(10)* pumping lemma

Closed Regular sets and pumping lemma

The regular sets are closed under set concatenation, set union, set intersection, set product and set closure. This means regular sets form boolean algebra.

The regular sets also satisfy another property called as pumping lemma.

This lemma says that for any sufficiently long string accepted by FSM, we can find a substring near the beginning of that string, that may be repeated or pumped as many times as you like and still the resulting string is accepted by FSM.

for e.g. $\xrightarrow[101011]{} \boxed{FSM}$

= string accepted
= language word accepted by FSM
= regular language
↓
set of words = regular set.

So we can use pumping lemma to ① prove that the certain set is not a regular set
② to know whether a lang. is finite or infinite

① If a lang. is regular it is accepted by a particular DFA. $M = (Q, \Sigma, \delta, q_0, F)$ with a particular number (finite) of states say $\boxed{n}$

$\boxed{q_0}$ = initial state.

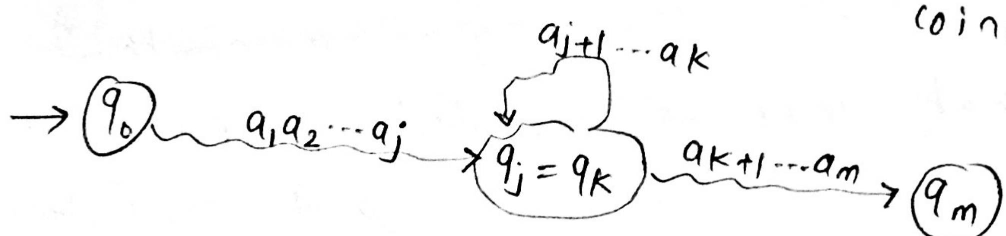
consider the input string of ~~m or~~ n or more no. of symbols.

ie $\boxed{a_1 a_2 a_3 \cdots a_m, \quad m \geq n,}$ for $i = 1, 2 \cdots m$.

let $\boxed{\delta(q_0, a_1 a_2 \cdots a_i) = q_i}$

② Then it is not possible for each of $n+1$ states $q_0 q_1 \cdots q_m$ to be distinct as there are only n different states. [ie at least 2 states coincide]

③



④ Thus there are two integers j and k such that $\boxed{0 \leq j < k \leq n \quad \text{and} \quad q_j = q_k}$

The path labelled $a_1 a_2 \cdots a_m$ in the transition diagram of machine is as shown in the figure.

⑤ since $j < k$ the string $\underbrace{a_{j+1} \cdots a_k}_{\text{at least}=1}$ is of the length at least 1.

since $k \leq n$ its length is not more than n.

for e.g.

n : No. of states = 3 [ie min no. of states from $\to q_0$ — (q_m)] to reach final

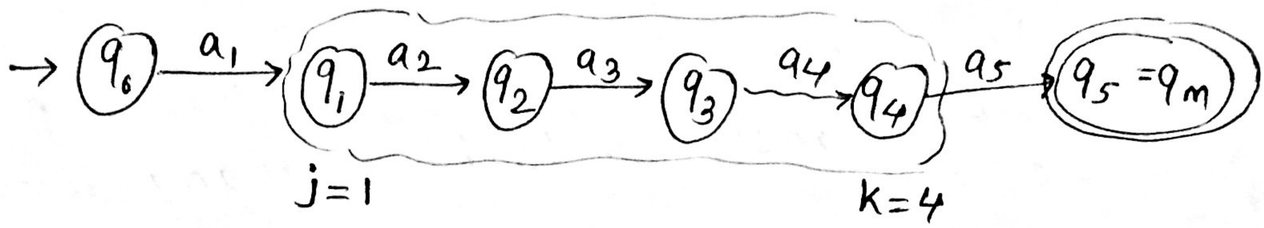
m : No. of symbols = 5

ie $a_1 a_2 a_3 a_4 a_5$

if $\delta(q_0, a_1 a_2 a_3) = q_3$

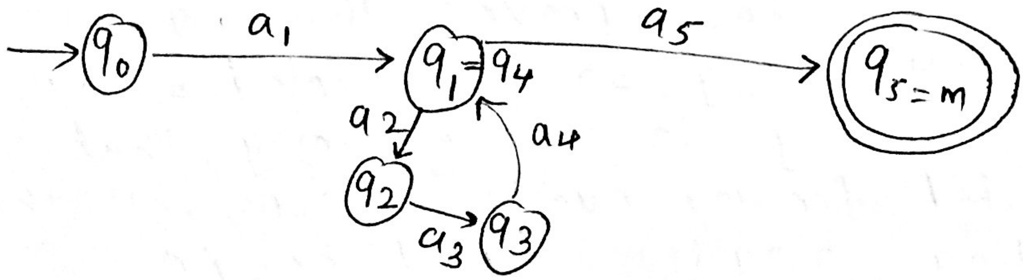
we assume $\delta(q_0, a_1 a_2 a_3 \cdots a_m) = (q_m)$ final state.

FA would be



but $n = 3$.

so we can say that [q_1 & q_4 coincide]



we got two numbers j & k such that

$j = 1$ & $k = 4$, $j \leq k$, $a_{j+1} \cdots a_k$

$a_2 \cdots a_4$ sub-string $\to$

$\to$ that can be repeated as many times as we like and still the resulting string is accepted by FSM.

we conclude

① if q_m is in F, $a_1 a_2 \cdots a_m$ is L(m)

then $a_1 a_2 \cdots a_j a_{k+1} \cdots a_m$ is also in L(m)

$$\delta(q_0, a_1 a_2 \cdots a_j a_{k+1} \cdots a_m) =$$

$$= \delta(\delta(q_0, a_1 a_2 \cdots a_j) a_{k+1} \cdots a_m)$$

$$= \delta(q_j, a_{k+1} \cdots a_m)$$

$$= \delta(q_k, a_{k+1} \cdots a_m)$$

$$= q_m$$

② similarly we can go around the loop more than once. in fact as many times as we like.

Thus $a_1 a_2 \cdots a_j (a_{j+1} \cdots a_k)^i a_{k+1} \cdots a_m$ is in L(m) for $i \geq 0$

Thus we have proved that given any sufficiently long string accepted by FA, there is a substring in the beginning, that can be repeated for any number times, still the resulting string is accepted by FA.

③ so if L = regular set

$\boxed{n}$ = pumping lemma constant.

z = any word in L

then $\boxed{|z| \geq n}$ length of z is $\geq$ pumping lemma constant

we can view z as = uvw $\boxed{uv \leq n}$

for all $i \geq 0$ uv^iw is in L. $\boxed{|v| \geq 1}$

1) Prove that $L = \{a^n b^n$ for $n = 0,1,2 \ldots\}$ is not regular.

1) Assume $L = \{a^n b^n \mid n = 0,1,2 \ldots\}$ is regular

2) It would follow pumping lemma.

3) word of the lang $= a^n b^n$

$= u v^i w$, $|uv| \leq n$ & $|v| \geq 1$

4) case 1 → $\dfrac{a^n b^n}{u v^i w}$

$a^{n-1}(a) b^n$ as per pumping lemma

if $a^{n-1}(a)^3 b^n$ (a) repeated for 3 times

resulting string $= a^{n-1} aaa b^n$

$= a^{n+2} b^n$ is not in L.

if $a^{n-1}(a)^0 b^n$ (a) repeated for zero times

resulting string $= a^{n-1} b^n$ is not in L.

so L does not follow pumping lemma

ie L is not a regular lang.

5) case 2 → word $\dfrac{a^n b^n}{u v^i w}$

$a^n (b) b^{n-1}$

if $= a^n (b)^3 b^{n-1}$ (b) repeated 3 tim.

$= a^n bbb b^{n-1} = a^n b^{n+2}$ is not in L

$a^n (b) b^{n-1}$ (b) repeated zero tin

$= a^n b^{n-1}$ not is in L.

6) case 3 → $a^n b^n$

$a^{n-1} (ab) b^{n-1}$ repeated 2 times

$a^{n-1} abab\, b^{n-1}$ not ture of $a^n b^n$

not in L.

Thus L does not follow pumping lemma

so L is not regular.

(2) $L = \{a^n b a^n \mid n = 0, 1, 2\}$ is not Regular

→ Assume that L is regular.

→ L follows pumping lemma.

→ word of the lang = $\underline{\underline{a^n b a^n}}$

of type $u v^i w$

case 1 : $w = a^n b a^n$

$u\,(v)\,w$ $\quad v = (b)$ repeated for zero / two times

$w = a^n a^n$ (zero times)

$= a^n bb\, a^n$ } not in L.

case 2 : $w = a^n b a^n$

$= \underline{a^{n-1}\ a\ ba^n}$

$u\ (v)^i\ w$

~~if (i = 0) then $a^{n-1} ba^n$ (a) zero times → word is not in L.~~

~~if $a^{n-1} (b)^2 a^n$ then $w = a^{n-1} bb\, a^n$ is not in L.~~

if $i = 0$ then $w = a^{n-1} ba^n$ is not in L

if $i = 2$ then $w = a^{n-1}\ aa\ ba^n$

$= a^{n+1} ba^n$ is not in L.

case 3 : $w = a^n \underline{\underline{ba}}\, a^{n-1}$

$u\ (v)^i\ w$

if $i = 0$ then $w = a^n a^{n-1}$ is not in L.

if $i = 3$ then $w = a^n bababa\, a^{n-1}$ is not in L.

Thus L does not follow pumping lemma.
so it is not regular lang.

3) $L = \{ 0^{i^2} \mid i$ is an integer, $i \geq 1 \}$

L is lang. consists of all strings of 0's whose length is a perfect square.

Prove L is not regular lang.

→ Assume L is a regular lang. then

→ It would follow pumping lemma.

→ n = pumping lemma constant

→ so we can say $z = 0^{n^2} = uvw$

$$\boxed{\begin{array}{l} z = 0^{n^2} = uvw \quad \text{ie } |uvw| = n^2 \\ \text{where } |uv| \leq n \ \& \ |v| \geq 1 \\ 1 \leq |v| \leq n \text{ and } uv^iw \text{ is in } L \text{ for all } i \end{array}}$$

$\therefore$ $1 + n^2 \leq |v + uvw| \leq n + n^2$

$\therefore$ $n^2 < |v + uvw| \leq n^2 + n < (n+1)^2$

$\therefore$ $n^2 < |uv^2w| < (n+1)^2$

$\therefore$ length of uv^2w lies in betn n^2 and $(n+1)^2$ ie it is not a perfect square

ie uv^2w is not in L.

ie L is not regular

④ lang which contains set of strings of balanced paranthesis is not regular

((()))

$L = \{ (^n)^n \mid n \geq 1 \}$

Assume L is regular. so it should follow pumping lemma

$w = (^n)^n$

$= u v^i w$

case 1 : $w = \underline{(^{n-1}}_u \; \underline{(}_v \; \underline{)^n}_w$

then w is not in L.

if v zero time
$w = (^{n-1})^n$
if v for 2 times
$w = (^{n-1} (( )^n$

case 2 : $(^n \;) \;)^{n-1}$
u v w

if v repeated zero time
$w = (^n)^{n-1}$ is not in L
$w = (^n)^2)^{n-1}$ if v 2 times
not in L.

case 3 : $(^{n-1} \underline{()})^{n-1}$
u (v) w

if (v) repeated zero times
$\therefore w = uw = (^{n-1})^{n-1}$ ✓
if (v) repeated 2 times
$\therefore w = (^{n-1} ()())^{n-1}$ ✓

from case 1 and 2
resulting word is not in L.
i.e L does not follow pumping lemma
so L is not regular