IMAGE RESTORATION USING GEOMETRICALLY STABILIZED REVERSE HEAT EQUATION

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ABSTRACT

Blind restoration of blurred images is a classical ill-posed problem. There has been considerable interest in the use of partial differential equations to solve this problem. The blurring of an image has traditionally been modeled by Witkin [10] and Koenderink [4] by the heat equation. This has been the basis of the Gaussian scale space. However, a similar theoretical formulation has not been possible for deblurring of images due to the ill-posed nature of the reverse heat equation. Here we consider the stabilization of the reverse heat equation. We do this by damping the distortion along the edges by adding a normal component of the heat equation in the forward direction. We use a stopping criterion based on the divergence of the curvature in the resulting reverse heat flow. The resulting stabilized reverse heat flow makes it possible to solve the challenging problem of blind space varying deconvolution. The method is justified by a varied set of experimental results.

Index Terms— Image restoration, image analysis, partial differential equations, diffusion equations

1. INTRODUCTION

The problem that is addressed in this paper is one of deblurring an image $Y(x)$ that has been blurred by a blurring kernel $h(x)$ representing some physical process. This problem is modeled by the convolution relation:

$$Y(x) = \int U(t)h(x-t)dt \tag{1}$$

where x can denote a $2D$ space in which case $U(x)$ might represent an image. As is normally assumed the function $h(x)$ has the properties that it is non-negative, and the integral of the function $h(x)$ is unity.

As shown by Guichard and Morel [3], the blurring of an image is proportional to its Laplacian. The process of blurring can be modeled by the heat equation as follows

$$\frac{\partial u}{\partial t} = c\triangle u,\ u(x,0) = I_0(x) \tag{2}$$

Here u represents the image being diffused using the heat equation, c is the diffusion coefficient, $\triangle u$ is the Laplacian of u and $I_0(x)$ is the initial deblurred image. The use of the heat equation has also been used by Witkin [10] and Koenderink [4] in the formation of the notion of scale space. An important work along these lines has been use of anisotropic diffusion for edge preserving denoising by Perona and Malik [6]. While there has been much work done on the forward aspect of heat diffusion [8], relatively less work has been done on the reverse aspect of the heat equation. The reverse heat equation is ill-posed and so its use has been limited. Osher and Rudin [9] in their work proposed the use of "shock" filters which are hyperbolic partial differential equations. These are stable and have good convergent properties. However, they provide piecewise constant results and do not achieve true deblurring. Another work has been the use of stabilized inverse diffusion equations [7] by Pollak *et al.*. They also use an approximation to the inverse diffusion which has a physical motivation. However, they also do not approach the true reverse heat equation. A very recent work [2], has explored the use of reverse heat equation with a non-local means based additional criterion. They perform alternating steps of reverse heat and non-local regularization. The alternate formulation that we provide is simpler. Here, we solve the problem of deblurring by using the reverse heat equation. Since the reverse heat equation is ill-posed we stabilize it by controlling the disruption of edges. This is achieved by adding a normal component of the heat equation in the forward direction. We also formulate a stopping criterion for terminating the reverse heat equation process when the deblurring of the image is completed. In the next section we discuss the reverse heat equation and its stabilization.

2. STABILIZED BACKWARD HEAT EQUATION

The reverse heat equation is given as

$$\begin{aligned} \frac{\partial u}{\partial t} &= c\triangle u \\ u(x,\tau) &= I(x) \end{aligned} \tag{3}$$

where $\triangle u$ denotes the Laplacian of u, $I(x)$ is the blurred observation and c is the diffusion coefficient. We have to find the solution

$$u(x, 0) = I_0(x). \tag{4}$$

This is achieved by reversing time in the heat equation

$$\frac{\partial u}{\partial t} = -c\triangle u,\ u(x, 0) = I(x). \tag{5}$$

However implementing eqn(5) can be done only for a few time steps and then the resulting image blows up due to the high pass nature of the resulting operation. It boosts the noise, especially along the edges where the Laplacian has high values. Explicit edge information can be considered in the heat equation by considering the geometric form of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \zeta^2} = u_{\eta\eta} + u_{\zeta\zeta}. \tag{6}$$

Here η refers to the normal and ζ to the tangential direction. The diffusion along the normal is given by

$$u_{\eta\eta} = \frac{u_{xx}u_x^2 + 2 * u_{xy}u_x u_y + u_{yy}u_y^2}{u_x^2 + u_y^2} \tag{7}$$

and the diffusion along the tangent is given by

$$u_{\zeta\zeta} = \frac{u_{xx}u_x^2 - 2 * u_{xy}u_x u_y + u_{yy}u_y^2}{u_x^2 + u_y^2} \tag{8}$$

Since the diffusion along the normal diffuses across the edges and diffusion along the tangent continues along the edges, the blurring in an image is caused more due to diffusion along the normal. Therefore in order to stabilize the reverse diffusion, the reverse diffusion across the edges has to be done at a slower rate as compared to reverse diffusion along the tangent. The diffusion along the normal is a more divergent process and has to be done at a slower rate. Thus in order to stabilize the reverse diffusion we add a forward component of diffusion along the normal. The resultant stabilized form of the heat equation is given by

$$u_t = -c\triangle u + \beta u_{\eta\eta} \tag{9}$$

Here we use $c > \beta$ in order to ensure the overall reverse nature of the diffusion. The diffusion is carried out until a stopping criterion is reached which corresponds to the initial required solution $I(x, 0)$.

3. RELATION TO OTHER TECHNIQUES

We now consider the analysis of shock filters and Kramer's algorithm as explored by Guichard and Morel [3]. Osher and Rudin in their "shock filter" formulation, proposed the following equation

$$\frac{\partial u}{\partial t} = -sign(\triangle u)|\nabla u| \tag{10}$$

where ∇u is the gradient of u. This equation enhances the Hildreth-Marr edges. Kramer defined a filter that sharpens blurred images by replacing the gray level value at a point x by either the minimum or the maximum of the gray level values in a circular neighborhood. Guichard and Morel [3] proved that the PDE underlying the Kramer filter is

$$\frac{\partial u}{\partial t} = -sign\left(\nabla^2 u(\nabla u, \nabla . u\right) \tag{11}$$

where ∇u is the gradient of u. This filter enhances the Canny edges. While, both these filters perform edge enhancement, they are not equivalent to the actual reverse heat equation, as compared to the proposed approach which is based on the reverse heat equation itself.

The non-local reverse heat equation proposed recently [2] is closely comparable to the proposed technique. The non-local reverse heat equation is given as

$$\frac{\partial u}{\partial t} = -\triangle u + \lambda NL_0 u \tag{12}$$

where

$$NL_0 u(x) = \frac{1}{C(x)} \int e^{\frac{G_\sigma * |u(X+) - u(Y+)|^2(0)}{h^2}} u(y)dy, \tag{13}$$

where $C(x)$ is the normalizing factor, h acts as a filtering parameter and G_σ is the Gaussian kernel with standard deviation σ. Here NL_0 is the non-local means filter [1] and it means that $u(x)$ is replaced by a weighted average of $u(y)$. The weights are significant only if a Gaussian window around y looks like the corresponding Gaussian window around x. This approach is certainly interesting. The main difference, as is evident by comparing eqns(9) and (12), is that in our approach we rely more on the local normal component of the heat equation to stabilize the equation as compared to the non-local component used by Buades *et al.*. Since the objective has been to closely approximate the reverse heat equation, the damping by using a normal component of the heat equation itself satisfies this criterion in a better way.

4. STOPPING CRITERION

Consider the eqn(3) using which we have to estimate the initial condition given in eqn(4), i.e. we have to estimate the value of $u(x, 0) = I_0(x)$. The eqn(5) has to be stopped when $u(x, t) = u(x, 0)$. However, here we do not know the value of $u(x, 0)$. An observation that can be used is that the eqn(3) is valid only till time $t = 0$ and it breaks down if we go beyond this time. The modified reverse heat equation given in eqn(9) will not be valid for the value of time $t < 0$. Hence, beyond this point the solution will degenerate rapidly. This observation can be used for stopping the reverse heat equation. If we consider the image as a manifold with at least C_2 continuity, the degeneration of the solution can be detected by the divergence of the curvature. In eqn (9), since the normal component is added, the tangential term is diffused in reverse direction more rapidly. The tangential term corresponds to curvature driven motion. Since, the curvature driven term is reversed at a faster rate, the degeneration in this term happens before degeneration in the normal component. Hence, the divergence of curvature can be used as an indicator that the image approximates the desired initial image. Further, when the divergence of the curvature happens, the degeneration in the normal component would happen in a few more time steps, based on the difference of the weightage given to them. And hence, the degeneration of the curvature is a good indicator for stopping. The curvature is given by

$$\kappa = \frac{u_{xx}u_x^2 - 2u_{xy}u_xu_y + u_{yy}u_y^2}{(u_x^2 + u_y^2)^{3/2}} \tag{14}$$

The eqn(9) is stopped when the change in curvature exceeds a threshold, i.e. $\kappa_t > \theta$. Comparatively, the shock filter formulation [5] is a convergent procedure and does not require a stopping criterion. In the non-local means based reverse heat equation [2], the authors suggest stopping the reverse heat equation when the value of the Laplacian exceeds twice the value of the initial Laplacian. But using this criterion results in certain artifacts being generated due to the degeneration of the solution.

5. IMPLEMENTATION DETAILS

In the implementation of the reverse heat equation, the boundary conditions were assumed to be Neumann boundary conditions, i.e. the gradient is zero along the boundary. We now consider the values of the various constants. In eqn(9) the values of c and β are chosen to be small and $c > \beta$. Additionally, they must be small enough to maintain CFL conditions. Here we have chosen values of c as 0.2 and β as 0.02. These values have been empirically chosen. The value of θ used for setting the threshold for change in curvature was 0.3

6. RESULTS

We first justify the use of the proposed technique by considering the performance of the reverse heat equation when used for deblurring without any modification. The results using the reverse heat equation are shown in fig. 1(c)&(d). Fig. 1(a) shows the original Lena image that is blurred with a constant Gaussian blur with standard deviation 3.0. Fig. 1(b) shows the blurred input image. Fig. 1(c) shows the result of using the reverse heat equation for 2 iterations. As seen in the figure use of reverse heat effectively starts deblurring the input. But as can be seen in fig. 1(d), which shows the resultant image after applying the reverse heat equation for 10 iterations, this equation is unstable and the values blow up quickly. We next evaluate the proposed technique by experimentally comparing the method with the shock filter method [5]. Note that the proposed technique as well as the shock filter method do not use any information about the nature of the blurring function and both perform blind deconvolution. Fig. 1(e) shows the result of applying the shock filter. As can be seen while the shock filter preserves the strong edges, the weak texture edges are strongly affected in this method. This is because the shock filter does not approximate the reverse heat equation appropriately. Fig. 1(f) shows the result using the proposed technique. As can be seen, the result achieves true deblurring as can be seen from the texture on the hat and hair. Here the method took 19 iterations (as compared to original reverse heat blowing up in 10 iterations) before the stopping criterion was satisfied over the entire image. The result is closely comparable to the original image. Quantitative comparison in terms of PSNR values establish around 1.5 db improvement over the input image and around 10db improvement over the shock filter method.

We next consider an experiment of performing blind *space varying* deblurring. We blur a sand texture image obtained from the Brodatz texture database with a Gaussian blur function with the standard deviation being increased from 1.0 to 1.5 from left to right. Fig. 2(a) shows the input image and Fig. 2(b) shows the blurred input image. The input image is restored using the shock filter and this result is shown in fig. 2(c). As can be seen, the shock filter method results in a piecewise constant resultant image where much of the texture information is lost. This particularly emphasizes the need for the proposed technique. Fig. 2(d) shows the result of the proposed technique. As can be seen, using the proposed technique one can obtain deblurring of the input image with very little artifacts being present. The results quantitatively show around 3db improvement over the input image and around 3.7db improvement over the shock filter method.

7. CONCLUSION

Here we present a technique which addresses the challenging problem of blind space varying deblurring. The problem is modeled using the heat equation and deblurring is framed as a problem of solving the reverse heat equation. The unstable nature of the reverse heat equation is addressed by adding the normal component of the reverse heat equation in the forward direction. A curvature based stopping criterion appropriately stops the reverse heat equation without artifacts being introduced in the solution. The results obtained justify the feasibility of the proposed theory.

8. REFERENCES

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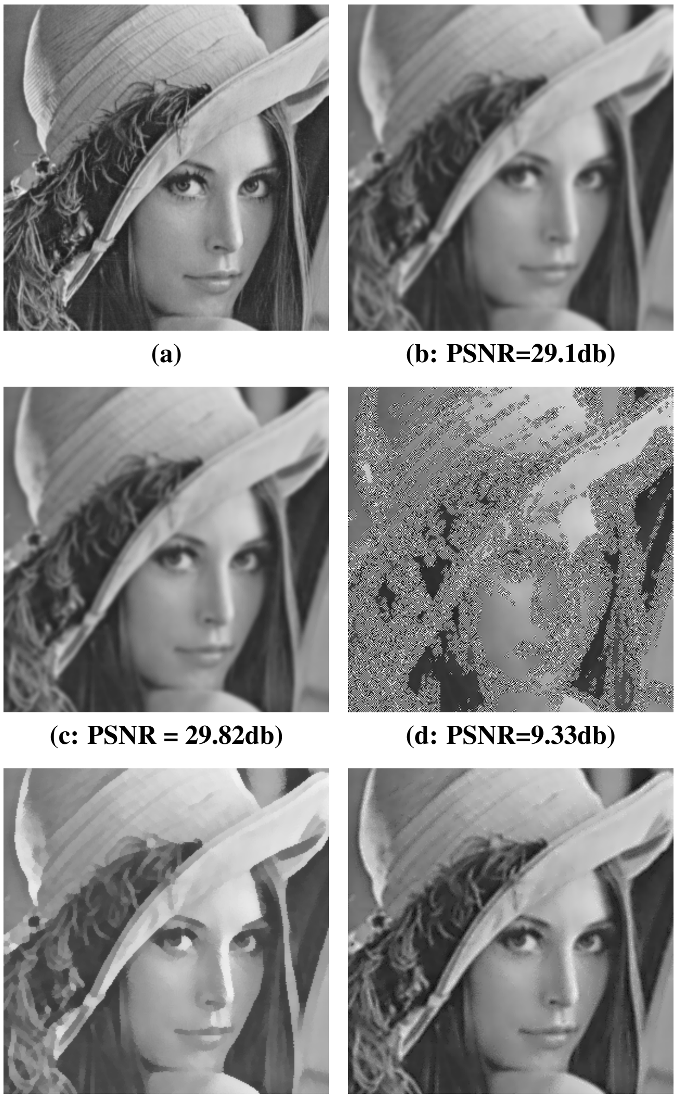
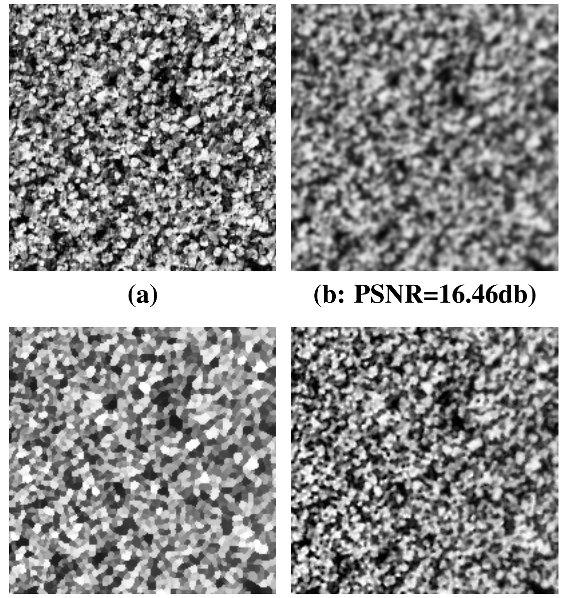


(a) | (b: PSNR=29.1db)

(c: PSNR = 29.82db) | (d: PSNR=9.33db)

(e: PSNR=21.56db) | (f: PSNR=30.48db)

Fig. 1. Deblurring results for a constant Gaussian blurred Lena image (a) is the original Lena image which is blurred as seen in (b) using constant Gaussian Blur. (c) shows the result of using the original reverse heat equation for 2 iterations and (d) shows the result on using 10 iterations of the reverse heat equation. This shows the instability of the reverse heat equation. (e) shows the result of applying the shock filter [5] and (f) shows the result by using the modified reverse heat equation.



(a) | (b: PSNR=16.46db)

(c: PSNR=15.94db) | (d: PSNR=19.64db)

Fig. 2. Deblurring of a space varying blurred input image where (a) is the input sand texture image that is blurred with a space varying (ramp) Gaussian blur. (c) is the result of applying the shock filter [5] and (d) is the result by using the proposed method.

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