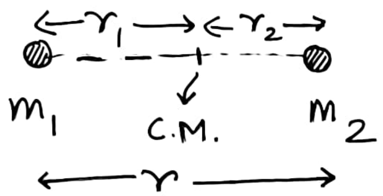
1.



$$r_1 = \frac{m_2}{m_1+m_2} r$$

$$r_2 = \frac{m_1}{m_1+m_2} r$$

$$m_1 \omega_1^2 r_1 = \frac{G m_1 m_2}{r^2}$$ (equating force of gravitation to centripetal force)

$$\Rightarrow \omega_1 = \sqrt{\frac{G(m_1+m_2)}{r^3}}$$

Similarly, $\omega_2 = \omega_1 = \sqrt{\frac{G(m_1+m_2)}{r^3}}$

Total energy of system

$= E_{kinetic} + E_{potential}$

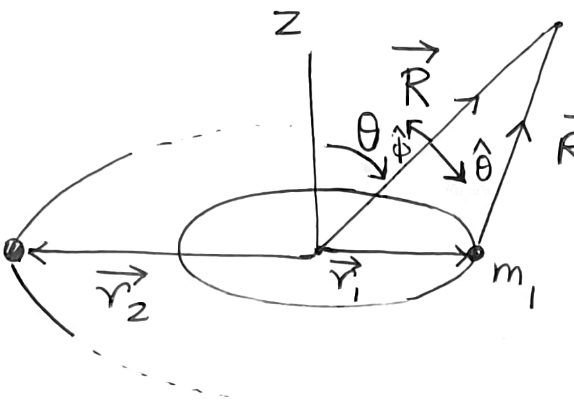
$$= \left(\frac{1}{2} m_1 \omega_1^2 r_1^2 + \frac{1}{2} m_2 \omega_2^2 r_2^2\right) - \frac{G m_1 m_2}{r}$$

$$= \frac{1}{2} \omega^{2/3} m_1 m_2 \frac{\cancel{(m_1+m_2)}}{(m_1+m_2)^{2 \cdot \frac{1}{3}}} G^{2/3} \cdot \cancel{(m_1+m_2)^{2/3}}$$

$$- G^{2/3} \omega^{2/3} \frac{m_1 m_2}{(m_1+m_2)^{1/3}}$$

$$= -\frac{1}{2} G^{2/3} \frac{m_1 m_2}{(m_1+m_2)^{1/3}} \omega^{2/3}$$

2.



$R \ggg r_1, r_2$

Just as accelerating charges radiate EM waves, accelerating masses radiate gravitational waves.

So, at this point we draw an analogy between EM and gravitational waves where $\frac{1}{4\pi\epsilon_0}$ and $-G$ are viewed as analogous to each other.

$$\frac{1}{4\pi\epsilon_0} \longrightarrow -G$$

$$\frac{\mu_0}{4\pi} \longrightarrow -\frac{G}{c^2}$$

In electromagnetism,

electric field $\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$

mag. field $\vec{B} = \vec{\nabla}\times\vec{A}$, ϕ is scalar potential

$\vec{A}$ is vector potential

Let us define gravitational vector and scalar potentials in a way similar to that in electromagnetism.

$$\vec{A}(\vec{R}, t) = \frac{\mu_0}{4\pi} \sum_i \frac{q_i v_i}{|\vec{R} - \vec{r}_i|}$$

$$\downarrow$$

$$\vec{A_G}(\vec{R}, t) = -\frac{G}{c^2} \sum_i \frac{m_i v_i}{|\vec{R} - \vec{r}_i|}$$

$$\phi_G(\vec{R}, t) = -G \sum_i \frac{m_i}{|\vec{R} - \vec{r}_i|}$$

In electromagnetism,

Poynting vector magnitude gives power per unit area in direction of propagation

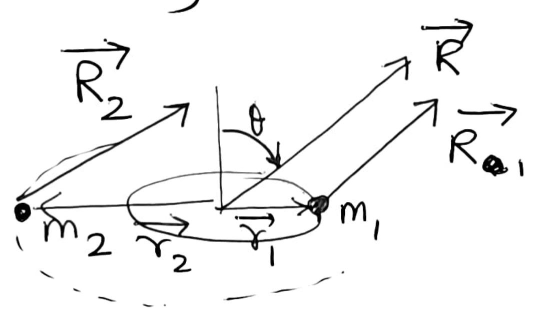
$$S_{EM} = \frac{c}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$$

Similarly, $S_G = \dfrac{c}{8\pi G}\left(E_g^2 + c^2 B_g^2\right)$

Also, $|E_g|^2 = c^2 |B_g|^2$

$$\Rightarrow \langle S_g \rangle = \frac{c}{4\pi G} \langle E_g^2(t) \rangle$$

Vector potential for a system of orbiting binary stars:



$$\vec{r} = \vec{r_1} - \vec{r_2}$$

$$\vec{A_G} = -\frac{G}{c^2}\frac{m_1 \vec{v_1}}{R_1} - \frac{G}{c^2}\frac{m_2 \vec{v_2}}{R_2} \quad \text{--- (1)}$$

$\vec{v_1}(t_{R_1})$, $\vec{v_2}(t_{R_2})$ [velocities are functions of retarded times]

Expressing velocities in terms of centre of mass retarded times,

$$t_R = t - R/c$$

$$t_{R_1} = t - R_1/c$$

$$t_{R_1} - t_R = \Delta t_1$$

$$\Rightarrow \Delta t_1 = \frac{1}{c}(R - R_1) \quad \text{--- (2)}$$

$v_1(t - R_1/c)$ $\longrightarrow$ Using Taylor expansion:

$$= v_1(t - R/c) + \left.\frac{dv_1}{dt}\right|_{t-R/c} \left(t - \frac{R_1}{c} - (t - R/c)\right) + \cdots$$

$$= v_1(t - R/c) + \left.\frac{dv_1}{dt}\right|_{t-R/c} \Delta t_1 \quad \text{—— (3)}$$

(ignoring higher order terms)

—— × ——

$$R_1 = (R^2 + r_1^2 - 2\vec{R}\cdot\vec{r_1})^{1/2}$$

$$= R\left(1 + \left(\frac{r_{a1}}{R}\right)^2 - 2\frac{\vec{R}\cdot\vec{r}_{a1}}{R^2}\right)^{1/2}$$

Using binomial expansion,

$$R_1 \approx R - \hat{R}\cdot\vec{r_1}$$

$$\Rightarrow R - R_1 = \hat{R}\cdot\vec{r_1} \quad \text{—— (4)}$$

—— × ——

Using (2) & (4),

$$\Delta t_1 = \frac{\hat{R}\cdot\vec{r_1}}{c}$$

Also, $\frac{dv_1}{dt} = -\omega^2 r_1$

(as shown in part-1)

Approximating $R_a \approx R_b \approx R$ in denominator of eq^n (1),

$$\vec{A}_q(R,t)$$

$$= -\frac{G}{c^2 R}\left(m_1 \vec{v}_1(t_R) + m_2 \vec{v}_2(t_R)\right)$$

$$+ \frac{G\omega^2}{c^3 R}\left(m_1 \vec{r}_1(\hat{R}\cdot\vec{r}_1) + m_2 \vec{r}_2(\hat{R}\cdot\vec{r}_2)\right)$$

Using results of part-1.

$$\left(r_1 = \frac{m_2 r}{m_1+m_2},\quad r_2 = \frac{m_1 r}{m_1+m_2},\quad \omega = \sqrt{\frac{G(m_1+m_2)}{r^3}}\right)$$

~~$\vec{A}_q$~~ & ~~ignoring~~ setting 1st term on RHS to $\vec{c}$

$\left(-\frac{G}{c^2 R}\left(m_1 \vec{v}_1(t_R) + m_2 \vec{v}_2(t_R)\right)\right.$, since net linear momentum is a constant, say $\left.\vec{c}\right)$.

$$\vec{A}_q(R,t) = \frac{G\omega^2}{c^3 R}\left(\frac{m_1 m_2}{m_1+m_2}\right)\left(\vec{r}\left(\hat{R}\cdot(\vec{r}_1-\vec{r}_2)\right)\right)$$

$$= \frac{G\omega^2}{c^3 R}\underbrace{\mu_{12}}_{\text{reduced mass of system}}\vec{r}(\hat{R}\cdot\vec{r}) + \vec{c}$$

$\vec{A}_q(R,t)$

$$= \frac{G\mu_{12}\,\omega^2 r^2}{c^3 R}\left(\hat{x}\cos\omega t_R + \hat{y}\sin\omega t_R\right)\left(\hat{R}\cdot\hat{x}\cos\omega t_R\right)$$

$\downarrow$

Converting to spherical coordinate system and taking only transverse component of $\vec{A}_q$,

$\vec{A}_{q(\text{trans})}$

$$= \frac{G\mu_{12}\,\omega^2 r^2}{2c^3 R}\sin\theta\left(\cos\theta\left(1+\cos 2\omega t_R\right)\hat{\theta} - \left(2\sin\theta\cos 2\omega t_R\right)\hat{\phi}\right)$$

Now, using Coulomb gauge instead of Lorentz gauge (setting divergence of vector potential $\vec{\nabla}\cdot\vec{A} = 0$),

$\vec{E}_g$ depends only on components of vector potential transverse to observation direction

$$\vec{E}_g = -\frac{\partial \vec{A}_{q(\text{trans})}}{\partial t}$$

$$\underset{\text{(radiated)}}{\vec{E}_g} = -\frac{\partial \vec{A}_{G(trans)}}{\partial t}$$

$$= \frac{G\mu_{12}\omega^3 r^2}{2c^3 R}\begin{pmatrix}\sin 2\theta \sin 2\omega t_R \,\hat{\theta} \\ -2\sin\theta \cos 2\omega t_R \,\hat{\phi}\end{pmatrix}$$

Angular Frequency of radiation is 2ω, twice that of orbital angular frequency ω.

[This is a characteristic of quadrupole radiation from an orbiting binary system; after half of the orbital period, the 2nd moment returns to initial value $\Rightarrow$ 2nd moment oscillates with twice the orbital frequency]

$$\langle S_G \rangle = \frac{c}{4\pi G}\left(\langle \vec{E}_g \cdot \vec{E}_g \rangle\right)$$

$$= \frac{G\omega^6 \mu_{12}^2 r^4}{32\pi c^5 R^2}\left(\sin^2 2\theta + 4\sin^2\theta\right)$$

For total gravitational power radiated integrating $\langle S_G \rangle$ over surface of sphere of radius R, centred at source.

$$dA = R^2 \sin\theta \, d\theta \, d\phi .$$

$$\therefore \; R^2 \int_0^{2\pi} d\phi \int_0^{\pi} (\sin^2 2\theta + 4\sin^2\theta) \sin\theta \, d\theta$$

$$= \frac{64\pi}{5} R^2$$

$\therefore$ Power radiated

$$= \frac{c}{32\pi G} \frac{G^2 (\mu_{12})^2 r^4 \omega^6}{c^6} \frac{64\pi}{5}$$

$$= \frac{2}{5} \frac{G \mu_{12}^2 r^4 \omega^6}{c^5} \quad \left(\text{Setting } \omega = \sqrt{\frac{G(m_1+m_2)}{r^3}} \right)$$

$$= \left(\frac{2}{5}\right) \frac{G^4 m_1^2 m_2^2 (m_1+m_2)}{5c^5 r^5}$$

The numerical factor is supposed to be $32/5$.

$\therefore$ Power radiated

$$= \frac{32}{5} \frac{G^4 m_1^2 m^2 (m_1+m_2)}{5c^5 r^5}$$

3. Using result of part (1),

$$r = \left(\frac{G(m_1+m_2)}{\omega^2}\right)^{1/3}$$

Using result of part (2),

$-\frac{dE}{dt}$ (power radiated)

$$= \frac{32\, G^4\, m_1^2 m_2^2 (m_1+m_2)}{5c^5 r^5}$$

Combining,

$$-\frac{dE}{dt} = \frac{32\, G^4\, m_1^2 m_2^2 (m_1+m_2)\, \omega^{10/3}}{5c^5\, G^{5/3} (m_1+m_2)^{5/3}}$$

Power radiated. $= \dfrac{32\, G^{7/3}\, m_1^2\, m_2^2}{5c^5 (m_1+m_2)^{2/3}}\, \omega^{10/3}$

4. Total energy lost per unit time = Power radiated.

$$E_{total} = \frac{1}{2} G^{2/3} \frac{M_1 m_2}{(m_1+m_2)^{1/3}} \omega^{2/3}$$ (from part 1)

$$-\frac{dE_{total}}{dt} = \frac{1}{2} G^{2/3} \frac{m_1 m_2}{(m_1+m_2)^{1/3}} \cdot \frac{2}{3} \omega^{-1/3} \frac{d\omega}{dt}$$

$$-\frac{dE_{total}}{dt} = \frac{32}{5} \cdot \frac{G^{7/3} \omega^{10/3}}{c^5} \cdot \frac{m_1^2 m_2^2}{(m_1+m_2)^{2/3}}$$

$$\Rightarrow \frac{1}{3} G^{2/3} \frac{m_1 m_2}{(m_1+m_2)^{1/3}} \omega^{-1/3} \frac{d\omega}{dt}$$

$$= \frac{32}{5} \frac{G^{7/3} \omega^{10/3}}{c^5} \frac{m_1^2 m_2^2}{(m_1+m_2)^{2/3}}$$

Bringing mass terms together on LHS and equating them with angular frequency, its derivative and constants to define

Chirp mass = M_{ch}

$$\frac{m_1 m_2}{(m_1+m_2)^{1/3}} = \frac{5}{96} \frac{c^5}{G^{5/3}} \omega^{-11/3} \frac{d\omega}{dt}$$

Chirp mass (m_{ch}) defined as

$$\frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

So raising the equation to power $3/5$,

$$m_{ch} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} = \left(\frac{5}{96}\right)^{3/5} \frac{c^3}{G} \omega^{-11/5} \left(\frac{d\omega}{dt}\right)^{3/5}$$

5. $\omega_{radiation} = 2\omega_{orbital}$

$\Rightarrow f_{radiation} = 2 f_{orbital}$.

$$f_{rad} = \frac{\omega_{rad}}{2\pi}$$

$$\Rightarrow f_{rad} = \frac{2\omega_{orb}}{2\pi} = \frac{\omega_{orb}}{\pi} = \frac{\omega}{\pi}$$

$$\frac{d\omega_{orb}}{dt} = \frac{1}{\pi} \frac{d f_{rad}}{dt}$$

$$\therefore m_{ch} = \left(\frac{5}{96}\right)^{3/5} \frac{c^3}{G} \pi^{-8/5} \left(f_{rad}\right)^{-11/5} \left(\frac{d f_{rad}}{dt}\right)^{3/5}$$