\frac{1}{9}\}$,$\{\frac{2}{9},\frac{1}{3}\}$, $\{\frac{2}{3},\frac{7}{9}\}$ and $\{\frac{8}{9},1\}$

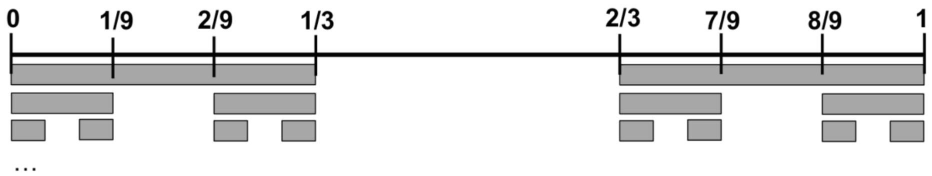


Figure 21: Cantor Set

Therefore, similarly S_n, will consists of 2^n intervals of length $(1/3)^n$. So if we pick $\varepsilon = (1/3)^n$, the number of intervals/boxes we will need will be $N(\varepsilon) = 2^n$. Thus,

$$d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = \lim_{n \to \infty} \frac{\ln(2^n)}{\ln(3^n)} = \frac{n \ln 2}{n \ln 3} = \frac{\ln 2}{\ln 3}.$$

This agrees with the similarity dimension. While the box dimension is formally defined using continuous $\varepsilon \to 0$, using the discrete sequence $\varepsilon = (1/3)^n$ ensures that it approaches 0.

The method for finding the pointwise dimension and the correlation dimension was given by **Peter Grassberger** and **Itamar Procaccia** in 1983 in their paper named *MEASURING THE STRANGENESS OF STRANGE ATTRACTORS*

7 Point-wise Dimension

While the box-counting dimension is a useful starting point for analysing fractals, it often falls short when we are dealing with real-world fractal structures or chaotic systems where the point distribution is highly irregular. It simply measures how many boxes are needed to cover a set as the box size shrinks, but it doesn't account for how points are distributed within those boxes. This is where the correlation dimension and pointwise dimension become essential. These dimensions reveal how complexity changes from point to point and how tightly points cluster at small scales, crucial for understanding the true nature of a fractal beyond its outer geometry.

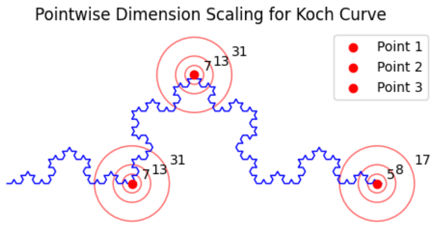
The pointwise dimension provides a local measure of the fractal geometry near a specific point on an attractor.

- To compute it, we select a point on the attractor and place a ball of radius ε around it.
- Then count how many other points lie within this ball.
- As ε decreases, this count, $N_x(\varepsilon)$, typically follows a power-law scaling of the form $N_x(\varepsilon) \propto \varepsilon^d$, where d is the pointwise dimension at that location.

Unlike the global box-counting dimension, which averages over the entire set, the pointwise dimension captures local scaling behaviour.

7.1 Example: Koch's Curve

Here, let us take an example of the Koch curve to see how the procedure works. Here we randomly took three points on the Koch curve, and then we took a circle around them for varying sizes and counted the vertices(since there are no points present in the curve)



(a) Koch's Curve of Order 4 with random points selected

Point	Radius	Points within Radius
Point 1 (x=0.333, y=0.000)	0.1	31
Point 1 (x=0.333, y=0.000)	0.05	13
Point 1 (x=0.333, y=0.000)	0.025	7
Point 2 (x=0.500, y=0.289)	0.1	31
Point 2 (x=0.500, y=0.289)	0.05	13
Point 2 (x=0.500, y=0.289)	0.025	7
Point 3 (x=0.988, y=0.000)	0.1	17
Point 3 (x=0.988, y=0.000)	0.05	8
Point 3 (x=0.988, y=0.000)	0.025	5

(b) This table contains the coordinates of the points and the radius, with the number of points falling in that radius

Figure 22: Pointwise Dimension Methodology in Koch Curve of Order 4

8 Correlation Dimension

One drawback with the pointwise dimension is that it depends significantly on x(the point that we chose at random); it will be smaller where the attractor has fewer points around it (i.e. rarefield) and higher where there are densely close points. To get an overall dimension of the attractor, one averages $N_x(\epsilon)$ over many x. This was the general idea, let us take a look at how Grassberger and Procaccia found the correlation dimension.

The correlation dimension estimates the fractal dimension of a set by evaluating how densely its points are distributed. This technique is especially useful in analysing strange attractors and chaotic systems in dynamical studies.

- A large number of trajectory points are collected from the system.
- For each scale ε, the number of point pairs that lie within a distance less than ε is counted. This is formalized using the *correlation sum*:

$$C(\varepsilon) = \frac{2}{N(N-1)} \sum_{i<j} H(\varepsilon - |x_i - x_j|)$$

where $H(x)$ is the **Heaviside function**, defined as:

$$H(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

- This sum effectively counts the fraction of point pairs that are separated by less than ε.
- By plotting $\log C(\varepsilon)$ against $\log \varepsilon$, a linear region often emerges.

To calculate it, The **slope** of this linear region provides an estimate of the correlation dimension, which captures the scaling behaviour of point-wise clustering in the dataset.

8.1 Example: Sierpinski Triangle

Let us first understand the algorithm by taking a look at a fractal that we already know. Here we generated multiple points from the Sierpinski triangle and applied the algorithm here

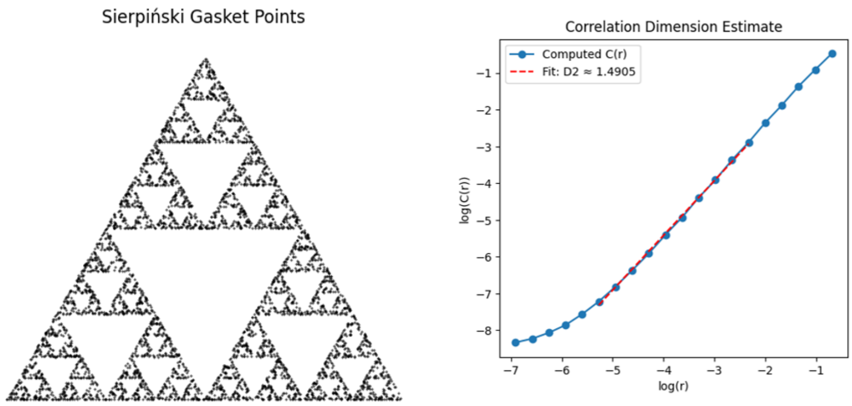


Figure 23: Seirpinski triangle for 4 iterations

8.2 Example: Logistic Map

Let us now apply the algorithm for the Logistic Map. The recursive equation for the logistic map is given by,

$$x_{n+1} = r x_n (1 - x_n)$$

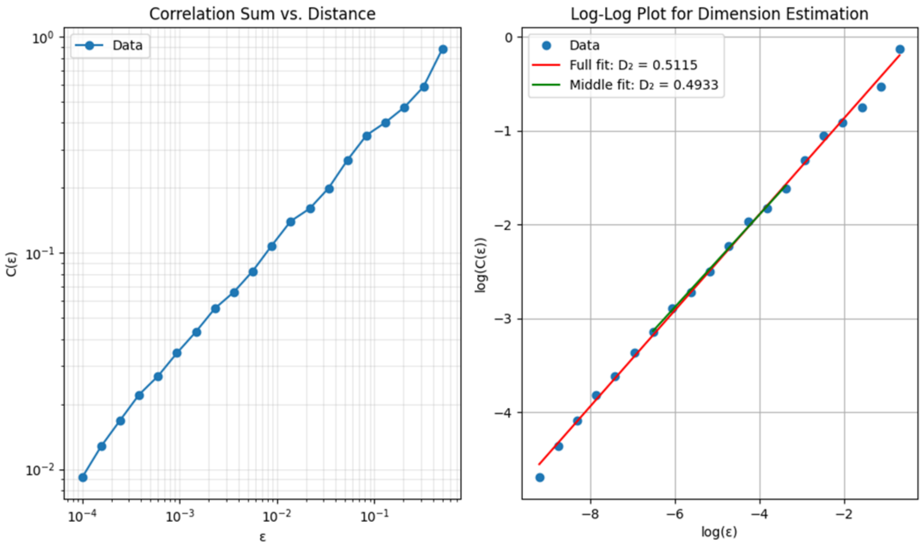


Figure 24: Dimension of Logistic Map using the Correlation Dimension

Here we will take the middle fit as it provides us with more precise and proper fit of the regression.

8.3 Example: Lorenz Equation

A dynamical system is a mathematical framework that describes how a point in a given space evolves, governed by specific rules or equations. Within such systems, attractors represent a set of states or points toward which nearby trajectories tend to evolve, regardless of their starting conditions. These attractors are the minimal sets in a phase portrait to which the system stabilises over time. Among these, a strange attractor is a particular type of attractor that not only draws in trajectories but also exhibits sensitive dependence on initial conditions, meaning small differences in starting points can lead to vastly different outcomes, a hallmark of chaotic behaviour.

The equations for the Lorenz equation are:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(r - z) - y$$

$$\frac{dz}{dt} = xy - bz$$

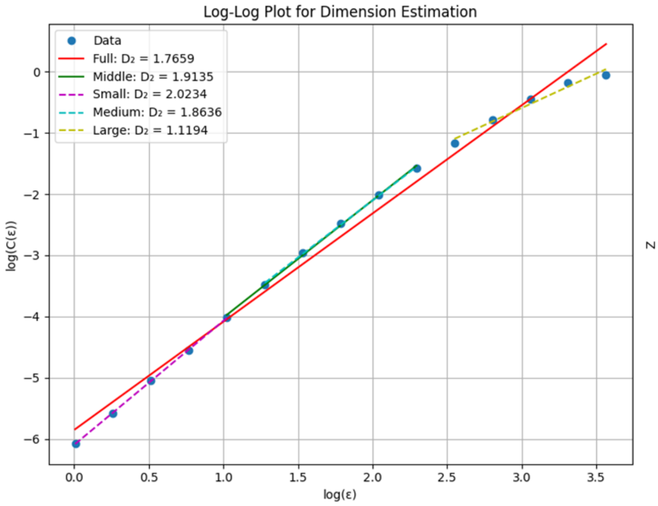


Figure 25: Correlation Fitting of the log-log plot the distances

The below figure contains the log values of the data points plotted in the figure above.

```
Data for log-log plot:
log(ε), log(C(ε))
0.005967, -6.074585
0.260303, -5.577088
0.514638, -5.041983
0.768974, -4.550955
1.023310, -4.014493
1.277645, -3.479018
1.531981, -2.960063
1.786316, -2.481434
2.040652, -2.018175
2.294988, -1.580015
2.549323, -1.174226
2.803659, -0.793594
3.057995, -0.444798
3.312330, -0.188539
3.566666, -0.053257
```

Figure 26: Data Points taken from the Lorenz Attractor

From the plot we obtained, we fitted the data points with various regression lines, but the small and middle fit give us the most nearly equal to dimension of the Lorenz attractor estimated by the Hausdorff Dimensio

9 Lyapunov Exponents and Dimension

Lyapunov exponents $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ are used to characterize the behavior of dynamical systems. These exponents are arranged in descending order: $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. They measure the rate of separation of infinitesimally close trajectories in the system, providing insight into the system's sensitivity to initial conditions.

A positive Lyapunov exponent ($\lambda > 0$) indicates chaotic behaviour, where trajectories diverge exponentially. A negative exponent ($\lambda < 0$) suggests convergence of nearby trajectories, while a zero exponent ($\lambda = 0$) implies neutral stability where trajectories neither converge nor diverge.

If two trajectories start very close to each other with an initial separation δ_0, then their separation at a later time t evolves approximately as $|\delta(t)| \approx e^{\lambda t}|\delta_0|$, where λ is the Lyapunov exponent. This exponential behaviour defines how sensitive the system is to small perturbations.

The maximal Lyapunov exponent can be calculated using the formula:

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \left(\frac{|\delta(t)|}{|\delta_0|} \right)$$

This measures the average exponential rate of divergence or convergence of nearby trajectories over time.

Every system can have multiple Lyapunov exponents, which together form the Lyapunov spectrum. The number of Lyapunov exponents is equal to the dimensionality of the system. Different initial directions of infinitesimal vectors result in different exponents, which together describe the stretching and contracting directions in phase space.

The dimension of a system's attractor can be estimated using the Kaplan-Yorke (Lyapunov) dimension. It is calculated using the formula:

$$D = D_{KY} = j + \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_j}{|\lambda_{j+1}|}$$

where j is the largest integer such that $\sum_{i=1}^{j} \lambda_i > 0$ and $\sum_{i=1}^{j+1} \lambda_i < 0$. This provides a way to estimate the fractal dimension of the attractor from the Lyapunov exponents.

The presence of a positive maximal Lyapunov exponent generally indicates chaos in the system, which is a hallmark of nonlinear dynamical behaviour.

10 Hausdorff Dimension

Since we have already talked about the Hausdorff dimension, let us see the formula to find it, too.
Let $X \subset \mathbb{R}^n$ be a subset of a metric space. The **Hausdorff dimension** of X is defined using the d-dimensional Hausdorff measure $\mathcal{H}^d(X)$.

Hausdorff Measure

$$\mathcal{H}^d(X) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d : \bigcup_{i=1}^{\infty} U_i \supset X,\ \operatorname{diam}(U_i) < \delta \right\}$$

Hausdorff Dimension

The **Hausdorff dimension** of X is given by:

$$\dim_{\mathrm{H}}(X) = \inf \left\{ d \geq 0 : \mathcal{H}^d(X) = 0 \right\}$$

Equivalently, it can also be expressed as:

$$\dim_{\mathrm{H}}(X) = \sup \left\{ d \geq 0 : \mathcal{H}^d(X) = \infty \right\}$$

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