Lecture 3: Kinematics: Rigid Motions and Homogeneous Transformations

- Composition of Rotations:
  - Rotations with Respect to the Current Frame
  - Rotations with Respect to the Fixed Frame
  - Example

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What Have We Learned:

Given N-frames in the 3-dimensional space

$$(o_0, x_0, y_0, z_0), \quad (o_1, x_1, y_1, z_1), \ \ldots \ (o_{N-1}, x_{N-1}, y_{N-1}, z_{N-1})$$

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If we are given $(N-1)$-rotation matrices

$$R_1^0, \quad R_2^1, \quad \ldots, \quad R_{(N-1)}^{(N-2)}$$

that represent consecutive rotation between the current frames

$$\left\{(x_0 y_0 z_0), (x_1 y_1 z_1)\right\}, \left\{(x_1 y_1 z_1), (x_2 y_2 z_2)\right\}, \ldots, \\ \left\{(x_{N-2} y_{N-2} z_{N-2}), (x_{N-1} y_{N-1} z_{N-1})\right\}$$

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The formula to compute the position of the point in the 0-frame having known its position in the 1-frame

$$p^0 = R_1^0 p^1$$

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The formula to compute the position of the point in the 0-frame having known its position in the 2-frame

$$p^0 = R_1^0 p^1, \quad p^1 = R_2^1 p^2$$

What Have We Learned:

Given N-frames in the 3-dimensional space

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The formula to compute the position of the point in the 0-frame having known its position in the $(N-1)$-frame

$$p^0 = R_1^0 p^1, \quad p^1 = R_2^1 p^2, \quad \dots, \quad p^{(N-2)} = R_{(N-1)}^{(N-2)} p^{(N-1)}$$

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The formula to compute the position of the point in the 0-frame having known its position in the $(N-1)$-frame is

$$\boxed{p^0 = R_1^0 \, R_2^1 \, R_3^2 \, \cdots \, R_{(N-1)}^{(N-2)} \, p^{(N-1)}}$$

Example 2.8:

Find the rotation $\boldsymbol{R}$ defined by the following basic rotations:

- 1: A rotation of $\boldsymbol{\theta}$ about the current axis $\boldsymbol{x}$;
- 2: A rotation of $\boldsymbol{\phi}$ about the current axis $\boldsymbol{z}$

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The total rotation will be then

$$\boldsymbol{R} = \boldsymbol{R}_{\boldsymbol{x},\boldsymbol{\theta}}\,\boldsymbol{R}_{\boldsymbol{z},\boldsymbol{\phi}}$$

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For any point of the $\mathbf{2}$-frame with coordinates $\boldsymbol{p}^2 = \left[x^2, y^2, z^2\right]^T$ its coordinates in the $\mathbf{0}$-frame are computed simply as

$$\boldsymbol{p}^0 = \boldsymbol{R}\,\boldsymbol{p}^2 = \left[\boldsymbol{R}_{x,\theta}\,\boldsymbol{R}_{z,\phi}\right]\,\boldsymbol{p}^2$$

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$$\boldsymbol{R} = \boldsymbol{R}_{x,\theta}\, \boldsymbol{R}_{z,\phi}\, \boldsymbol{R}_3$$

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$$R = R_{x,\theta}\, R_{z,\phi}\, R_3$$

We have computed this rotation as

$$R_3 = \left[R_{x,\theta}\, R_{z,\phi}\right]^{-1} \cdot R_{z,\alpha} \cdot \left[R_{x,\theta}\, R_{z,\phi}\right]$$

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- 4: A rotation of $\boldsymbol{\beta}$ about the current axis $\boldsymbol{y}$

The total rotation will be then

$$\boldsymbol{R} = \boldsymbol{R}_{x,\theta}\,\boldsymbol{R}_{z,\phi}\,\boldsymbol{R}_{3}\,\boldsymbol{R}_{y,\beta} = \left[\boldsymbol{R}_{z,\alpha}\,\boldsymbol{R}_{x,\theta}\,\boldsymbol{R}_{z,\phi}\right]\boldsymbol{R}_{y,\beta}$$

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- 5: A rotation of $\boldsymbol{\delta}$ about the fixed axis $\boldsymbol{x}$

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- 4: A rotation of β about the current axis y
- 5: A rotation of δ about the fixed axis x

The total rotation will be then

$$\boldsymbol{R} = \boldsymbol{R}_{z,\alpha}\, \boldsymbol{R}_{x,\theta}\, \boldsymbol{R}_{z,\phi}\, \boldsymbol{R}_{y,\beta}\, \boldsymbol{R}_{\mathbf{5}}$$

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- 3: A rotation of α about the fixed axis z
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The total rotation will be then

$$R = R_{z,\alpha}\, R_{x,\theta}\, R_{z,\phi}\, R_{y,\beta}\, R_5$$

We have computed this rotation as

$$R_5 = \left[R_{z,\alpha}\, R_{x,\theta}\, R_{z,\phi}\, R_{y,\beta}\right]^{-1} \cdot R_{x,\delta} \cdot \left[R_{z,\alpha}\, R_{x,\theta}\, R_{z,\phi}\, R_{y,\beta}\right]$$

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The total rotation will be then

$$\boxed{\boldsymbol{R} = \boldsymbol{R}_{x,\delta} \cdot \boldsymbol{R}_{z,\alpha} \cdot \boldsymbol{R}_{x,\theta} \cdot \boldsymbol{R}_{z,\phi} \cdot \boldsymbol{R}_{y,\beta}}$$

We have computed this rotation as

$$\boldsymbol{R}_5 = \left[\boldsymbol{R}_{z,\alpha}\,\boldsymbol{R}_{x,\theta}\,\boldsymbol{R}_{z,\phi}\,\boldsymbol{R}_{y,\beta}\right]^{-1} \cdot \boldsymbol{R}_{x,\delta} \cdot \left[\boldsymbol{R}_{z,\alpha}\,\boldsymbol{R}_{x,\theta}\,\boldsymbol{R}_{z,\phi}\,\boldsymbol{R}_{y,\beta}\right]$$

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Parameterizations of Rotations:

Any rotation matrix $\boldsymbol{R}$ is

- of dimension $\mathbf{3} \times \mathbf{3}$, i.e. it is $\mathbf{9}$-numbers
- belongs to $\boldsymbol{\mathcal{SO}}(\mathbf{3})$, i.e.
  - its 3 columns are vectors of length 1 (3 equations)
  - its 3 columns are orthogonal to each other (3 equations)

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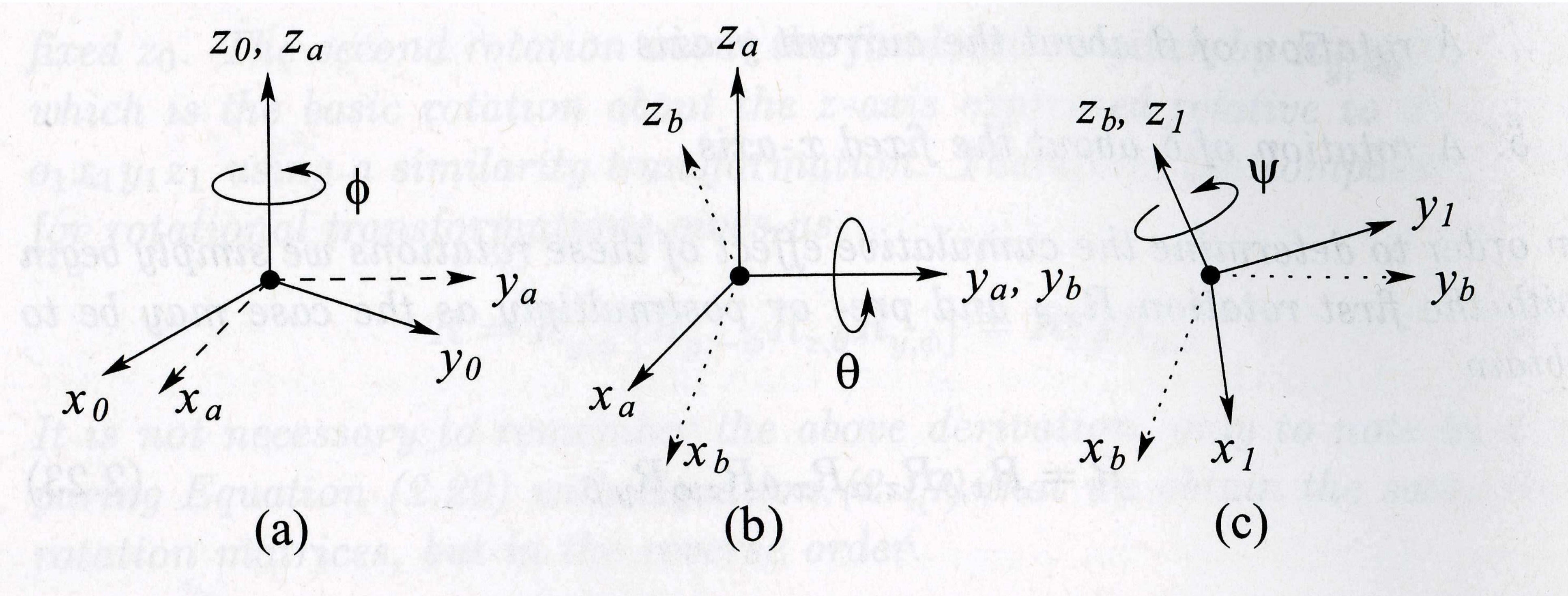
Except the particular cases, only 3 of 9 numbers that parameterize the rotation matrix, can be assigned freely!

Euler Angles:



Euler angles are angles of 3 rotations about current axes

$$R_{ZYZ} := R_{z,\phi} \cdot R_{y,\theta} \cdot R_{z,\psi}$$

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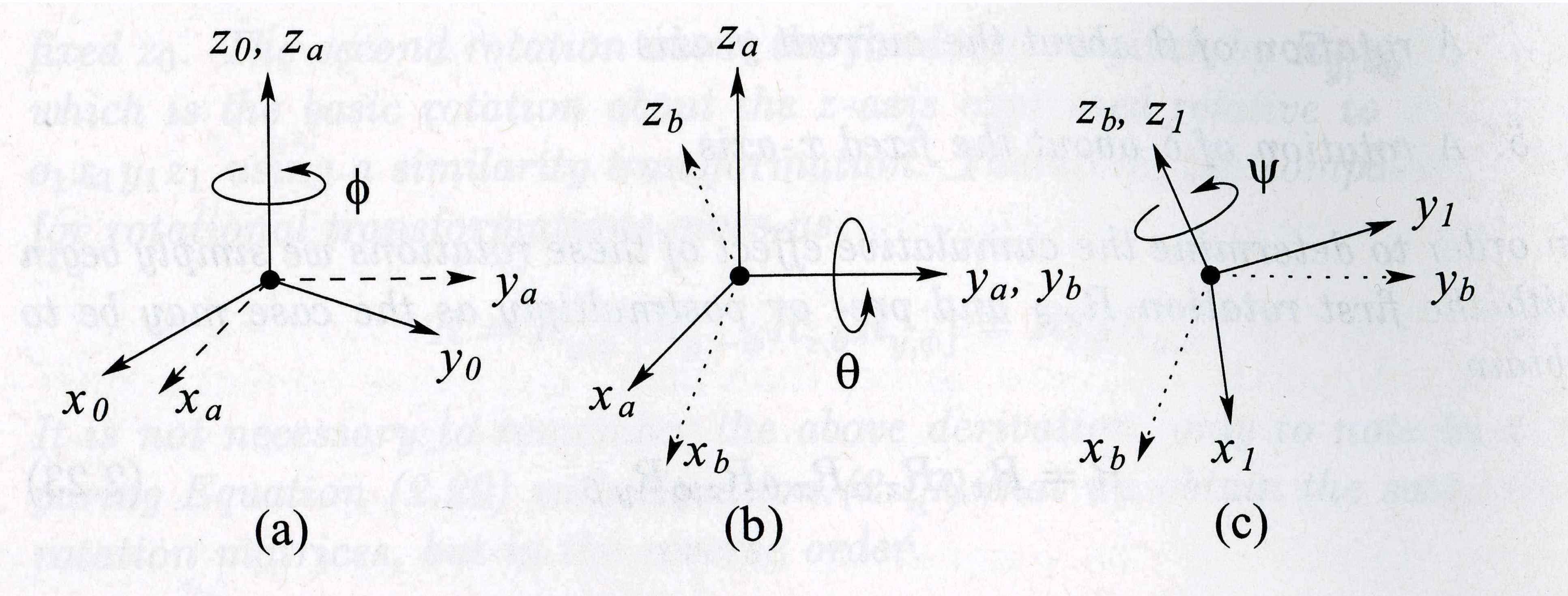

Euler angles are angles of $\mathbf{3}$ rotations about current axes

$$R_{ZYZ} := \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot R_{y,\theta} \cdot R_{z,\psi}$$

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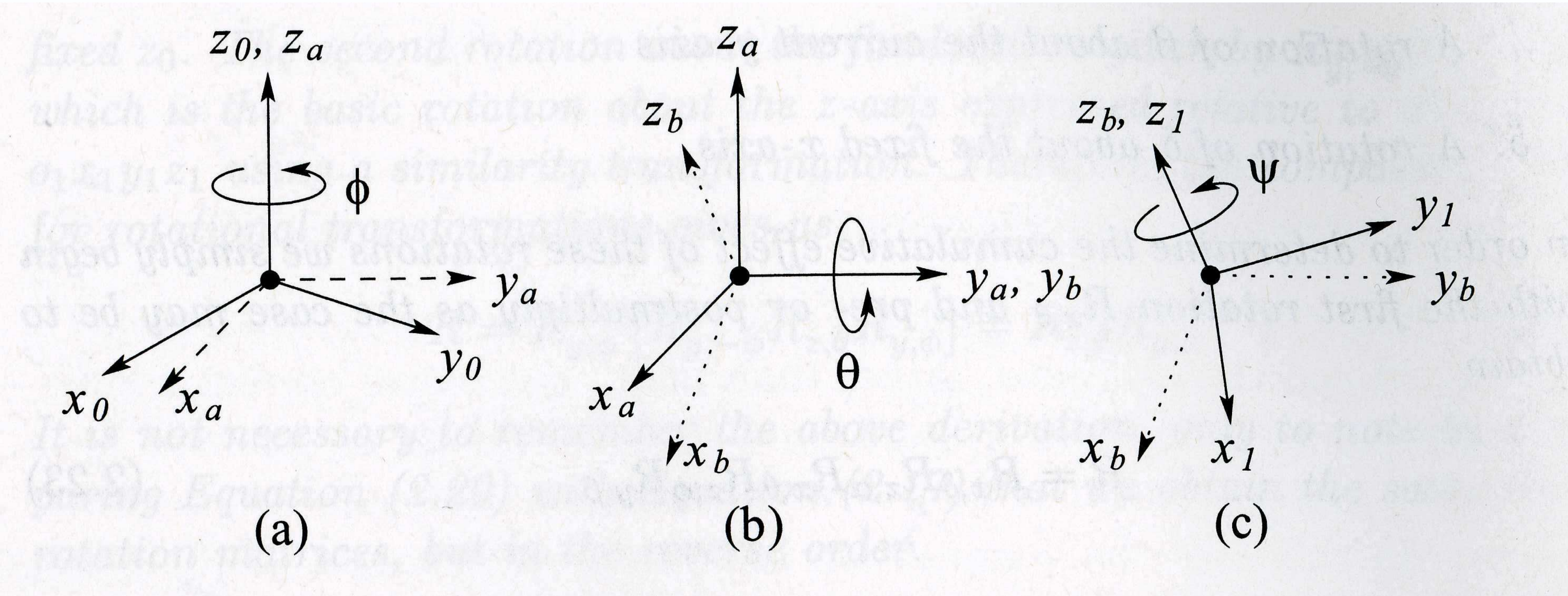

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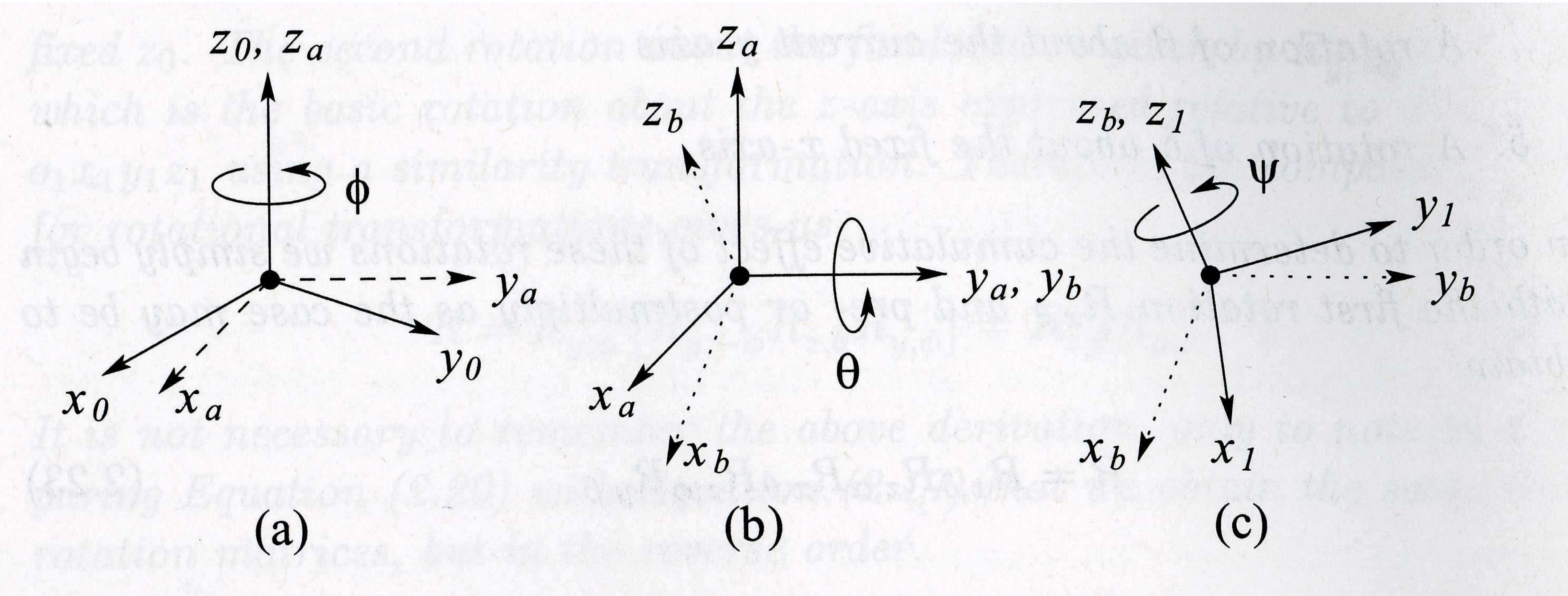

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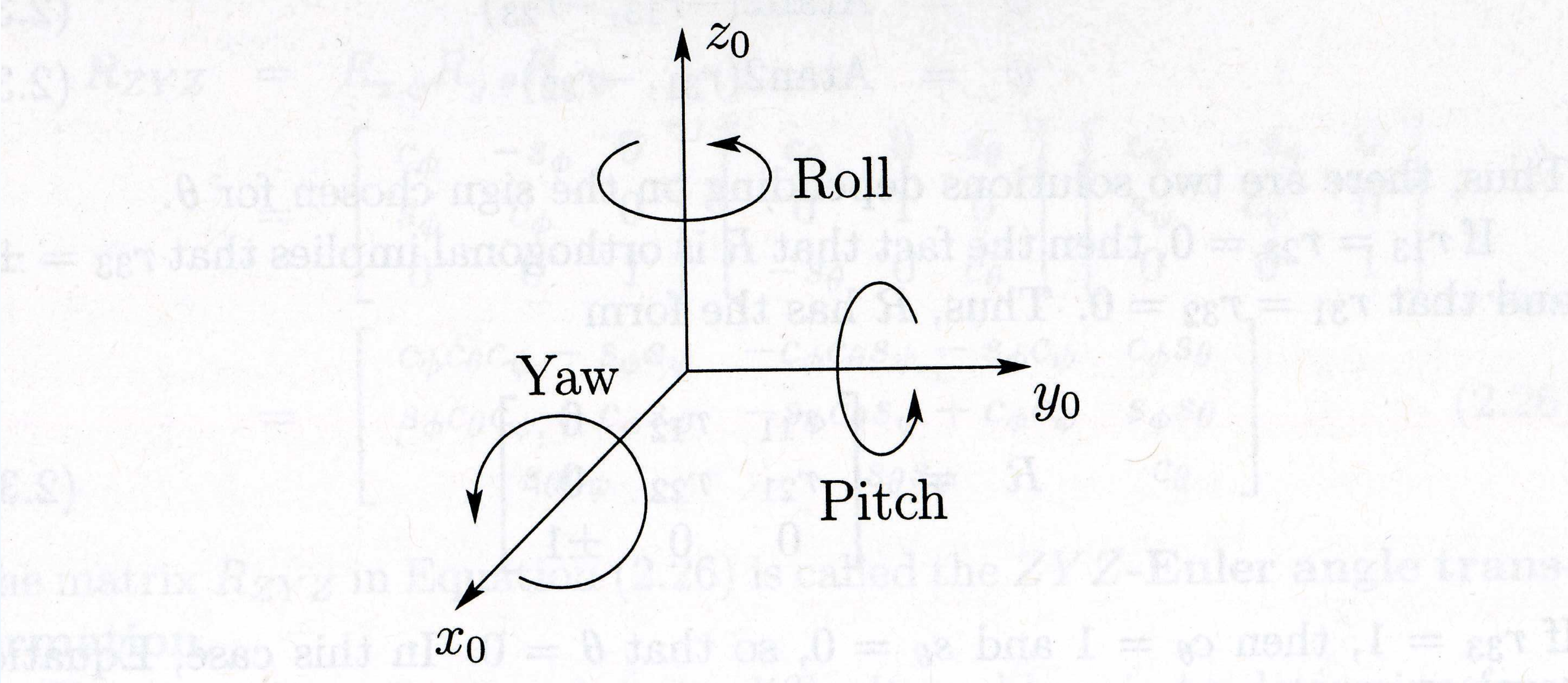
Roll, Pitch Yaw Angles:



Roll, pitch and yaw angles are angles of 3 rotations about the fixed axes x, y and z

$$R_{xyz} := R_{z,\phi} \cdot R_{y,\theta} \cdot R_{x,\psi}$$

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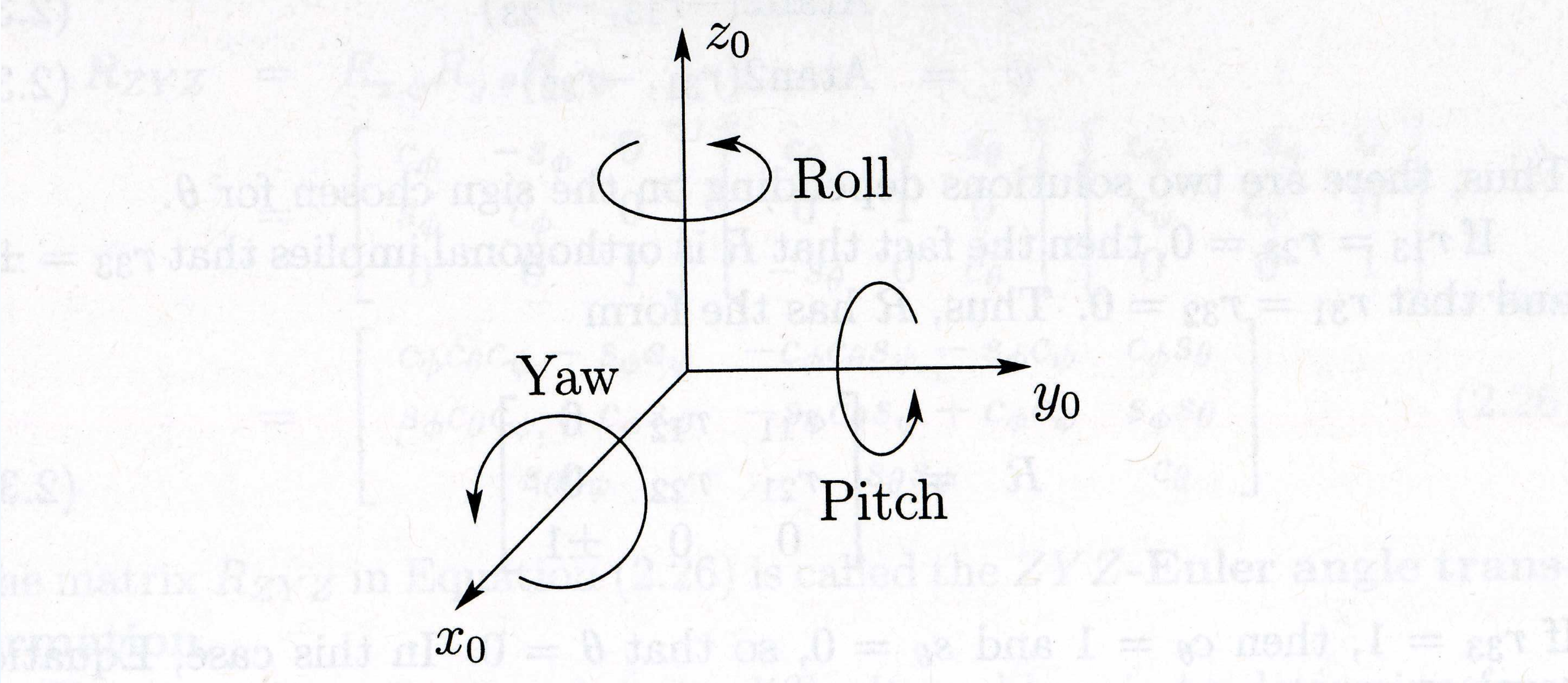
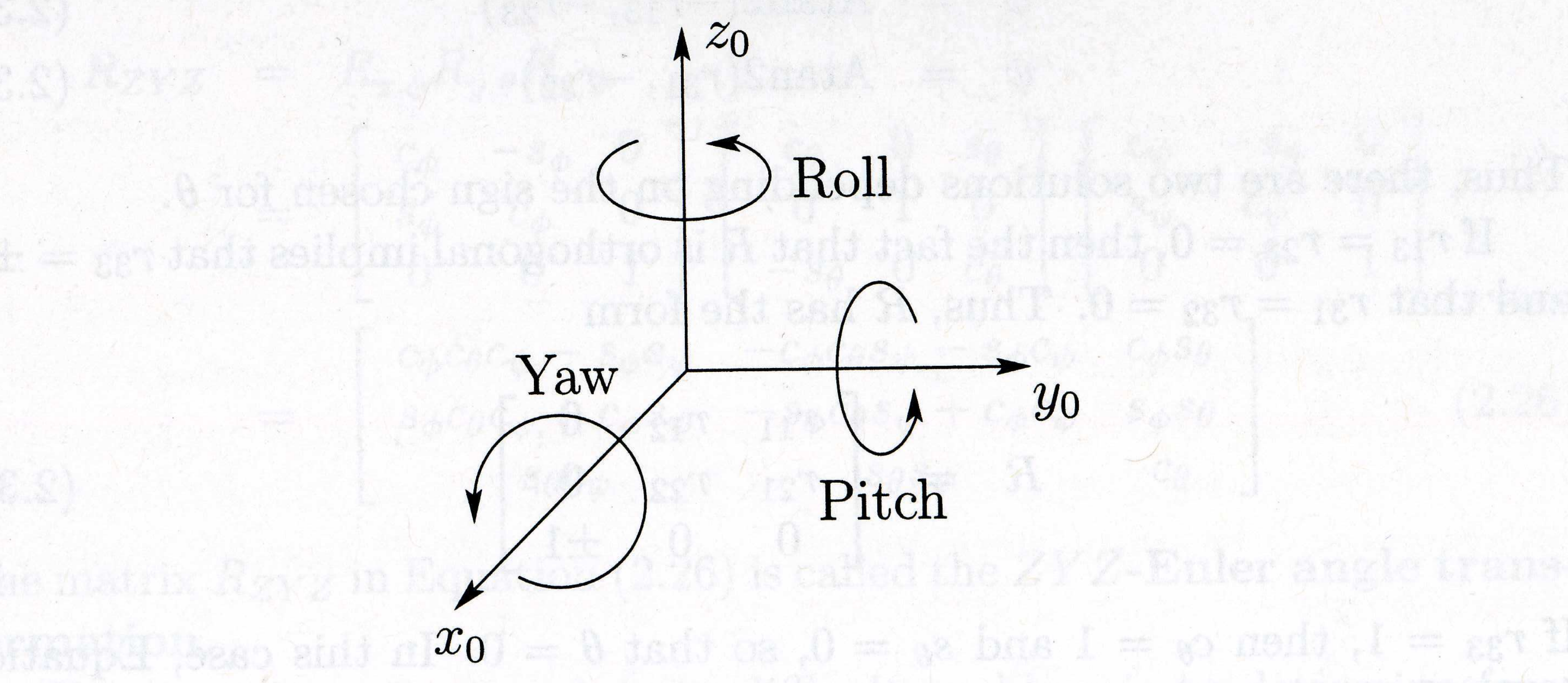


Roll, pitch and yaw angles are angles of 3 rotations about the fixed axes x, y and z

$$R_{xyz} := R_{z,\phi} \cdot R_{y,\theta} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$

Roll, Pitch Yaw Angles:



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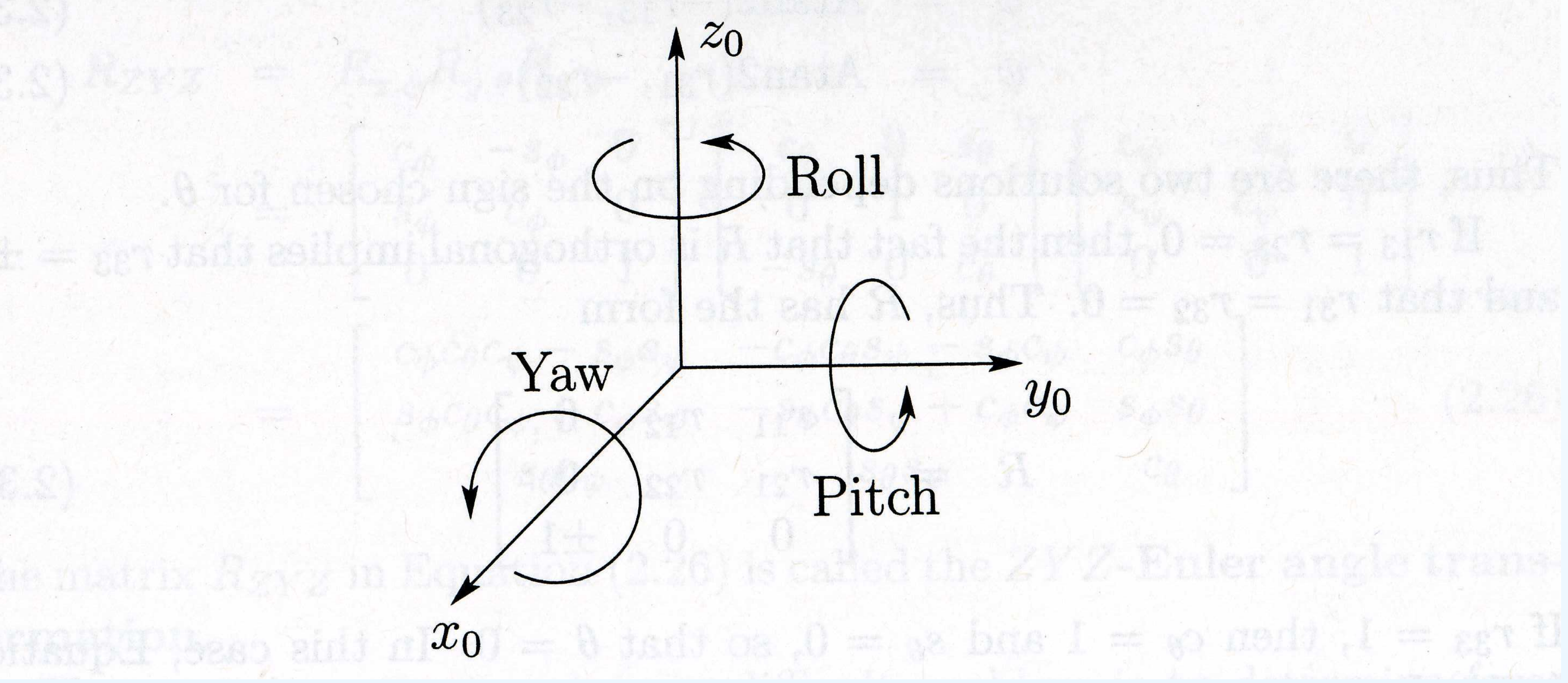

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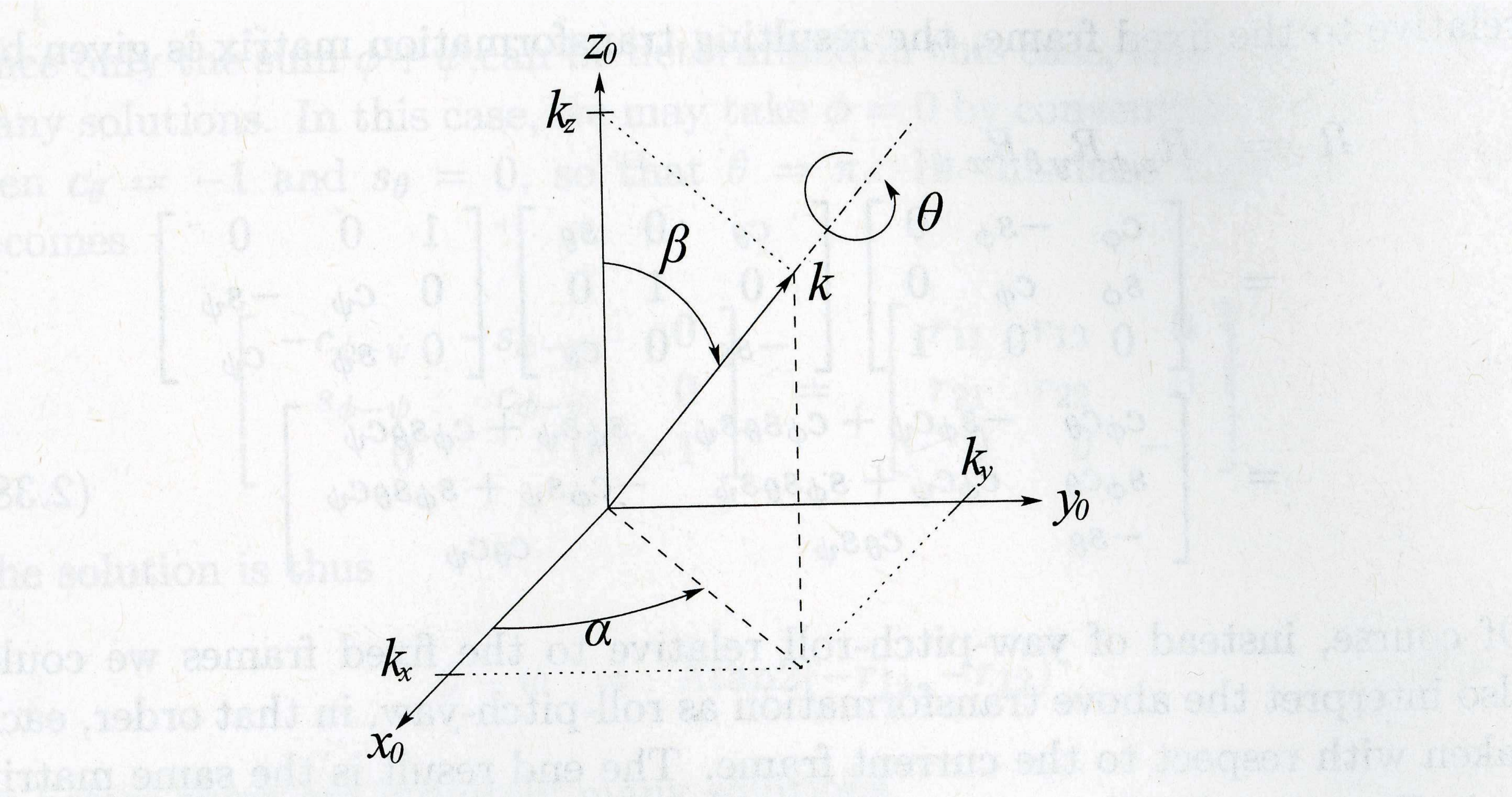
$$R_{xyz} := \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$

Axis/Angle Representation for a Rotation Matrix:



Any rotational matrix can be expressed as a rotation of angle $\boldsymbol{\theta}$ about an axis $\boldsymbol{k} = [\boldsymbol{k_x},\ \boldsymbol{k_y},\ \boldsymbol{k_z}]^T$

$$\boldsymbol{R_{\vec{k},\theta}} = \boldsymbol{R} \cdot \boldsymbol{R_{z,\theta}} \cdot \boldsymbol{R}^{-1}, \qquad \boldsymbol{R} = \boldsymbol{R_{z,\alpha}} \cdot \boldsymbol{R_{y,\beta}}$$

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Rigid Motions

A rigid motion is an ordered pair $(\boldsymbol{R}, \boldsymbol{d})$, where $\boldsymbol{R} \in \boldsymbol{\mathcal{SO}(3)}$ and $\boldsymbol{d} \in \boldsymbol{R^3}$. The group of all rigid motions is known as Special Euclidean Group denoted by $\boldsymbol{\mathcal{SE}(3)}$.

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$$p^0 = R_1^0 p^1 + d^0$$

If there are 3 frames corresponding to 2 rigid motions

$$\begin{aligned} p^1 &= R_2^1 p^2 + d_2^1 \\ p^0 &= R_1^0 p^1 + d_1^0 \end{aligned}$$

then the overall motion is

$$\boxed{p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0}$$

Concept of Homogeneous Transformation

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Given two rigid motions (R_1^0, d_1^0) and (R_2^1, d_2^1), consider the product of two matrices

$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\ 0_{1\times 3} & 1 \end{bmatrix}$$

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Given a rigid motion $(R, d) \in \mathcal{SE}(3)$, the 4×4-matrix

$$H = \begin{bmatrix} R & d \\ 0_{1\times 3} & 1 \end{bmatrix}$$

is called homogeneous transformation associated with (R, d)

Concept of Homogeneous Transformation

To use HTs in computing coordinates of points, we need to extend the vectors $\boldsymbol{p}^0$ and $\boldsymbol{p}^1$ by one coordinate. Namely

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Then

$$\boldsymbol{P}^0 = \begin{bmatrix} \boldsymbol{p}^0 \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 \boldsymbol{p}^1 + \boldsymbol{d}^0 \\ \mathbf{1} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{d}^0 \\ \mathbf{0}_{1\times 3} & \mathbf{1} \end{bmatrix}}_{\boldsymbol{H}_1^0} \underbrace{\begin{bmatrix} \boldsymbol{p}^1 \\ \mathbf{1} \end{bmatrix}}_{\boldsymbol{P}^1}$$

that is in short

$$\boldsymbol{P}^0 = \boldsymbol{H}_1^0 \boldsymbol{P}^1$$