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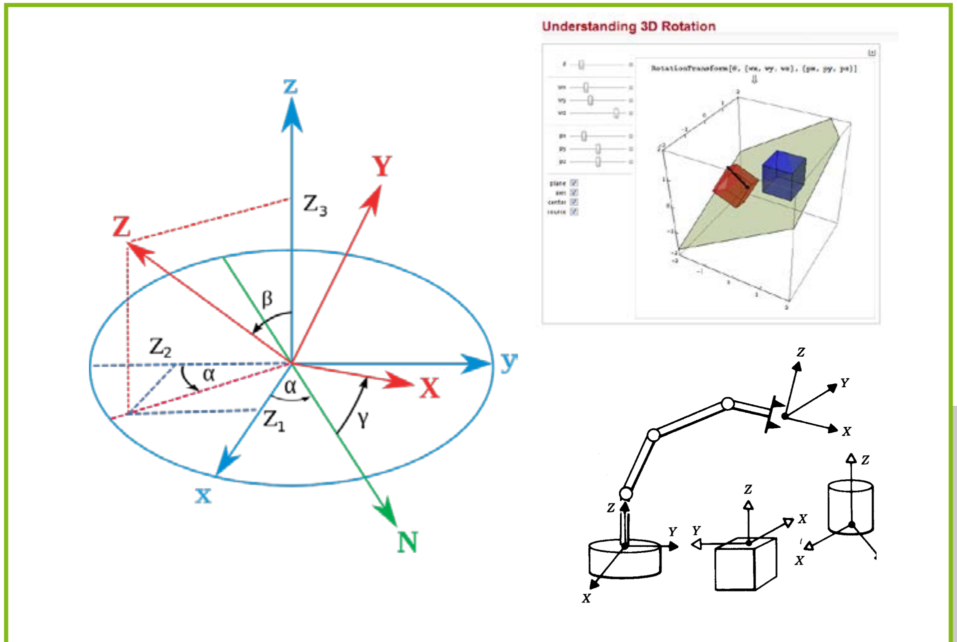


Summer Term 2019

version 09.04.2019

Modeling and Control of Robotic Manipulators

Modellierung und Regelung von Robotern

Spatial Transformations

apl. Prof. Dr. rer. nat. Frank Hoffmann

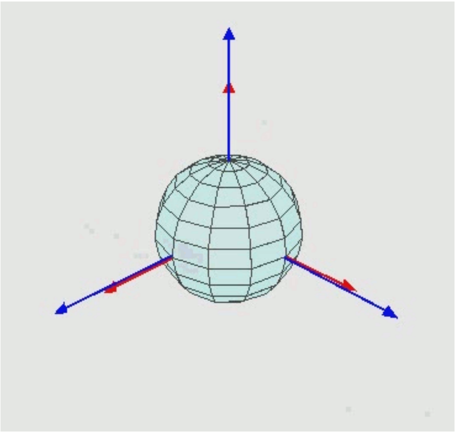
Lehrstuhl für Regelungssystemtechnik

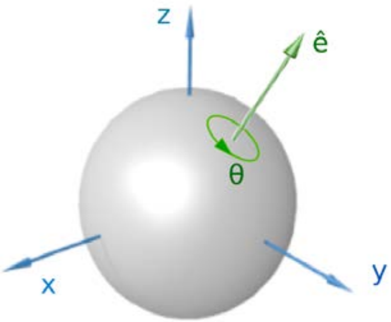
Spatial Transformations

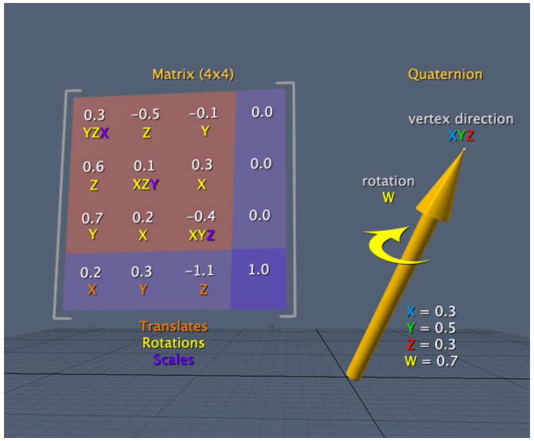
- rotation matrices
- Euler angles
- angle-axis
- unit quaternions
- homogeneous transformations

$$\mathbf{R} = \begin{bmatrix} \mathbf{x}'^T\mathbf{x} & \mathbf{y}'^T\mathbf{x} & \mathbf{z}'^T\mathbf{x} \\ \mathbf{x}'^T\mathbf{y} & \mathbf{y}'^T\mathbf{y} & \mathbf{z}'^T\mathbf{y} \\ \mathbf{x}'^T\mathbf{z} & \mathbf{y}'^T\mathbf{z} & \mathbf{z}'^T\mathbf{z} \end{bmatrix}$$

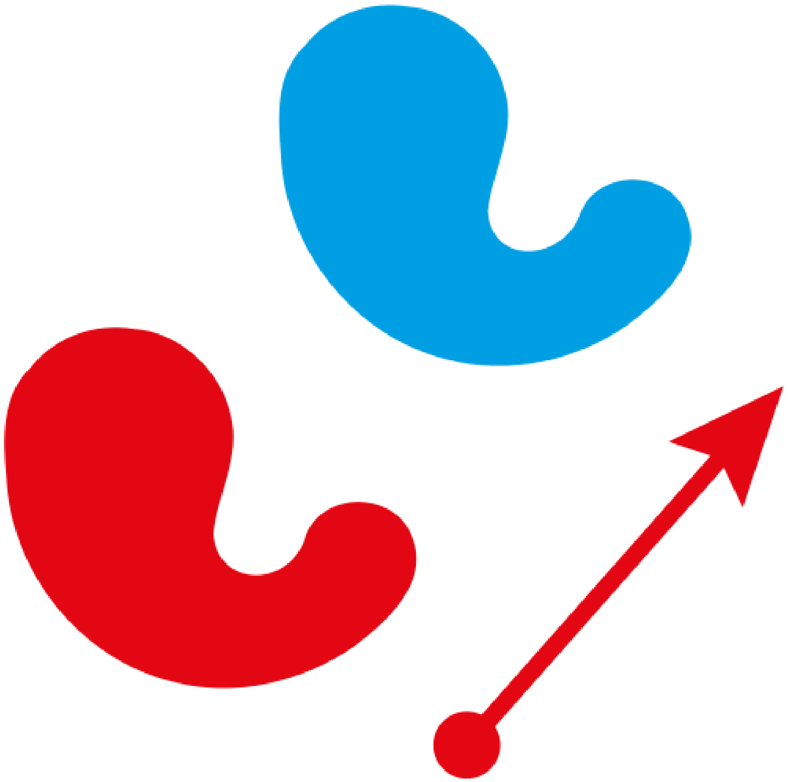
$$\mathbf{A}_1^0 = \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{o}_1^0 \\ \mathbf{0}^T & 1 \end{bmatrix}$$







Translation



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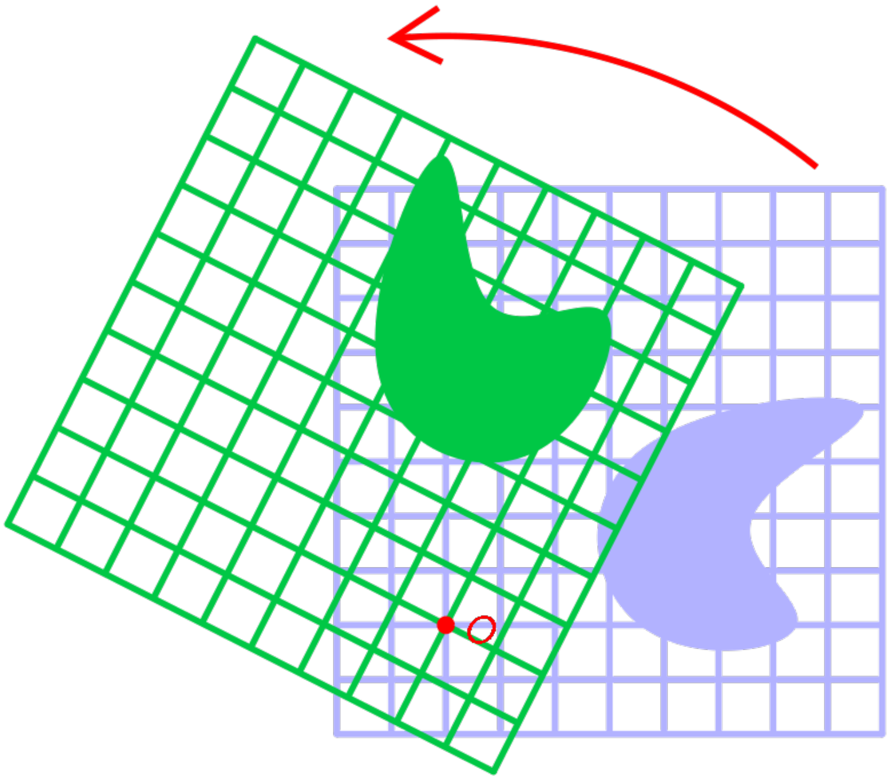
https://commons.wikimedia.org/w/index.php?curid=35017753

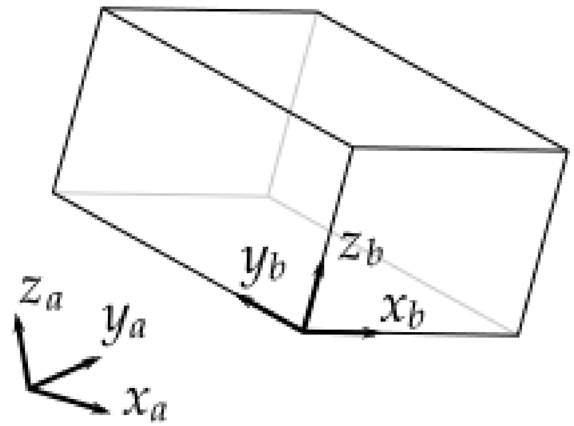
Rotation



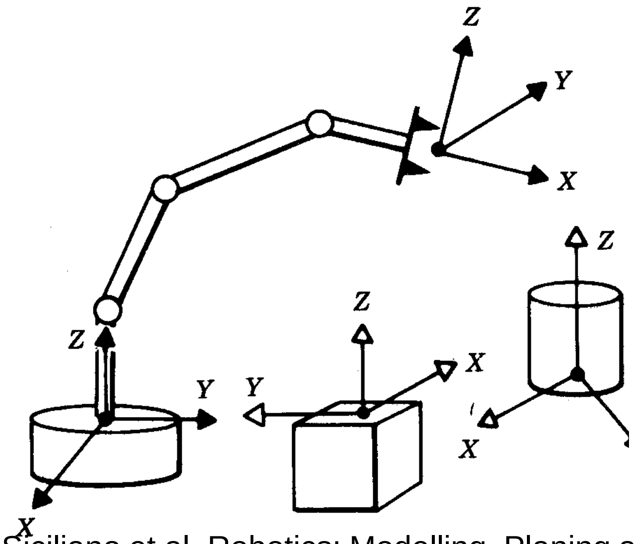
By Oleg Alexandrov - self-made, with MATLAB, then tweaked with en:Inkscape, Public Domain, https://commons.wikimedia.org/w/index.php?curid=2220861

Representation of Pose and Orientation of Rigid Body

- Transformation among two reference frames
  - translation (3 DOF)
  - rotation (3 DOF)
- translation
  - 3D-vector
- rotation
  - Rotation matrix
  - Euler angles
  - Angle-axis
  - Quaternions
- translation and rotation
  - homogeneous matrix
  - screws
  - 6D-vector

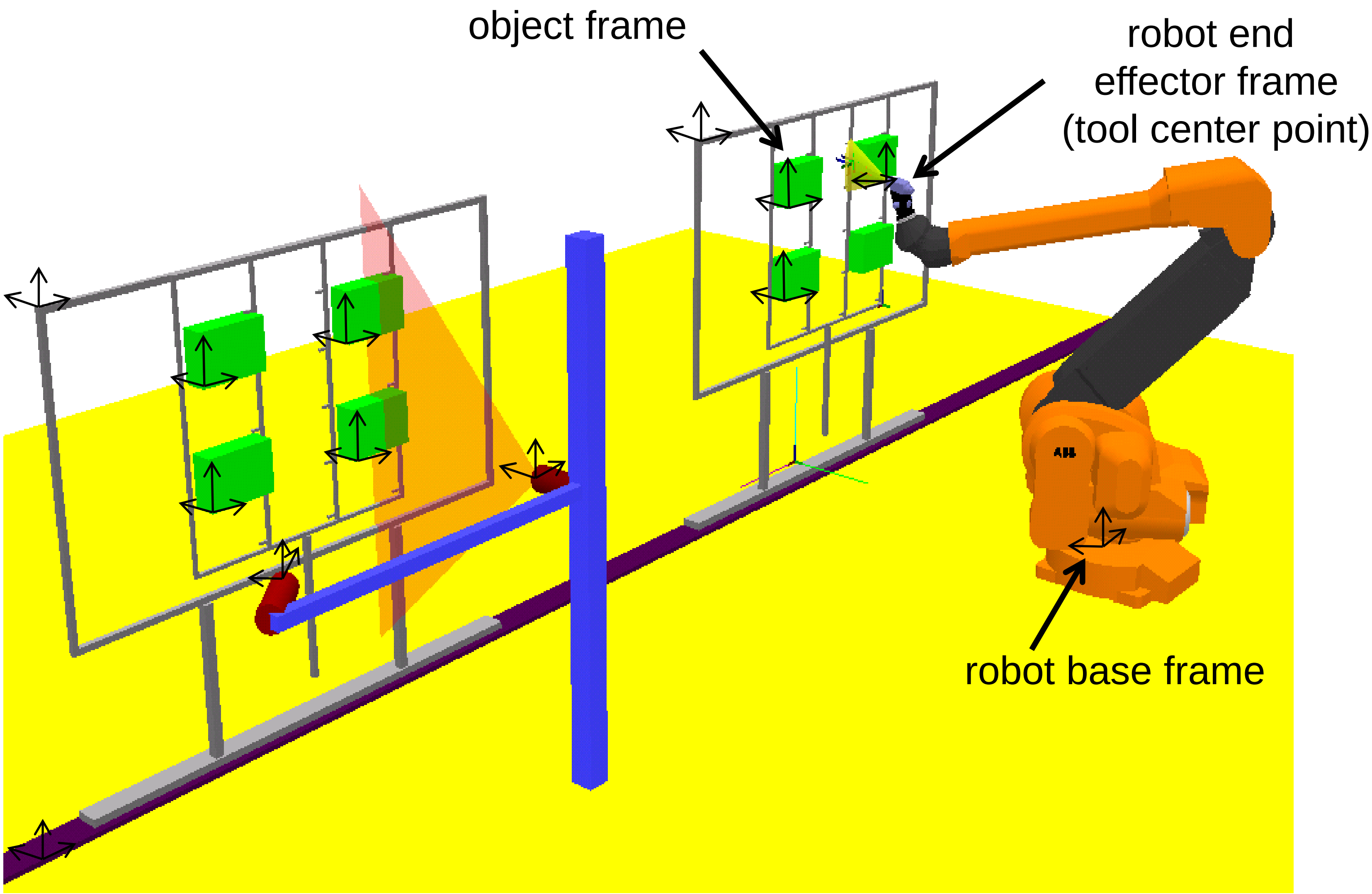


https://en.wikibooks.org/wiki/File:Rigid_body_attached_frame.svg



Siciliano et al, Robotics: Modelling, Planing and Control

Robotic Reference Frames

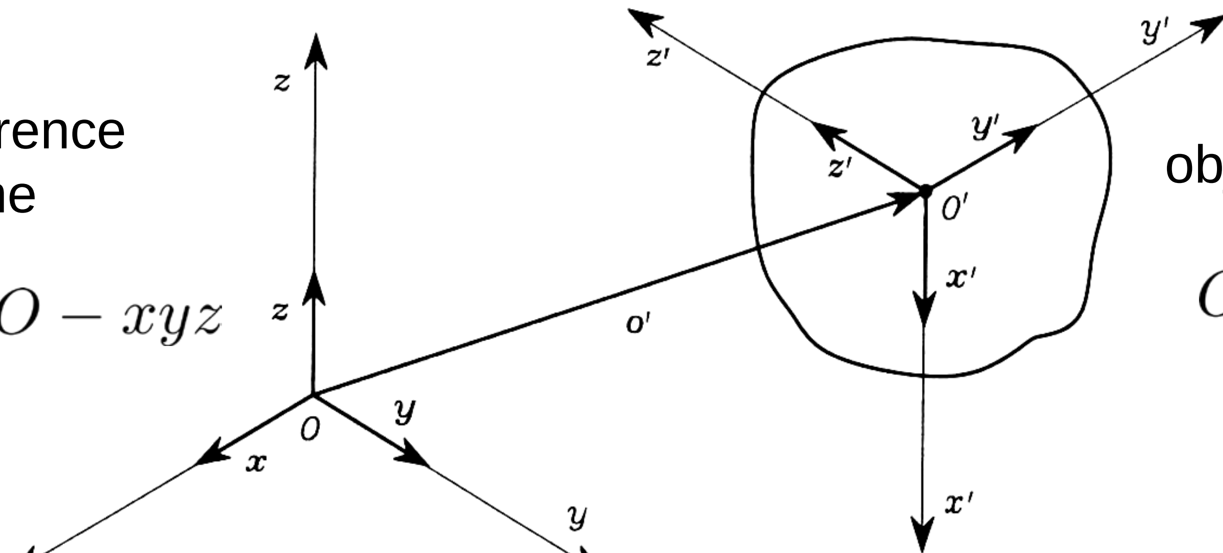


Position and Orientation of a Rigid Body

Pose : position + orientation

reference
frame

$O - xyz$



object frame

$O' - x'y'z'$

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$$\mathbf{o}' = \mathbf{o}'_x\mathbf{x} + \mathbf{o}'_y\mathbf{y} + \mathbf{o}'_z\mathbf{z}$$

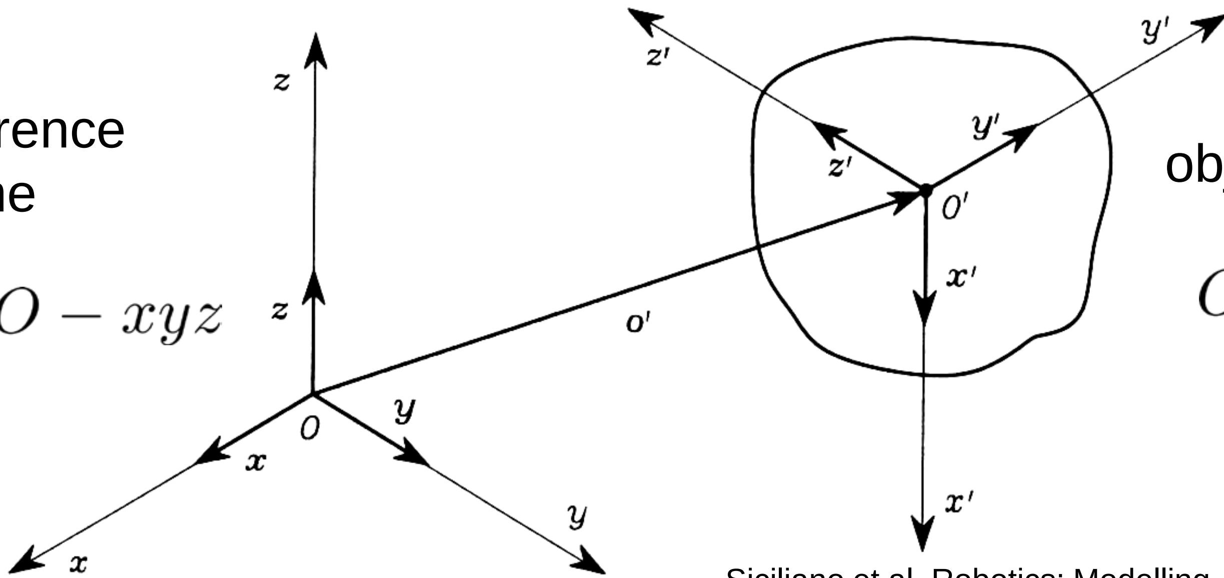
position: $$\mathbf{o}' = \begin{pmatrix} \mathbf{o}'_x \\ \mathbf{o}'_y \\ \mathbf{o}'_z \end{pmatrix}$$

orientation
$$\mathbf{x}' = \mathbf{x}'_x\mathbf{x} + \mathbf{x}'_y\mathbf{y} + \mathbf{x}'_z\mathbf{z}$$
$$\mathbf{y}' = \mathbf{y}'_x\mathbf{x} + \mathbf{y}'_y\mathbf{y} + \mathbf{y}'_z\mathbf{z}$$
$$\mathbf{z}' = \mathbf{z}'_x\mathbf{x} + \mathbf{z}'_y\mathbf{y} + \mathbf{z}'_z\mathbf{z}$$

Translation of a Rigid Body

reference frame

$O - xyz$



object frame

$O' - x'y'z'$

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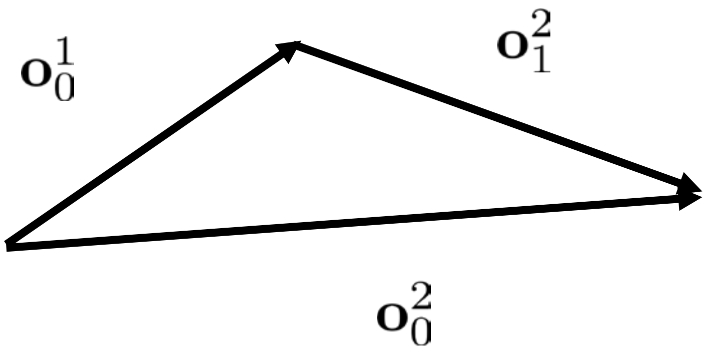
$$\mathbf{o}' = \mathbf{o}'_x \mathbf{x} + \mathbf{o}'_y \mathbf{y} + \mathbf{o}'_z \mathbf{z}$$

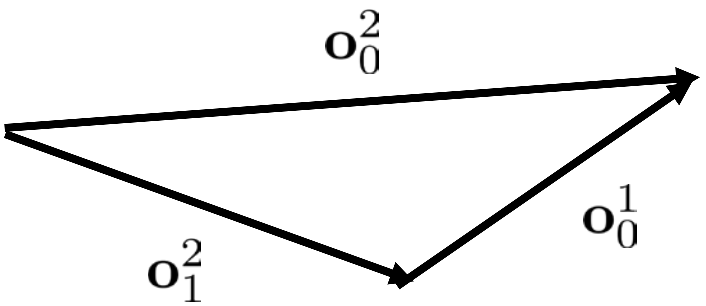
$$\mathbf{o}' = \begin{pmatrix} \mathbf{o}'_x \\ \mathbf{o}'_y \\ \mathbf{o}'_z \end{pmatrix}$$

Composition of Translations

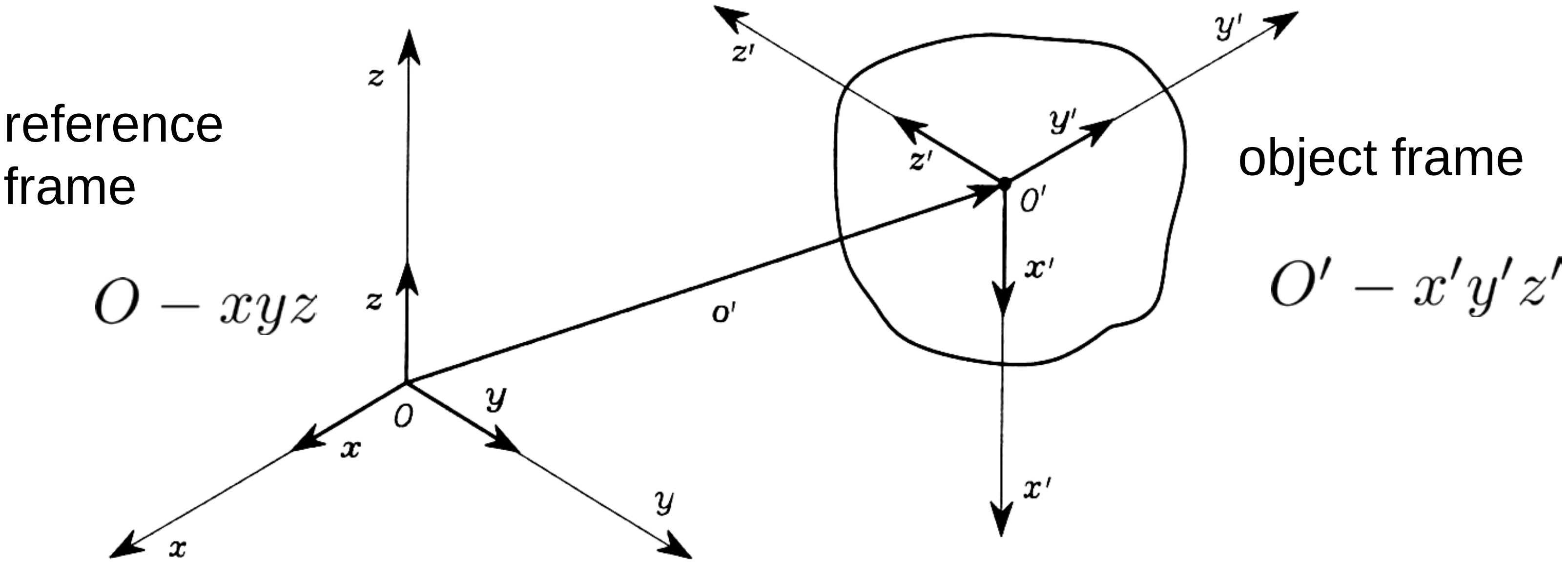
$$\mathbf{o}_0^2 = \mathbf{o}_0^1 + \mathbf{o}_1^2$$

$$\mathbf{o}_0^2 = \mathbf{o}_1^2 + \mathbf{o}_0^1$$





Orientation of a Rigid Body



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$$\mathbf{x}' = \mathbf{x}'_x \mathbf{x} + \mathbf{x}'_y \mathbf{y} + \mathbf{x}'_z \mathbf{z}$$
$$\mathbf{y}' = \mathbf{y}'_x \mathbf{x} + \mathbf{y}'_y \mathbf{y} + \mathbf{y}'_z \mathbf{z}$$
$$\mathbf{z}' = \mathbf{z}'_x \mathbf{x} + \mathbf{z}'_y \mathbf{y} + \mathbf{z}'_z \mathbf{z}$$

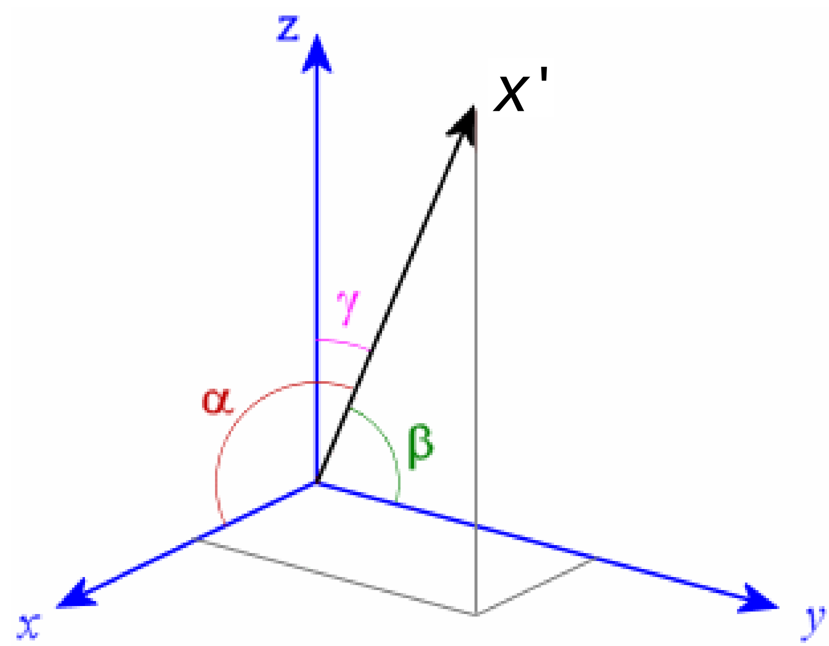
Rotation Matrix

$$\mathbf{R} = \begin{bmatrix}\mathbf{x}' & \mathbf{y}' & \mathbf{z}'\end{bmatrix} = \begin{bmatrix} \mathbf{x}'_x & \mathbf{y}'_x & \mathbf{z}'_x \\ \mathbf{x}'_y & \mathbf{y}'_y & \mathbf{z}'_y \\ \mathbf{x}'_z & \mathbf{y}'_z & \mathbf{z}'_z \end{bmatrix} = \begin{bmatrix} \mathbf{x}'^T\mathbf{x} & \mathbf{y}'^T\mathbf{x} & \mathbf{z}'^T\mathbf{x} \\ \mathbf{x}'^T\mathbf{y} & \mathbf{y}'^T\mathbf{y} & \mathbf{z}'^T\mathbf{y} \\ \mathbf{x}'^T\mathbf{z} & \mathbf{y}'^T\mathbf{z} & \mathbf{z}'^T\mathbf{z} \end{bmatrix}$$

Rotation matrix R captures the relative orientation among two frames

$x'_x, x'_y, \ldots, z'_z$ are the direction cosines of the axes of frame O´-x´y´z with respect to O-xyz

$$x'_x = \cos(\angle \mathbf{x}'\mathbf{x}) = \mathbf{x}' \circ \mathbf{x}$$
$$x'_y = \cos(\angle \mathbf{x}'\mathbf{y}) = \mathbf{x}' \circ \mathbf{y}$$
$$\ldots$$
$$z'_z = \cos(\angle \mathbf{z}'\mathbf{z}) = \mathbf{z}' \circ \mathbf{z}$$



http://intmstat.com/vectors/cosines.gif

Properties of a Rotation Matrix

$\mathbf{R}$ is an **orthogonal 3x3** matrix $\mathbf{R}^T\mathbf{R} = \mathbf{I}_3$

$$\mathbf{x}'^T\mathbf{y}' = \mathbf{y}'^T\mathbf{z}' = \mathbf{z}'^T\mathbf{x}' = 0$$

$$\mathbf{x}'^T\mathbf{x}' = \mathbf{y}'^T\mathbf{y}' = \mathbf{z}'^T\mathbf{z}' = 1$$

Inverse of a rotation matrix : $\mathbf{R}^{-1} = \mathbf{R}^T$

A **square matrix** $\mathbf{R}$ is a rotation matrix if

$$\mathbf{R}^T\mathbf{R} = \mathbf{I}_3$$

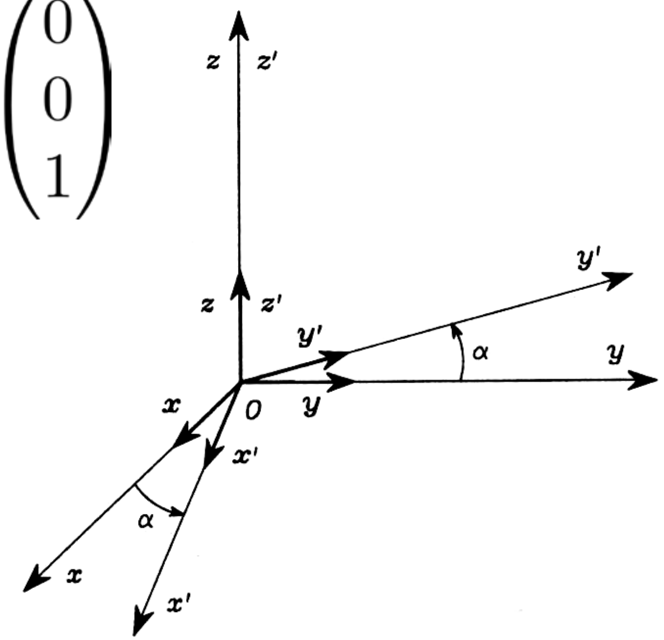
$$\det(\mathbf{R}) = 1$$

Elementary Rotations

- Rotation of frame O-xyz by angle α along z-axis

$$\mathbf{x}' = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} \quad \mathbf{y}' = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix} \quad \mathbf{z}' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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Elementary Rotations

- Rotation by angle β along y-axis

$$\mathbf{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

- Rotation by angle γ along x-axis

$$\mathbf{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

Rotation Matrices

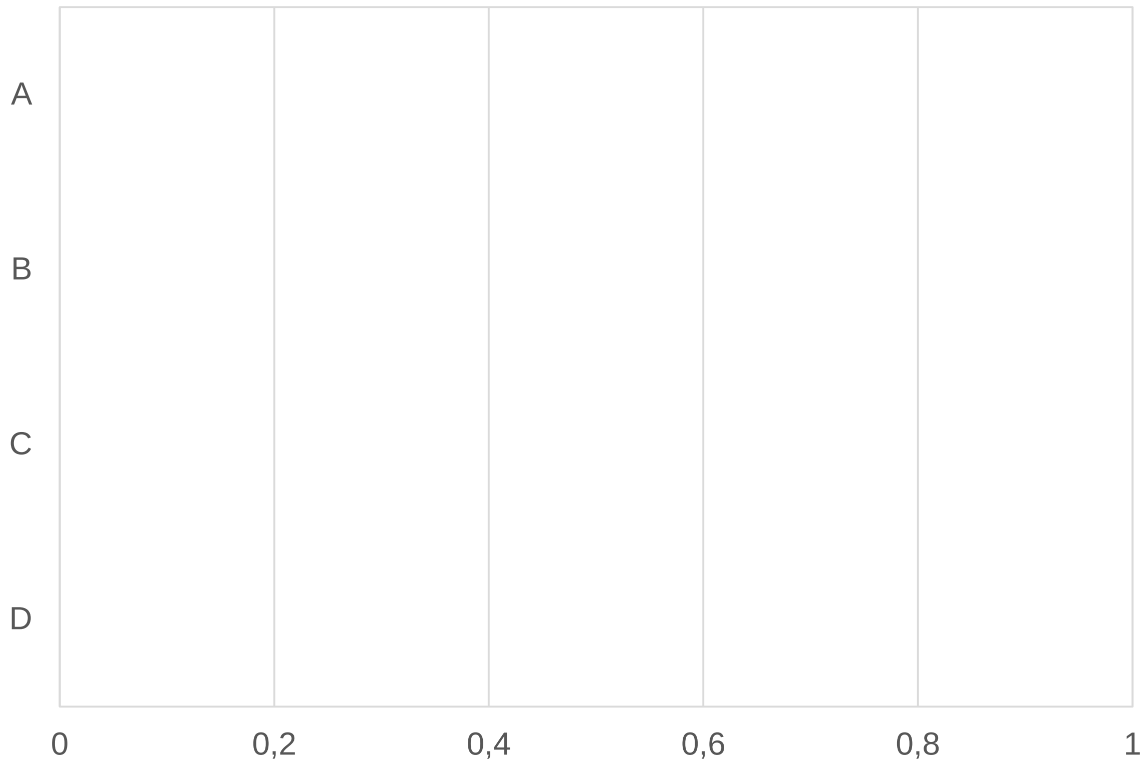
Which properties apply to rotation matrices?

$$\mathbf{R}^T\mathbf{R} = \mathbf{I}$$

$$\mathbf{R}^T = -\mathbf{R}$$

$$\det(\mathbf{R}) = 1$$

$$||\mathbf{x}^T\mathbf{R}\mathbf{x}|| \geq 0$$



Umfrage starten

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Umfrage noch nicht gestartet

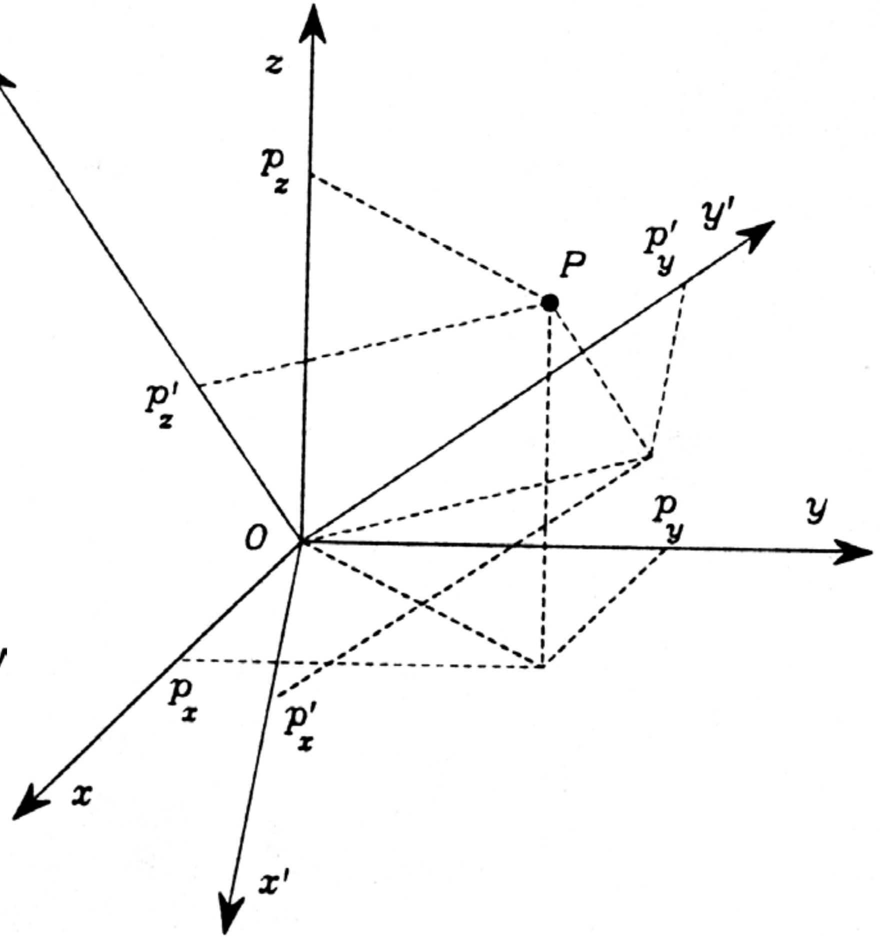
Representation of a Vector in Rotated Frames

A point P in space is represented w.r.t. *O-xyz* by coordinates

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = p_x\mathbf{x} + p_y\mathbf{y} + p_z\mathbf{z}$$

and w.r.t. frame *O-x'y'z'* by coordinates

$$\mathbf{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = p'_x\mathbf{x}' + p'_y\mathbf{y}' + p'_z\mathbf{z}'$$



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Representation of a Vector in Rotated Frames

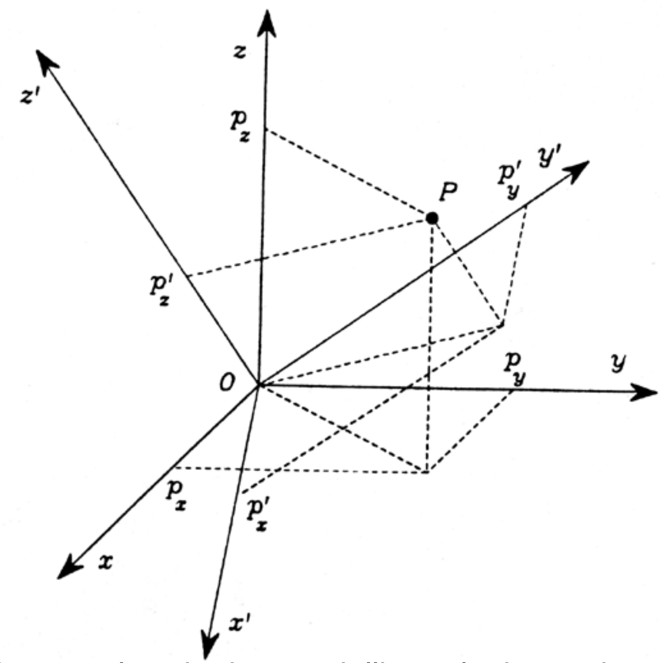
$$\mathbf{p} = p'_x \mathbf{x}' + p'_y \mathbf{y}' + p'_z \mathbf{z}' = \begin{bmatrix} \mathbf{x}' & \mathbf{y}' & \mathbf{z}' \end{bmatrix} \mathbf{p}'$$

Rotation matrix **R** : **transformation matrix** to map coordinates of vector **p’** in frame *O-x’y’z’* into the coordinates of the same vector **p** in frame *O-xyz*.

$$\mathbf{p} = \mathbf{R}\mathbf{p}'$$

The inverse transformation is given by

$$\mathbf{p}' = \mathbf{R}^{\mathrm{T}}\mathbf{p}$$



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Representation of a Vector in Rotated Frames

- two frames with identical origin (pure rotation)
- rotation by α along Z-axis.
- $\mathbf{p}$: coordinates of P w.r.t. O-xyz
  $\mathbf{p}'$: coordinates of P w.r.t. O´-x´y´z´

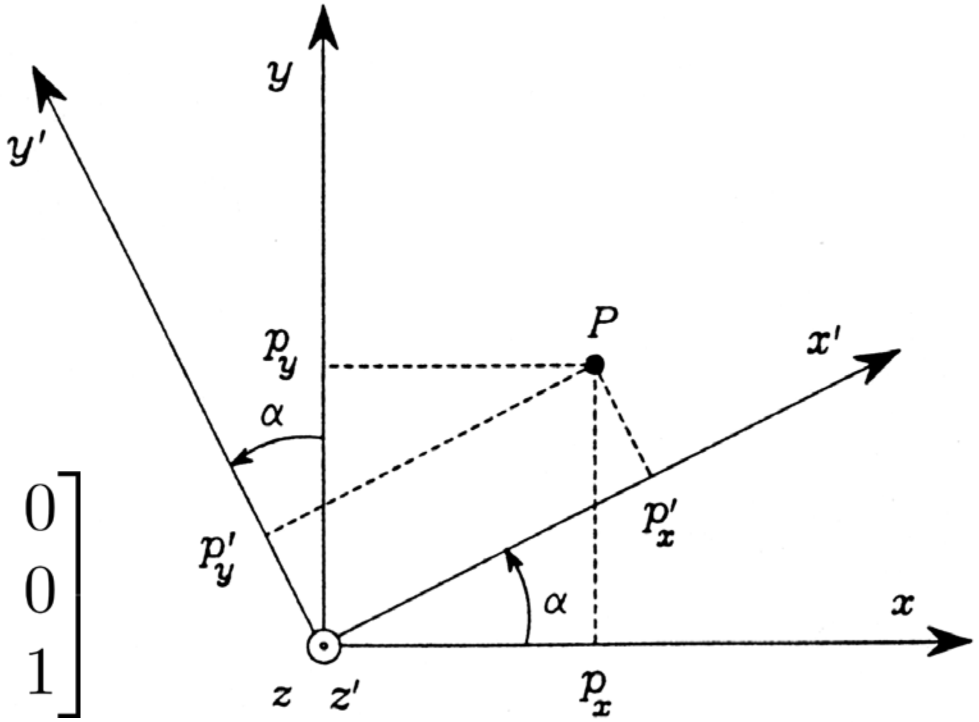
$$p_x = p'_x \cos\alpha - p'_y \sin\alpha$$

$$p_y = p'_x \sin\alpha + p'_y \cos\alpha$$

$$p_z = p'_z$$

$$\mathbf{p} = \mathbf{R}_z(\alpha)\mathbf{p}'$$

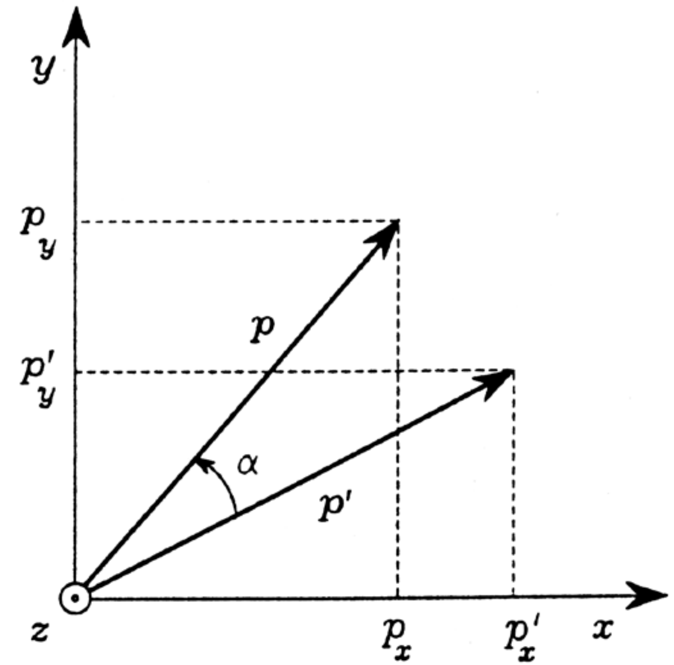
$$\mathbf{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation of a Vector

- **p´** vector in reference frame O-xyz.

- **Rp´** yields a vector **p**
  - same length as **p´**
  - rotated bei **R** w.r.t. **p´**



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$$p_x = p'_x \cos\alpha - p'_y \sin\alpha$$
$$p_y = p'_x \sin\alpha + p'_y \cos\alpha$$
$$p_z = p'_z$$

$$\mathbf{p} = \mathbf{R}_z(\alpha)\mathbf{p}' \qquad \mathbf{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Geometrical Interpretation of a Rotation Matrix

- mutual orientation between two frames

column vectors are the direction cosines of the rotated axis w.r.t. the original frame

$$\mathbf{R} = \begin{bmatrix} \mathbf{x}' & \mathbf{y}' & \mathbf{z}' \end{bmatrix} = \begin{bmatrix} \mathbf{x}'_x & \mathbf{y}'_x & \mathbf{z}'_x \\ \mathbf{x}'_y & \mathbf{y}'_y & \mathbf{z}'_y \\ \mathbf{x}'_z & \mathbf{y}'_z & \mathbf{z}'_z \end{bmatrix} = \begin{bmatrix} \mathbf{x}'^T\mathbf{x} & \mathbf{y}'^T\mathbf{x} & \mathbf{z}'^T\mathbf{x} \\ \mathbf{x}'^T\mathbf{y} & \mathbf{y}'^T\mathbf{y} & \mathbf{z}'^T\mathbf{y} \\ \mathbf{x}'^T\mathbf{z} & \mathbf{y}'^T\mathbf{z} & \mathbf{z}'^T\mathbf{z} \end{bmatrix}$$

- coordinate transformation between the coordinates of a vector expressed in two different frames

$$\mathbf{p} = p'_x\mathbf{x}' + p'_y\mathbf{y}' + p'_z\mathbf{z}' = \begin{bmatrix} \mathbf{x}' & \mathbf{y}' & \mathbf{z}' \end{bmatrix} \mathbf{p}'$$

- operator that generates a rotation of a vector in the same frame

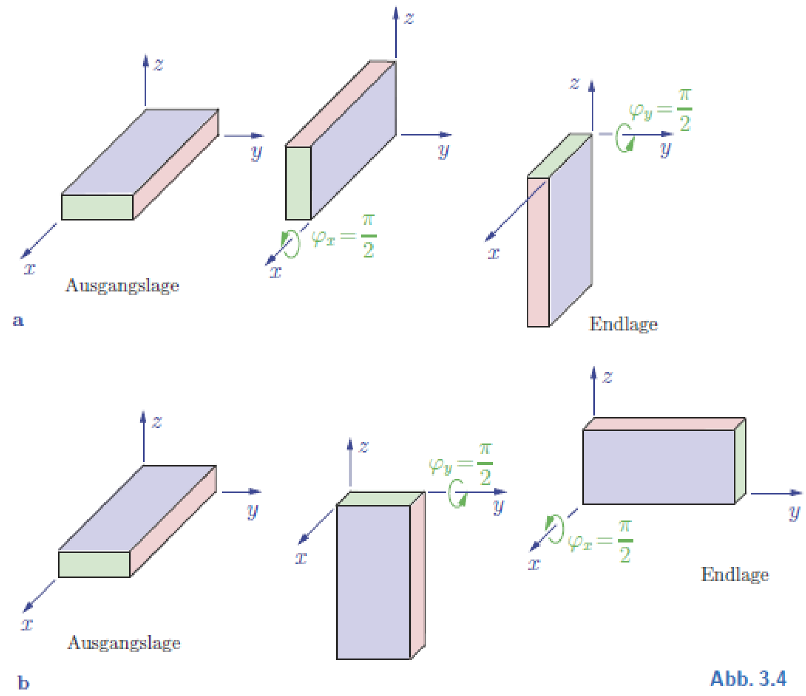
$$\mathbf{p} = \mathbf{R}_z(\alpha)\mathbf{p}'$$

Composition of Rotations

- Infinitesimal rotations and angular velocities are described by *vectors*
- Finite rotations are described by *matrices*
- Composition of rotations by matrix multiplication is <u>non-commutative</u>

first rotate along x-axis
then along y-axis

first rotate along y-axis
then along x-axis

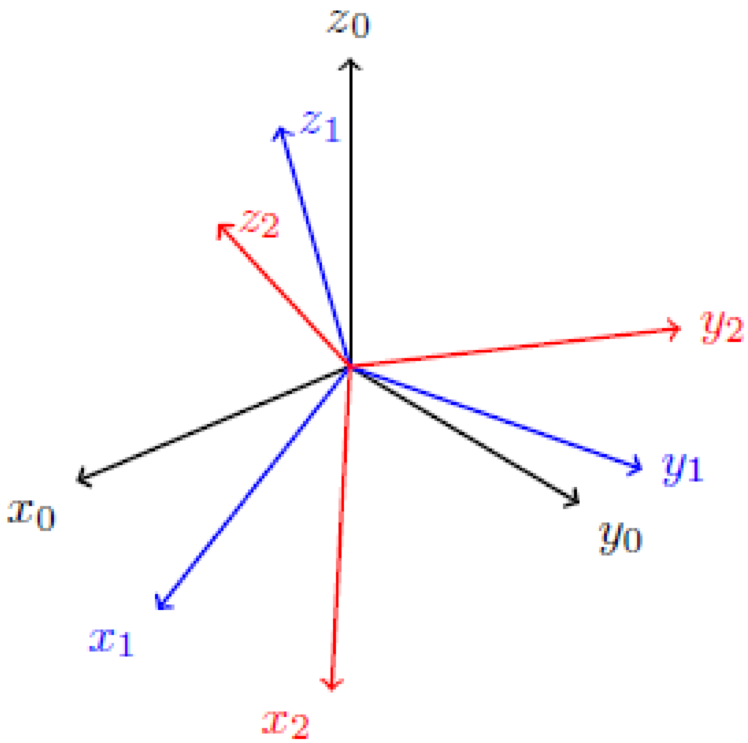


Gross, Hauger,Technische Mechanik

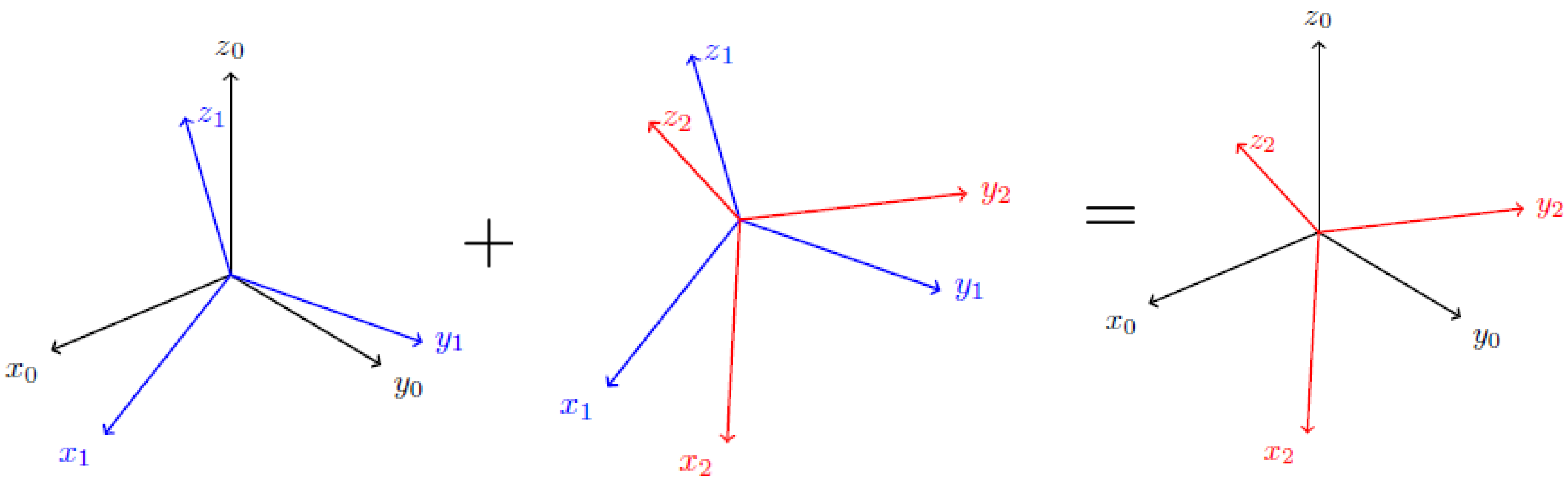
Composition of Rotation Matrices

Three frames with common origin

$$O - x_0 y_0 z_0 \qquad O - x_1 y_1 z_1 \qquad O - x_2 y_2 z_2$$



Composition of Rotation Matrices



$\mathbf{R}_1^0$ $\mathbf{R}_2^1$ $\mathbf{R}_2^0$

Composition of Rotation Matrices

vector $\mathbf{p}$ in the frames $O - x_i y_i z_i$

$$\mathbf{p}^0, \mathbf{p}^1, \mathbf{p}^2$$

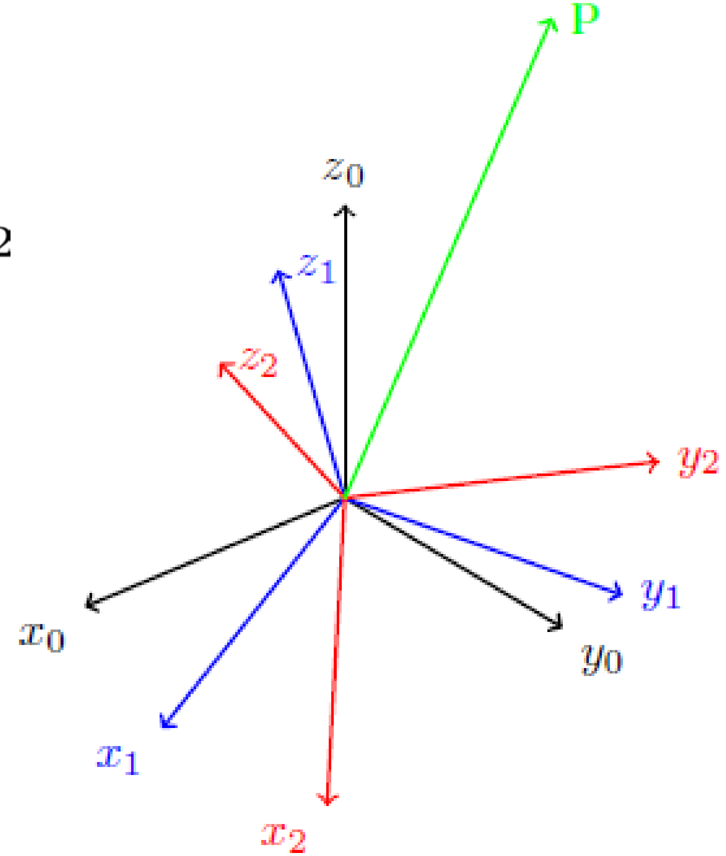
$$\mathbf{p}^1 = \mathbf{R}_2^1 \mathbf{p}^2$$

$$\mathbf{p}^0 = \mathbf{R}_1^0 \mathbf{p}^1$$

$$\mathbf{p}^0 = \mathbf{R}_2^0 \mathbf{p}^2$$

$$\mathbf{p}^0 = \mathbf{R}_1^0 \mathbf{R}_2^1 \mathbf{p}^2$$

$$\mathbf{R}_2^0 = \mathbf{R}_1^0 \mathbf{R}_2^1$$



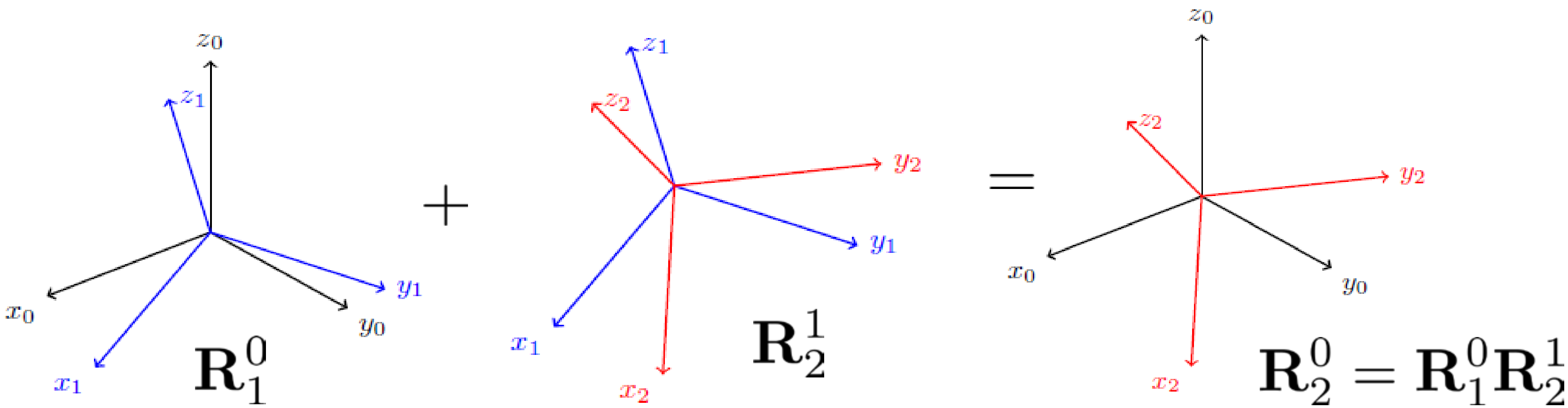
Composition of Rotations w.r.t. Current Frame

Composition of rotations by **post multiplication** of rotation matrices

$$\mathbf{R}_2^0 = \mathbf{R}_1^0 \mathbf{R}_2^1$$

rotation $\mathbf{R}_2^0$ obtained in two steps

- first rotate frame $O - x_0 y_0 z_0$ by $\mathbf{R}_1^0$ to align it with $O - x_1 y_1 z_1$
- then rotate the frame by $\mathbf{R}_2^1$ to align it with $O - x_2 y_2 z_2$



rotation w.r.t. ***current frame*** $\longrightarrow$ postmultiplication of $\mathbf{R}_j^i$

Rotations w.r.t. to Axes of Current Frame

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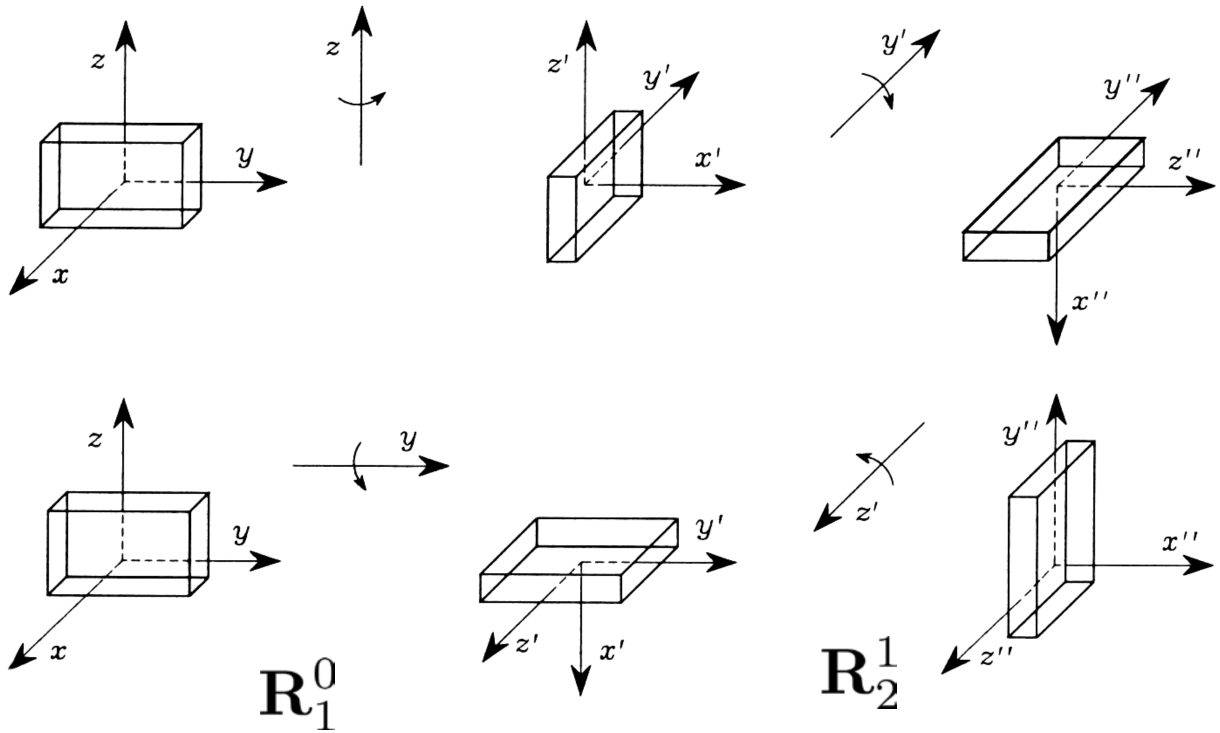

$O - x_0 y_0 z_0$ $\quad$ $O - x_1 y_1 z_1$ $\quad$ $O - x_2 y_2 z_2$

$$\mathbf{R}_2^0 = \mathbf{R}_1^0 \mathbf{R}_2^1$$

Composition of Rotations w.r.t. Fixed Frame

$\mathbf{R}_1^0$ rotation matrix of frame $O - x_1y_1z_1$ w.r.t. fixed frame $O - x_0y_0z_0$
$\bar{\mathbf{R}}_2^0$ rotation matrix of frame $O - x_2y_2z_2$ w.r.t. fixed frame $O - x_0y_0z_0$

- Realign Frame 1 with Frame 0 by means of $\mathbf{R}_0^1$
- Make the rotation expressed by $\bar{\mathbf{R}}_2^1$ w.r.t. current frame
- Compensate for the initial realignment $\mathbf{R}_0^1$ by means of the inverse rotation $\mathbf{R}_1^0$
- Composition rule for rotations w.r.t. to current frame

$$\bar{\mathbf{R}}_2^0 = \mathbf{R}_1^0\mathbf{R}_0^1\bar{\mathbf{R}}_2^1\mathbf{R}_0^1$$

- Since $\mathbf{R}_1^0\mathbf{R}_0^1 = \mathbf{I}$

$$\bar{\mathbf{R}}_2^0 = \bar{\mathbf{R}}_2^1\mathbf{R}_0^1$$

- Premultiplication of rotation matrices

30

Rotations w.r.t to Axes of Fixed Frame

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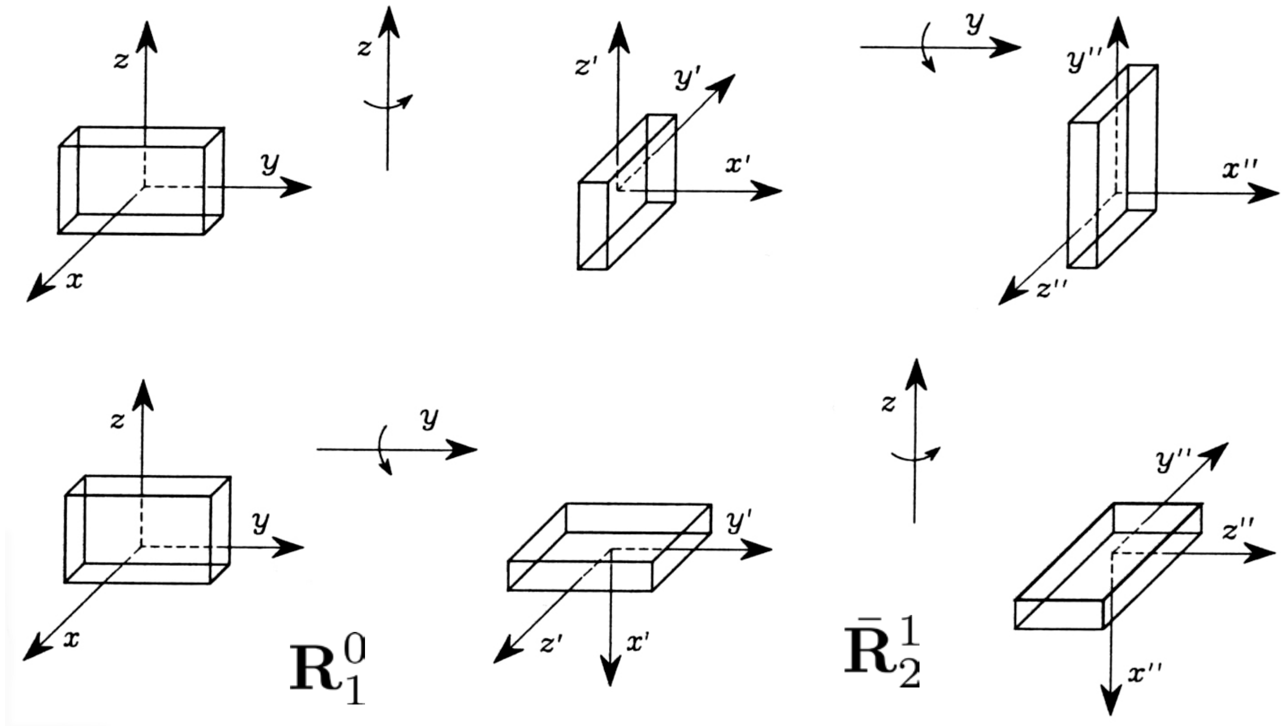


$O - x_0y_0z_0$ $\qquad$ $O - x_1y_1z_1$ $\qquad$ $O - x_2y_2z_2$

$$\bar{\mathbf{R}}_2^0 = \bar{\mathbf{R}}_2^1\mathbf{R}_0^1$$

Rotation Matrices

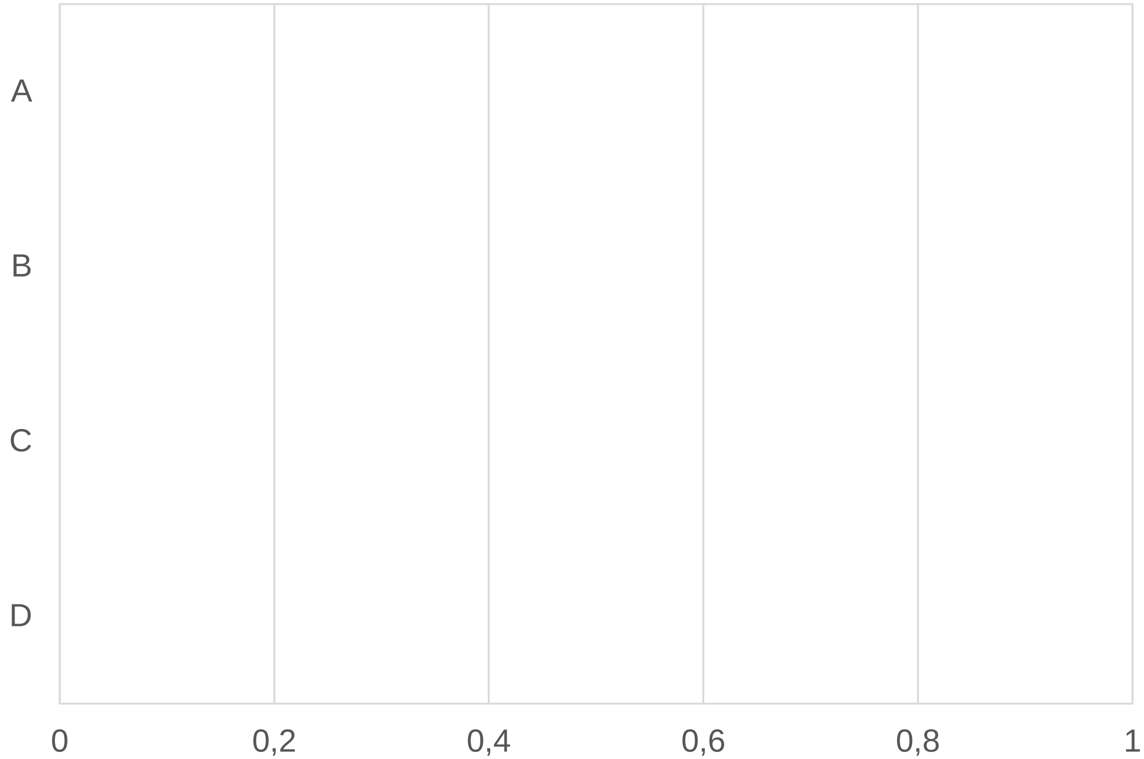
The composition of rotations R01 and R12 w.r.t. current frame is given by?

$$\mathbf{R}_2^0 = \mathbf{R}_2^1 \mathbf{R}_1^0$$

$$\mathbf{R}_2^0 = {\mathbf{R}_2^1}^T {\mathbf{R}_1^0}^T$$

$$\mathbf{R}_2^0 = \mathbf{R}_1^0 \mathbf{R}_2^1$$

$$\mathbf{R}_2^0 = \mathbf{R}_2^1 + \mathbf{R}_1^0$$



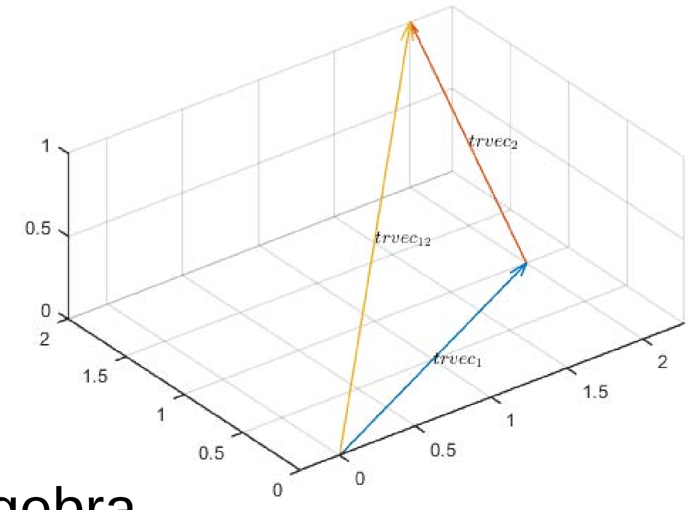
Umfrage starten

ID = frank.hoffmann@tu-dortmund.de
Umfrage noch nicht gestartet

Translations in Robotics System Toolbox

- Translations are ordinary 1-by-3 vectors

```
trvec = [3 0 2.5];
```



- Composition of translations by vector algebra

```
trvec1 = [2 1 0];
trvec2 = [0 1 1];
trvec12=trvec1+trvec2;
clf;
hold on;
quiver3(0,0,0,trvec1(1),trvec1(2),trvec1(3),0);
quiver3(trvec1(1),trvec1(2),trvec1(3), trvec2(1), trvec2(2), trvec2(3),0);
quiver3(0,0,0,trvec12(1),trvec12(2),trvec12(3),0);
```

Rotations in Robotics System Toolbox

- Define rotation matrix component wise

```
theta = pi/2;
% elemental rotation about the x-axis
rotmx = [1 0 0; 0 cos(theta) -sin(theta); 0 sin(theta) cos(theta)];
% elemental rotation about the y-axis
rotmy = [cos(theta) 0 sin(theta) 0; 0 1 0; -sin(theta) 0 cos(theta)];
% elemental rotation about the z-axis
rotmz = [cos(theta) -sin(theta) 0; sin(theta) cos(theta) 0; 0 0 1];
```

- Define elemental rotations by angle axis notation

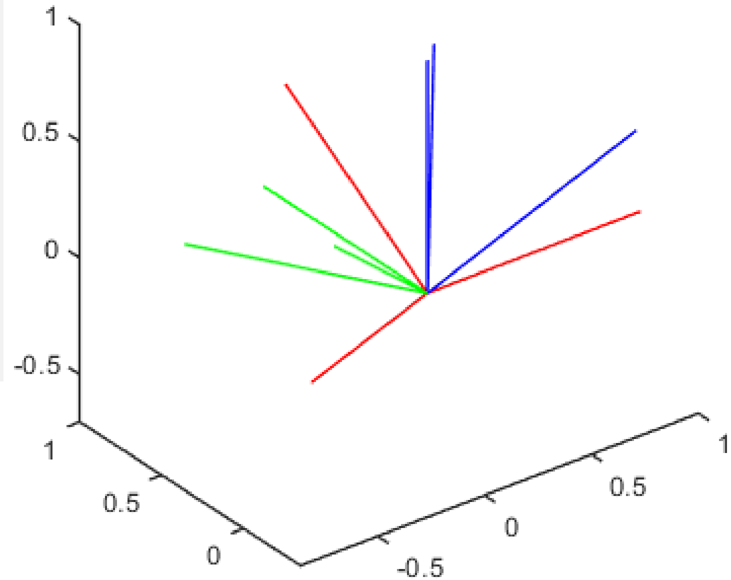
```
theta = pi/2;
% elemental rotation about the x-axis
rotmx = axang2rotm([1 0 0 theta]);
% elemental rotation about the y-axis
rotmy = axang2rotm([0 1 0 theta]);
% elemental rotation about the z-axis
rotmz = axang2rotm([0 0 1 theta]);
```

Sequence of Rotations in Robotics System Toolbox

- Composition w.r.t. current frame

$$\mathbf{R} = \mathbf{R}_z(\alpha)\,\mathbf{R}_y(\beta)\,\mathbf{R}_x(\gamma)$$

```
alpha = pi/2;
beta=pi/4;
gamma=pi/6;
rotmz = axang2rotm([0 0 1 alpha]);
rotmy = axang2rotm([0 1 0 beta]);
rotmx = axang2rotm([1 0 0 gamma]);
rotrpy= rotmz*rotmy*rotmx;
```



- Composition w.r.t. fixed frame

$$\mathbf{R} = \mathbf{R}_x(\gamma)\,\mathbf{R}_y(\beta)\,\mathbf{R}_z(\alpha)$$

```
roteul= rotmx*rotmy*rotmz;
```

```
plotTransforms(zeros(3,3),[1 0 0 0; rotm2quat(rotrpy); rotm2quat(roteul)]);
axis equal;
```

Apply Translation and Rotation to a 3DVector

- First apply rotation then translation $\mathbf{p}^0 = \mathbf{R}_1^0\mathbf{p}^1 + \mathbf{t}_1^0$

```
p1=[1 1 2];
trvec01=[1 0 -1];
rotmx01=axang2rotm([1 0 0 pi/2]);
p0=rotmx01*p1'+trvec01';
```

- First apply translation then rotation $\mathbf{p}^0 = \mathbf{R}_1^0(\mathbf{p}^1 + \mathbf{t}_1^0)$

```
p1=[1 1 2];
trvec01=[1 0 -1];
rotmx01=axang2rotm([1 0 0 pi/2]);
p0=rotmx01*(p1+trvec01)';
```

Matlab Grader Assignment

Your Script

```
%% 1.1 translations along x axis by a=1 and z-axis by d=2
a=1;
d=2;

```

Your Script

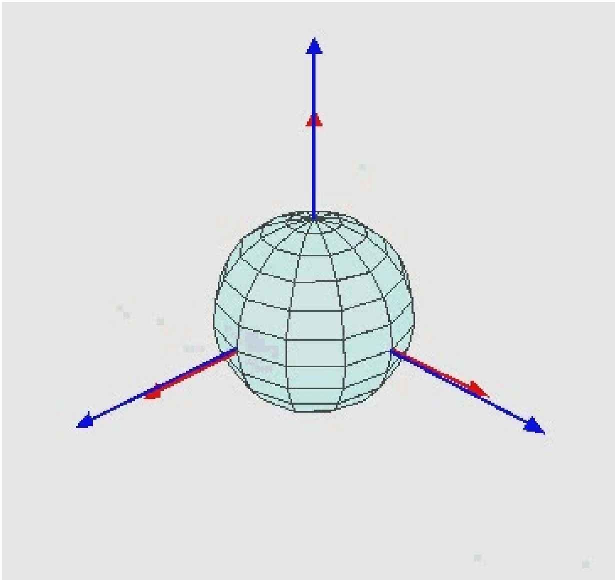
```
%% 1.2 rotations rotmx about x-axis by alpha = pi/2  and rotmz z-axis by gamma= pi/4
alpha=pi/2;
gamma=pi/4;
rotmx=
rotmz=

%% 1.3 rotations rotmc w.r.t. current and rotmf w.r.t. fixed frame
rotmc=
rotmf=

%% 1.4 apply rotations to 3D vector p
p=[1 2 0]';
pc=
pf=

```

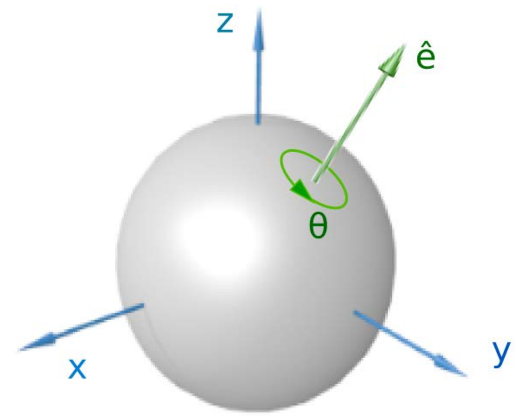
Minimal Representation

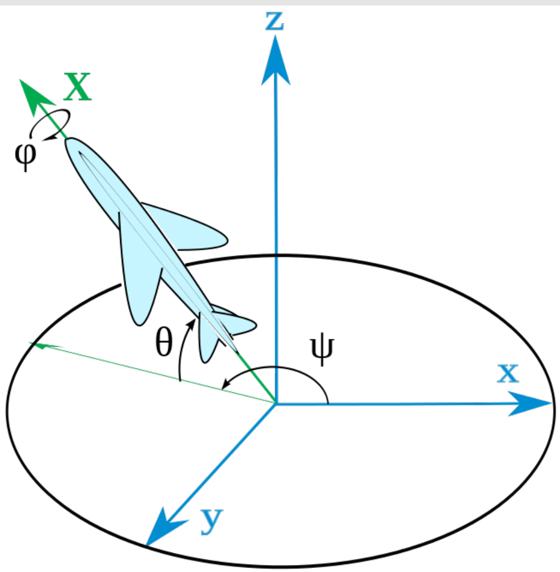
$$\mathbf{R} = \begin{bmatrix}\mathbf{x}' & \mathbf{y}' & \mathbf{z}'\end{bmatrix} = \begin{bmatrix} \mathbf{x}'_x & \mathbf{y}'_x & \mathbf{z}'_x \\ \mathbf{x}'_y & \mathbf{y}'_y & \mathbf{z}'_y \\ \mathbf{x}'_z & \mathbf{y}'_z & \mathbf{z}'_z \end{bmatrix} = \begin{bmatrix} \mathbf{x}'^T\mathbf{x} & \mathbf{y}'^T\mathbf{x} & \mathbf{z}'^T\mathbf{x} \\ \mathbf{x}'^T\mathbf{y} & \mathbf{y}'^T\mathbf{y} & \mathbf{z}'^T\mathbf{y} \\ \mathbf{x}'^T\mathbf{z} & \mathbf{y}'^T\mathbf{z} & \mathbf{z}'^T\mathbf{z} \end{bmatrix}$$

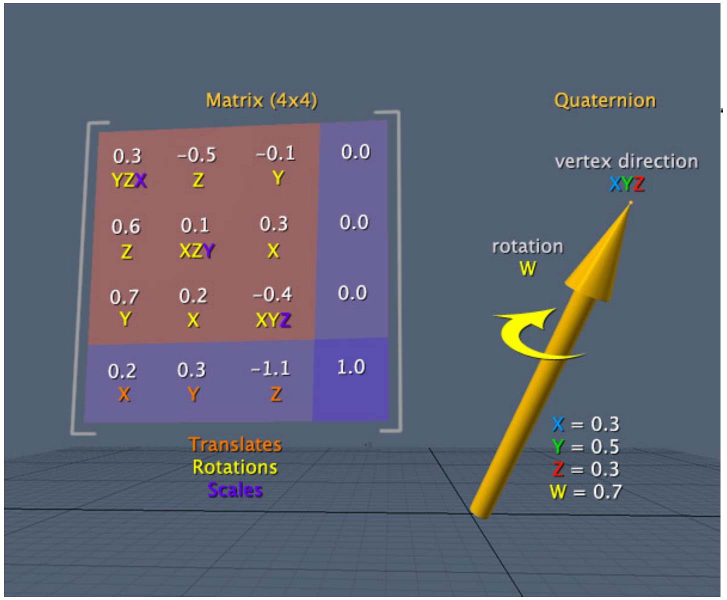
- Rotation matrix has ***redundant*** parameters
- $\mathbf{R}$ is a 3x3 matrix with *nine* components $\mathbf{R}_{ij}$
- $\mathbf{R}$ is constrained by *six* orthogonality constraints $\mathbf{R}^T\mathbf{R} = \mathbf{I}_3$
- *Three independent* parameter are sufficient to determine orientation of a rigid body
- Euler angles provide such a *minimal representation*.

Rotation Representations

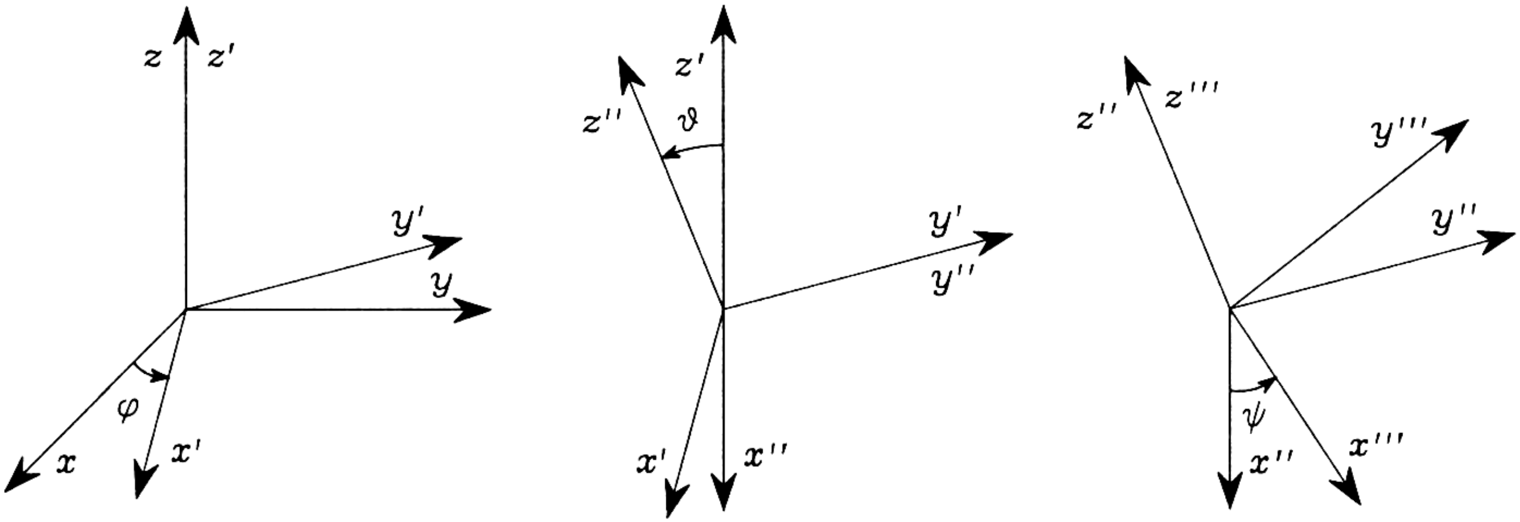








Euler Angles

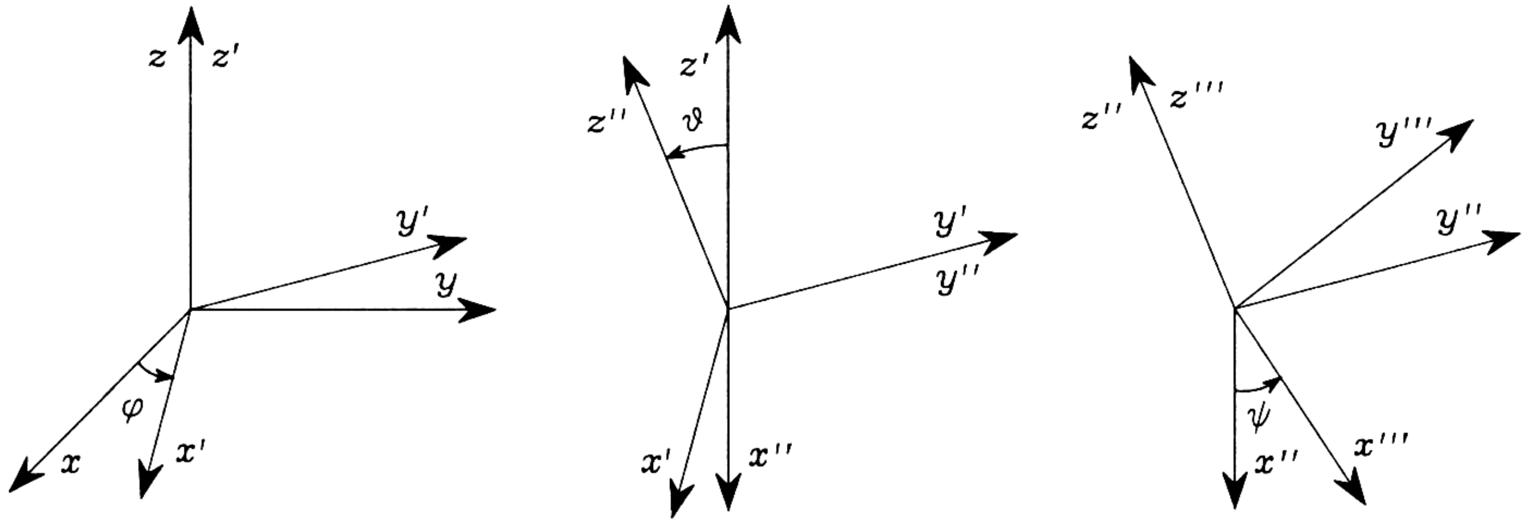


Siciliano et al, Robotics: Modelling, Planing and Control

Euler angles (*z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y*)

Tait–Bryan angles (*x-y-z, y-z-x, z-x-y, x-z-y, z-y-x, y-x-z*).

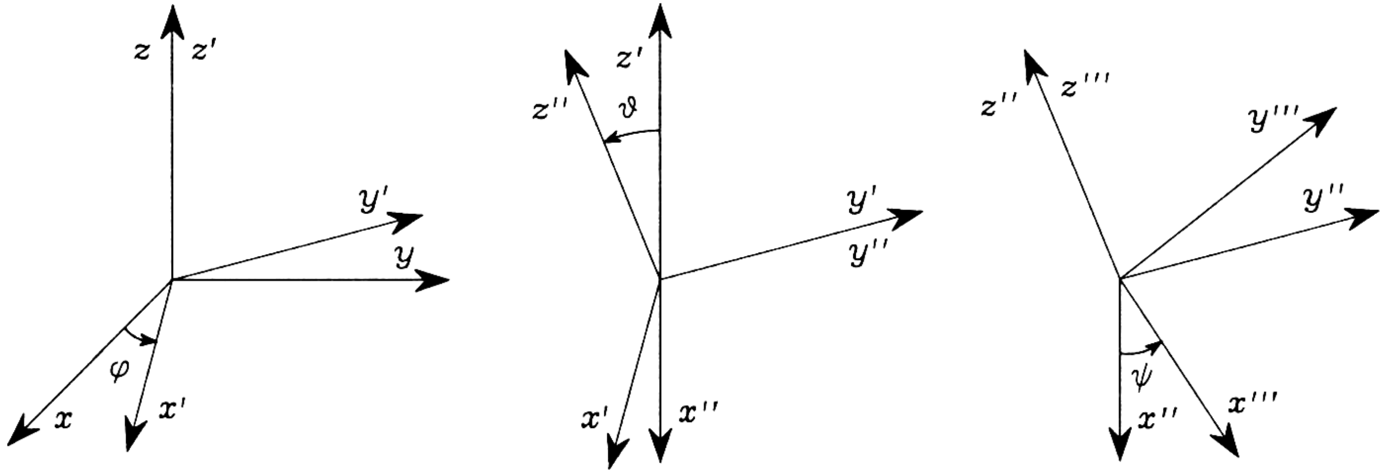
Euler Angles ZYZ



Siciliano et al, Robotics: Modelling, Planing and Control

- Rotate the reference frame by the angle φ about z-axis
- Rotate the current frame by the angle ϑ about y'-axis
- Rotate the current frame by the angle ψ about z''-axis

Euler Angles ZYZ



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$$\mathbf{R}_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R}_{y'}(\vartheta) = \begin{bmatrix} \cos\vartheta & 0 & -\sin\vartheta \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{bmatrix}$$

$$\mathbf{R}_{z''}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles ZYZ

- composition of rotations w.r.t. current frame

$$\mathbf{R}(\phi) = \mathbf{R}_z(\varphi)\mathbf{R}_{y'}(\vartheta)\mathbf{R}_{z''}(\psi)$$

$$= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}$$

Inverse Solution to Euler Angles

- Euler angles φ, γ, ψ that correspond to a rotation matrix **R**

$$\mathbf{R}(\phi) = \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}$$

$$\varphi = \mathrm{atan2}(r_{23}, r_{13})$$

$$\vartheta = \mathrm{atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

$$\psi = \mathrm{atan2}(r_{32}, -r_{31})$$

Inverse Solution to Euler Angles

$$\mathbf{R}(\phi) = \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}$$

$$\varphi = \operatorname{atan2}(r_{23}, r_{13})$$

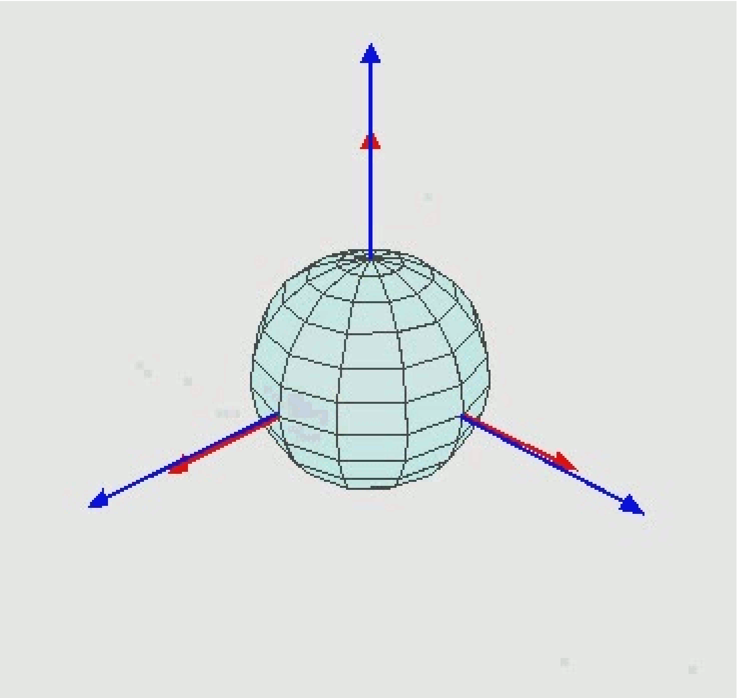
$$\vartheta = \operatorname{atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

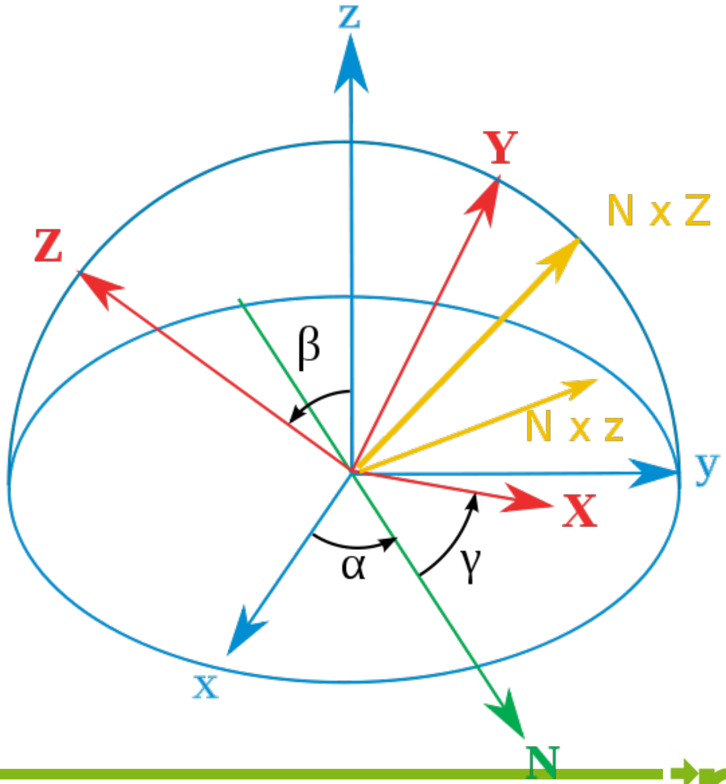
$$\psi = \operatorname{atan2}(r_{32}, -r_{31})$$

Singularity for $\sin\vartheta = 0$

Euler Angles ZXZ

- start with xyz-frame and XYZ-frame aligned
- rotate *XYZ*-frame along current *Z*-axis by α.
- rotate *XYZ*-frame w.r.t. to current *X*-axis by β.
- rotate *XYZ*-frame w.r.t. to current Z-axis by γ.

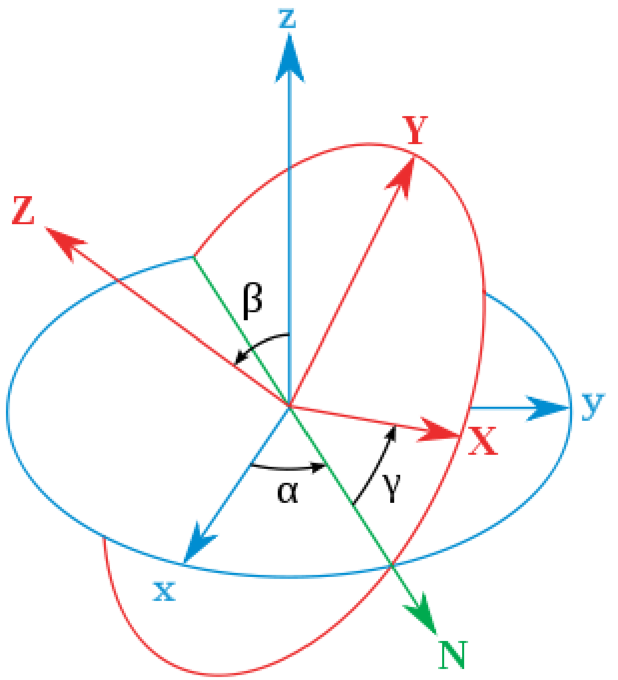




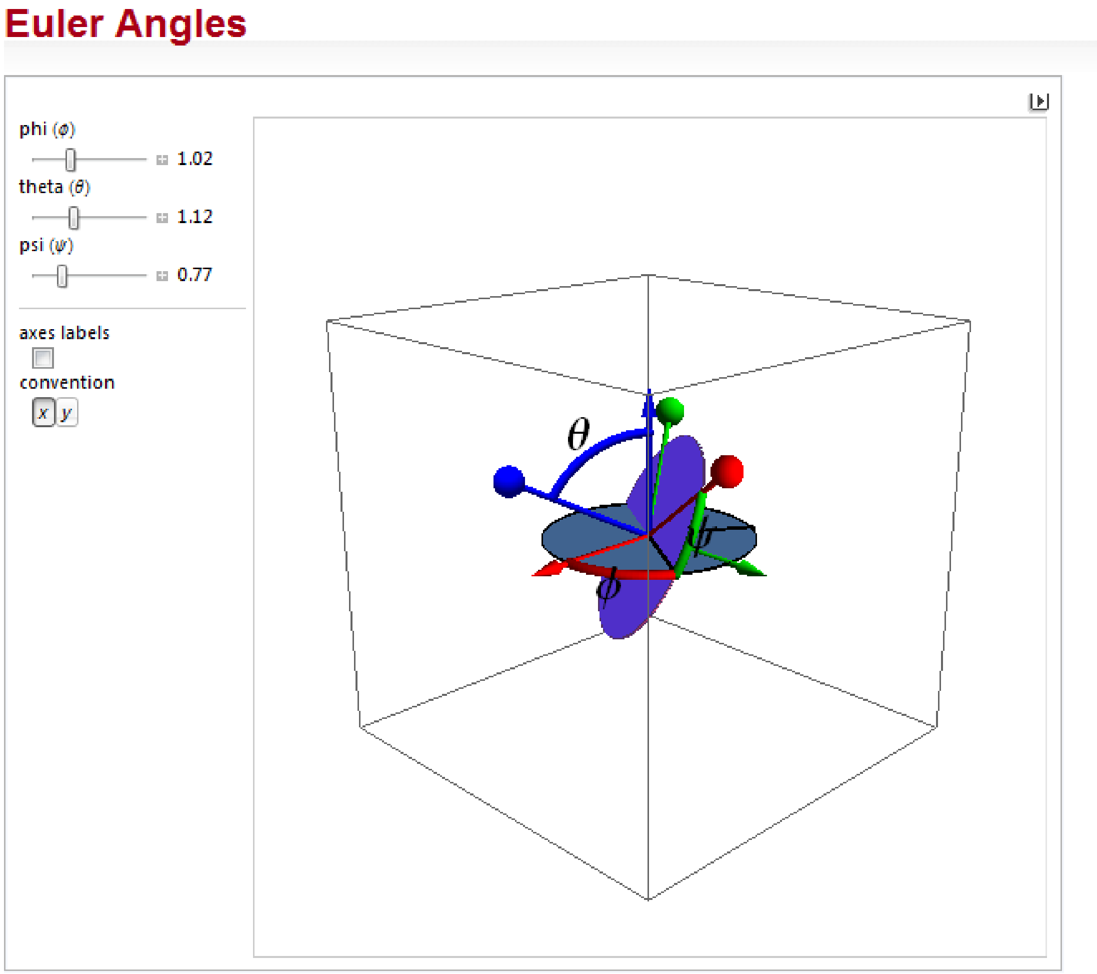
Euler Angles ZXZ

- intersection line N between xy and XY plane
- fixed frame xyz
- rotated frame XYZ

- α (φ) is the angle between *x*-axis and N
- β (θ) is the angle between *z*-axis and *Z*-axis
- γ (ψ) is the angle between N and *X*-axis.



Euler Angles Wolfram Demonstration



http://demonstrations.wolfram.com/EulerAngles/

Gimbal Lock Euler Angles

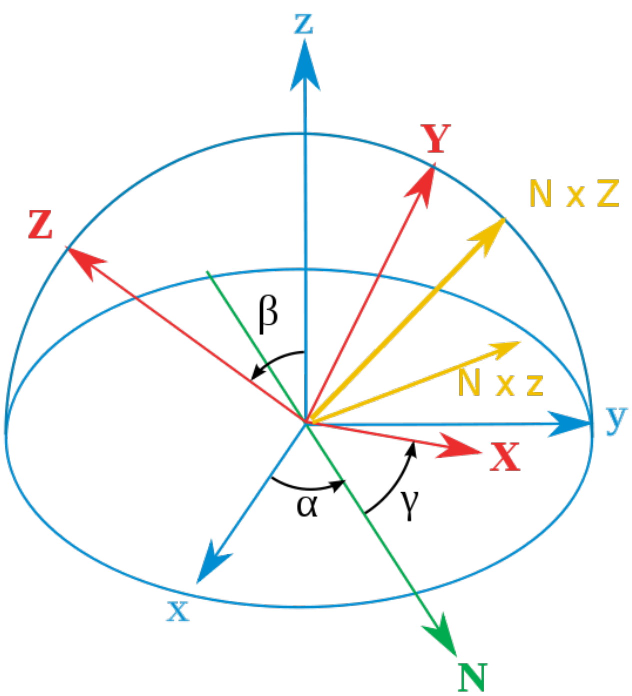
- singularity (gimbal lock)
- for $\vartheta,\beta=0$ the z- and Z-axis become parallel
- ambigious solutions for φ and ψ
- inverse solution degenerates for $\vartheta=0$ since $s_\vartheta=0$, $r_{13},r_{23},r_{32},r_{31}=0$

$$\boldsymbol{R}=\begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}$$

$$\varphi = \mathrm{atan2}(r_{23}, r_{13})$$

$$\vartheta = \mathrm{atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

$$\psi = \mathrm{atan2}(r_{32}, -r_{31})$$



Euler Angles

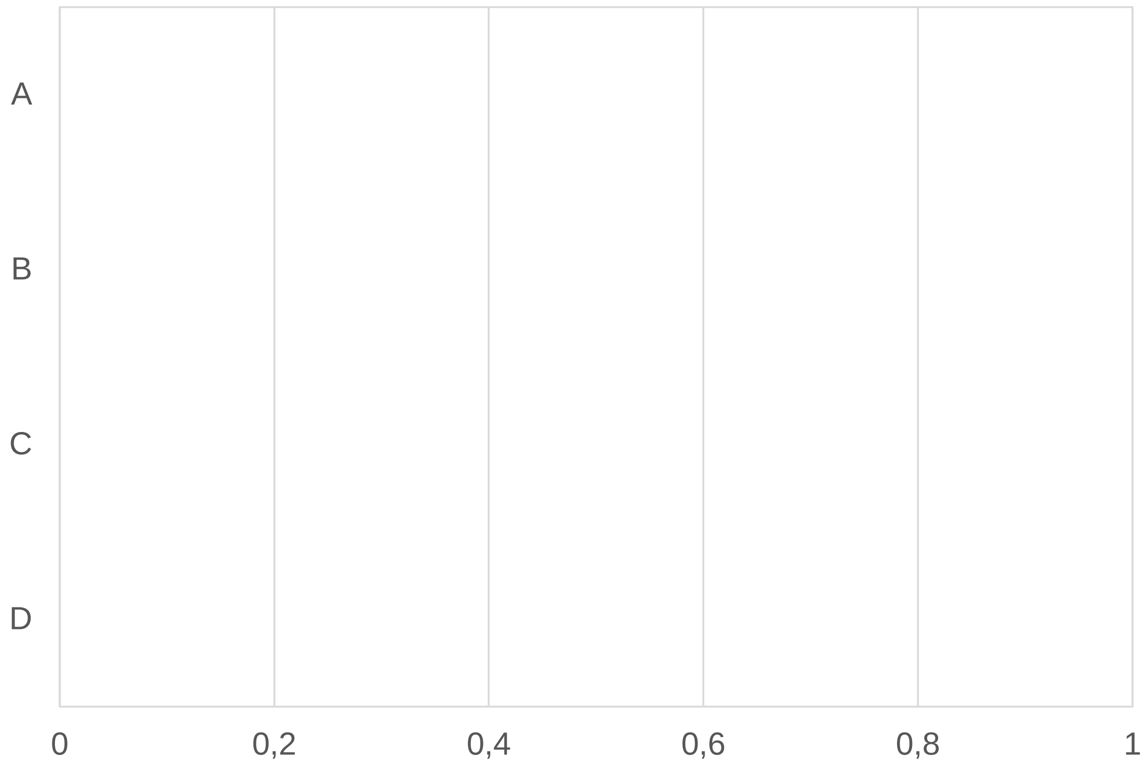
What is the most common order of axis in Euler angles?

XYZ

YZY

ZYX

ZYZ



Umfrage starten

ID = frank.hoffmann@tu-dortmund.de
Umfrage noch nicht gestartet

Representation of Rotations

Z-Y-X Euler angles (α, β, γ):

$$
{}^j\boldsymbol{R}_i = \begin{pmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{pmatrix}
$$

X-Y-Z fixed angles (ψ, θ, ϕ):

$$
{}^j\boldsymbol{R}_i = \begin{pmatrix} c_\phi c_\theta & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ s_\phi c_\theta & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{pmatrix}
$$

Angle-axis $\theta\hat{\boldsymbol{w}}$:

$$
{}^j\boldsymbol{R}_i = \begin{pmatrix} w_x^2 v_\theta + c_\theta & w_x w_y v_\theta - w_z s_\theta & w_x w_z v_\theta + w_y s_\theta \\ w_x w_y v_\theta + w_z s_\theta & w_y^2 v_\theta + c_\theta & w_y w_z v_\theta - w_x s_\theta \\ w_x w_z v_\theta - w_y s_\theta & w_y w_z v_\theta + w_x s_\theta & w_z^2 v_\theta + c_\theta \end{pmatrix}
$$

Unit quaternions $(\epsilon_0\ \epsilon_1\ \epsilon_2\ \epsilon_3)^{\mathrm{T}}$:

$$
{}^j\boldsymbol{R}_i = \begin{pmatrix} 1-2(\epsilon_2^2+\epsilon_3^2) & 2(\epsilon_1\epsilon_2 - \epsilon_0\epsilon_3) & 2(\epsilon_1\epsilon_3 + \epsilon_0\epsilon_2) \\ 2(\epsilon_1\epsilon_2 + \epsilon_0\epsilon_3) & 1-2(\epsilon_1^2+\epsilon_3^2) & 2(\epsilon_2\epsilon_3 - \epsilon_0\epsilon_1) \\ 2(\epsilon_1\epsilon_3 - \epsilon_0\epsilon_2) & 2(\epsilon_2\epsilon_3 + \epsilon_0\epsilon_1) & 1-2(\epsilon_1^2+\epsilon_2^2) \end{pmatrix}
$$

Rotation matrix:

$$
{}^j\boldsymbol{R}_i = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}
$$

Z-Y-X Euler angles (α, β, γ):

$$\beta = \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\alpha = \mathrm{Atan2}\left(\frac{r_{21}}{\cos\beta}, \frac{r_{11}}{\cos\beta}\right)$$

$$\gamma = \mathrm{Atan2}\left(\frac{r_{32}}{\cos\beta}, \frac{r_{33}}{\cos\beta}\right)$$

X-Y-Z fixed angles (ψ, θ, ϕ):

$$\theta = \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\psi = \mathrm{Atan2}\left(\frac{r_{21}}{\cos\theta}, \frac{r_{11}}{\cos\theta}\right)$$

$$\phi = \mathrm{Atan2}\left(\frac{r_{32}}{\cos\theta}, \frac{r_{33}}{\cos\theta}\right)$$

Angle axis $\theta\hat{\boldsymbol{w}}$:

$$\theta = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)$$

$$\hat{\boldsymbol{w}} = \frac{1}{2\sin\theta}\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

Unit quaternions $(\epsilon_0\ \epsilon_1\ \epsilon_2\ \epsilon_3)^{\mathrm{T}}$:

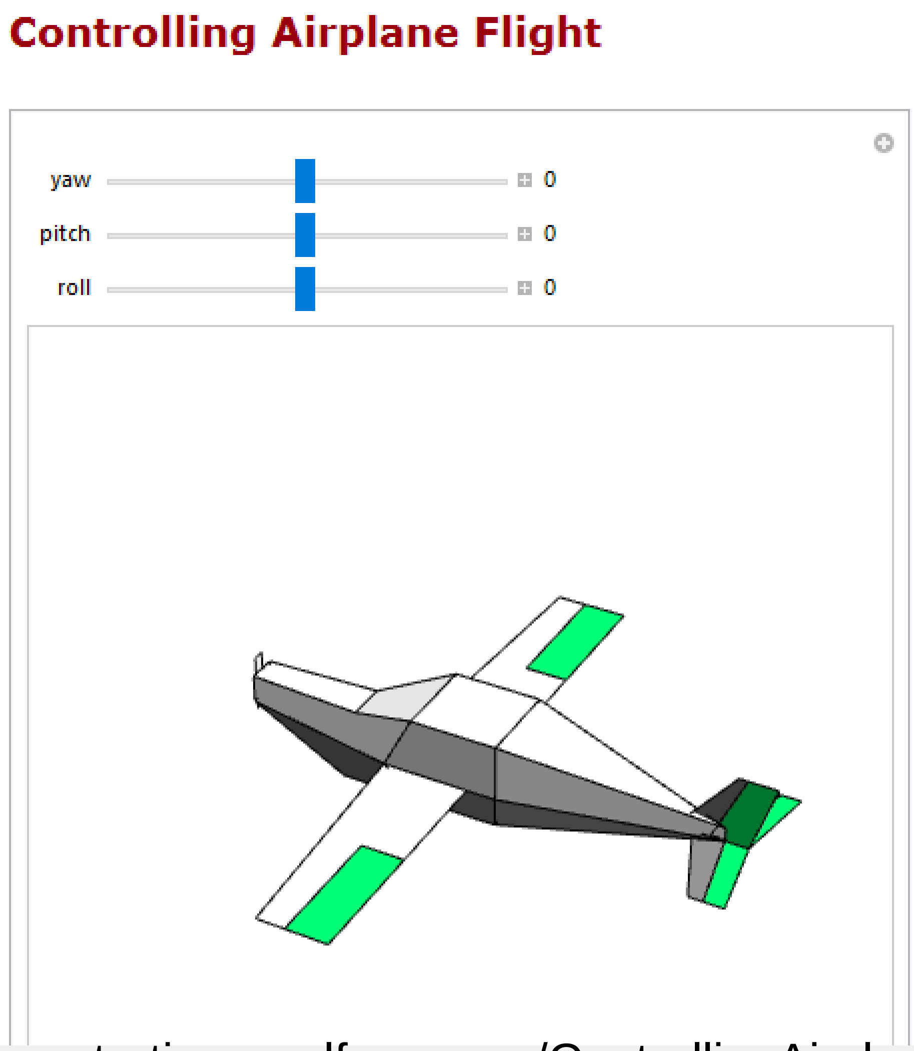
$$\epsilon_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

$$\epsilon_1 = \frac{r_{32} - r_{23}}{4\epsilon_0}$$

$$\epsilon_2 = \frac{r_{13} - r_{31}}{4\epsilon_0}$$

$$\epsilon_3 = \frac{r_{21} - r_{12}}{4\epsilon_0}$$

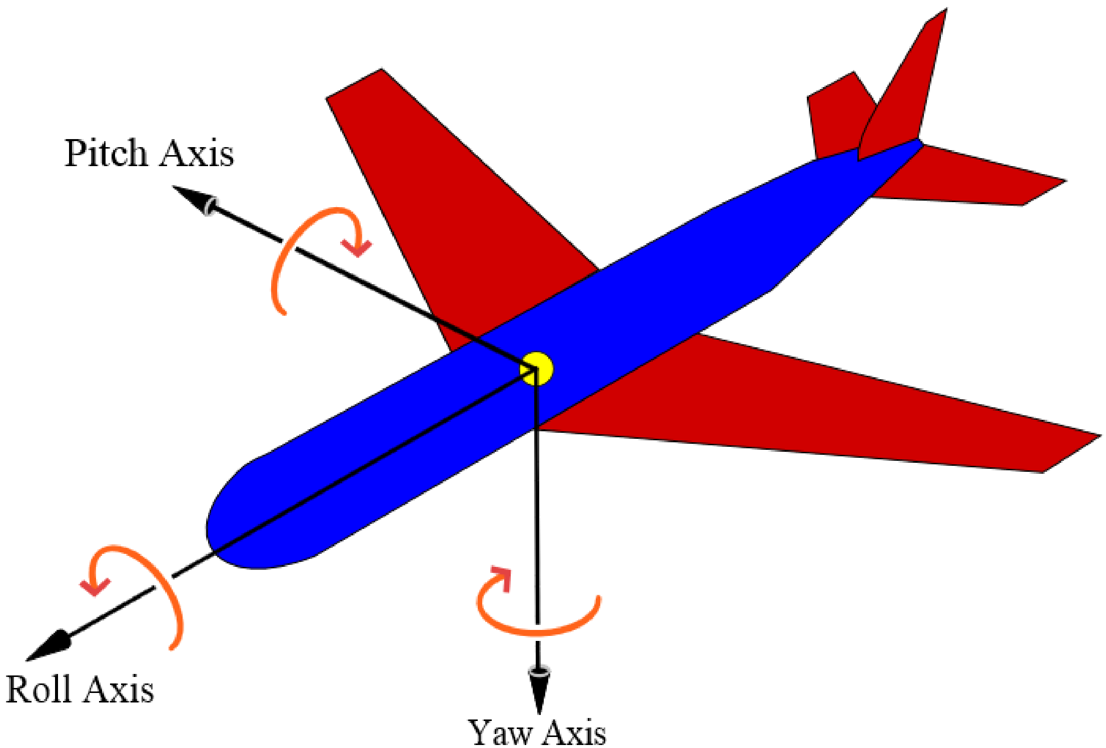
Roll Pitch Yaw



http://demonstrations.wolfram.com/ControllingAirplaneFlight/

Roll Pitch Yaw (Tait Bryan) Angles

- Aeronautic and automotive applications
- Example: attitiude of an aircraft

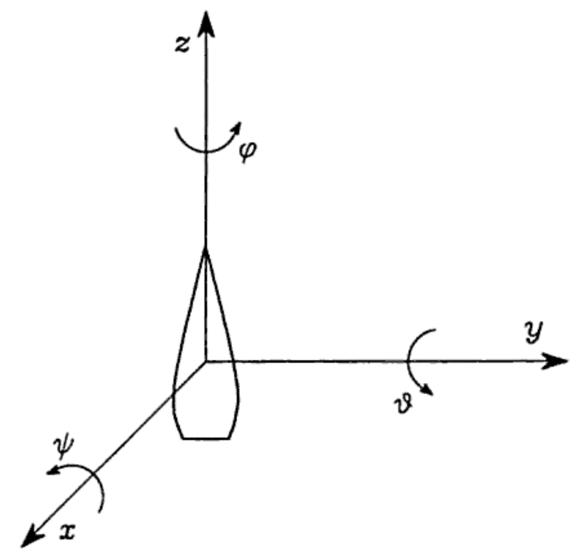


Roll Pitch Yaw Angles

- Rotate the reference frame by the angle ψ about x-axis (roll)
- Rotate the reference frame by the angle ϑ about y-axis (pitch)
- Rotate the reference frame by the angle φ about z'-axis (yaw)
- Rotations typically defined w.r.t fixed frame (extrinsic)

$$\phi = (\psi, \vartheta, \varphi) \qquad R(\phi) = R_z(\varphi) R_y(\vartheta) R_x(\psi)$$

$$\boldsymbol{R}(\phi) = \begin{bmatrix} c_\varphi c_\vartheta & -c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & -s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix}$$



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Roll Pitch Yaw Angles Inverse Mapping

$$\boldsymbol{R}(\phi) = \begin{bmatrix} c_\varphi c_\vartheta & -c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & -s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix}$$

$$\phi = (\psi, \vartheta, \varphi) \qquad R(\phi) = R_z(\varphi) R_y(\vartheta) R_x(\psi)$$

$$\varphi = \text{atan2}(r_{21}, r_{11})$$

$$\vartheta = \text{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\psi = \text{atan2}(r_{32}, r_{33})$$

RPY Angles

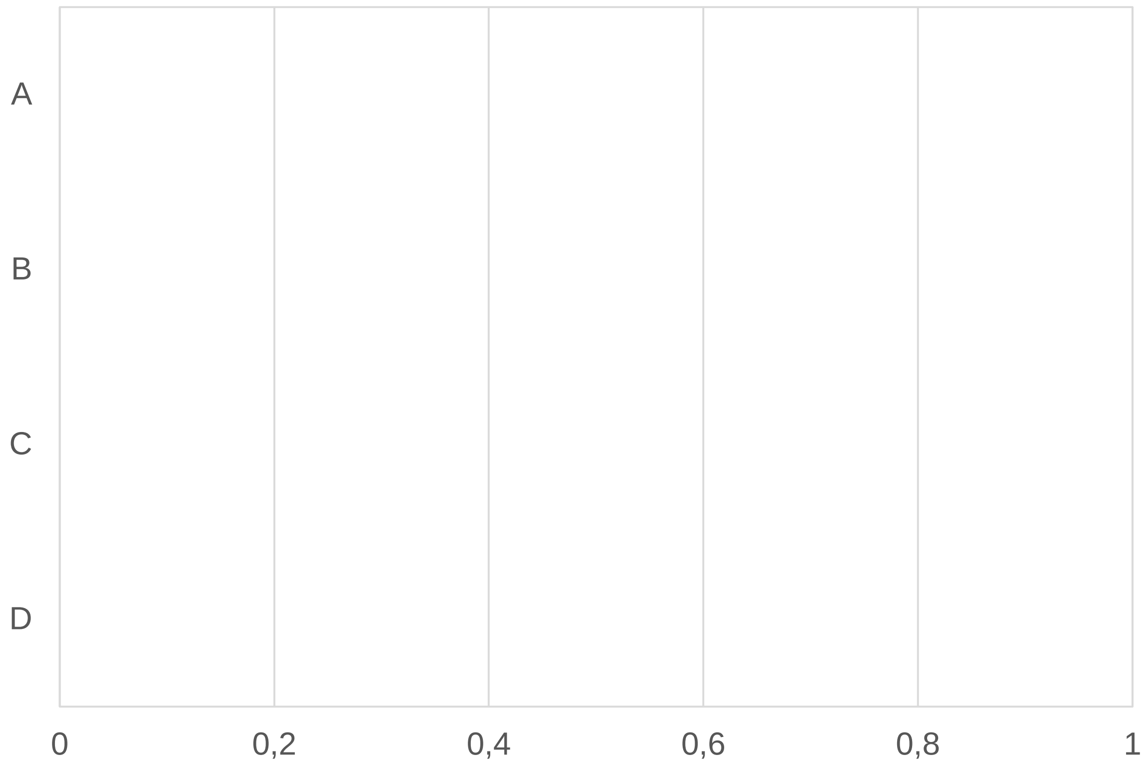
What is the most common order of axis in RPY angles?

XYZ

YZY

ZYX

ZYZ



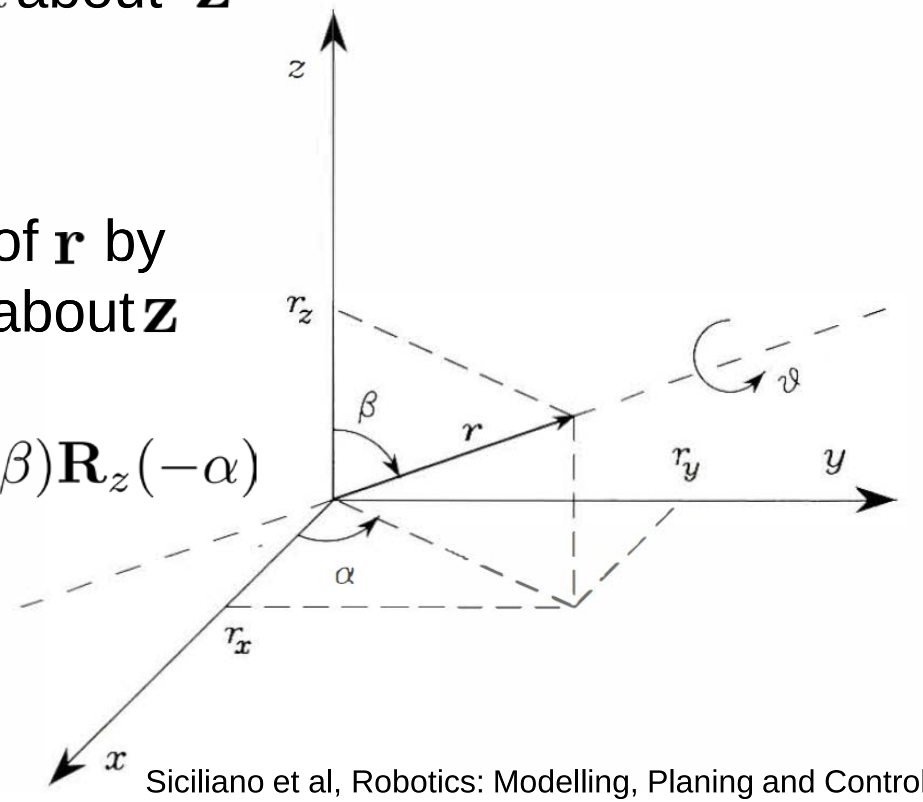
Umfrage starten

ID = frank.hoffmann@tu-dortmund.de
Umfrage noch nicht gestartet

Angle - Axis

- Unit vector of rotation axis $\mathbf{r} = [r_x \ r_y \ r_z]$ $\qquad r_x^2 + r_y^2 + r_z^2 = 1$
- rotation matrix $\mathbf{R}(\vartheta, \mathbf{r})$
- align $\mathbf{r}$ with $\mathbf{z}$ by rotating by $-\alpha$ about $\mathbf{z}$ and by $-\beta$ about $\mathbf{y}$.
- rotate by ϑ about $\mathbf{z}$
- realign with the initial direction of $\mathbf{r}$ by rotating by β about $\mathbf{y}$ and by α about $\mathbf{z}$

$$\mathbf{R}(\vartheta, \mathbf{r}) = \mathbf{R}_z(\alpha)\mathbf{R}_y(\beta)\mathbf{R}_z(\gamma)\mathbf{R}_y(-\beta)\mathbf{R}_z(-\alpha)$$



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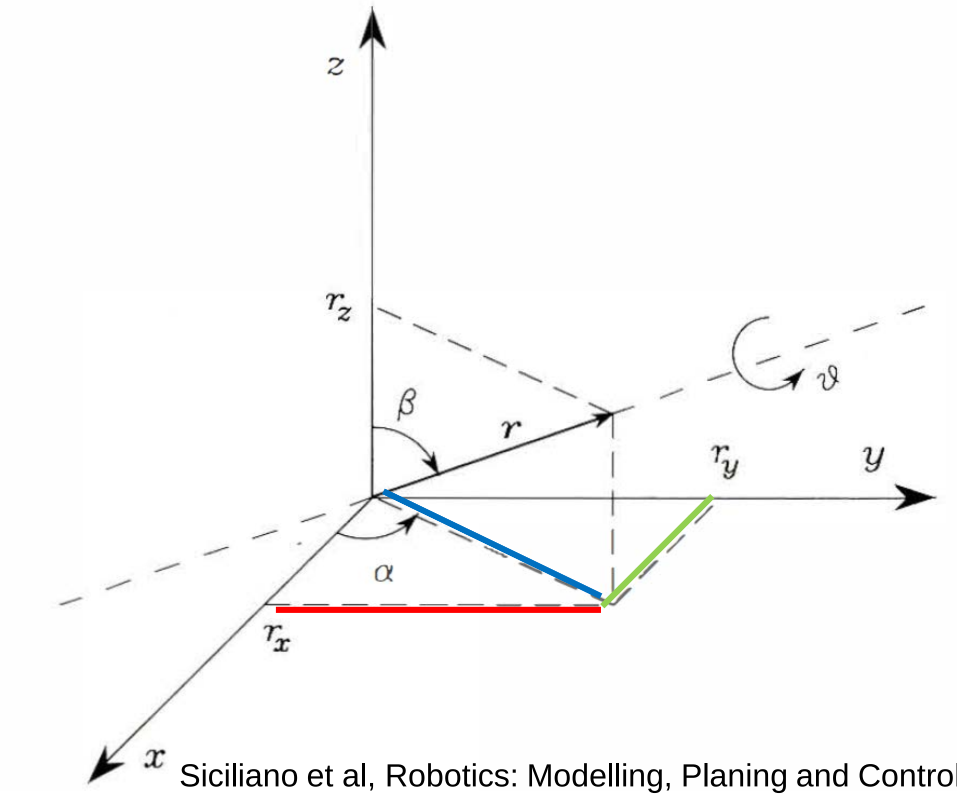
Angle - Axis

$$\mathbf{R}(\vartheta, \mathbf{r}) = \mathbf{R}_z(\alpha)\mathbf{R}_y(\beta)\mathbf{R}_z(\gamma)\mathbf{R}_y(-\beta)\mathbf{R}_z(-\alpha)$$

$$\sin\alpha = \frac{\mathbf{r}_x}{\sqrt{\mathbf{r}_x^2 + \mathbf{r}_y^2}}$$

$$\cos\alpha = \frac{\mathbf{r}_y}{\sqrt{\mathbf{r}_x^2 + \mathbf{r}_y^2}}$$



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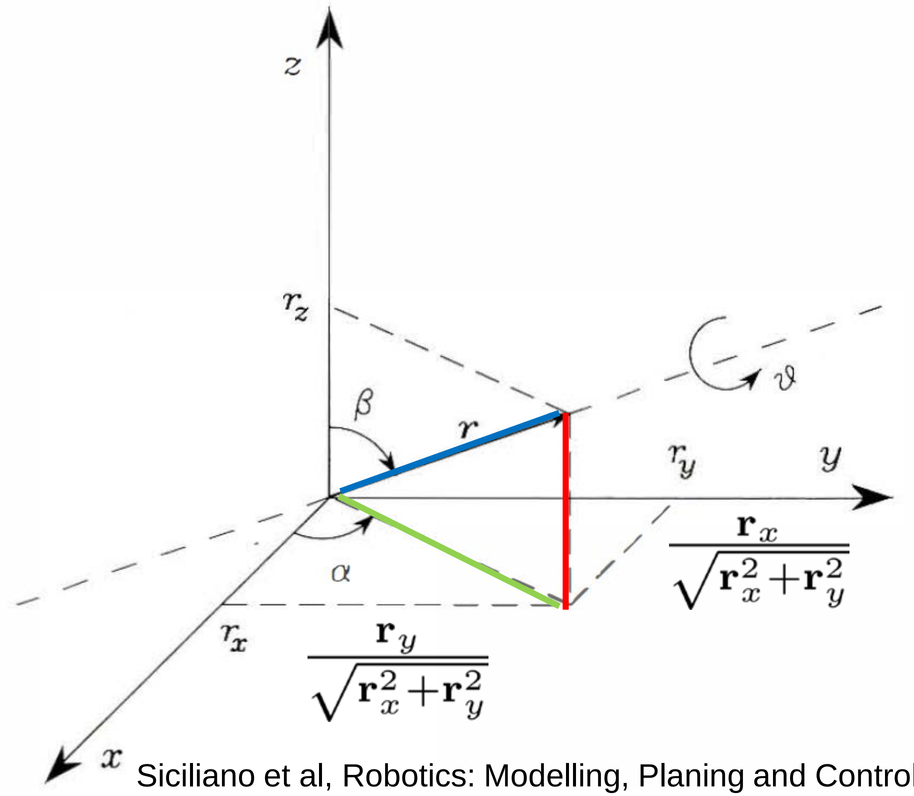
Angle - Axis

$$\mathbf{R}(\vartheta, \mathbf{r}) = \mathbf{R}_z(\alpha)\mathbf{R}_y(\beta)\mathbf{R}_z(\gamma)\mathbf{R}_y(-\beta)\mathbf{R}_z(-\alpha)$$

$$\sin\beta = \sqrt{\mathbf{r}_x^2 + \mathbf{r}_y^2}$$

$$\cos\beta = \mathbf{r}_z$$

$$\sqrt{\mathbf{r}_x^2 + \mathbf{r}_y^2 + \mathbf{r}_z^2} = 1$$



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Angle - Axis

$$R(\vartheta,\mathbf{r}) = R_z(\alpha)R_y(\beta)R_z(\vartheta)R_y(-\beta)R_z(-\alpha)$$

$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2 + r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin\beta = \sqrt{r_x^2 + r_y^2} \qquad \cos\beta = r_z$$

$$\mathbf{R}(\vartheta,\mathbf{r}) = \begin{pmatrix} r_x^2(1-c_\vartheta)+c_\vartheta & r_x r_y(1-c_\vartheta)-r_z s_\vartheta & r_x r_z(1-c_\vartheta)+r_y s_\vartheta \\ r_x r_y(1-c_\vartheta)+r_z s_\vartheta & r_y^2(1-c_\vartheta)+c_\vartheta & r_x r_z(1-c_\vartheta)-r_x s_\vartheta \\ r_x r_z(1-c_\vartheta)-r_y s_\vartheta & r_y r_z(1-c_\vartheta)+r_x s_\vartheta & r_z^2(1-c_\vartheta)+c_\vartheta \end{pmatrix}$$

$$\mathbf{R}(\vartheta,\mathbf{r}) = \mathbf{R}(-\vartheta,-\mathbf{r})$$

inverse solution

$$\vartheta = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)$$

$$\mathbf{r} = \frac{1}{2\sin\vartheta}\begin{pmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{pmatrix} \qquad r_x^2 + r_y^2 + r_z^2 = 1$$

Rodrigues Rotation Formula (Euler Parameter)

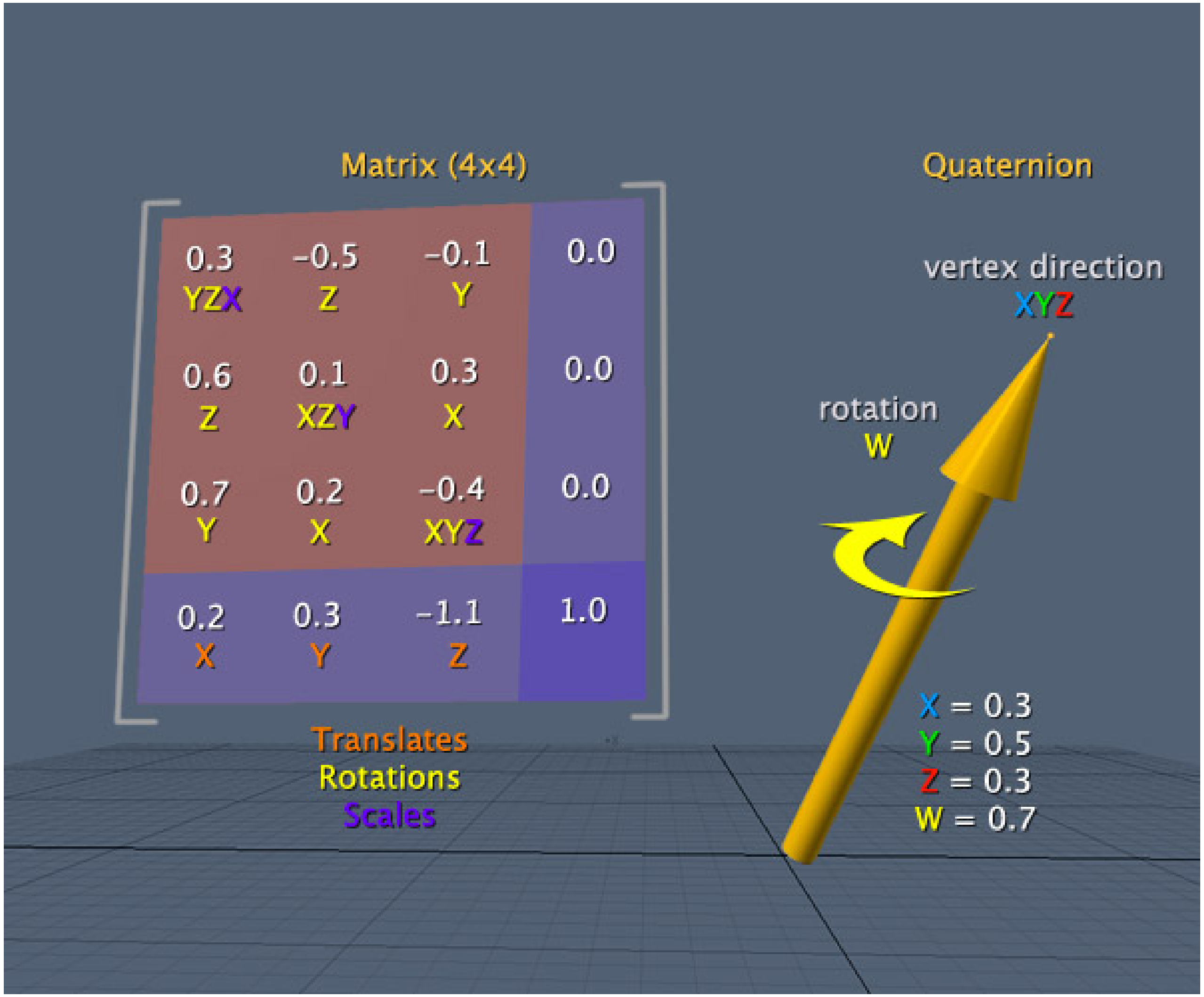
$$\mathbf{p}' = (\cos\vartheta)\mathbf{p} + \sin\vartheta(\mathbf{r}\times\mathbf{p}) + (1-\cos\vartheta)(\mathbf{r}\circ\mathbf{p})\mathbf{r}$$

$$\mathbf{p}' = \mathbf{R}\mathbf{p} \qquad\qquad \mathbf{R} = \mathbf{I}\cos\vartheta + \sin\vartheta[\mathbf{r}]_{\mathbf{x}} + (1-\cos\vartheta)\mathbf{r}\otimes\mathbf{r}$$

$$[\mathbf{r}]_{\times} = \mathbf{r}\times\ldots = \begin{pmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{pmatrix} \qquad \mathbf{r}\otimes\mathbf{r} = \begin{pmatrix} r_x{}^2(1-c_\vartheta) & r_x r_y(1-c_\vartheta) & r_x r_z(1-c_\vartheta) \\ r_x r_y(1-c_\vartheta) & r_y{}^2(1-c_\vartheta) & r_x r_z(1-c_\vartheta) \\ r_x r_z(1-c_\vartheta) & r_y r_z(1-c_\vartheta) & r_z{}^2(1-c_\vartheta) \end{pmatrix}$$

$$\mathbf{R}(\vartheta,\mathbf{r}) = \begin{pmatrix} r_x{}^2(1-c_\vartheta)+c_\vartheta & r_x r_y(1-c_\vartheta)-r_z s_\vartheta & r_x r_z(1-c_\vartheta)+r_y s_\vartheta \\ r_x r_y(1-c_\vartheta)+r_z s_\vartheta & r_y{}^2(1-c_\vartheta)+c_\vartheta & r_x r_z(1-c_\vartheta)-r_x s_\vartheta \\ r_x r_z(1-c_\vartheta)-r_y s_\vartheta & r_y r_z(1-c_\vartheta)+r_x s_\vartheta & r_z{}^2(1-c_\vartheta)+c_\vartheta \end{pmatrix}$$

Unit Quaternion



http://vignette4.wikia.nocookie.net/science/images/1/1e/Quaternion-03-goog.jpg/revision/latest?cb=20131024191311&path-prefix=el

Rotations and Quaternions

3D world
Euclidean space

4D space
(one real and
three imaginary
dimensions)

Rotation matrix to quaternion

3D rotation

$$\mathbf{p}' = \mathbf{R}\mathbf{p}$$

matrix
product

$$\mathbf{R}_2^0 = \mathbf{R}_1^0 \mathbf{R}_2^1$$

Quaternion
product

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$$

$$\mathbf{p} = [0 \quad \mathbf{p}_x \quad \mathbf{p}_y \quad \mathbf{p}_z]$$

Quaternion
product

$$\mathbf{q}_2^0 = \mathbf{q}_1^0 \mathbf{q}_2^1$$

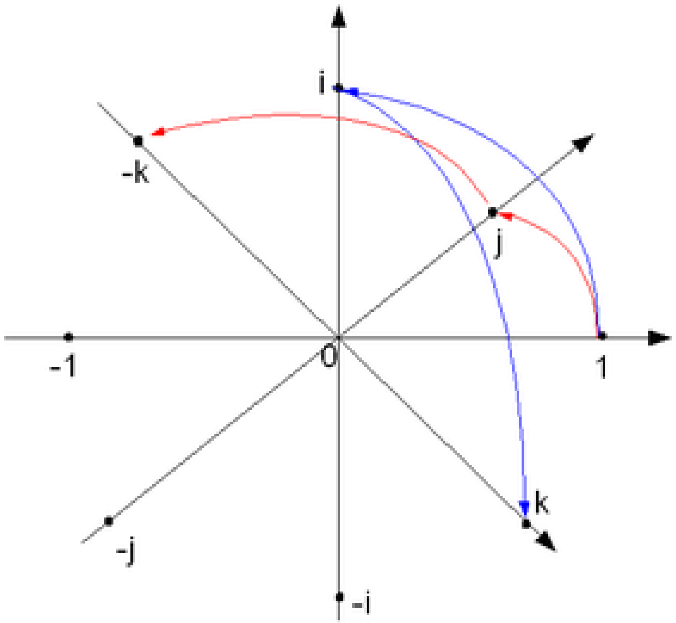
Quaternion to rotation matrix

Quaternion

$$\mathbf{q} = \{\eta, \boldsymbol{\varepsilon}\} = [\eta \quad \varepsilon_x \quad \varepsilon_y \quad \varepsilon_z]$$

$$= [\eta \quad \varepsilon_x \mathbf{i} \quad \varepsilon_y \mathbf{j} \quad \varepsilon_z \mathbf{k}]$$

×	1	*i*	*j*	*k*
1	1	*i*	*j*	*k*
i	*i*	−1	*k*	−*j*
j	*j*	−*k*	−1	*i*
k	*k*	*j*	−*i*	−1



Graphical representation of quaternion units product as 90°-rotation in 4D-space

ij = k
ji = -k
ij = -ji

- Unit quaternion

$$\eta^2 + \varepsilon_x^{\ 2} + \varepsilon_y^{\ 2} + \varepsilon_z^{\ 2} = 1$$

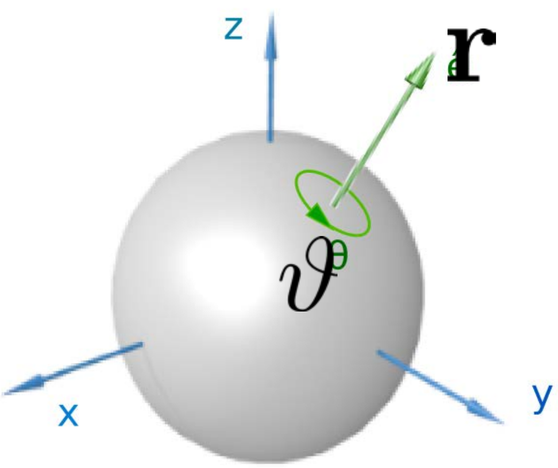
By ProkopHapala at English Wikipedia - Transferred from en.wikipedia to Commons by Ebe123., Public Domain, https://commons.wikimedia.org/w/index.php?curid=16097554

Eulers Formula

Euclidean 3D vector $\mathbf{r} = [r_x, r_y, r_z] = r_x\mathbf{i} + r_y\mathbf{j} + r_x\mathbf{k}$

$$\mathbf{q} = e^{\frac{\vartheta}{2}(r_x\mathbf{i}+r_y\mathbf{j}+r_z\mathbf{k})} = \cos\frac{\vartheta}{2} + (r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k})\sin\frac{\vartheta}{2}$$

$$\mathbf{q}^{-1} = e^{-\frac{\vartheta}{2}(r_x\mathbf{i}+r_y\mathbf{j}+r_z\mathbf{k})} = \cos\frac{\vartheta}{2} - (r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k})\sin\frac{\vartheta}{2}$$



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Unit Quaternion and Angle Axis

- Quaternion $$\mathbf{q} = \{\eta, \boldsymbol{\varepsilon}\} = [\eta \quad \boldsymbol{\varepsilon}_x \quad \boldsymbol{\varepsilon}_y \quad \boldsymbol{\varepsilon}_z]$$

- Relation to angle-axis $$\underbrace{\eta}_{\text{scalar part}} = \cos\frac{\vartheta}{2}, \quad \underbrace{\boldsymbol{\varepsilon}}_{\text{vector part}} = \begin{bmatrix} \boldsymbol{\varepsilon}_x & \boldsymbol{\varepsilon}_y & \boldsymbol{\varepsilon}_z \end{bmatrix} = \sin\frac{\vartheta}{2}\mathbf{r}$$

$$\mathbf{q} = \{\eta, \boldsymbol{\varepsilon}\} = [\cos\frac{\vartheta}{2} \quad \sin\frac{\vartheta}{2} r_x \quad \sin\frac{\vartheta}{2} r_y \quad \sin\frac{\vartheta}{2} r_z]$$

- Constraint $$\eta^2 + \varepsilon_x{}^2 + \varepsilon_y{}^2 + \varepsilon_z{}^2 = 1$$

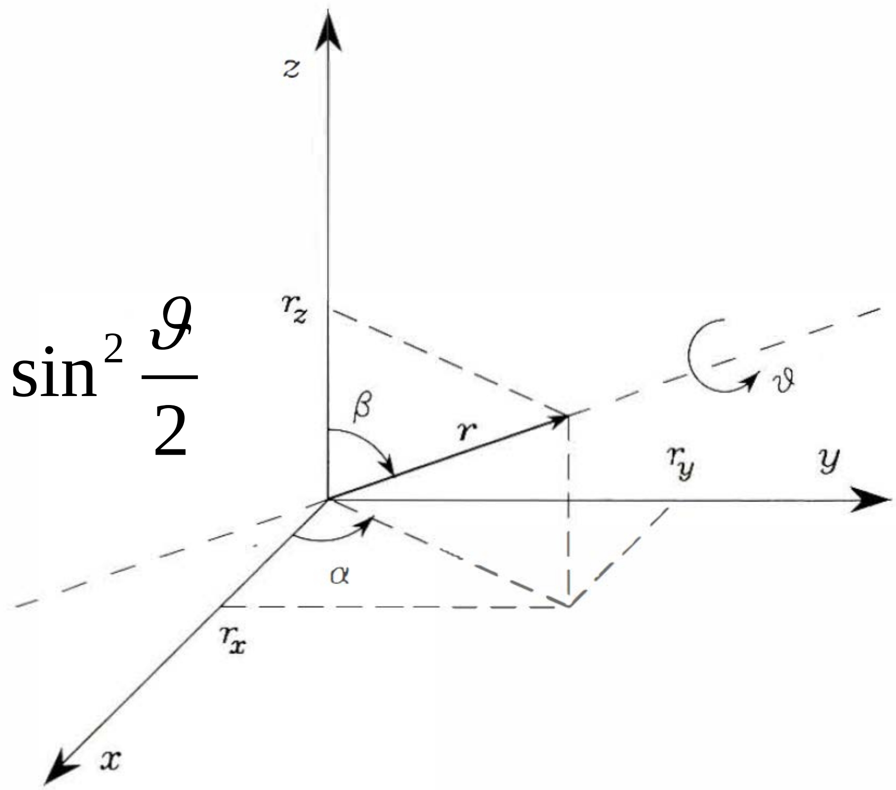
Unit Quaternion to Angle Axis

- Quaternion $\mathbf{q} = \{\eta, \boldsymbol{\varepsilon}\} = [\eta \quad \boldsymbol{\varepsilon}_x \quad \boldsymbol{\varepsilon}_y \quad \boldsymbol{\varepsilon}_z]$

- Rotation axis $\mathbf{r} = \dfrac{\boldsymbol{\varepsilon}}{\|\boldsymbol{\varepsilon}\|}$

- Rotation angle

$$\cos\vartheta = \eta^2 - \|\boldsymbol{\varepsilon}\|^2 = \cos^2\frac{\vartheta}{2} - \sin^2\frac{\vartheta}{2}$$



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Unit Quaternion to Rotation Matrix

- Quaternion $\mathbf{q} = \{\eta, \boldsymbol{\varepsilon}\}$

- Rotation matrix

$$\mathbf{R}(\eta,\boldsymbol{\varepsilon}) = \begin{pmatrix} 2(\eta^2+\varepsilon_x{}^2)-1 & 2(\varepsilon_x\varepsilon_y-\eta\varepsilon_z) & 2(\varepsilon_x\varepsilon_z-\eta\varepsilon_y) \\ 2(\varepsilon_x\varepsilon_y+\eta\varepsilon_z) & 2(\eta^2+\varepsilon_y{}^2)-1 & 2(\varepsilon_y\varepsilon_z-\eta\varepsilon_x) \\ 2(\varepsilon_x\varepsilon_z-\eta\varepsilon_y) & 2(\varepsilon_y\varepsilon_z+\eta\varepsilon_x) & 2(\eta^2+\varepsilon_z{}^2)-1 \end{pmatrix}$$

$$\mathbf{R}^{-1}(\eta,\boldsymbol{\varepsilon}) = \mathbf{R}^T(\eta,\boldsymbol{\varepsilon}) \longrightarrow \mathbf{q}^{-1} = \{\eta, -\boldsymbol{\varepsilon}\}$$

Rotation Matrix to Unit Quaternion

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\eta = \frac{1}{2}\sqrt{r_{11}{}^2 + r_{22}{}^2 + r_{33}{}^2}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\begin{bmatrix} \operatorname{sgn}(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1} \\ \operatorname{sgn}(r_{13} - r_{31})\sqrt{r_{22} - r_{22} - r_{11} + 1} \\ \operatorname{sgn}(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1} \end{bmatrix}$$

Unit Quaternion

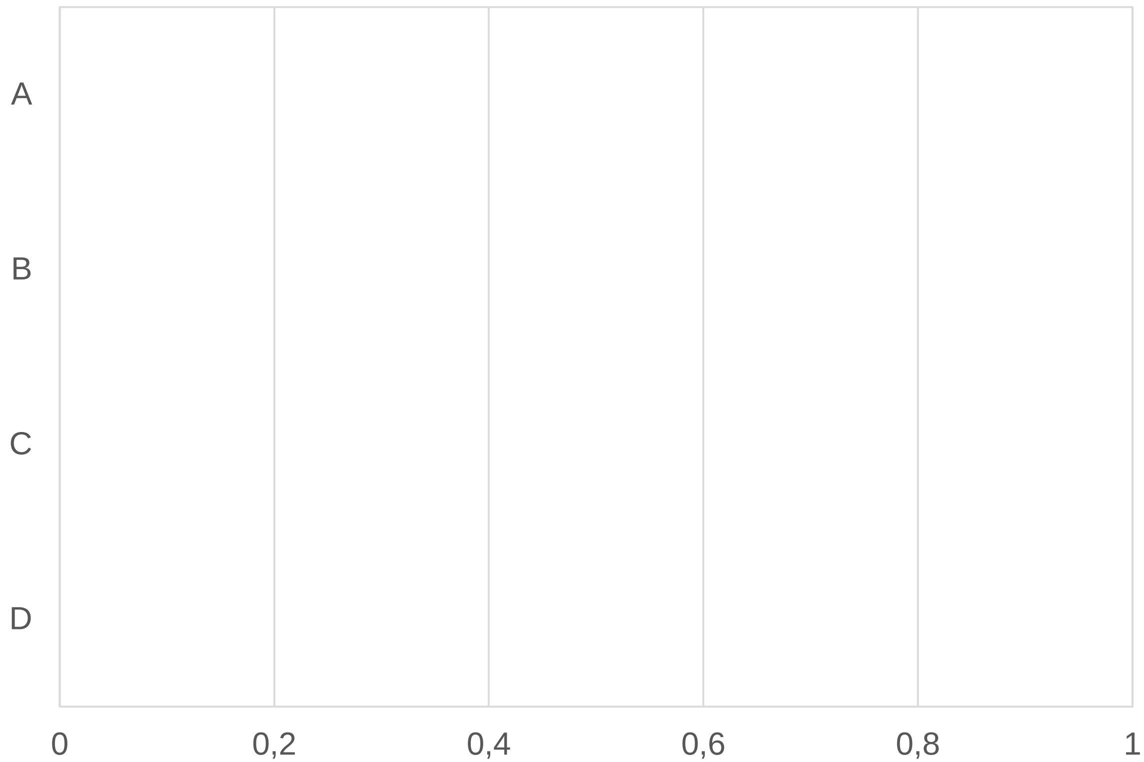
Which relations between unit quaternion and angle axis are correct?

$$\boldsymbol{\epsilon} = \sin(\vartheta)\mathbf{r}/2$$

$$\boldsymbol{\epsilon} = \sin(\vartheta/2)\mathbf{r}$$

$$\eta = \cos(\vartheta/2)$$

$$\eta = \sin(\vartheta)$$



Umfrage starten

ID = frank.hoffmann@tu-dortmund.de
Umfrage noch nicht gestartet

Product of Unit Quaternions

- Complex number $z = a + b\mathbf{i}$
- Product of complex numbers

$$(a+b\mathbf{i})(c+d\mathbf{i}) = (ac-bd)+(bc+ad)\mathbf{i}$$

- Quaternion: scalar + vector $\eta + \varepsilon_x\mathbf{i} + \varepsilon_y\mathbf{j} + \varepsilon_z\mathbf{k} = (\eta, \boldsymbol{\varepsilon})$
- Product of quaternions

$$(\eta^1 + \varepsilon_x{}^1\mathbf{i} + \varepsilon_y{}^1\mathbf{j} + \varepsilon_z{}^1\mathbf{k})(\eta^2 + \varepsilon_x{}^2\mathbf{i} + \varepsilon_y{}^2\mathbf{j} + \varepsilon_z{}^2\mathbf{k}) =$$

$$(\eta^1\eta^2 - \varepsilon_x{}^1\varepsilon_x{}^2 - \varepsilon_y{}^1\varepsilon_y{}^2 - \varepsilon_z{}^1\varepsilon_z{}^2) +$$

$$(\eta^1\varepsilon_x{}^2 + \varepsilon_x{}^1\eta^2 + \varepsilon_y{}^1\varepsilon_z{}^2 - \varepsilon_z{}^1\varepsilon_y{}^2)\mathbf{i} +$$

$$(\eta^1\varepsilon_y{}^2 - \varepsilon_x{}^1\varepsilon_z{}^2 + \varepsilon_y{}^1\eta^2 + \varepsilon_z{}^1\varepsilon_x{}^2)\mathbf{j} +$$

$$(\eta^1\varepsilon_z{}^2 + \varepsilon_x{}^1\varepsilon_y{}^2 - \varepsilon_y{}^1\varepsilon_x{}^2 + \varepsilon_z{}^1\eta^2)\mathbf{k}$$

$\mathbf{ij} = \mathbf{k}$ $\mathbf{ji} = -\mathbf{k}$

$\mathbf{jk} = \mathbf{i}$ $\mathbf{kj} = -\mathbf{i}$

$\mathbf{ki} = \mathbf{j}$ $\mathbf{ik} = -\mathbf{j}$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

Composition of Rotations with Unit Quaternions

- Quaternions $q_1 = \{\eta_1, \boldsymbol{\varepsilon}_1\}, q_2 = \{\eta_2, \boldsymbol{\varepsilon}_2\}$ corresponding to rotation matrices $\mathbf{R}_1, \mathbf{R}_2$

- Quaternion corresponding to product $\mathbf{R} = \mathbf{R}_1 \mathbf{R}_2$

$$q_1 * q_2 = \{\eta_1\eta_2 - \boldsymbol{\varepsilon}_1{}^T\boldsymbol{\varepsilon}_2, \eta_1\boldsymbol{\varepsilon}_2 + \eta_2\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2\}$$

$$(\eta^1 + \varepsilon_x{}^1\mathbf{i} + \varepsilon_y{}^1\mathbf{j} + \varepsilon_z{}^1\mathbf{k})(\eta^2 + \varepsilon_x{}^2\mathbf{i} + \varepsilon_y{}^2\mathbf{j} + \varepsilon_z{}^2\mathbf{k}) =$$

$$(\eta^1\eta^2 - \varepsilon_x{}^1\varepsilon_x{}^2 - \varepsilon_y{}^1\varepsilon_y{}^2 - \varepsilon_z{}^1\varepsilon_z{}^2) +$$

$$(\eta^1\varepsilon_x{}^2 + \varepsilon_x{}^1\eta^2 + \varepsilon_y{}^1\varepsilon_z{}^2 - \varepsilon_z{}^1\varepsilon_y{}^2)\mathbf{i} +$$

$$(\eta^1\varepsilon_y{}^2 - \varepsilon_x{}^1\varepsilon_z{}^2 + \varepsilon_y{}^1\eta^2 + \varepsilon_z{}^1\varepsilon_x{}^2)\mathbf{j} +$$

$$(\eta^1\varepsilon_z{}^2 + \varepsilon_x{}^1\varepsilon_y{}^2 - \varepsilon_y{}^1\varepsilon_x{}^2 + \varepsilon_z{}^1\eta^2)\mathbf{k}$$

Rotation of 3D Vectors Using Quaternions

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$$

$$\mathbf{p} = [0 \quad \mathbf{p}_x \quad \mathbf{p}_y \quad \mathbf{p}_z]$$

$$\mathbf{q} = \{\eta, \boldsymbol{\varepsilon}\} = [\eta \quad \boldsymbol{\varepsilon}_x \quad \boldsymbol{\varepsilon}_y \quad \boldsymbol{\varepsilon}_z]$$

$$\mathbf{q}^{-1} = \{\eta, -\boldsymbol{\varepsilon}\} = [\eta \quad -\boldsymbol{\varepsilon}_x \quad -\boldsymbol{\varepsilon}_y \quad -\boldsymbol{\varepsilon}_z]$$

$$\mathbf{q} = e^{\frac{\vartheta}{2}(r_x\mathbf{i}+r_y\mathbf{j}+r_x\mathbf{k})} = \cos\frac{\vartheta}{2} + (r_x\mathbf{i} + r_y\mathbf{j} + r_x\mathbf{k})\sin\frac{\vartheta}{2}$$

$$\mathbf{q}^{-1} = e^{-\frac{\vartheta}{2}(r_x\mathbf{i}+r_y\mathbf{j}+r_x\mathbf{k})} = \cos\frac{\vartheta}{2} - (r_x\mathbf{i} + r_y\mathbf{j} + r_x\mathbf{k})\sin\frac{\vartheta}{2}$$

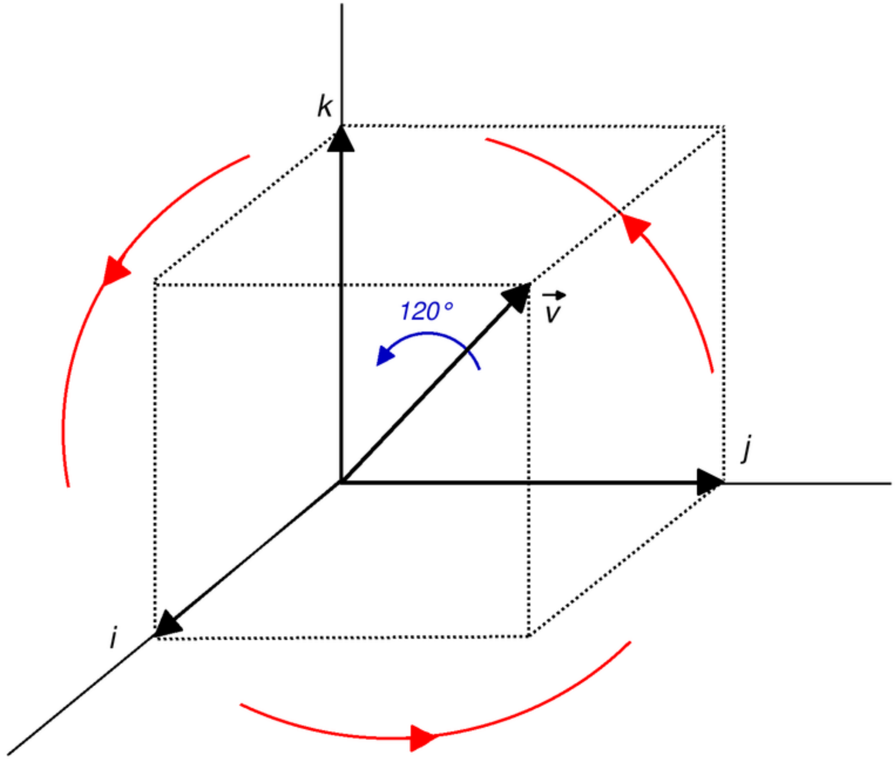
Rotation of 3D Vectors Using Quaternions

$$\vec{\mathbf{v}} = \begin{bmatrix}1 & 1 & 1\end{bmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \qquad \vartheta = \frac{2\pi}{3}$$

$$\mathbf{q} = \frac{1 + \mathbf{i} + \mathbf{j} + \mathbf{k}}{2}$$

$$\mathbf{q}^{-1} = \frac{1 - \mathbf{i} - \mathbf{j} - \mathbf{k}}{2}$$

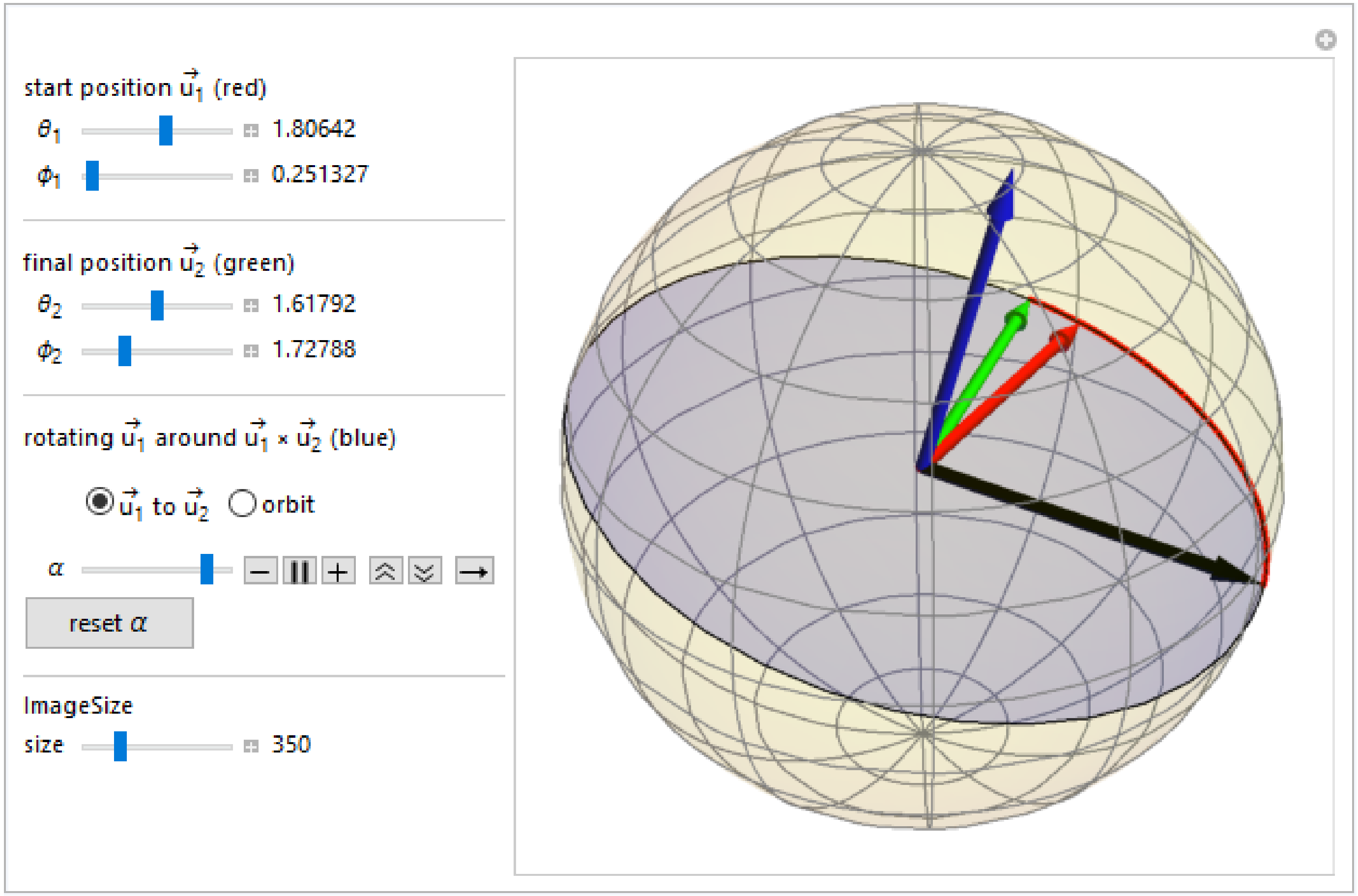
$$\mathbf{p} = \begin{bmatrix}p_x & p_x & p_x\end{bmatrix} = 0 + p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}$$



$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1} = \frac{1 + \mathbf{i} + \mathbf{j} + \mathbf{k}}{2}(0 + p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k})\frac{1 - \mathbf{i} - \mathbf{j} - \mathbf{k}}{2}$$

$$= p_z\mathbf{i} + p_x\mathbf{j} + p_y\mathbf{k}$$

Rotating a Unit Vector in 3D Using Quaternions



Quaternion Arithmetic

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1} \qquad \mathbf{p} = [0 \quad \mathbf{p}_x \quad \mathbf{p}_y \quad \mathbf{p}_z]$$

$$\mathbf{q} = e^{\frac{\vartheta}{2}(r_x\mathbf{i}+r_y\mathbf{j}+r_x\mathbf{k})} = \cos\frac{\vartheta}{2} + (r_x\mathbf{i} + r_y\mathbf{j} + r_x\mathbf{k})\sin\frac{\vartheta}{2}$$

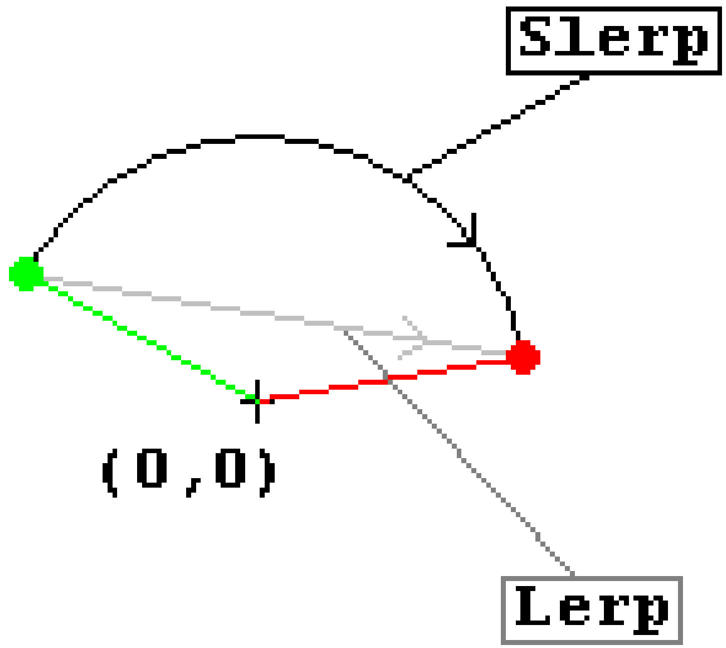
$$\mathbf{q}^{-1} = e^{-\frac{\vartheta}{2}(r_x\mathbf{i}+r_y\mathbf{j}+r_x\mathbf{k})} = \cos\frac{\vartheta}{2} - (r_x\mathbf{i} + r_y\mathbf{j} + r_x\mathbf{k})\sin\frac{\vartheta}{2}$$

$$\mathbf{R}(\vartheta, \mathbf{r}) = \begin{pmatrix} r_x^{\,2}(1-c_\vartheta)+c_\vartheta & r_x r_y(1-c_\vartheta)-r_z s_\vartheta & r_x r_z(1-c_\vartheta)+r_y s_\vartheta \\ r_x r_y(1-c_\vartheta)+r_z s_\vartheta & r_y^{\,2}(1-c_\vartheta)+c_\vartheta & r_x r_z(1-c_\vartheta)-r_x s_\vartheta \\ r_x r_z(1-c_\vartheta)-r_y s_\vartheta & r_y r_z(1-c_\vartheta)+r_x s_\vartheta & r_z^{\,2}(1-c_\vartheta)+c_\vartheta \end{pmatrix}$$

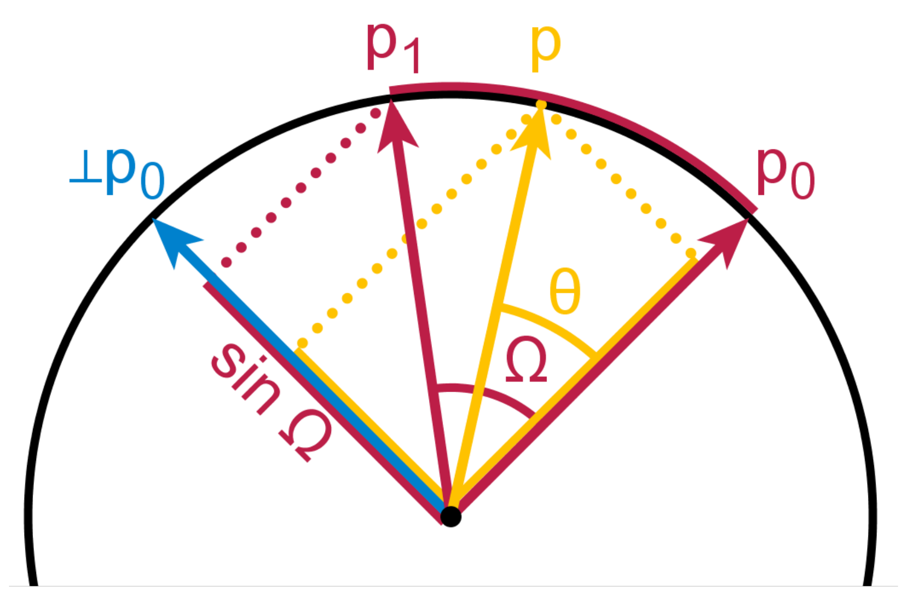
Spherical Linear Interpolation

$$\mathrm{lerp}(\mathbf{p}_0, \mathbf{p}_1, t) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathrm{slerp}(\mathbf{p}_0, \mathbf{p}_1, t) = \frac{sin((1-t)\Omega)}{\sin\Omega}\mathbf{p}_0 + \frac{sin(t\Omega)}{\sin\Omega}\mathbf{p}_1$$

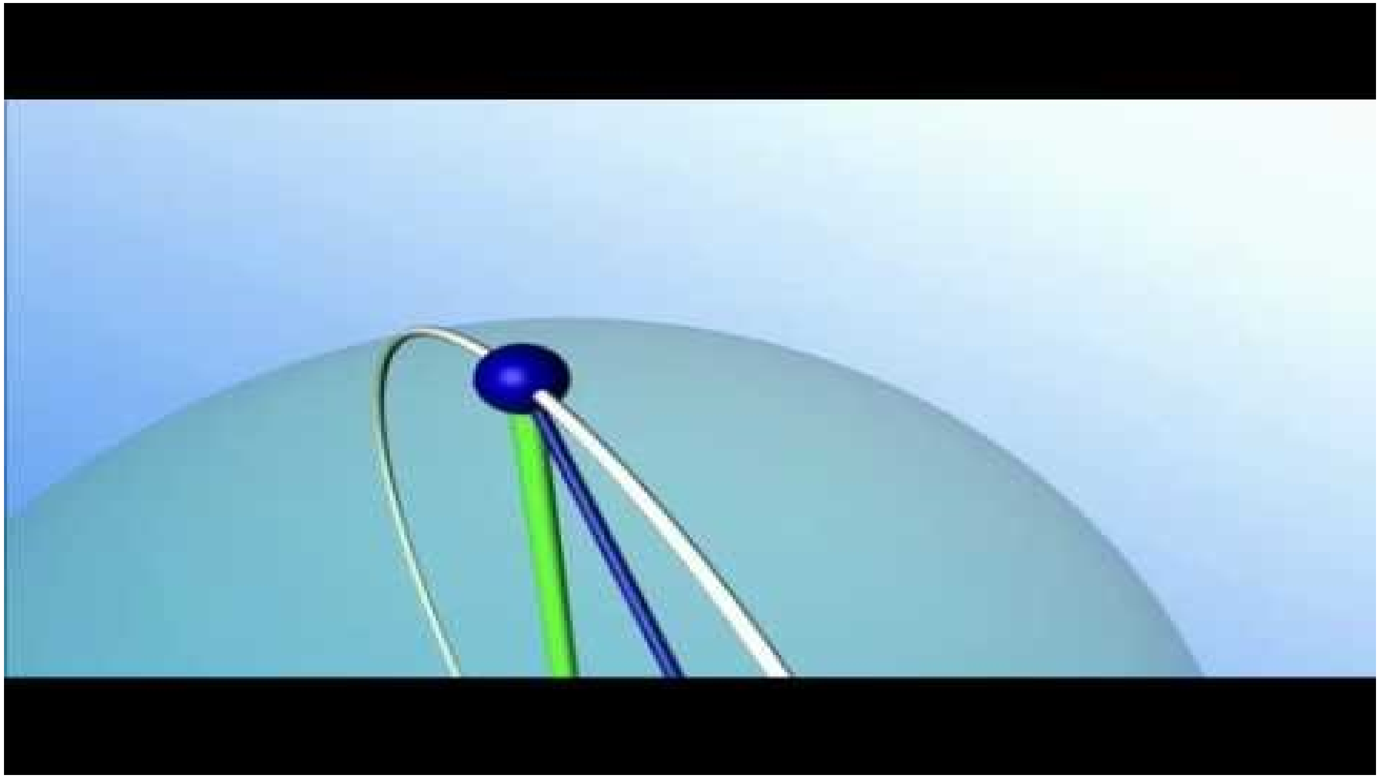


http://answers.unity3d.com/storage/temp/7325-slerp.png



https://commons.wikimedia.org/w/index.php?curid=267089

Slerp vs. Lerp

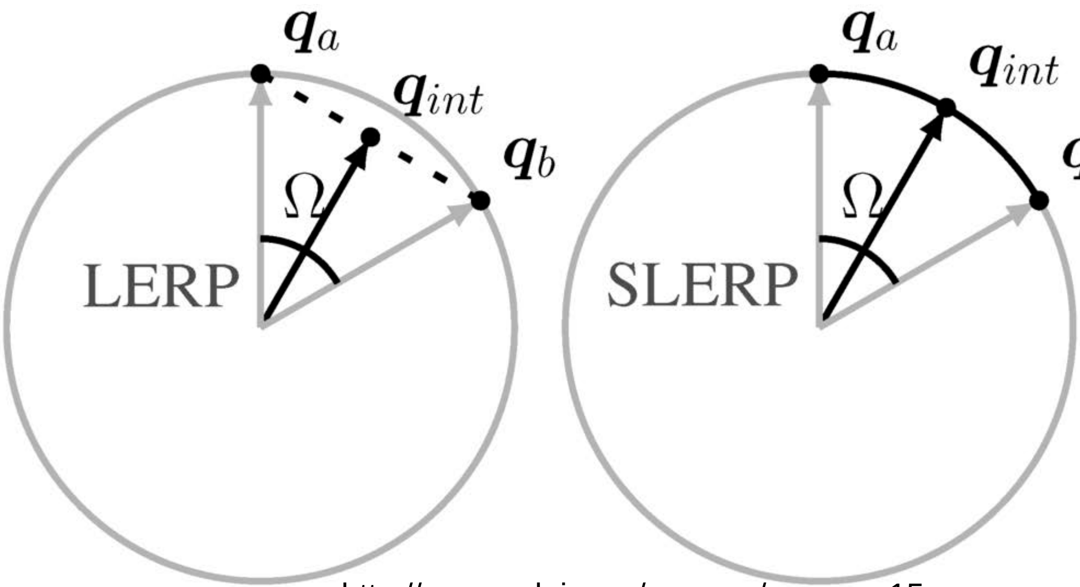


https://www.youtube.com/watch?v=uNHIPVOnt-Y

Quaternion Slerp

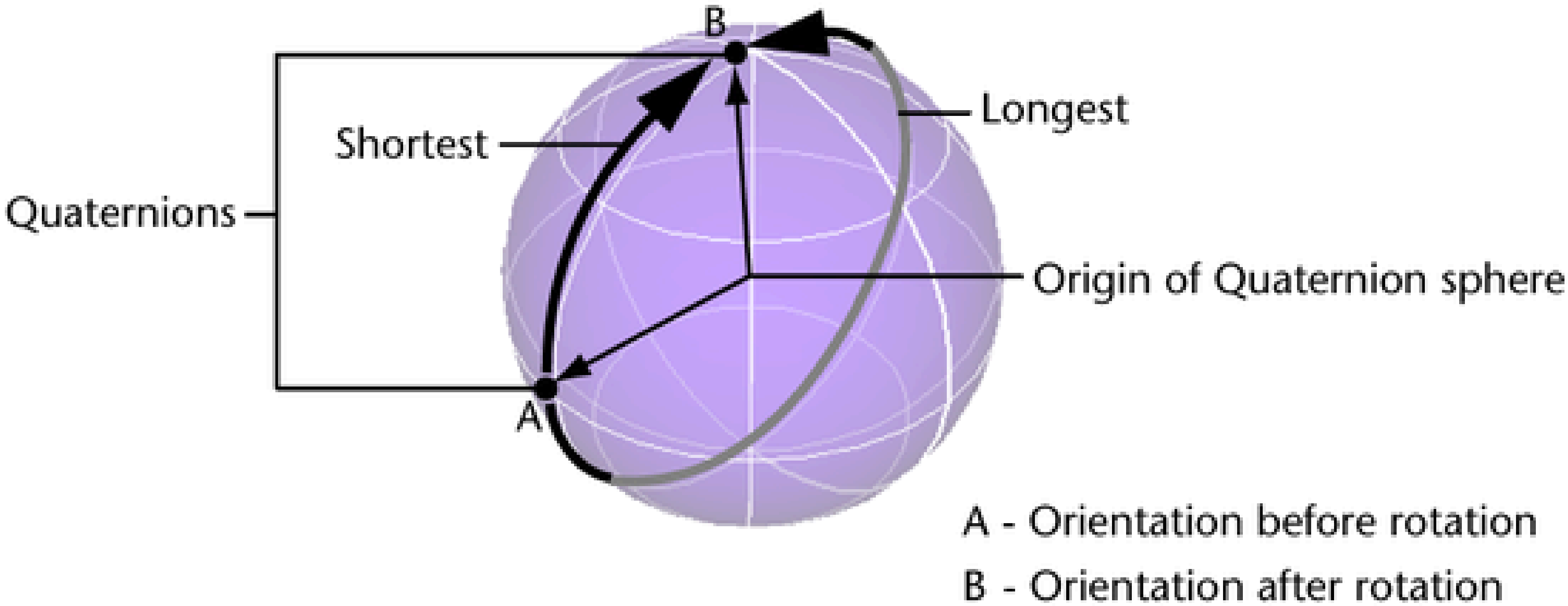
- Power series $$e^q = 1 + \frac{q^2}{2} + \frac{q^3}{6} + \ldots + \frac{q^n}{n!}$$
- Versor form : $$q^t = \cos\Omega t + \mathbf{v}\sin\Omega t \qquad e^{\mathbf{v}\Omega} = q$$

$$\mathbf{q} = \mathbf{q}_1\mathbf{q}_0^{-1} \qquad \mathrm{slerp}(\mathbf{q}_0, \mathbf{q}_1, t) = \mathbf{q}_0(\mathbf{q}_0^{-1}\mathbf{q}_1)^t$$



http://www.mdpi.com/sensors/sensors-15-19302/article_deploy/html/images/sensors-15-19302-g004-1024.png

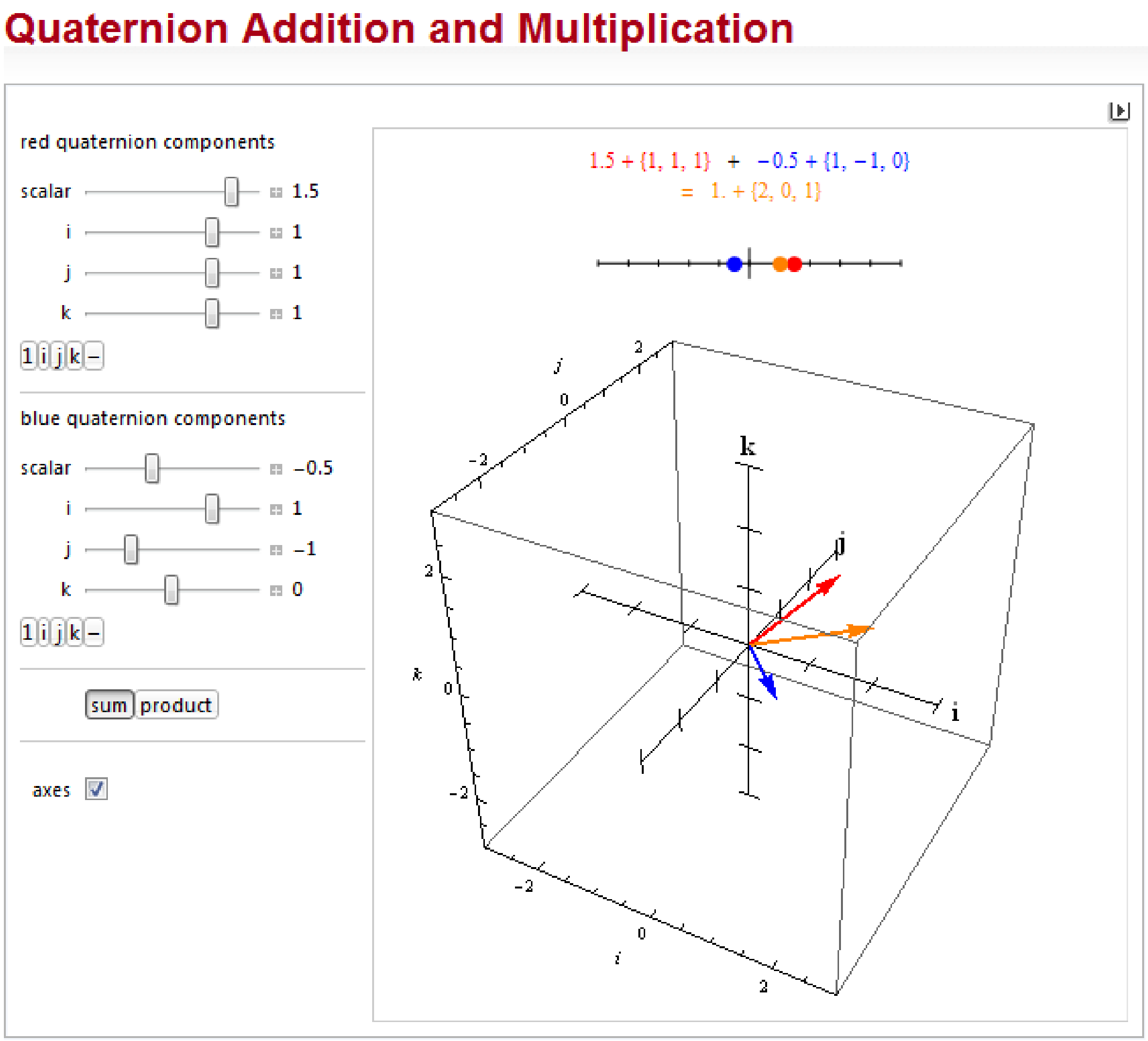
Quaternion Rotation Interpolation



Quaternion rotation interpolation

http://help.autodesk.com/cloudhelp/2015/ENU/Maya/images/GUI
D-D6410250-2B26-4B80-B810-C2D9AAE79F9E.png

Quaternion



Quaternion Addition and Multiplication
red quaternion components
scalar 1.5
i 1
j 1
k 1
1 i j k –
blue quaternion components
scalar −0.5
i 1
j −1
k 0
1 i j k –
sum product
axes
1.5 + {1, 1, 1} + −0.5 + {1, −1, 0}
= 1. + {2, 0, 1}
k
j
i

Spatial Transformations

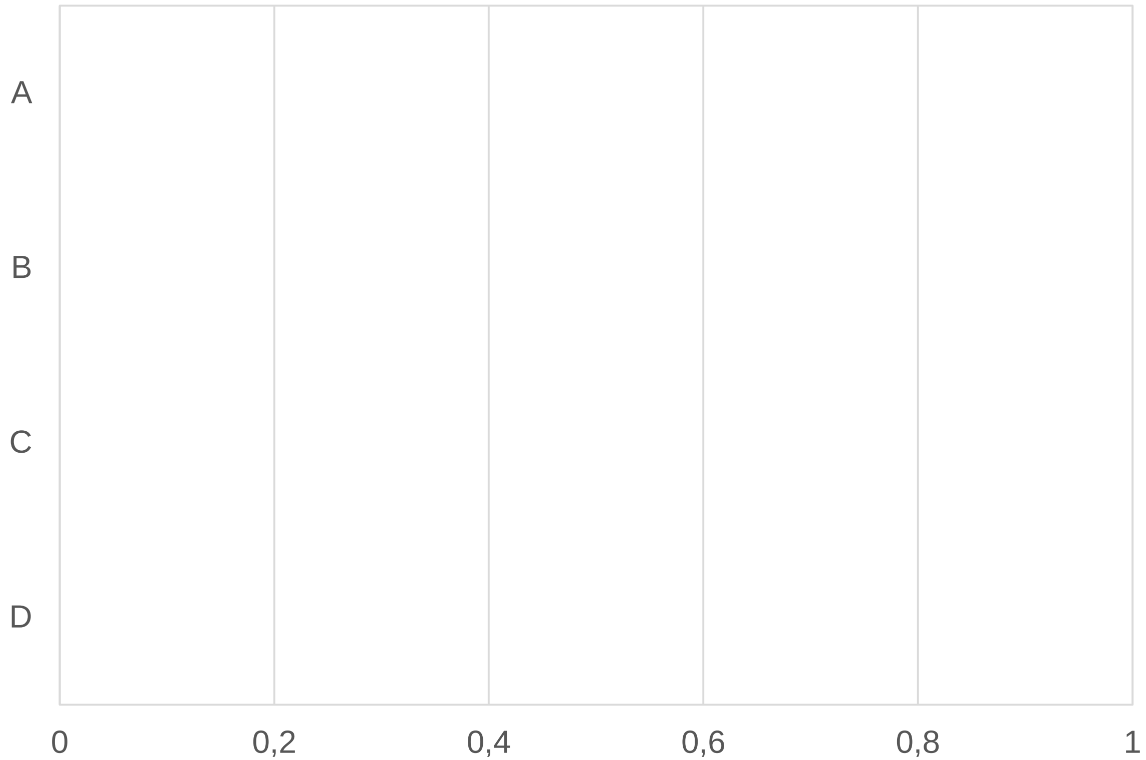
Which representations of rotations have redundant parameters?

Rotation matrix

Euler angles

Unit quaternion

Angle axis



Umfrage starten

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Umfrage noch nicht gestartet

Homogeneous Transformations

Position of point P w.r.t. the reference frame

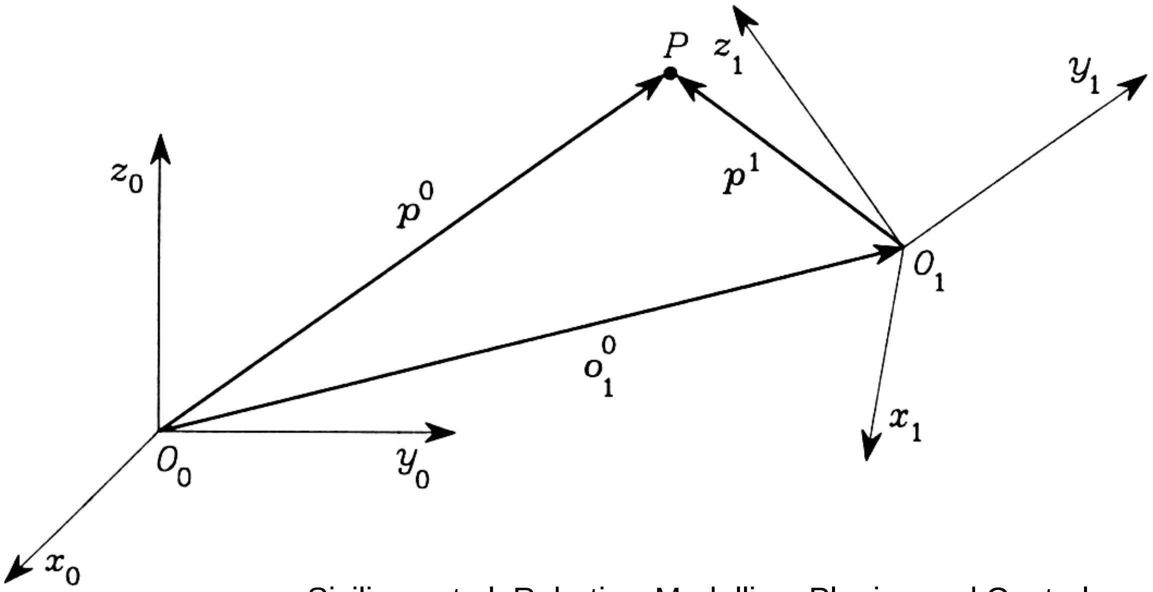
$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1.$$

coordinate transformation (*translation* + *rotation*) of a bound vector between two frames.

Inverse transformation

$$\boldsymbol{p}^1 = -\boldsymbol{R}_1^{0^T} \boldsymbol{o}_1^0 + \boldsymbol{R}_1^{0^T} \boldsymbol{p}^0$$

$$\boldsymbol{p}^1 = -\boldsymbol{R}_0^1 \boldsymbol{o}_1^0 + \boldsymbol{R}_0^1 \boldsymbol{p}^0$$



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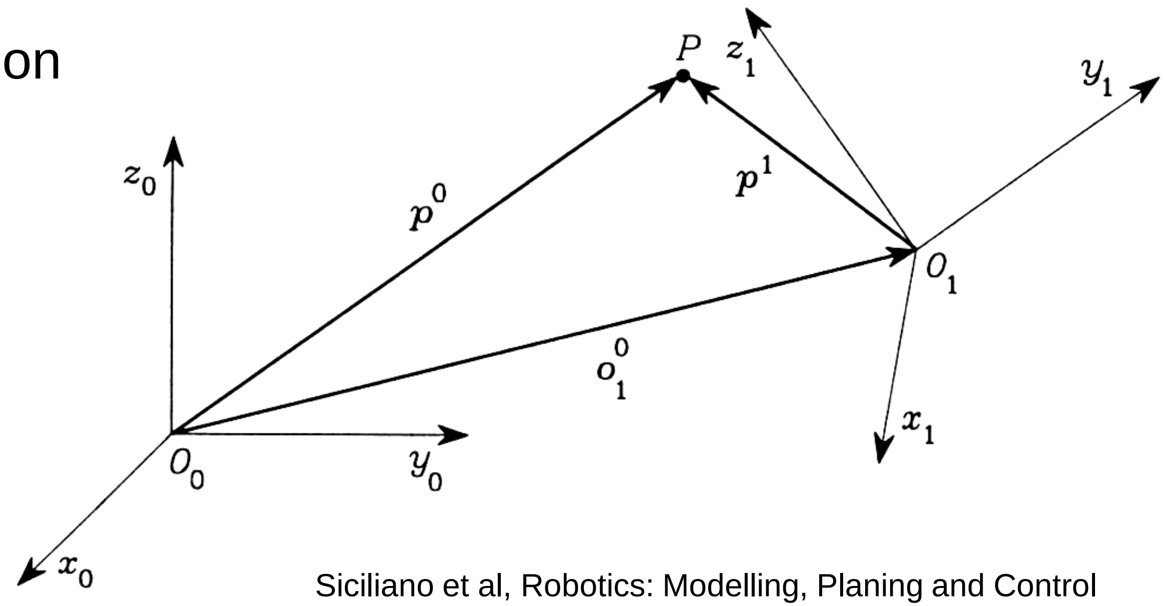
Homogeneous Transformations

Homogeneous representation of a vector

$$\tilde{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix}$$

$$\tilde{\boldsymbol{p}}_0 = \begin{bmatrix} \boldsymbol{p}_0 \\ 1 \end{bmatrix} \qquad \tilde{\boldsymbol{p}}_1 = \begin{bmatrix} \boldsymbol{p}_1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{A}_1^0 = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}.$$



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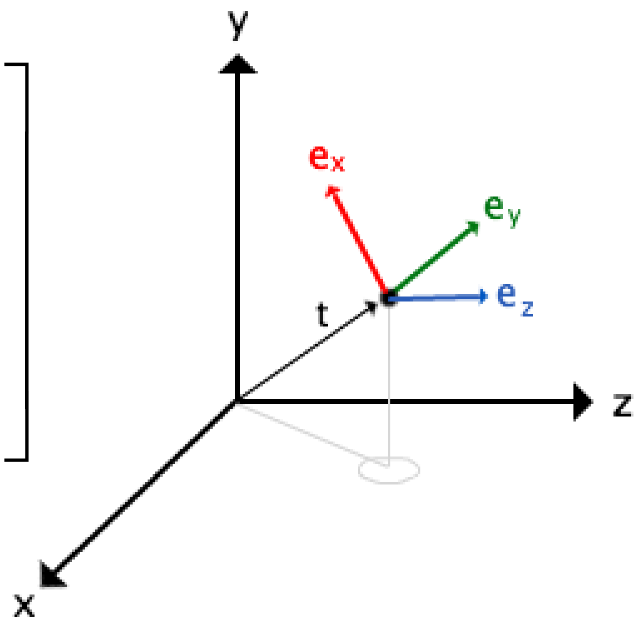
$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1 \longrightarrow \tilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \tilde{\boldsymbol{p}}^1$$

$$\mathbf{o}_1^0 \in \mathbb{R}^3, \mathbf{R}_1^0 \in SO(3)$$
$$\mathbf{A}_1^0 \in SE(3) = \mathbb{R}^3 \times SO(3)$$

Homogeneous Transformations

$$H = \begin{bmatrix} e_{xx} & e_{yx} & e_{zx} & t_x \\ e_{xy} & e_{yy} & e_{zy} & t_y \\ e_{xz} & e_{yz} & e_{zz} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



http://mabulous.com/wp-content/uploads/2013/10/TransformationMatrixBasisVectors.png

Inverse of Homogeneous Transformation

$$\tilde{\mathbf{p}}^1 = \mathbf{A}_0^1 \tilde{\mathbf{p}}^0 = (\mathbf{A}_1^0)^{-1} \tilde{\mathbf{p}}^0$$

$$\mathbf{A}_0^1 = \begin{bmatrix} \mathbf{R}_1^{0\,T} & -\mathbf{R}_1^{0\,T} \mathbf{o}_1^0 \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_0^1 & -\mathbf{R}_0^1 \mathbf{o}_1^0 \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

$$\mathbf{A}^{-1} \neq \mathbf{A}^T$$

Homogeneous transformation has 6 independent parameters

- 3 for rotation R (3x3-Matrix with 6 constraints)
- 3 for translation o

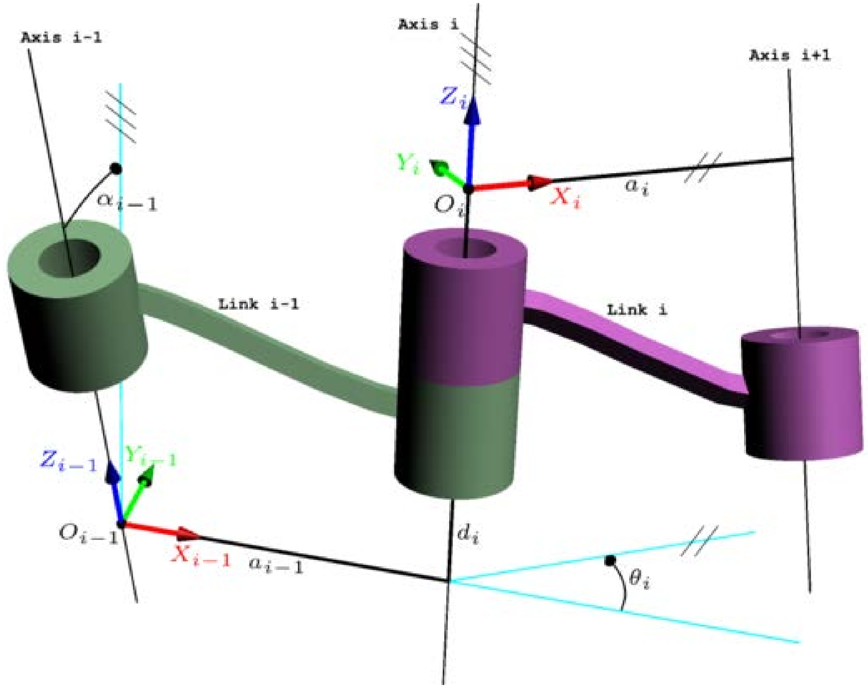
Composition of Homogeneous Transformation

$$\mathbf{A}_1^0 = \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{o}_1^0 \\ \mathbf{0}^T & 1 \end{bmatrix}. \quad \mathbf{A}_2^1 = \begin{bmatrix} \mathbf{R}_2^1 & \mathbf{o}_2^1 \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

$$\mathbf{A}_2^0 = \mathbf{A}_1^0 \mathbf{A}_2^1 = \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{o}_1^0 \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_2^1 & \mathbf{o}_2^1 \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_1^0 \mathbf{R}_2^1 & \mathbf{R}_1^0 \mathbf{o}_2^1 + \mathbf{o}_1^0 \\ \mathbf{0}^T & 1 \end{bmatrix}$$

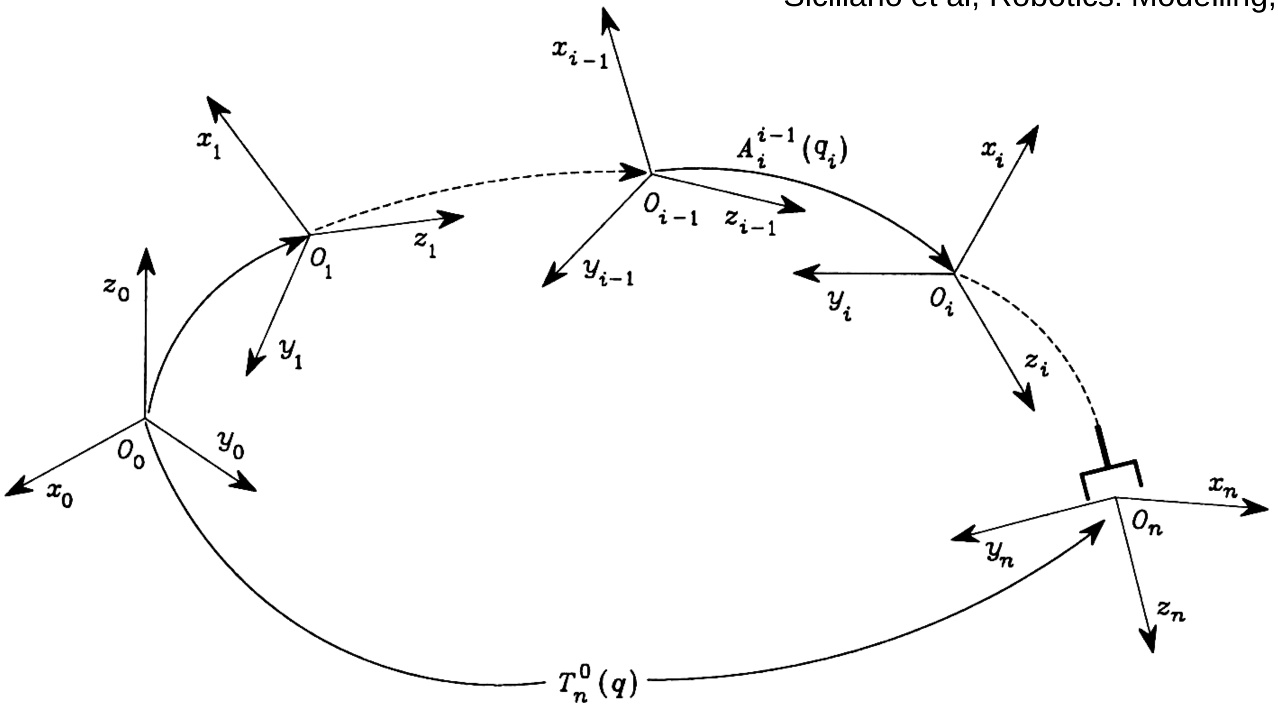
$$\tilde{\mathbf{p}}^0 = \mathbf{A}_1^0 \mathbf{A}_2^1 \tilde{\mathbf{p}}^2$$



https://upload.wikimedia.org/wikipedia/commons/thumb/d/d8/DHParameter.png/519px-DHParameter.png

Composition of Homogeneous Transformations

Siciliano et al, Robotics: Modelling, Planing and Control



Siciliano et al, Robotics: Modelling, Planing and Control

$$\tilde{\mathbf{p}}^0 = \mathbf{A}_1^0 \mathbf{A}_2^1 \ldots \mathbf{A}_n^{n-1} \tilde{\mathbf{p}}^n$$

Spatial Transformations Robotics System Toolbox

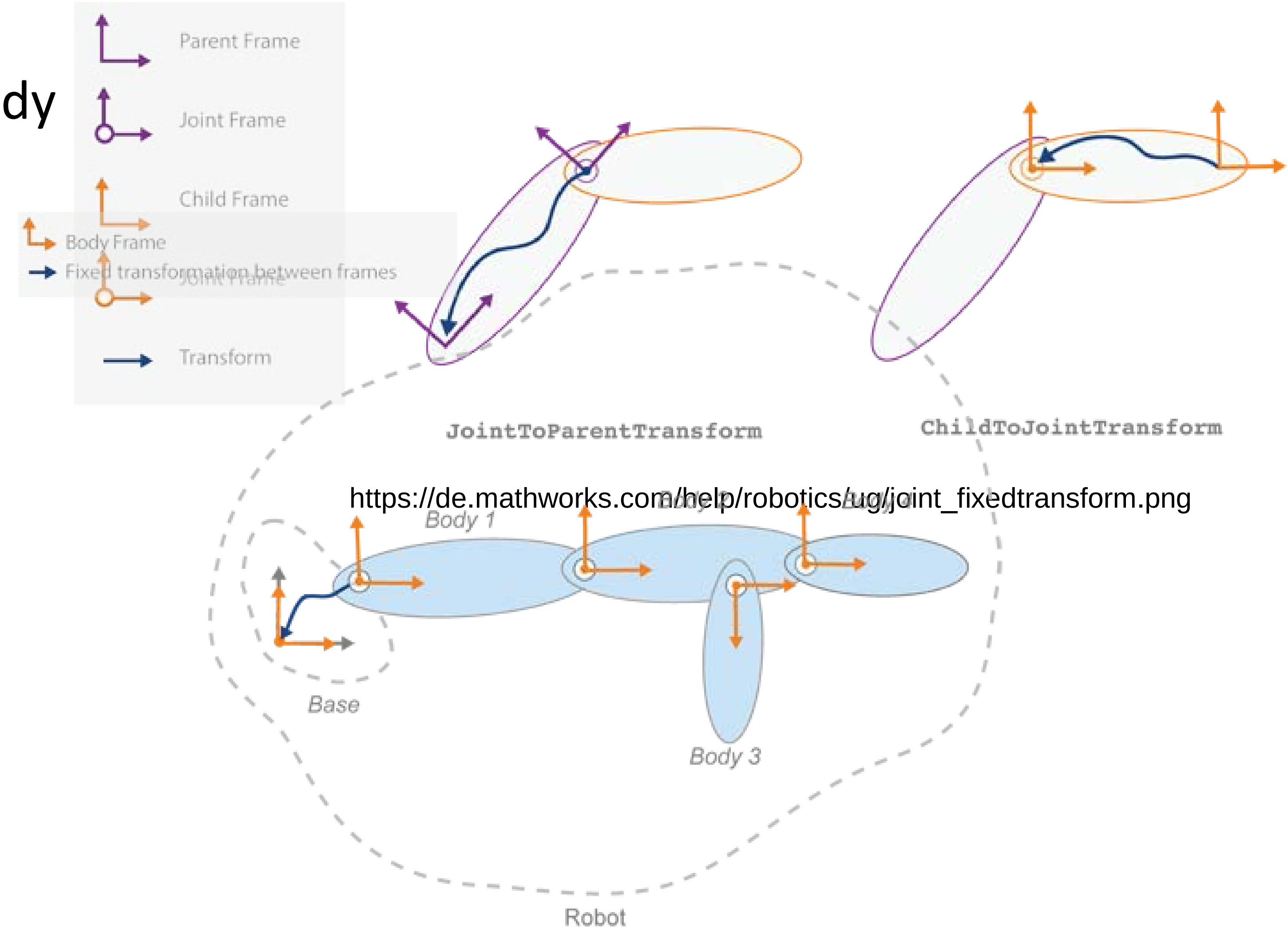
- axang
- eul
- quat
- rotm
- tform
- trvec

- axang2quat
- trvec2tform
- <a>2<b>

Converting To / Converting From	Axis-Angle (axang)	Euler Angles (eul)	Quaternion (quat)	Rotation Matrix (rotm)	Homogeneous Transformation (tform)	Translation Vector (trvec)
Axis-Angle (axang)						
Euler Angles (eul)						
Quaternion (quat)						
Rotation Matrix (rotm)						
Homogeneous Transformation (tform)						
Translation Vector (trvec)						

Rigid Body Tree

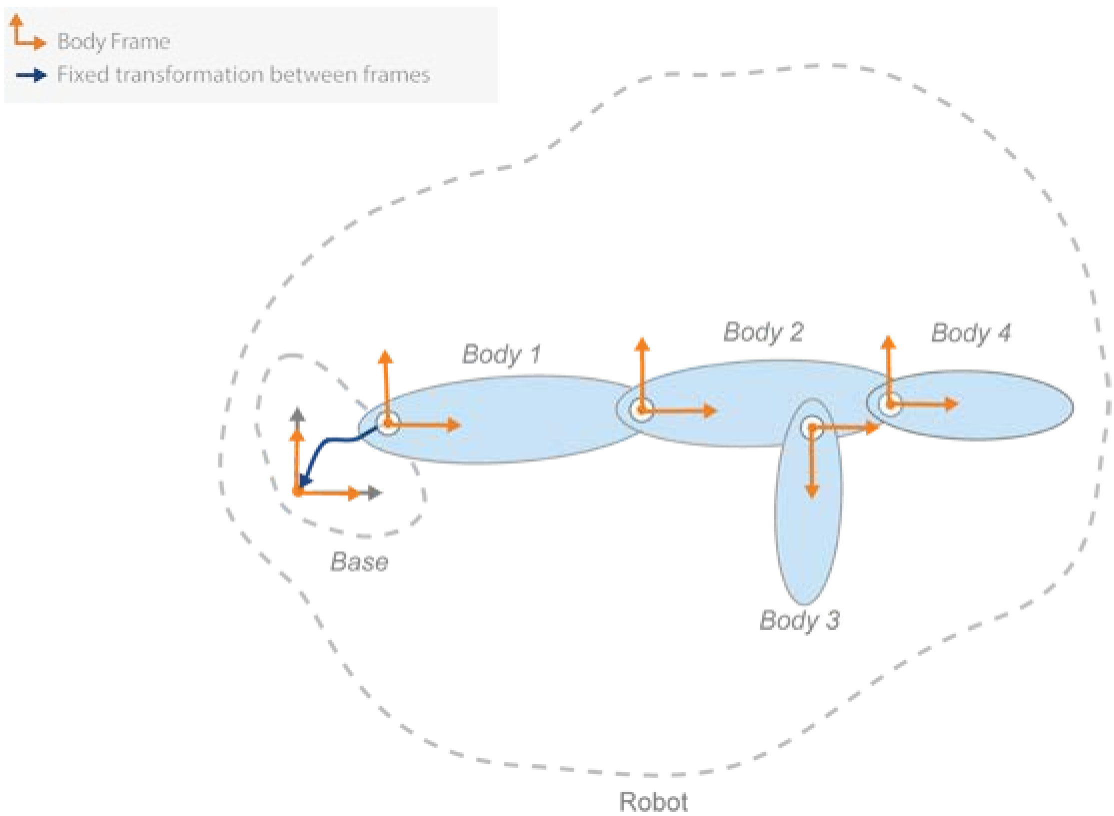
- Base
- Rigid Body
- Joint



https://de.mathworks.com/help/robotics/ug/joint_fixedtransform.png

Rigid Body Tree

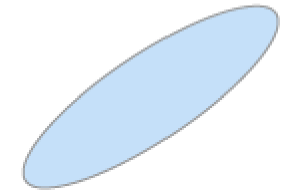
- Base
- Rigid Body
- Joint

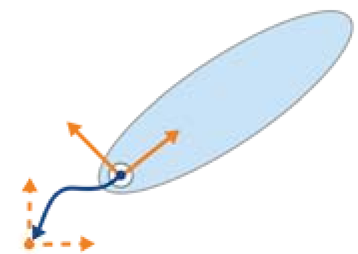


https://de.mathworks.com/help/robotics/ug/rigidbodytree_homeconfig.png

Rigid Body Tree

```
body1 = robotics.RigidBody('body1');
```

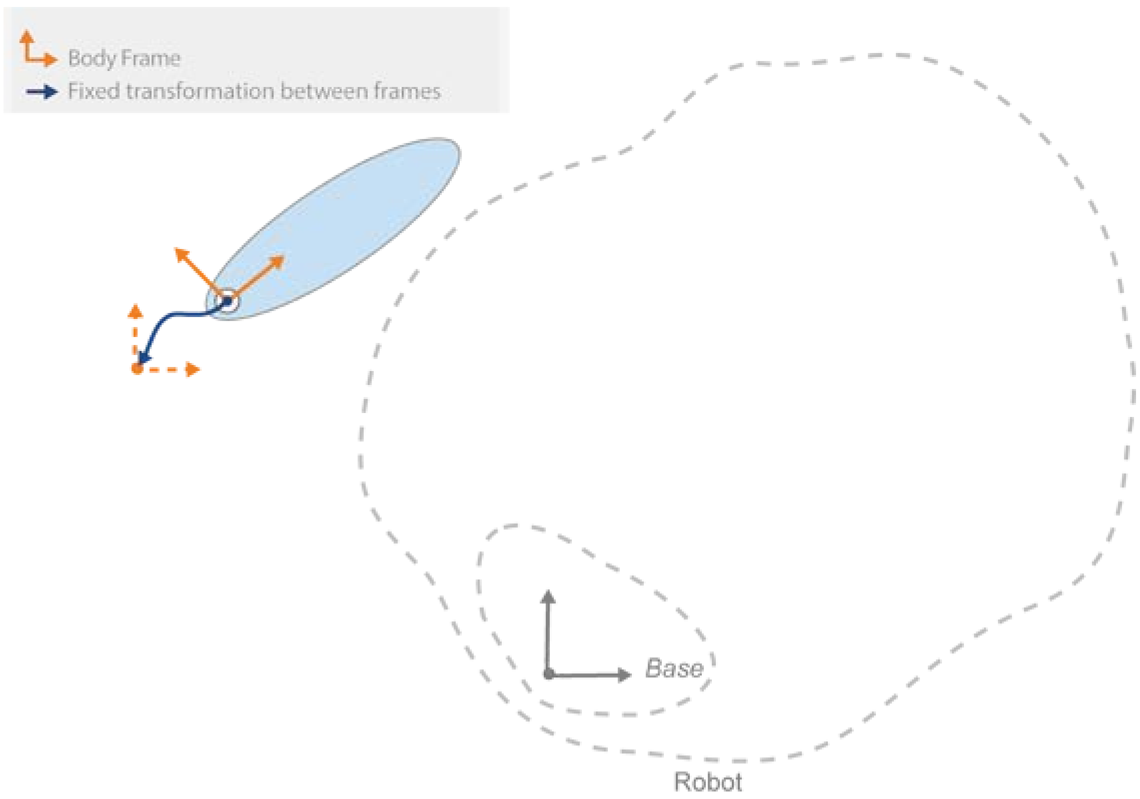




```
jnt1 = robotics.Joint('jnt1','revolute');
jnt1.HomePosition = pi/4;
tform = trvec2tform([0.25, 0.25, 0]); % User defined
setFixedTransform(jnt1,tform);
body1.Joint = jnt1;
```

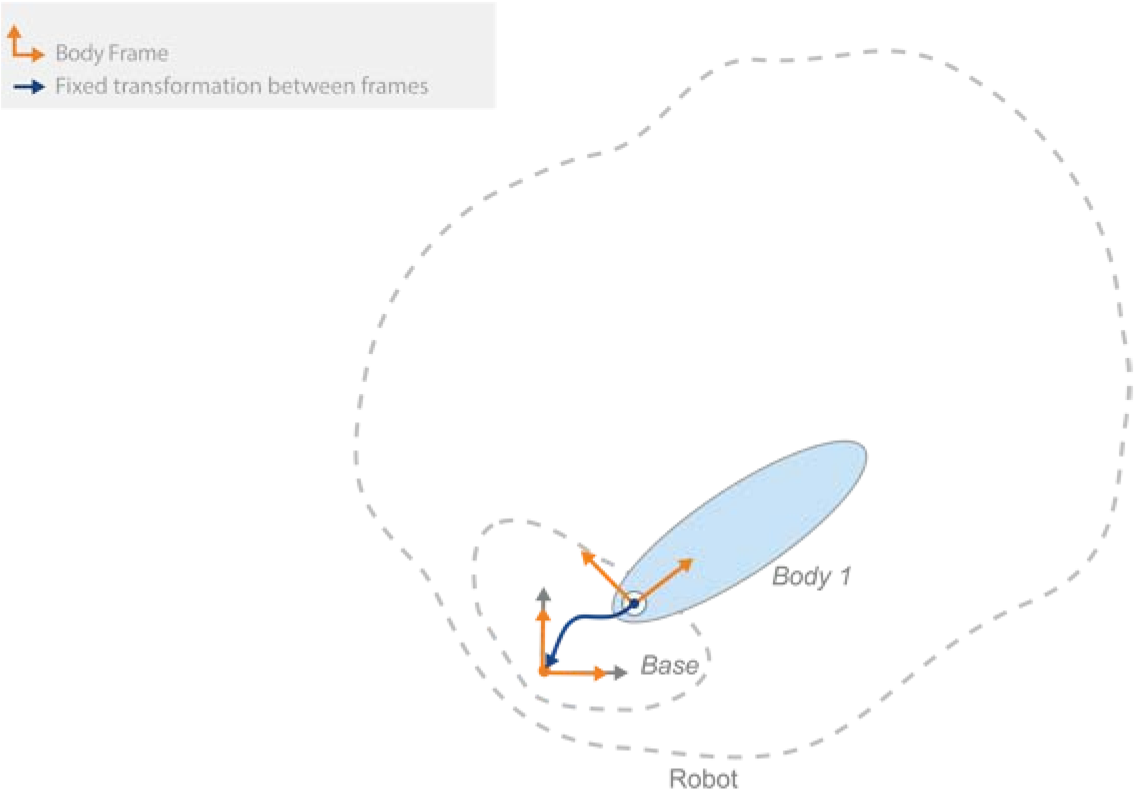
Rigid Body Tree

robot = robotics.RigidBodyTree;



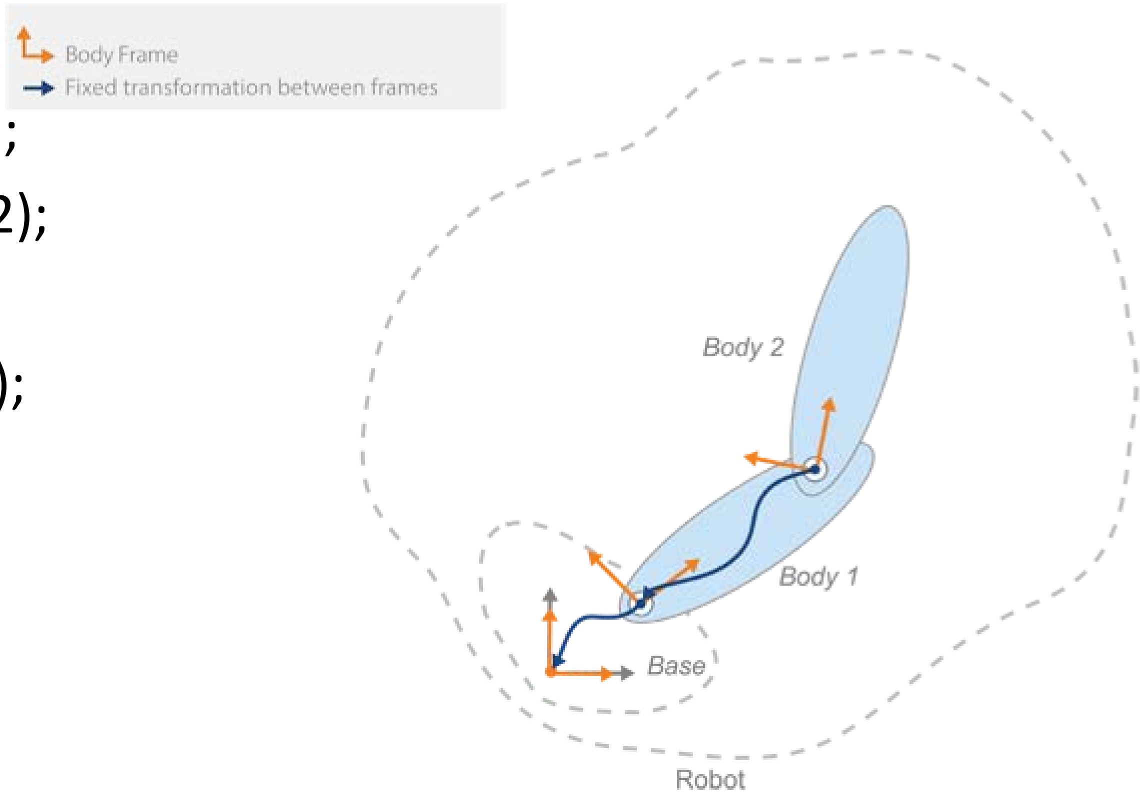
Rigid Body Tree

addBody(robot,body1,'base')



Rigid Body Tree

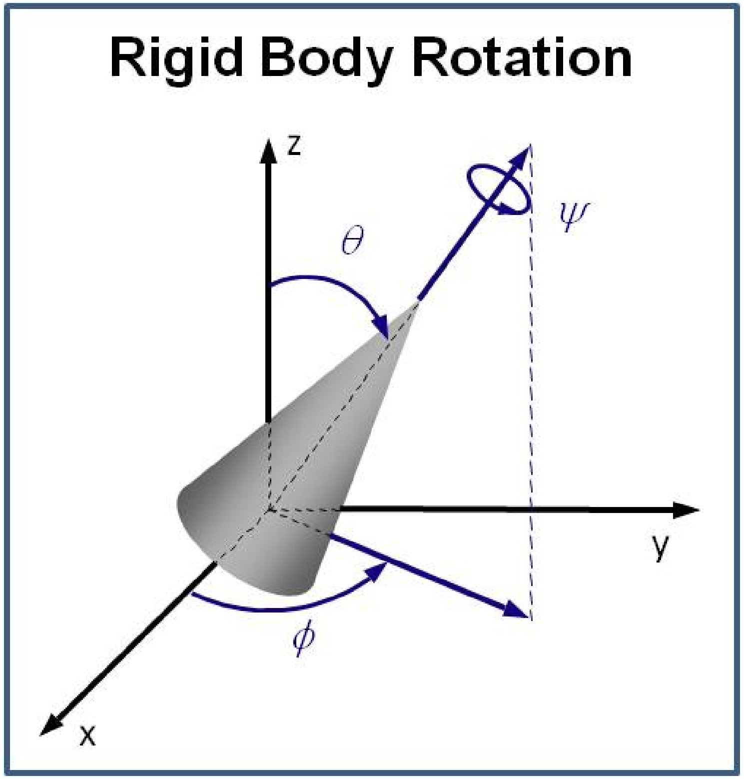
```
body2 = robotics.RigidBody('body2');
jnt2 = robotics.Joint('jnt2','revolute');
jnt2.HomePosition = pi/6;
tform2 = trvec2tform([1, 0, 0]);
setFixedTransform(jnt2,tform2);
body2.Joint = jnt2;
addBody(robot,body2,'body1');
```



Quaternion Class

- https://de.mathworks.com/matlabcentral/fileexchange/33341-quaternion-m

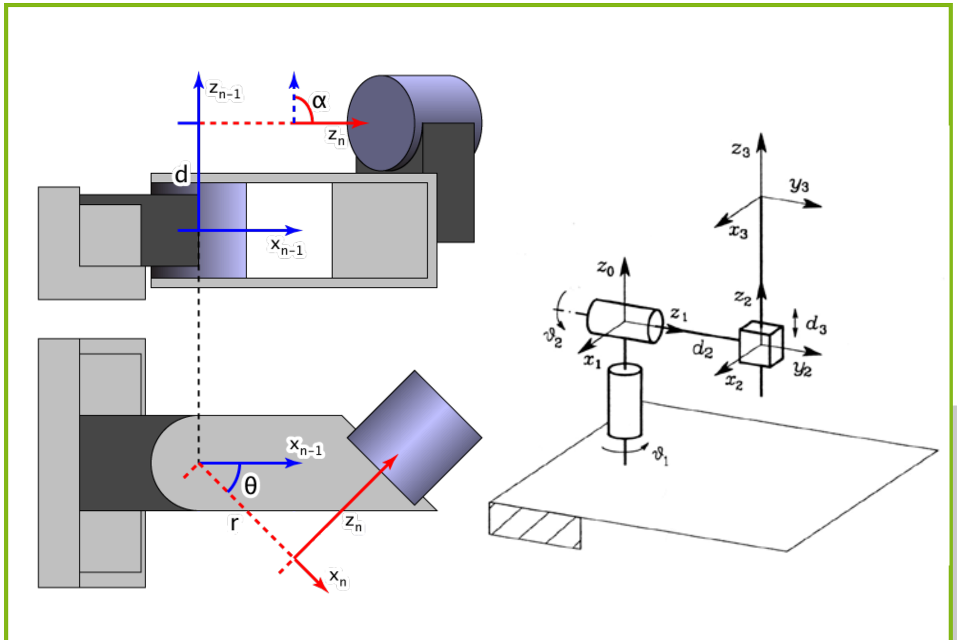
- vp = RotateVectorQ(q,v,dim);
- qc = conj(q);
- qe = exp(q);
- ql = log(q);
- q3 = mtimes(q1,q2);
- qp = power(q,p);
- qi = inverse(q) ;
- qi = interp1(t,q,ti,method);

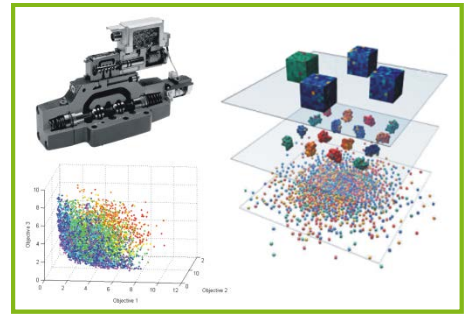


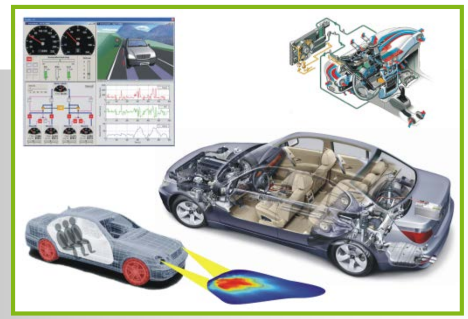
https://de.mathworks.com/matlabcentral/mlc-downloads/downloads/submissions/33341/versions/8/screenshot.jpg

Recommended Literature

- Robotics Modelling, Planning and Control, Chapter Kinematics, sections 2.1-2.7







What is next?
Kinematics

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