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Complex Variables

Analytic Function

(I) ① If $f(z) = u + iv$ is analytic with constant modulus, show that $f(z)$ is constant.

② If $f(z)$ is analytic with constant argument, show that $f(z)$ is constant.

③ If $u+iv$ and $u-iv$ are analytic, show that both are constants.

④ If $u+iv$ and $v+iu$ are analytic, show that both are constants.

(II) If $f(z)$ is analytic, then show that

① $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^n = n^2 |f(z)|^{n-2} |f'(z)|^2$

Hence deduce the result for $n = 2$

② $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^n) = n(n-1)\, u^{n-2} |f'(z)|^2$

Hence deduce the result for $n = 3$

③ $\nabla^2 [\log |f'(z)|] = 0$ $\left[\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$

④ $\left[\frac{\partial}{\partial x} |f(z)|\right]^2 + \left[\frac{\partial}{\partial y} |f(z)|\right]^2 = |f'(z)|^2$

(III) ① Find the values of a, b, c, d so that the funⁿ.

$f(z) = (x^2 + axy + by^2) + i\,(cx^2 + dxy + y^2)$ is analytic

② Find 'p' such that

$f(z) = \frac{1}{2} \log (x^2+y^2) + i \tan^{-1}\left(\frac{px}{y}\right)$ is analytic

③ Find a, b so that

$f(z) = \cos x\,[\cosh(y) + a \sinh(y)] + i \sin x\,[\cosh(y) + b \sinh(y)]$ is analytic

④ Find 'b' such that $u = e^{bx} \cos(5y)$ is harmonic

Ans. (1) $a = 2 = d$; $b = -1 = c$ (2) $p = -1$ (3) $a = b = -1$, (4) $b = \pm 5$ P.T.O.

⑤ Find a, b, c, d, e if

$$f(z) = (ax^3 + bxy^2 + 3x^2 + cy^2 + x) + i(dx^2y - 2y^3 + exy + y)$$

is analytic. Ans: $\{2, -6, -3, 6, 6\}$

⑥ Find a, b, c, d, e if

$$f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$$

is analytic. Ans: $\{1, -6, 1, 2, 4\}$

⑦ Find a & b such that

$$f(z) = (-r^2 \sin(a\theta) + r\sin\theta) + i(r^2\cos(2\theta) - r\cos(b\theta) + 2)$$

is analytic. $\{a = 2, b = 1\}$

IV ⓐ S.T. the following funⁿˢ $u(x,y)$ / $v(x,y)$ are harmonic [$u(x,y)$ can be a real part of an ana. funⁿ]

ⓑ Construct the corresponding analytic funⁿ

ⓒ Determine the harmonic conjugate & hence the analytic funⁿ. $f(z)$

ⓓ Obtain the orthogonal trajectories of the families of curves $u(x,y) = \text{const}^t$.

	Ans:-
① $u = x^3y - xy^3$	① $v = \frac{3x^2y^2}{2} - \frac{y^4}{4} - \frac{x^4}{4} + c'$ $f(z) = -i\frac{z^4}{4} + c$
② $u = x^2 - y^2 - y$	② $v = 2xy + x + c'$ $f(z) = z^2 + iz + c$
③ $v = x^2 - y^2 + \frac{1}{x^2+y^2}$	③ $u = -2xy + \frac{y}{x^2+y^2} + c'$ $f(z) = i\left[z^2 + \frac{1}{z}\right] + c$
④ $u = 3xy^2 - x^3$	④ $v = y^3 - 3x^2y + c'$ $f(z) = -z^3 + c$
⑤ $u = e^{-x}(x\sin y - y\cos y)$	⑤ $v = e^{-x}(y\sin y + x\cos y) + c'$ $f(z) = ize^{-z} + c$
⑥ $u = \frac{x^2 - y^2}{(x^2+y^2)^2}$	⑥ $v = \frac{-2xy}{(x^2+y^2)^2} + c'$ $f(z) = \frac{1}{z^2} + c$
⑦ $u = \frac{1}{2}\log(x^2+y^2)$	⑦ $v = \tan^{-1}(y/x) + c'$ $f(z) = \log z + c$

(04)

⑧ $u = e^{-2xy} \sin(x^2-y^2)$

⑧ $v = -e^{-2xy}\cos(x^2-y^2) + c$ (03)

$f(z) = -i e^{iz^2} + c$

⑨ $u = \dfrac{\sin(2x)}{\cosh(2y) - \cos(2x)}$

⑨ $v = \dfrac{-\sinh(2y)}{\cosh(2y) - \cos(2x)}$

$f(z) = \cot z + c$

⑩ If $u = \dfrac{x}{2}\log(x^2+y^2) - y\tan^{-1}\left(\dfrac{y}{x}\right) + \sin x \cosh(y)$

find the ana. funⁿ. $f(z)$

⑩ $f(z) = z\log z + \sin z + c$

⑪ $v = e^{-x}(x\sin y + y\cos y)$

⑪ $f(z) = z e^{-z} + c$

⑫ $v = \sinh(x)\cos y$

⑫ $f(z) = i\sinh(z) + c$

⑬ $v = \cos x \cosh(y)$

⑬ $f(z) = i\cos z + c$

⑭ If $v = \dfrac{x}{x^2+y^2} + \cosh(x)\cos y$,

find the ana. funⁿ $f(z)$

⑭ $f(z) = i\left[\dfrac{1}{z} + \cosh(z)\right] + c$

⑮ If $v = e^{-x}\left[2xy\cos y + (y^2-x^2)\sin y\right]$

find the ana. funⁿ. $f(z)$

⑮ $f(z) = e^{-z} z^2 + c$

(V) Verify whether the following funⁿˢ. can be the real part / imag. part of an analytic funⁿ. If so, find the corresponding ana. funⁿ.

① $u = e^x\cos y + x^3 - 3xy^2$

① Yes

② $u = x + e^{xy} + y + e^{-xy}$

② No

③ $u = e^x\cos y$

③ Yes ; $f(z) = e^z + c$

④ $u = 3x^2y + 2x^2 - y^3 - 2y^2$

④ Yes

$f(z) = 2z^2 - iz^3 + c$

⑤ $u = x^2 - y^2 - 2xy - 2x + 3y$

⑤ Yes

$f(z) = (z^2 - 2z) + i(z^2 - 3z) + c$

⑥ $u = (x-1)^3 - 3xy^2 + 3y^2$, find v & hence $f(z)$

⑥ Yes; $f(z) = (z-1)^3 + c$

$v = 3x^2y - 6xy + 3y - y^3 + c'$

⑦ $u = 3x^3y + 2x^2 - y^3 - 2y^2$

⑦ No

(P.T.O)

(8) $u(r,\theta) = r^2\cos(2\theta) - r\sin\theta$

(9) $u(r,\theta) = r^n\cos(n\theta)$

(10) $u(x,y) = e^{x^2-y^2}\cos(2xy)$

(11) $v(x,y) = \log\left[(x-1)^2+(y-2)^2\right]$

[except at $(1,2)$]

(12) $v = e^{2x}\left[x\cos(2y) - y\sin(2y)\right]$

(13) $u = 3x^2 + \sin x + y^2 + 5y + 4$; (14) $u = 3x^2 - 2x^2y + y^2$

(VI) Find the ana. fun^n $f(z) = u + iv$ given

(1) $u - v = (x-y)(x^2+4xy+y^2)$

(2) $u - v = x^3 + x^2 - 3xy^2 - y^2 - 3x^2y + y^3 - 2xy$

(3)(a) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ subject to $f(\frac{\pi}{2}) = 0$

(b) $u - v = \dfrac{e^{y} - \cos x + \sin x}{\cosh(y) - \cos x}$; $f(\frac{\pi}{2}) = \frac{3-i}{2}$

(4) $u + v = \dfrac{x}{x^2+y^2}$

(5) $u + v = e^x(\cos y + \sin y) + \dfrac{x-y}{x^2+y^2}$

(6)(a) $u + v = \dfrac{2x}{x^2+y^2}$; $f(1) = i$ (b) $u+v = \dfrac{2\sin(2x)}{e^{2y}+e^{-2y}-2\cos(2x)}$

(7) $u - v = \dfrac{\sin(2x)}{-\cos(2x)+\cosh(2y)}$

(8) $v = \dfrac{\sin x\sinh(y)}{\cos(2x)+\cosh(2y)}$; $f(0) = 1$

(9) $v = e^{-2y}\left[y\cos(2x) + x\sin(2x)\right]$ find $f(z)$ first & hence find 'u'

(10) $3u + 2v = y^2 - x^2 + 16xy$

[Hint: Diff. p.w.r.t. x; p.w.r.t. y, solve for u_x, v_x]

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(8) Yes, $f(z) = z^2 + z + \ldots$

(9) Yes; $f(z) = z^n + c$

(10) Yes, $f(z) = e^{z^2} + c$

(11) Yes, $f(z) = 2i\log(z-1-2i) + c$

(12) Yes; $f(z) = ize^{2z} + c$

(13) No (14) No

(1) $f(z) = -iz^3 + c$

(2) $f(z) = z^3 + z^2 + c$

(3)(a) $f(z) = -\frac{1}{2}\cot(\frac{z}{2}) + c$, $c = \frac{1}{2}$

(b) $f(z) = \cot(\frac{z}{2})$; $c = \frac{1-i}{2}$

(4) $f(z) = \left(\dfrac{i}{1+i}\right)\left(\dfrac{1}{z}\right) + c$

(5) $f(z) = e^z + \frac{1}{z} + c$

(6)(a) $f(z) = \dfrac{1+i}{z} - 1$

(b) $f(z) = \dfrac{i}{1+i}\cot z$

$f(z) = \dfrac{i\cot z}{1+i} + c$

(8) $f(z) = \frac{1}{2}(1+\sec z) + c$, $c = 0$

(9) $f(z) = ze^{i(2z)} + c$

$u = e^{-2y}\left[x\cos(2x) - y\sin(2x)\right]$

(10) $f'(z) = 2z - 4iz$

$\Rightarrow f(z) = (1-2i)z^2 + c$

VII) Verify whether the foll.. funⁿ.. are analytic. If so find $f'(z)$ in two ways. [Hint: Verify C.R. equⁿ.. & use: $f'(z) = u_x + iv_x$] & verify using direct differentiation

1. $f(z) = \cosh(z)$ $[= \cos(iz) = \cos[ix - y] = \quad]$ — Yes; $f'(z) = \sinh(z)$
2. $\cos(z)$ — Yes, $f'(z) = -\sin z$
3. $e^{x}[\cos y - i\sin y]$ — No
4. $e^{-x}[\cos y - i\sin y]$ — Yes; $f'(z) = -e^{-\bar{z}}$
5. $x^2 - y^2 + 2ixy$ — Yes; $f'(z) = 2z$
6. $(x^3 - 3xy^2 + 3x) + i(3x^2y - y^3 + 3y)$ — Yes, $f'(z) = 3z^2 + 3$
7. ze^{2z} — Yes; $f'(z) = \cancel{2z}e^{2z} + e^{2z} = e^{2z}(2z+1)$
8. e^{2z} — Yes; $f'(z) = 2e^{2z}$

VIII. Find the orthogonal trajectories of

1. $3xy^2 + 2x^2 - x^3 - 2y^2 = \text{const}$
2. $e^{x}\cos y - xy = c$
3. $x^2 - y^2 - 2xy + 2x - 3y = c$

Answers:

1. $4xy - 3x^2y + y^3 = c'$
2. $e^{x}\sin y + \frac{1}{2}(x^2 - y^2) = c'$

$\downarrow$

$3x + 2xy + 2y + x^2 - y^2 = c$

IX. Find the values of z for which the foll. funⁿ.. ceases to be analytic [Hint: Find 'z' for which $f'(z) \to \infty$]

1. $f(z) = \dfrac{z}{z^2 - 1}$ — $z = \pm 1$
2. $\dfrac{z^2 - 4}{z^2 + 1}$ — $z = \pm i$
3. $\dfrac{z + i}{(z - i)^2}$ — $z = i$
4. $z^3 - 4z - 1$ — analytic everywheere