Variational and Quantum Foundations of Circuit Theory

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8th August 2025

Abstract

This thesis develops a unified variational and quantum-scattering framework for circuit theory, spanning classical resistive networks and quantum coherent transport. We derive Ohm's law and Kirchhoff's circuit laws from a power-minimisation variational principle, formulating resistive networks as convex optimisation problems. The existence and uniqueness of solutions are established via graph Laplacians and dual formulations. In the quantum regime, we derive coherent transport using scattering theory and the Landauer–Büttiker formalism. We also explain time-reversal symmetry and Onsager reciprocity in mesoscopic conductors and draw analogies between classical network theory and quantum transport.

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1 Introduction

Circuit theory traditionally deals with the relationship between voltages, currents and resistances in electrical networks. Kirchhoff's laws and Ohm's law provide the fundamental equations for static circuits. In this work, we revisit these classical results from a variational perspective: the steady-state current distribution in a resistive network minimises the total power dissipation. This minimisation yields Kirchhoff's current law as an Euler–Lagrange condition. We also formulate the problem as a convex optimisation, yielding a dual formulation. In parallel, we develop a scattering theory approach to quantum coherent transport in mesoscopic systems. We derive the Landauer–Büttiker formula for conductance from wavefunction continuity and current conservation, and discuss implications of time-reversal symmetry (Onsager relations). Finally, we highlight analogies between classical graph-theoretic Laplacians and quantum conductance matrices. **It is important to note that symbolic derivations are assisted by computational tools; all physical reasoning and interpretation have been provided by the author.**

2 Classical Resistive Networks

2.1 Ohm's Law and Power Dissipation

We consider a network of linear resistors. Ohm's law states that the voltage drop V_{ij} across a resistor of resistance R_{ij} between nodes i and j is proportional to the current I_{ij} flowing from i to j:

$$V_{ij} = R_{ij} I_{ij} \,,$$

or equivalently $I_{ij} = G_{ij}(V_i - V_j)$ where $G_{ij} = 1/R_{ij}$ is the conductance. The instantaneous power dissipated in that resistor is $P_{ij} = I_{ij} V_{ij} = I_{ij}^2 R_{ij} = (V_i - V_j)^2 G_{ij}$. Summing over all resistors yields the total Joule dissipation

$$P = \sum_{(i,j)\in E} I_{ij} V_{ij} = \sum_{(i,j)\in E} I_{ij}^2 R_{ij} = \sum_{(i,j)\in E} (V_i - V_j)^2 G_{ij},$$

where E is the set of edges (branches) of the network. This power P serves as an energy functional to be minimised in the variational formulation.

2.2 Kirchhoff's Laws

Kirchhoff's current law (KCL) states that at each node the sum of currents flowing out equals any external injection:

$$\sum_{j:(i,j)\in E} I_{ij} = I_i^{\text{ext}} \,.$$

Kirchhoff's voltage law (KVL) states that the directed sum of voltage drops around any closed loop is zero (energy conservation). In algebraic form,

$$\sum_{(i,j)\in \text{loop}} V_{ij} = 0 \,.$$

These laws embody charge and energy conservation in the network. Tellegen's theorem provides a general energy-balance statement: in any network obeying Kirchhoff's laws the sum of instantaneous powers is zero. That is, for the instantaneous voltages V_{ij} and currents I_{ij} in each branch,

$$\sum_{(i,j)\in E} V_{ij} I_{ij} = 0 \,.$$

This theorem (Tellegen, 1952) implies that the imposition of KCL and KVL automatically enforces energy conservation, and vice versa.

2.3 Variational Principle and Thomson's Principle

Instead of using Kirchhoff's laws directly, we derive them by minimising total power dissipation under fixed boundary conditions. Thomson's principle of minimum energy dissipation (Thomson, 1847) asserts that among all admissible current flows delivering a given net current between terminals, the actual electrical flow minimises the total Joule dissipation:

$$P[I] = \sum_{(i,j)\in E} I_{ij}^2 R_{ij}.$$

Equivalently, one can minimise the potential energy functional

$$E[V] = \frac{1}{2} \sum_{(i,j)\in E} G_{ij}(V_i - V_j)^2$$

over node potentials V_i with specified values on boundary nodes. By taking variations of $E[V]$ with respect to an infinitesimal perturbation of potentials, one obtains at each interior node

$$\frac{\partial E}{\partial V_k} = \sum_{j:(k,j)\in E} G_{kj}(V_k - V_j) \;=\; \sum_j I_{kj},$$

and setting $\delta E = 0$ yields $\sum_j I_{kj} = 0$ (Kirchhoff's current law). Similarly, the condition that V_i be single-valued ensures KVL. In summary, the Euler–Lagrange equations of the variational problem enforce Ohm's law together with KCL and KVL.

Theorem 1 (Thomson's Minimum Dissipation Principle)**.** *Among all flows that carry the prescribed currents between specified terminals, the actual electrical current flow minimises the total power dissipation. Equivalently, the vector of node potentials V that minimises*

$$E[V] = \frac{1}{2} \sum_{(i,j)\in E} G_{ij}(V_i - V_j)^2,$$

subject to fixed boundary potentials, yields the unique physical solution satisfying Ohm's law and Kirchhoff's laws (Thomson, 1847).

Proof. The functional $E[V]$ is convex quadratic in the potentials. Varying V_k at an interior node k with no boundary condition yields

$$\delta E = \sum_{j:(k,j)\in E} G_{kj}(V_k - V_j)\,\delta V_k = \sum_j I_{kj}\,\delta V_k,$$

since $I_{kj} = G_{kj}(V_k - V_j)$ by Ohm's law. Requiring $\delta E = 0$ for arbitrary δV_k forces $\sum_j I_{kj} = 0$, which is KCL. A similar argument (or Kirchhoff's law) enforces consistency of potential around loops (KVL). Strict convexity of E (for a connected network) ensures that the minimiser is unique. Thus the variational condition recovers exactly the circuit equations. (See Tellegen 1952 for a discussion of related energy principles.) □

2.4 Convex Optimisation and Duality

The power minimisation can be viewed as a convex quadratic optimisation. Define the node potentials vector $V \in \mathbb{R}^n$ and let L be the weighted graph Laplacian: $L_{ii} = \sum_j G_{ij}$, $L_{ij} = -G_{ij}$ for $i \neq j$. Then $E[V] = \frac{1}{2} V^T L V$ (up to irrelevant constants). The Euler–Lagrange conditions become $LV = I^{\text{ext}}$ subject to fixed boundary potentials. Since L is positive semidefinite, this linear system has a unique solution up to an additive constant in each connected component (enforced by the boundary constraints). By convex duality (Rockafellar, 1970), one may also formulate a dual problem in terms of edge currents, which yields equivalent equations. The uniqueness and optimality follow from the strict convexity of $E[V]$ restricted to fixed potentials.

2.5 Graph Laplacians and Matrix Tree Theorem

The matrix L is known as the graph Laplacian of the network graph. It satisfies $L\mathbf{1} = 0$ (each row sums to zero) and is positive semidefinite. In a connected network, L has rank $n-1$ with a one-dimensional nullspace spanned by the all-ones vector. Kirchhoff's matrix-tree theorem relates determinants of cofactors of L to spanning-tree counts, yielding formulas for effective resistances. For example, the effective resistance R_{ab} between nodes a and b can be written in terms of cofactors of L. These results mirror the theory of continuous Laplace equations (Courant and Hilbert, 1953).

2.6 Tellegen's Theorem

Tellegen's theorem (Tellegen, 1952) provides a general energy-balance relation: for any arbitrary currents and voltages in a network satisfying Kirchhoff's laws, the sum of instantaneous powers is zero. In algebraic form,

$$\sum_{(i,j)\in E} V_{ij} I_{ij} = 0.$$

This theorem follows from network topology and Kirchhoff's laws, and shows that any two of (energy conservation, KCL, KVL) imply the third. It underpins the variational formulation and guarantees consistency of the power balance.

Figure 1: A simple three-node resistive network. Each edge has a resistor R_{ij} and node A is connected to an external voltage source.

3 Quantum Coherent Transport and Scattering Theory

3.1 Scattering Formalism and S-Matrix

In the quantum regime, electronic transport through a small conductor connected to leads is described by scattering theory. Electrons in each lead are treated as incoming and outgoing waves. Label the leads by $i = 1, \ldots, N$, each held at electrochemical potential μ_i. The scattering region is characterized by an $N \times N$ unitary scattering matrix $S(E)$, relating incoming wave amplitudes to outgoing ones:

$$b_i = \sum_j S_{ij}(E)\, a_j,$$

where $|S_{ij}|^2 = T_{ij}$ is the transmission probability from channel j to i. Unitarity ($S^\dagger S = I$) ensures current conservation across the scatterer.

3.2 Two-Terminal Landauer Formula

Consider the simplest two-terminal conductor (left lead and right lead). At zero temperature and small bias V, the chemical potentials are $\mu_L = eV/2$ and $\mu_R = -eV/2$. The

net current from left to right is given by the difference in flux of occupied states times charge. A standard result is the Landauer formula for conductance:

$$G = \frac{2e^2}{h}\, T,$$

where $T = |S_{21}(E_F)|^2$ is the transmission probability of the scatterer at the Fermi energy E_F. The factor $2e^2/h$ is the conductance quantum (including spin degeneracy) (Datta, 1995). Landauer first recognized this relation between transmission and conductance (Landauer, 1957). Physically, the resistance arises from quantum scattering in the conductor, rather than from dissipative processes inside it.

3.3 Multi-Terminal Landauer–Büttiker Formalism

Büttiker generalized the Landauer approach to an arbitrary multi-terminal conductor (Büttiker, 1986). In linear response, the current in lead i is given by

$$I_i = \frac{2e}{h} \sum_{j=1}^{N} \big(T_{ji}\, \mu_i - T_{ij}\, \mu_j \big),$$

where μ_j are the reservoir potentials. Equivalently, writing the conductance matrix elements G_{ij}, one has

$$I_i = \sum_j G_{ij} V_j, \quad G_{ij} = \frac{2e^2}{h} (\delta_{ij} \sum_k T_{ki} - T_{ij}).$$

Current conservation ($\sum_i I_i = 0$) follows from unitarity of S. The Landauer–Büttiker formula is derived by expressing the current from each lead in terms of incoming and outgoing wave amplitudes, using continuity of the wavefunction and current at the scatterer.

3.4 Time-Reversal Symmetry and Onsager Relations

In the absence of magnetic fields, the microscopic dynamics are time-reversal invariant. This implies a symmetry of the scattering matrix $S_{ij}(B) = S_{ji}(-B)$. Hence at $B = 0$, $T_{ij} = T_{ji}$. This leads to Onsager reciprocity for the conductance matrix: $G_{ij}(B) = G_{ji}(-B)$ (Onsager, 1931). In particular, for zero magnetic field $G_{ij} = G_{ji}$. When a magnetic field breaks time-reversal symmetry, these reciprocal relations no longer hold, reflecting non-reciprocal transport.

3.5 Analogy to Graph Laplacians

There is a formal analogy between the multi-terminal conductance matrix and a graph Laplacian. Indeed, G_{ij} satisfies $\sum_j G_{ij} = 0$, just like the Laplacian L of a network. One can interpret the transmissions T_{ij} as weights on a directed graph of leads. In certain regimes (e.g. with dephasing), the quantum network behaves like a classical resistor network (Datta, 1995). Thus both classical and quantum transport problems reduce to solving linear equations of Kirchhoff type, with conductance matrices encoding connectivity. The continuous Laplace operator in a conductor has a discrete analogue in the graph Laplacian of the scatterer network.

Figure 2: Schematic of a two-terminal quantum conductor. Electrons incident from the left (L) and right (R) leads scatter through a central region (quantum dot), with transmission probability T. The Landauer formula relates T to the conductance between the leads.

4 Conclusion

This thesis has demonstrated that both classical and quantum conductance problems can be understood via variational and scattering frameworks. In classical resistor networks, the minimisation of power dissipation (Thomson's principle) yields Kirchhoff's circuit laws. This is a convex optimisation problem whose solution is unique and governed by the graph Laplacian. In the quantum regime, the Landauer–Büttiker scattering formalism gives currents in terms of transmission probabilities of the S-matrix. Current conservation and unitarity of S ensure physically sensible results. Onsager reciprocity (time-reversal symmetry) in linear transport emerges naturally in the scattering picture. Both frameworks exhibit Laplacian-like structure: in circuits $I = LV$, while in quantum transport $I = GV$ with G a conductance matrix having row-sum zero.

- **Classical networks:** Thomson's principle ensures the actual flow minimises energy, leading to KCL/KVL and to $LV = I$. The problem is quadratic and convex, with uniqueness ensured by positive definiteness of L.
- **Quantum conductors:** The Landauer–Büttiker formula expresses current in terms of the scattering matrix of the device. Quantum phases and resonances control transmission T_{ij}, and unitary symmetry yields current conservation.
- **Reciprocity:** Time-reversal symmetry implies Onsager relations $G_{ij}(B) = G_{ji}(-B)$, a generalisation of symmetric conductance in classical circuits. Breaking B-symmetry breaks reciprocity (non-reciprocal transport).
- **Analogy:** In both pictures currents satisfy linear Kirchhoff-like equations. The conductance matrix in the Landauer formula plays a role analogous to the graph Laplacian. This analogy suggests that methods from graph theory and optimization may extend between classical circuits and mesoscopic transport.

Possible extensions include incorporating nonlinearity (e.g. diodes, active elements), electron-electron interactions, and finite-temperature effects. The analogy also suggests studying quantum networks via graph-theoretic tools (quantum graphs), and exploring thermoelectric or spin transport analogues. Finally, while the present theory assumes phase-coherent, elastic scattering, real devices may require inclusion of dephasing, inelastic scattering and many-body effects, which form the frontier of modern mesoscopic physics.

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