Z- Transform

Note: (1) General term of the sequence $\{f(k)\}$ is specified as a function of k, which is an ordered list of real or complex numbers.

(2) (e.g) (i) $f(k) = \dfrac{1}{3^k} = \left\{ \cdots\cdots, \dfrac{1}{3^{-3}}, \dfrac{1}{3^{-2}}, \dfrac{1}{3^{-1}}, \underset{\uparrow}{1}, \dfrac{1}{3}, \dfrac{1}{3^2}, \dfrac{1}{3^3}, \cdots \right\}$

$(k = -\infty \text{ to } +\infty)$

(3) $\uparrow$ denotes the term in zero position (ie) $k=0$

(4) k is an index of position of a term in the sequence $\{f(k)\}$

(5) (e.g) $f(k) = \begin{cases} 0 & ; k<0 \\ \cos(k/2); & k \geq 0 \end{cases}$

$\Rightarrow f(k) = \left\{ \cdots\cdots 0, 0, 0, \underset{\uparrow}{1}, \cos(1/2); \cos(1), \cos(3/2), \cdots \right\}$

Z- Transform : Def^n:

Z- transform of the sequence $\{f(k)\}$ is defined as

$$\mathbf{Z}\left[\{f(k)\}\right] = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} \quad [z \text{ is a complex no}]$$

Properties of Z-transform.

Prop : (I) Linearity Property : $Z[a\{f(k)\} \pm b\{g(k)\}] = a\,Z\{f(k)\} \pm b\,Z\{g(k)\}$

(a, b are constants)

Problems. Find the Z-transform of the following sequences.

(1) $f(k) = \begin{cases} 5^k ; & k < 0 \\ 3^k ; & k \geq 0 \end{cases}$

Solution. $Z[\{f(k)\}] = \sum_{k=-\infty}^{-1} 5^k z^{-k} + \sum_{k=0}^{\infty} 3^k z^{-k}$

$= [\cdots + 5^{-3}z^{+3} + 5^{-2}z^{2} + 5^{-1}z]$

$+ [1 + 3z^{-1} + 3^2 z^{-2} + 3^3 z^{-3} + \cdots]$

$= \left[\frac{z}{5} + \frac{z^2}{5^2} + \frac{z^3}{5^3} + \cdots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots\right]$

$= \left(\frac{z}{5}\right)\left[1 + \frac{z}{5} + \frac{z^2}{5^2} + \cdots\right] + [\quad]$

$= \left(\frac{z}{5}\right)\left(\frac{1}{1-\frac{z}{5}}\right) + \left(\frac{1}{1-\frac{3}{z}}\right)$ $\left[\because S_\infty = \frac{a}{1-r},\ |r|<1\right]$

$= \left(\frac{z}{5}\right)\left(\frac{5}{5-z}\right) + \frac{z}{z-3}$ $\longrightarrow$ for $\left|\frac{z}{5}\right| < 1$ & $\left|\frac{3}{z}\right| < 1$

$|z| < 5 ; 3 < |z|$

$\Rightarrow F(z) = \frac{z}{5-z} + \frac{z}{z-3}$ $\longrightarrow$ i.e., $3 < |z| < 5$ series is convergent

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(2) $\{f(k)\} = \{a^{|k|}\}$; Deduce for $\{f(k)\} = \left\{\left(\frac{1}{2}\right)^{|k|}\right\}$ Note: $|k| = -k,\ k<0$; $= +k,\ k \geq 0$

Solution, $Z[\{f(k)\}] = \sum_{k=-\infty}^{\infty} f(k) z^{-k} =$

$= \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$

$= [\cdots + a^3 z^3 + a^2 z^2 + az] \;\; (az)\left(\frac{1}{1-az}\right)$

$+ \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots\right] \;\; + \frac{z}{z-a}$

(Convergent for $|az| < 1$ & $\left|\frac{a}{z}\right| < 1$)

$\Rightarrow F(z) = \frac{az}{1-az} + \frac{z}{z-a} = \frac{z - a^2 z}{(1-az)(z-a)}$

Deduction: $a = \frac{1}{2} \Rightarrow F(z) = \frac{3z}{\cdots}$ P.T.O.

③ S.T. Z transform of $\{\sin(\alpha k)\}$; $\boxed{k \geq 0} = \dfrac{z \sin(\alpha)}{z^2 - 2z\cos\alpha + 1}$ (03)

Solution: $\sin(\alpha k) = \dfrac{e^{i\alpha k} - e^{-i\alpha k}}{2i}$

$$Z[\{\sin(\alpha k)\}] = \sum_{k=-\infty}^{\infty} \sin(\alpha k) z^{-k} = \frac{1}{2i}\left[\sum_{k=0}^{\infty} e^{i\alpha k} z^{-k} - \sum_{k=0}^{\infty} e^{-i\alpha k} z^{-k}\right]$$

$$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} (e^{i\alpha} z^{-1})^k - \sum_{k=0}^{\infty} (e^{-i\alpha} z^{-1})^k\right]$$

$$= \frac{1}{2i}\left[\{1 + e^{i\alpha} z^{-1} + (e^{i\alpha} z^{-1})^2 + \cdots\} - \{1 + e^{-i\alpha} z^{-1} + (e^{-i\alpha} z^{-1})^2 + \cdots\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{1 - e^{i\alpha} z^{-1}} - \frac{1}{1 - e^{-i\alpha} z^{-1}}\right]$$

convergent for $\left|\frac{e^{i\alpha}}{z}\right| < 1$

i.e. $|e^{i\alpha}| < |z|$

i.e. $1 < |z|$ or $|z| > 1$

& $\left|\frac{e^{-i\alpha}}{z}\right| < 1$

i.e. $1 < |z|$

i.e. $|z| > 1$

$$= \frac{1}{2i}\left[\frac{z}{z - e^{i\alpha}} - \frac{z}{z - e^{-i\alpha}}\right]$$

$$= \frac{1}{2i}\left[\frac{(z^2 - z e^{-i\alpha}) - (z^2 - e^{i\alpha})}{(z - e^{i\alpha})(z - e^{-i\alpha})}\right]$$

$$= \frac{1}{2i}\left[\frac{z(z - e^{-i\alpha} - z + e^{i\alpha})}{z^2 - z(e^{i\alpha} + e^{-i\alpha}) + 1}\right]$$

$$= \frac{1}{2i}\left[\frac{z(e^{i\alpha} - e^{-i\alpha})}{z^2 - z(2\cos\alpha) + 1}\right]$$

$$\Rightarrow Z[\{\sin(\alpha k)\}] = \frac{1}{2i}\left[\frac{z(2i\sin\alpha)}{z^2 - z(2\cos\alpha) + 1}\right] = \frac{z \sin(\alpha)}{z^2 - 2z\cos\alpha + 1}$$ //

④ S.T $Z[\{\cos(\alpha k)\}] = \dfrac{z^2 - z\cos(\alpha)}{z^2 - 2z\cos\alpha + 1}$ $(k \geq 0)$ | Hint: $\cos(\alpha k) = \dfrac{e^{i\alpha k} + e^{-i\alpha k}}{2}$

⑤ S.T $Z[\{\cosh(\alpha k)\}] = \dfrac{z(z - \cosh(\alpha))}{z^2 - 2z\cosh(\alpha) + 1}$ $(k \geq 0)$ | Hint: $\cosh(\alpha k) = \dfrac{e^{\alpha k} + e^{-\alpha k}}{2}$

⑥ S.T $Z[\{\sin(3k+5)\}] = \dfrac{z^2 \sin(5) - z\sin(2)}{z^2 - 2z\cos 3 + 1}$; $|z| > 1$ $(k \geq 0)$

Hint: Method ① $\underbrace{\sin(3k+5)}_{x} = \dfrac{e^{ix} - e^{-ix}}{2i}$

Aliter: $\sin(3k+5) = \sin(3k)\cos 5 + \cos(3k)\sin(5)$

(PTO)

(7) If $\{f(k)\} = \left\{\frac{a^k}{k!}\right\}$; $k \geq 0$; find $Z[\{f(k)\}]$

Solution :-

$$Z[\{f(k)\}] = \sum_{k=0}^{\infty}\left(\frac{a^k}{k!}\right)z^{-k} = 1 + \frac{az^{-1}}{1!} + \frac{(az^{-1})^2}{2!} + \frac{(az^{-1})^3}{3} + \cdots$$

$$\left[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots\right]$$

$$\Rightarrow Z[\{f(k)\}] = e^{a/z}$$

(8) If $\{f(k)\} = \{{}^nc_k\}$; $0 \leq k \leq n$

Solution :-

$$Z[\{f(k)\}] = \sum_{k=0}^{n} {}^nc_k\, z^{-k} = 1 + {}^nc_1 z^{-1} + {}^nc_2 z^{-2} + {}^nc_3 z^{-3} + \cdots + {}^nc_{n-1} z^{-(n-1)} + {}^nc_n z^{-n}$$

$$= (1 + z^{-1})^n = \left(1 + \frac{1}{z}\right)^n = \left(\frac{z+1}{z}\right)^n$$

[Binomial expn: $(1+x)^n = 1 + {}^nc_1 x + {}^nc_2 x^2 + \cdots + {}^nc_{n-1} x^{n-1} + {}^nc_n x^n$]

(9) $\{f(k)\} = \{{}^{k+n}c_n\}$; $k \geq 0$

Solution:

$$Z[\{f(k)\}] = \sum_{k=0}^{\infty}\left({}^{k+n}c_n\right)z^{-k} = {}^nc_n(1) + \left({}^{n+1}c_n\right)z^{-1} + \left({}^{n+2}c_n\right)z^{-2} + \left({}^{n+3}c_n\right)z^{-3} + \cdots$$

$$\left[{}^nc_r = \frac{n!}{r!(n-r)!}\right]$$

$$= 1 + \frac{(n+1)!}{(n!)(1!)}z^{-1} + \frac{(n+2)!}{(n!)(2!)}z^{-2} + \frac{(n+3)!}{(n!)(3!)}z^{-3} + \cdots$$

$$= 1 + (n+1)z^{-1} + \frac{(n+1)(n+2)}{2!}z^{-2} + \frac{(n+1)(n+2)(n+3)}{3!} + \cdots$$

$$\left[1 + \frac{mx^1}{1!} + \frac{m(m+1)}{2!}x^2 + \frac{m(m+1)(m+2)}{3!}x^3 + \cdots\right.$$

$$\left.= (1-x)^{-m} \quad ; m = n+1 \ ; x = z^{-1}\right]$$

$$\Rightarrow Z[\{f(k)\}] = (1 - z^{-1})^{-(n+1)} \;//$$

(10) $\{f(k)\} : \left\{\frac{1}{k}\right\}$; $k > 0 \Rightarrow Z\left[\left\{\frac{1}{k}\right\}\right] = \sum_{k=1}^{\infty}\left(\frac{1}{k}\right)z^{-k}$

$$= (1)z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \frac{z^{-4}}{4} + \cdots = \frac{z^{-1}}{1} + \frac{z^{-2}}{2} + \cdots + \frac{z^{-k}}{k}$$

$$= -\log[1 - z^{-1}] \quad \left[\text{Use: } \log\frac{1}{(1-x)} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} ; x = z^{-1}\right]$$

Prop: (II) **Change of scale:**

If $Z[\{f(k)\}] = F(z)$, then $\| Z[\{a^k f(k)\}] = F(z)\Big|_{z\to z/a} = F(z/a) \Big]$ ※

Problems: ① Find $Z[\{c^k \sin(\alpha k)\}]$; $k \geq 0$

Solution: $Z[\{\underbrace{\sin(\alpha k)}_{f(k)}\}] = \dfrac{z\sin(\alpha)}{z^2 - 2z\cos(\alpha)+1} = F(z)$ [Pb (3) under prop ①]

$$\Rightarrow Z[\{c^k \sin(\alpha k)\}] = F(z)\Big|_{z\to z/c} = \frac{\frac{z}{c}\sin(\alpha)}{\frac{z^2}{c^2} - 2\frac{z}{c}\cos(\alpha)+1}$$
$(a=c)$

$$= \frac{zc\sin(\alpha)}{z^2 - 2zc\cos(\alpha) + c^2} \,//$$

Prop: (III) **Shifting Property:-**

If $Z[\{f(k)\}] = F(z)$, then $\| Z[\{f(k\pm n)\}] = z^{\pm n}F(z) \|$ ※

eg: for $k\geq 1$, $Z[\{\frac{1}{k+1}\}] = z^1 Z[\{\frac{1}{k}\}] = z\left[-\log(1-\frac{1}{z})\right]$; $|z|>1$
$(n=+1)$ ↳ Pb (10) under prop ①

$$= -z\log\left(1-\frac{1}{z}\right) \; ; \; |z|>1$$

Note: ① Special case: For casual sequence (i.e. $\{f(k)\}$ defined for only $k\geq 0$) & $f(k)=0$ for $k<0$
[unilateral or one-sided]

(eg) $f(k) = \begin{cases} 0 & , k<0 \\ \frac{1}{2^k} & ; k\geq 0 \end{cases}$

Result: ① $Z[\{f(k+1)\}] = zF(z) - zf(0)$; $f(0) = f(k)\big|_{k=0}$

$Z[\{f(k+2)\}] = z^2F(z) - z^2 f(0) - zf(1)$

⋮

$Z[\{f(k+n)\}] = z^n F(z) - z^n f(0) - z^{n-1}f(1) - z^{n-2}f(2) - \cdots - z^{n-(n-1)}f(n-1)$

$\Rightarrow Z[\{f(k+n)\}] = z^n F(z) - z^n f(0) - z^{n-1}f(1) - \cdots - zf(n-1)$

② $Z[\{f(k-n)\}] = z^{-n}F(z)$ [∵ $f(-1), f(-2) \ldots = 0$]

Note ② Unit Impulse fun^n: $\delta(k) = \begin{cases} 1 ; & k=0 \\ 0 ; & k\neq 0\end{cases}$ | $Z[\{\delta(k)\}] = 1$

③ Discrete Unit step fun^n: $U(k) = \begin{cases} 0 ; & k<0 \\ 1 ; & k\geq 0\end{cases}$ | $Z[\{U(k)\}] = \dfrac{z}{z-1}$ } Use Def^n of $Z[\{f(k)\}]$

(P.T.O)

Prop (V): Convolution:

Let $\{f(k)\}$ and $\{g(k)\}$ be two sequences.
Convolution of $\{f(k)\}$ and $\{g(k)\}$ is denoted as

$$\{f(k)\} * \{g(k)\} = \{h(k)\} = \sum_{n=-\infty}^{\infty} f(n)\, g(k-n)$$

$$= \sum_{n=-\infty}^{\infty} g(n)\, f(k-n) = \{g(k)\} * \{f(k)\}$$

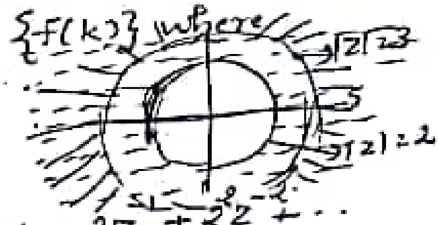
$Z[\{h(k)\}] = F(z)\, G(z)$ where

$F(z) = Z[\{f(k)\}]$ & $G(z) = Z[\{g(k)\}]$

i.e. $H(z) = F(z)\, G(z)$

Note: Region of convergence [ROC] of $H(z)$ is the common region of convergence of $F(z)$ & $G(z)$.

Problem: Find the Z-transform of $\{f(k)\}$ where



$$\{f(k)\} = \sum_{k=0}^{\infty} 2^k \sum_{k=0}^{\infty} 3^k$$

Solution: $Z[\{2^k\}] = \sum_{k=0}^{\infty} 2^k z^{-k} = 1 + 2z + 2^2 z + \cdots$

$$= \frac{1}{1-2z^{-1}} \; ; \; |2z^{-1}| < 1$$

i.e. $\left|\frac{2}{z}\right| < 1$

i.e. $|z| > 2$

$|||^{ly}$ $Z[\{3^k\}] = \frac{1}{1-3z^{-1}}$; $|z| > 3$

Using Prop (IV), $Z[\{f(k)\}] = \left(\frac{1}{1-2z^{-1}}\right)\left(\frac{1}{1-3z^{-1}}\right)$; $|z| > 3$

Result: Partial Sum: If $Z[\{f(k)\}] = F(z)$, then

$$Z\left[\left\{\sum_{n=-\infty}^{k} f(n)\right\}\right] = \frac{F(z)}{1-z^{-1}} \; ; \; \left|\frac{1}{z}\right| < 1 \text{ or } |z| > 1.$$

Prop (V): Multiplication by k^n; $(n \geq 0)$

If $Z[\{f(k)\}] = F(z)$, then

$$Z[\{k^n f(k)\}] = \left(-z\frac{d}{dz}\right)^n F(z) \; ; (n \geq 0)$$

Problem (1) Find $Z[\{k^2\}]$:

Solution: $Z[\{k^2\}] = Z[\{k^2(1)\}] = Z[\{k^2 f(k)\}]$

where $f(k) = 1$

$$\Rightarrow Z[\{f(k)\}] = Z[\{1\}] = \sum_{k=-\infty}^{\infty} (1) z^{-k}$$

$$= 1 + z^{-1} + z^{-2} + \cdots$$

$$\Rightarrow F(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1} \; ; \; |z^{-1}| < 1$$

or $|z| > 1$

$$\Rightarrow Z[\{k^2 f(k)\}]_{(n=2)} = \left(-z\frac{d}{dz}\right)^2 F(z)$$

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right)$$

$$= \left(-z\frac{d}{dz}\right)\left[-z\frac{d}{dz}\left\{\frac{z}{z-1}\right\}\right]$$

$$= \left(-z\frac{d}{dz}\right)\left[(-z)\left\{\frac{-1}{(z-1)^2}\right\}\right]$$

$$= (-z)\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = (-z)\left[\frac{(z-1)^2(1) - z(2)(z-1)}{(z-1)^4}\right]$$

$$= (-z)\left[\frac{(z-1)\{z-1-2z\}}{(z-1)^4}\right]$$

$$\Rightarrow Z[\{k^2\}] = \frac{z(z+1)}{(z-1)^3} \;//$$

(2) Find $Z[\{k\}]$:-

$f(k) = 1$, $n = 1 \Rightarrow Z[\{k\}] = \left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}$

Prop (VI): $Z\left[\left\{\frac{f(k)}{k}\right\}\right] = \int_z^{\infty} z^{-1} F(z)\, dz.$