$$\Phi(t) \geq \delta(M_1 e)^{-1}$$

and therefore

$$\int_0^\infty \Phi(t)^p dt \geq \sum_j \int_{\Delta_j} \Phi(t) dt \geq \left(\frac{\delta}{M_1 e}\right) \sum_j \frac{1}{\omega} = \infty$$

contradicts (2).

Hence there exists $M > 0$ such that $\forall t \in [0, \infty)$, $\Phi(t) \leq M$.

Let $0 < \rho < M^{-1}$.

Define

$$t(\rho) = \sup\{t | \Phi(s) \geq \rho, 0 \leq s \leq t\}.$$

$$t(\rho)\rho^P \leq \int_0^{t(\rho)} \Phi(t)^P dt \leq \int_0^\infty \Phi(t)^P dt \leq M_2^P,$$

implies

$$t(\rho) \leq \left(\frac{M_2}{\rho}\right)^P = t_0$$

then $\forall t \geq t_0$

$$\Phi(t) \leq \Phi(t - t(\rho))\Phi(t) \leq M\rho < 1.$$

Finally let $t_1 > t_0$ be fixed, and let $t = nt_1 + s$, $0 \leq s \leq t_1$, then

$$\Phi(t) \leq \Phi(s)\Phi(nt_1) \leq M(\Phi(t_1))^n \leq M\beta^n \leq M'e^{-\mu t},$$

where $\beta = \Phi(t_1)$, $M' = M\beta^{-1}$ and $\mu = -(1/t_1) \ln \beta > 0$.

Exponential Stabilization of a Wheeled Mobile Robot Via Discontinuous Control

A. Astolfi [1]

In the present work the problem of exponential stabilization of the kinematic and dynamic model of a simple wheeled mobile robot is addressed and solved using a discontinuous, bounded, time invariant, state feedback control law. The properties of the closed-loop system are studied in detail and its performance in presence of model errors and noisy measurements are evaluated and discussed.

1 Introduction

The most studied nonholonomic systems are, probably, two-degrees-of-freedom, wheeled mobile robots, see [13, 8, 15, 16, 14, 9]. Traditionally mobile robots have been modeled in a cartesian coordinates system while, more recently, a polar representation has been introduced [6]. The polar representation allows to overcome the obstruction to stabilizability contained in the Theorem of Brockett and, at the same time, to design exponentially stabilizing, smooth, state feedback control laws. It must be noticed that, throughout the paper, the exponential stability property will be always referred to the transformed system, i.e., to the system in polar coordinates; moreover, the stabilizing control laws are smooth in the polar coordinates system, whereas are discontinuous, although bounded (see Remark 5) in the cartesian one. The potentiality of the polar description have been partially exploited in the work of Badreddin and Mansour [6], Astolfi [2, 4, 3], and Casalino et al. [9]. The former introduced for the first time the polar representation and proposed, for the kinematic model, a linear, fuzzy-tuned, state feedback control guarantying local asymptotic stability. The second exploited in detail the peculiarity of the kinematic polar representation and pointed out some geometrical properties of such a description and why the Theorem of Brockett does not apply to it. The latter designed, always for the kinematic system, a globally exponentially stabilizing, nonlinear, state feedback control law and discussed some geometrical issues of the closed-loop trajectories, e.g., the curvature of the floor paths. Further results can be also found in [11]. One of the features of the nonlinear control law presented in [9], as well as of other control laws (see e.g. [8, 15, 14, 17]) is that, when initialized in certain configurations, the robot is not able to reach the final position without inverting the direction of its motion, thus moving along discontinuous paths. Moreover, almost all the aforementioned results make use of kinematic models, rather than dynamic ones. This raises the questions about the applicability of such kinematic controllers in real applications.

In this work, we present globally exponentially stabilizing, state feedback control laws for the kinematic and the dynamic model of a simple wheeled mobile robot and we discuss in detail their performance. In particular, we show that the motion of the robot from any starting point to its final position is smooth, i.e., the floor path does not contain cusps. Moreover, we evaluate the closed-loop performance not only for the nominal system, but also in presence of noisy measurement and model mismatches. Finally, experimental results are also displayed. The paper is organized as follows. In Section 2 we derive the kinematic equations of the mobile robot in polar coordinates, whereas in Section 3 we propose a linear control law for such a kinematic model and we discuss the local and global stability issues of the closed-loop system. Section 4 deals with the problem of exponential stabilization of the dynamic extended model of the mobile robot. Finally, Sections 5 and 6 contain simulations, experimental results, and concluding remarks.

2 System Description

Consider the kinematic equations describing a wheeled mobile robot (see [8] for detail), i.e.,

$$\dot{x} = \cos\theta\, \nu$$

$$\dot{y} = \sin\theta\, \nu$$

$$\dot{\theta} = \omega. \tag{1}$$

Let α denote the angle between the y-axis of a reference frame fixed with the robot (Y_R) and the vector connecting the center of the axle of the wheels with the final position ($\hat{Y}$), see Fig. 1.

If $\alpha(0) \in I_1$, where

$$I_1 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{2}$$

consider the coordinates transformation

$$\rho = \sqrt{x^2 + y^2}$$

$$\alpha = -\theta + \arctan\left(\frac{-y}{-x}\right) \quad \bmod \left(\frac{\pi}{2}\right)$$

[1] Department of Electrical and Electronic Engineering, Imperial College, Exhibition Road, London SW7 2BT, UK. e-mail: a.astolfi@ic.ac.uk

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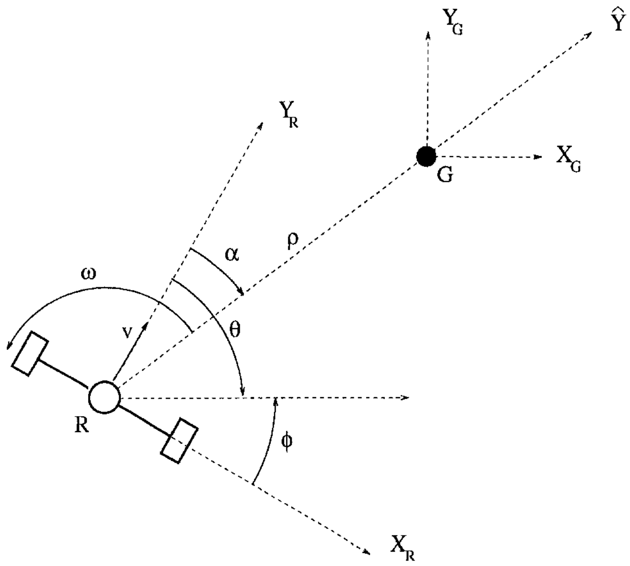

Fig. 1 Robot kinematics

$$\phi = \frac{\pi}{2} - \theta. \tag{3}$$

Remark 1. The coordinates transformation (3) is not defined at $x = y = 0$ as, in such a point, the determinant of the Jacobian matrix of the coordinates transformation is not defined, i.e., it is unbounded.

In the new coordinates (3) system (1) is described by equations of the form

$$\dot{\rho} = -\cos \alpha \nu$$
$$\dot{\alpha} = \frac{\sin \alpha}{\rho} \nu - \omega$$
$$\dot{\phi} = -\omega, \tag{4}$$

where ρ is the distance between the center of the axle of the wheels of the mobile robot and the goal position, ϕ denotes the angle between the x-axis of the robot reference frame (X_R) and the x-axis associated with the final (desired) position (X_G) and ν and ω are the tangent and the angular velocity, respectively. On the other hand, if $\alpha(0) \in I_2$, where

$$I_2 = \left(-\pi, -\frac{\pi}{2}\right] \cup \left(\frac{\pi}{2}, \pi\right],$$

redefining the forward direction of the robot, by setting $\nu = -\nu$, and applying a coordinates transformation similar to (3), we obtain a system described by equations of the form

$$\dot{\rho} = \cos \alpha \nu$$
$$\dot{\alpha} = -\frac{\sin \alpha}{\rho} \nu - \omega$$
$$\dot{\phi} = -\omega, \tag{5}$$

with, again, $\alpha(0) \in I_1$.

Remark 2. An alternative description of the system (4) can be given in terms of the differential one form which describes the velocity constraint along the direction of the two wheels (X_R), i.e., $(\sin \alpha) d\rho + (\rho \cos \alpha) d\alpha - (\rho \cos \alpha) d\phi = 0$.

Remark 3. If $\alpha \in I_1$, the forward direction of the robot points toward the origin of the coordinates system; whereas if $\alpha \in I_2$, the backward direction of the robot points toward the origin. In the first situation the robot will park in forward direction, i.e., with a positive tangential velocity ν; whereas in the second case the parking maneuver is reverse.

Remark 4. As already pointed out, by properly defining the forward direction of the robot at its initial configuration, it is always possible to have $\alpha \in I_1$ at $t = 0$. Obviously, this does not mean that α remains in I_1 for all t. Hence, to avoid the use of two models, it is necessary to determine, if possible, the control signals ν and ω in such a way that $\alpha \in I_1$ for all t, whenever $\alpha(0) \in I_1$.

In the sequel we restrict our analysis to system (4), as system (5) can be dealt with exactly in the same manner. The control signals ν and ω must be designed to drive the robot from its initial configuration, say $(\rho_0, \alpha_0, \phi_0)$, to the origin of the state space. It is immediately seen that the equations (4) present a discontinuity at $\rho = 0$, thus, as already pointed out in [2, 9], the Theorem of Brockett [7] does not yield any obstruction to smooth stabilizability.

3 The Control Law

Following the approach of [6, 2] we consider the linear control law

$$\nu = k_\rho \rho$$
$$\omega = k_\alpha \alpha + k_\phi \phi, \tag{6}$$

yielding a closed-loop system described by equations of the form

$$\dot{\rho} = -k_\rho \rho \cos \alpha$$
$$\dot{\alpha} = -k_\alpha \alpha - k_\phi \phi + k_\rho \sin \alpha$$
$$\dot{\phi} = -k_\alpha \alpha - k_\phi \phi. \tag{7}$$

System (7) does not have any singularity at $\rho = 0$ and has the unique equilibrium point $(\rho, \alpha, \phi) = (0, 0, 0)$ (recall that $\alpha \in I_1$).

Remark 5. In the cartesian coordinates system the control law (6) is described by equations of the form

$$\nu = k_\rho \sqrt{x^2 + y^2}$$
$$\omega = k_\alpha \left(\arctan\left(\frac{y}{x}\right) - \theta \right) + k_\phi \left(\frac{\pi}{2} - \theta \right) \tag{8}$$

and it is not defined at $x = y = 0$. However, such an indeterminacy can be resolved setting $\nu|_{(0,0,\theta)} = 0$ and $\omega|_{(0,0,\theta)} = (k_\alpha + k_\phi)(\pi/2 - \theta)$. We stress that the indeterminacy of the control law at $x = y = 0$, as well as the discontinuity in the second of equations (4), namely the equation of $\dot{\alpha}$, are due to the singularity of the cartesian to polar transformation at $x = y = 0$, or equivalently at $\rho = 0$. Observe finally that the control law (8) is always bounded, despite its discontinuous nature.

Remark 6. Observe that the control signal ν has always constant sign, i.e., it is positive whenever $\alpha(0) \in I_1$ and is negative otherwise. This implies that the robot performs its parking maneuver without ever inverting its motion. Hence, the floor trajectories do not contain cusps.

3.1 The Local Stability Issue. We study now the asymptotic properties of the closed-loop system (7). For, it is possible to prove the following interesting results.

Proposition 1. System (7) is locally exponentially stable iff

$$k_\rho > 0$$
$$k_\phi < 0$$
$$k_\alpha + k_\phi - k_\rho > 0. \tag{9}$$

Proof. System (7) can be written as

$$\begin{bmatrix} \dot\rho \\ \dot\alpha \\ \dot\phi \end{bmatrix} = \begin{bmatrix} -k_\rho & 0 & 0 \\ 0 & -(k_\alpha - k_\rho) & -k_\phi \\ 0 & -k_\alpha & -k_\phi \end{bmatrix} \begin{bmatrix} \rho \\ \alpha \\ \phi \end{bmatrix} + \text{h.o.t.},$$

hence it is locally exponentially stable iff the eigenvalues of the matrix describing the linear approximation of system (7) have all negative real part. Such eigenvalues are the roots of the polynomial $p_A(\lambda) = (\lambda + k_\rho)(\lambda^2 + \lambda(k_\alpha + k_\phi - k_\rho) - k_\rho k_\phi)$, which has all roots with negative real part iff conditions (9) hold.

Proposition 2. Consider the system (7) and assume that $\alpha(0) \in I_1$ and $\phi(t) \in (-n\pi, n\pi]$ for all t. Then, if

$$k_\alpha + 2nk_\phi - \frac{2}{\pi} k_\rho > 0 \tag{10}$$

one has $\alpha(t) \in \mathring{I}_1$[2] for all $t > 0$.

Proof. It suffices to show that if (10) holds for a given n the region I_1 is a trapping region, i.e., no trajectory of system (7) can leave it, and that $\dot\alpha$ is nonzero at $\alpha = \pm\pi/2$, for all $\phi \in (-n\pi, n\pi]$ and all $\rho \geq 0$. For, observe that if (10) holds, $\dot\alpha$ is strictly negative at $\alpha = \pi/2$ for all $\phi \in (-n\pi, n\pi]$ and all $\rho \geq 0$, whereas $\dot\alpha$ is strictly positive at $\alpha = -\pi/2$ for all $\phi \in (-n\pi, n\pi]$ and all $\rho \geq 0$.

Remark 7. Conditions (10) and the first of conditions (9) have an interesting geometric interpretation. They ensure that $\alpha(t) \in \mathring{I}_1$ for all $t > 0$ and that $\dot\rho$ is strictly negative for all $t > 0$. Hence, the mobile robot approaches the goal position monotonically.

Remark 8. It must be noticed that if $\phi \in (-n\pi, n\pi]$ a transition between $-n\pi$ and $n\pi$ (or vice-versa) results in a discontinuity in the control law. We do not study such a discontinuity here as we will show in Proposition (3) that if $n = 2$, i.e., $\phi \in (-2\pi, 2\pi]$, and $\alpha(0) \in I_1$ such a transition cannot occur.

3.2 The Global Stability Issue. We now discuss the global stability issue for system (7). An interesting preliminary result is contained in the following statement.

Proposition 3. Consider the reduced system

$$\dot\alpha = f_1(\alpha, \phi) = -k_\alpha\alpha - k_\phi\phi + k_\rho \sin\alpha$$
$$\dot\phi = f_2(\alpha, \phi) = -k_\alpha\alpha - k_\phi\phi. \tag{11}$$

Assume that conditions (9) and (10) with $n = 2$ hold. Suppose moreover that $\alpha(0) \in I_1$.

Then any trajectory of the system starting in the set $D_1 = \{(\alpha, \phi) \in \mathbb{R}^2 | \alpha \in I_1, \phi \in (-\pi, \pi]\}$ is contained in $D_2 = \{(\alpha, \phi) \in \mathbb{R}^2 | \alpha \in I_1, \phi \in (-2\pi, 2\pi]\}$ and converges asymptotically to the origin.

Proof. Consider the candidate Lyapunov function

$$V = (-k_\phi\alpha + k_\phi\phi)^2 + 2k_\phi k_\rho(\cos\alpha - 1),$$

which is positive definite, by conditions (9), in all D_2. Simple calculations yield

$$\dot V = 2k_\phi k_\rho \alpha \sin\alpha \left(k_\phi + k_\alpha - k_\rho \frac{\sin\alpha}{\alpha} \right),$$

which is nonpositive in D_2 by condition (9). We now show that any trajectory starting in D_1 remains in D_2 for all $t > 0$. For, observe that the set

$$M = \{(\alpha, \phi) \in D_2 | V \leq \tfrac{9}{4} k_\phi^2 \pi^2 - 2k_\phi k_r\}$$

contains D_1, is contained in D_2 and is (positively) invariant. Hence, M is a positively invariant set, which contains only the equilibrium (0, 0) and no closed orbit. Therefore, the claim is a consequence of Poincare-Bendixson Theorem, see [19, p. 49].

We now summarize the above discussion with a formal statement, which provides an interesting global stability result.

Proposition 4. Consider the closed-loop system (7). Assume that conditions (9) and (10) with $n = 2$ hold. Suppose, moreover, that $\alpha(0) \in I_1$.

Then the region of attraction of the equilibrium (0, 0, 0) contains the region $\Omega_r = \{(\rho, \alpha, \phi) \in \mathbb{R}^3 | \rho \geq 0, \alpha \in I_1, \phi \in (-\pi, \pi]\}$.

Remark 9. We yield only a sufficient condition of global stability as we consider a particular Lyapunov function, thus other conditions, possibly less stringent, can be obtained using different candidate Lyapunov functions. However, it must be noticed that only condition (10) can be weakened, whereas conditions (9) are necessary to have asymptotic stability.

4 Stabilization of the Dynamic Extended Model

In this section, we describe a possible approach for extending the proposed kinematic control law, which commands input velocity, to a control law which commands input torque. Such a problem has been recently addressed only for some classes of nonholonomic systems, see [18, 10, 12, 2], as traditionally, stabilization of nonholonomic control systems has concentrated on the use of kinematic models, rather than dynamic ones. This raises the questions about the applicability of such kinematic controllers to physical systems, in which the torques, and not the velocities, are the control inputs. We consider the equations describing the dynamic extended model of a mobile robot in polar coordinates with the simplifying assumptions that the center of gravity of the robot is on the center of the wheel base and that the motor dynamics have been compensated, i.e.

$$\dot\nu = T_1$$
$$\dot\omega = T_2$$
$$\dot\rho = -\nu\cos\alpha$$
$$\dot\alpha = \frac{\nu}{\rho}\sin\alpha - \omega$$
$$\dot\phi = -\omega, \tag{12}$$

and we are interested in finding a static, state feedback control law which renders the closed loop system locally asymptotically stable. It must be noticed that standard back-stepping theory, as presented e.g. in [1], cannot be directly used as the system under consideration is not continuous; hence a stand alone theory must be developed. Observe, moreover, that the stability results discussed in Section 3 rely on the fact that, despite the discontinuous nature of the open-loop description (4), the closed-loop system is smooth, i.e., it is possible to cancel the singularity in the equation of $\dot\alpha$ by means of the velocity control ν. Such cancellation cannot take place if the dynamic model is considered, as ν is now a state variable and not a control input. Nevertheless, it is still possible to find a control law which render the discontinuous closed-loop system well posed and

[2] $\mathring{U}$ denotes the interior of U.

cancel the discontinuity *asymptotically.* For, we prove the following preliminary results.

Lemma 1. Consider the time varying system

$$\dot{\nu} = T_1$$
$$\dot{\rho} = \nu \cos \alpha(t). \tag{13}$$

Assume that, for all $t > 0$, $\alpha(t) \in \mathring{I}_1$, where the set I_1 is defined by Eq. (2). Suppose, moreover, that $\nu(0) = 0$ and that $\rho(0) \neq 0$. Consider the control law

$$T_1 = -k_\rho \nu \cos \alpha(t) - \lambda_1(\nu - k_\rho \rho) \tag{14}$$

with $k_\rho > 0$ and $\lambda_1 > k_\rho$.

Then the ratio $\xi(t) = \nu(t)/\rho(t)$ is well defined and bounded for all $t \geq 0$. Moreover, $\lim_{t\to\infty} \xi(t) = k_\rho$, $\lim_{t\to\infty} \nu(t) = 0$ and $\lim_{t\to\infty} \rho(t) = 0$.

Proof. Consider the variable $\xi = \nu/\rho$. Simple calculations show that $\dot{\xi} = (\xi \cos \alpha - \lambda_1)(\xi - k_\rho)$. It follows that the equilibrium $\xi = k_\rho$ is locally exponentially stable and that the point $\xi = 0$ belongs to its basin of attraction. Moreover, the bound $0 = \xi(0) \leq \xi(t) \leq k_\rho$ holds for every $t \geq 0$. Finally, to show asymptotic stability of the closed-loop system (13)–(14) it suffices to use the Lyapunov function $W(\nu, \rho) = 2\lambda_1 k_\rho \rho^2 + (\nu - k_\rho \rho)^2$.

Lemma 2. Consider the system (12) and assume that $\alpha(0) \in I_1$, $\rho(0) \neq 0$, $\nu(0) = 0$ and $\omega(0) = 0$. Set

$$T_1 = -k_\rho \nu \cos \alpha - \lambda_1(\nu - k_\rho \rho)$$
$$T_2 = k_\alpha\left(\frac{\nu}{\rho} \sin \alpha - \omega\right) - k_\phi \omega - \lambda_2(\omega - k_\alpha \alpha - k_\phi \phi), \tag{15}$$

with $\lambda_1 > k_\rho$ and $\lambda_2 > 0$. Suppose, moreover, that the gains k_ρ, k_α and k_ϕ fulfill the conditions (9) and (10) with $n = 2$. Then $\alpha(t) \in \mathring{I}_1$, for all $t > 0$.

Proof. Even though more involved, the proof is similar to that of Proposition 2; hence it is omitted.

We conclude with the following statement, which turns out to be a simple consequence of the results established so far.

Proposition 5. Consider the system (12) and the control law (15). Assume the following.

(i) The gains k_ρ, k_α and k_ϕ fulfill the conditions (9) and (10) with $n = 2$.
(ii) $\lambda_1 > k_\rho$ and $\lambda_2 > 0$.
(iii) $\alpha(0) \in I_1$, $\rho(0) \neq 0$, $\nu(0) = 0$ and $\omega(0) = 0$.

Then every initial condition in the region

$$\Omega_{rd} = \{(\nu, \omega, \rho, \alpha, \phi) \in \mathbb{R}^5 | \nu = 0, \omega = 0, \rho \geq 0, \alpha \in I_1, \phi \in (-\pi, \pi]\}$$

converges asymptotically to the origin.

Remark 10. The state variables ν and ω represent the velocities of the mobile robot, hence the assumption $\nu(0) = 0$ and $\omega(0) = 0$ means that the robot is at rest. This is without lack of generality, as we can always apply an open-loop control law which stops the robot in arbitrarily small time.

Remark 11. Note that the point (0, 0, 0, 0, 0) is not an equilibrium for the closed-loop system (12)–(15), as the system is not defined at $\rho = 0$. Nevertheless, if the hypotheses of Lemma 1 hold we can conclude that the closed loop system has a well defined forward solution; whereas, if the hypotheses of Proposition 5 hold, we can conclude that the point (0, 0, 0, 0, 0) is the unique ω-limit point of any trajectory starting in Ω_{rd}.

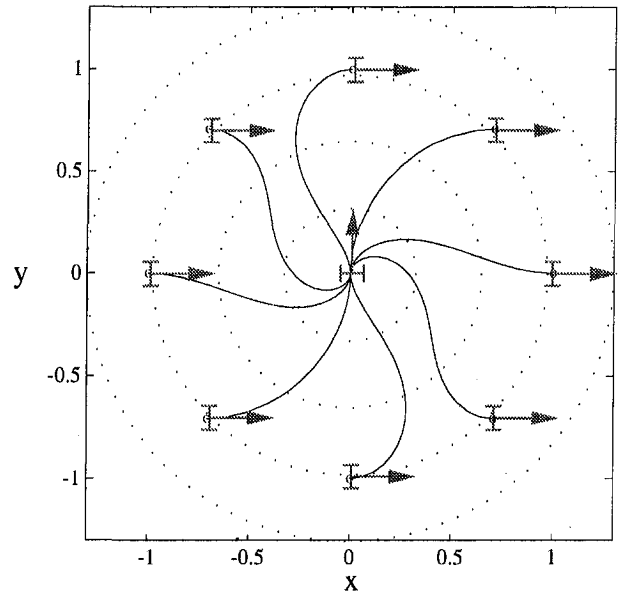

Fig. 2 Resulting paths when the robot is initially on the unit circle in the *xy*-plane

5 Results

In this section, we present a certain number of simulated and measured results showing the behavior of a mobile robot controlled with the proposed control laws. Simulations have been carried setting $k = (k_\rho, k_\alpha, k_\phi) = (1, 8, -1.5)$ which guarantees global asymptotic stability of the closed-loop system (7), i.e. such a k fulfills all the conditions of Proposition 4. Figure 2 shows the resulting paths in the xy-plane of the controlled mobile robot in the case of no disturbances or model errors. We see that the cart approaches the origin of the coordinates system without ever inverting its motion.

Simulations have also been done in presence of measurement noise. We have added a zero mean, normally distributed, white noise, to the states x, y and ϕ. The effect of the measurement noise can be evaluated considering the equation governing ρ, namely,

$$\dot{\rho} = -k_\rho \cos \alpha \sqrt{\rho^2 - 2\rho(\sin \alpha \Delta x - \cos \alpha \Delta y) + \Delta x^2 + \Delta y^2}, \tag{16}$$

where Δx and Δy represent the noise contributions. Equation (16) can be written as

$$\dot{\rho} = -k_\rho \rho \cos \alpha + k_\rho \cos \alpha \sin \alpha \Delta x - k_\rho \cos^2 \alpha \Delta y + \cos \alpha \frac{o^2(\Delta x, \Delta y)}{\rho}.$$

Thus, it is clear that when ρ is big, i.e., the robot is far from the final configuration, the nominal behavior dominates the effect of the disturbances and only near the origin their effect becomes significant. Figure 3 (left) shows the behavior of the robot around the origin assuming that the disturbances Δx, Δy and $\Delta\theta$ are uncorrelated white noise with normal distribution and variance of 0.5; while Fig. 3 (right) shows the difference between the nominal behavior of ρ and that with noisy measurements. We use a logarithmic scale to show when the effect of the noise becomes relevant. Note that if ρ is bigger than 0.1 the effect of the noise is neglectable, while for value of ρ smaller than 0.01 the behavior of the system is completely determined by the noise.

One of the most common model error is due to wheels unbalancing, i.e., the two wheels of the mobile robot do not have the

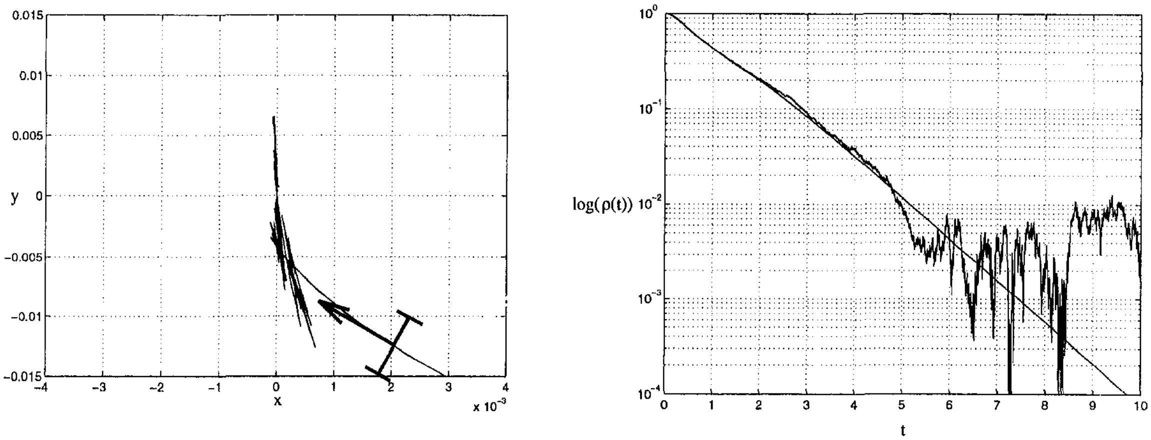

Fig. 3 Floor path of the mobile robot with noisy measurements in the *xy*-plane, note the extreme small scale of the figure (left); and time histories of *p*(*t*) with and without noisy measurements (right)

same radius. The easiest way to introduce such a mismatch in the model of the mobile robot (see [16] for further detail) is to multiply the control signal with a symmetric, diagonal dominant matrix of the form

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12} & \Lambda_{22} \end{bmatrix}.$$

Note that in the case of perfectly balanced wheels Λ reduces to the identity matrix. A parallel parking maneuver has been simulated for $\Lambda_{11} = 1.1$, $\Lambda_{12} = 0.1$, $\Lambda_{22} = 1.1$. The resulting path is displayed in Fig. 4 (solid line). It must be noted that the gains k_ρ, k_α, and k_ϕ are the same for all the simulation displayed in Fig. 4. A parallel parking maneuver for the dynamic system has also been simulated. The corresponding path, with zero initial velocities, is displayed in Fig. 4 (dashed line). Finally, the control law (15) has been implemented on the mobile robot RAMSIS [5], built at the Automatic Control Laboratory of the ETH Zürich. Figure 4 (dash-dotted line) shows a measured path; note how the real trajectory matches the simulated one.

6 Conclusions

Discontinuous control laws yielding exponential convergence for the kinematic and dynamic models of a mobile robot has been proposed. The controllers have been designed starting from the polar description of the kinematics of the robot, in contrast with the standard approach based on the cartesian description. It has been shown that the use of the polar description allows to treat the issues of local and global stability in a straightforward way. One particularity of the proposed controllers is that non-smooth trajectories are avoided. The robot *exponentially* converges to the goal position, i.e., the origin of the coordinates system, from any configuration. Moreover, the derivation of a stabilizing controller for the dynamic model allows a direct implementation of the proposed control law on real systems.

Simulations and real measures have shown the main features of the resulting paths, which differ substantially from those yielded by other approaches [8, 15, 14, 9], and the ability of the control law to counteract measurement noise and model mismatches.

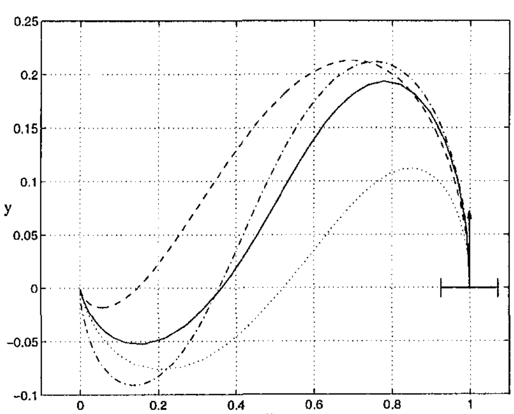

Fig. 4 Parallel parking maneuver; nominal kinematic model (dotted), kinematic model with wheels unbalancing (solid), nominal dynamic model (dashed), real robot (dash-dotted)

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Tangential-Contouring Controller for Biaxial Motion Control

Chih-Ching Lo[1] and Ching-Yei Chung[2]

In biaxial contour tracking applications, the main concerns are the tangential and the contour errors. This paper presents a tangential-contouring controller to achieve effect and decoupled control of these concerns. The proposed controller is based on a coordinate transformation between the X-Y frame and a tangential-contouring (T-C) frame that is defined along the contour. Experimental evaluation for the proposed controller is conducted on a biaxial positioning table.

1 Introduction

For biaxial contour tracking applications, the machines (e.g., machine tools) are driven to track a desired trajectory, such as lines, arcs, or other curves. The tracking is generally not so perfect that it may result in position errors. These errors can be represented in several different coordinate frames. The most popular one is the *X-Y* frame. However, the major concerns in contour tracking are not defined along the *X* and *Y* axes. A typical biaxial tracking process is depicted in Fig. 1. As can be seen, the tangential error (ϵ_t) and the contour error (ϵ_c), which are, respectively, the error components tangent and perpendicular to the contour, are our main concerns. As shown in Fig. 1, we can define a tangential-contouring (T-C) coordinate frame, of which the T and C axes are, respectively, tangent and perpendicular to the contour, so that ϵ_t and ϵ_c are the axial error components in this frame.

Conventional design of motion control systems focuses on elimination of the *X*- and *Y*-axis errors, E_x and E_y, and expects a reduction in ϵ_t and ϵ_c. Many feedback control (Boucher et al., 1990) and feedforward control (Tomizuka, 1987; Tsao and Tomizuka, 1987) methods have been developed to achieve this purpose. In practice, the reduction of axial errors may not mean the reduction of the contour error (Koren and Lo, 1992). To achieve an accurate contour tracking, many cross-coupling controllers have been developed and proven effective in elimination of the contour error (Koren and Lo, 1991; Kulkarni and Srinivasan, 1989, 1990). However, to minimize the contour error as well as the tangential error, proper designs for both the axial controllers and the cross-coupling controller are required. In practice, the coupled effect between the axial controllers and the cross-coupling controller may cause a degradation in the tracking performance, and consequently bring a difficulty in the designs of these controllers (Lo, 1997).

To cope with the above problem, a tangential-contouring controller is proposed in this paper. The proposed controller focuses on direct and decoupled control of the concerned error components, ϵ_t and ϵ_c, through a transformation between the *X-Y* and the T-C coordinate frames. In this paper, it can be proven that if the mismatch between the *X*- and *Y*-axis dynamics is small enough, the controls of the tangential error and the contour error can be decoupled.

2 Tangential-Contouring Controller

A schematic description for the proposed biaxial tangential-contouring controller is shown in Fig. 2. As can be seen, the direct elimination of the tangential and the contour errors, ϵ_t and ϵ_c, is conducted through the coordinate conversion between the *X-Y* and the T-C coordinate frames. Here, the coordinate conversion is represented by a matrix $[\Phi]$ and its transpose $[\Phi]'$. Let θ be the inclination angle between the two frames, then we can have

$$[\Phi] = \begin{bmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{bmatrix}, \tag{1}$$

where $c_\theta = \cos(\theta)$ and $s_\theta = \sin(\theta)$.

Let's denote (R_t, R_c) and (P_t, P_c) as the incremental reference and output positions in the T-C basis, respectively. Then, we may have

$$\begin{bmatrix} R_x \\ R_y \end{bmatrix} = [\Phi]' \begin{bmatrix} R_t \\ R_c \end{bmatrix}, \tag{2}$$

$$\begin{bmatrix} P_t \\ P_c \end{bmatrix} = [\Phi] \begin{bmatrix} P_x \\ P_y \end{bmatrix}. \tag{3}$$

Note that the incremental reference command in the contouring direction is zero, i.e., $R_c = 0$.

Based on Eqs. (1)–(3) and the block diagram shown in Fig. 2, the transfer function matrix in the T-C basis can be formulated as

$$\begin{bmatrix} P_t \\ P_c \end{bmatrix} = \begin{bmatrix} G_{tt}(z) & G_{tc}(z) \\ G_{ct}(z) & G_{cc}(z) \end{bmatrix} \begin{bmatrix} R_t \\ R_c \end{bmatrix}, \tag{4}$$

where

$$G_{tt}(z) = \frac{1}{\Delta(z)} \{ c_\theta^2 T(z) G_x(z) + s_\theta^2 T(z) G_y(z) + T(z) C(z) G_x(z) G_y(z) \}, \tag{5}$$

$$G_{ct}(z) = G_{tc}(z) = \frac{1}{\Delta(z)} \{ c_\theta s_\theta C(z) [G_x(z) - G_y(z)] \}, \tag{6}$$

$$G_{cc}(z) = \frac{1}{\Delta(z)} \{ s_\theta^2 C(z) G_x(z) + s_\theta^2 C(z) G_y(z) + T(z) C(z) G_x(z) G_y(z) \}, \tag{7}$$

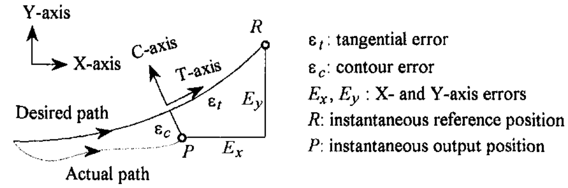

Fig. 1 Position errors in biaxial contour tracking

[1] Department of Mechanical Engineering, Feng Chia University, Taichung, Taiwan.
[2] Department of Automatic Control Engineering, Feng Chia University, Taichung, Taiwan.

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