23-08-2014 Question Bank for Unit Test - 01

Applied Mathematics - III [Laplace Transform]

S.E. Production & S.E. Electronics ; Computer ; I.T

(1) Find the L.T. of $f(t)$ =

(a) (1) $\begin{cases} 3 & ; 0<t<5 \\ 0 & ; t>5 \end{cases}$

(2) $\begin{cases} (t-2)^2 & ; t>2 \\ 0 & ; 0<t<2 \end{cases}$

(3) $\begin{cases} \cos t & ; 0<t<\pi \\ \sin t & ; t>\pi \end{cases}$

(4) $\begin{cases} 1 & ; 0<t<1 \\ e^t & ; 1<t<4 \\ 0 & ; t>4 \end{cases}$

(5) $\begin{cases} \cos(t-a) & ; t>a \\ 0 & ; t<a \end{cases}$ Second Shifting

(6) $\begin{cases} 0 & ; 0<t<\pi \\ \sin^2(t-\pi) & ; t>\pi \end{cases}$

[using basic def^n. of L.T.]

(b) (1) $\cos t \cos(2t)\cos(3t)$

(2) $\cosh^5(t)$

(3) $[\sin(2t)-\cos(2t)]^2$

(4) $\sin^5(t)$

(5) $\frac{\cos\sqrt{t}}{\sqrt{t}}$ Ans $\sqrt{\frac{\pi}{s}}\, e^{-\left(\frac{1}{4s}\right)}$

(6) S.T. $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-(1/4s)}$

(7) If $J_0(t) = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(r!)^2}\right]\left(\frac{t}{2}\right)^{2r}$, show that $L[J_0(t)] = \frac{1}{\sqrt{1+s^2}}$

(c) (1) $J_0(3t)$

(2) $\sin 2\sqrt{t}$ } Use (b)(6) & (b)(7)

(3) $\operatorname{erf} 3\sqrt{t}$ [given $L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$]

[Change of scale]

(d) (1) $e^{4t}\sin^3 t$

(2) $e^t(1+\sqrt{t})^4$

(3) $e^{-3t}\cosh(t)\sin(3t)$ ✓

(4) $\sinh\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}t\right)$

(5) $\sin(2t)\cos(t)\cosh(2t)$

(6) $(1+te^{-t})^3$

[First Shifting]

(e) (1) $\begin{cases} \cos\left(t-\frac{2\pi}{3}\right) & ; t>\frac{2\pi}{3} \\ 0 & ; t<\frac{2\pi}{3} \end{cases}$

(2) $\begin{cases} 5\sin\left[3\left(t-\frac{\pi}{4}\right)\right] & ; t>\frac{\pi}{4} \\ 0 & ; t<\frac{\pi}{4} \end{cases}$

(3) $\begin{cases} e^{t-a} & ; t>a \\ 0 & ; t<a \end{cases}$

(4) $\begin{cases} (t-2)^2 & ; t>2 \\ 0 & ; t<2 \end{cases}$

(Second Shifting)

Hint: $g(t) = \begin{cases} f(t-a) & ; t>a \\ 0 & ; t<a \end{cases}$ then $L[g(t)] = e^{-as}L[f(t)]$

P.T.O.

(f)
1. $t \sin(2t) \cosh(t)$
2. $t\, e^{3t} \sin t$
3. $t\sqrt{1+\sin(2t)}$
4. $t\left(\dfrac{\sin t}{e^t}\right)^2$
5. $t^2 \cos^2(2t)$
6. $\left[1+te^{-t}\right]^3$

(Multiplication by t^n)

(g)
1. $\dfrac{\cos(2t)-\cos(3t)}{t}$
2. $\dfrac{e^{-at}-e^{-bt}}{t}$
3. $\dfrac{\sin(2t)\cosh(2t)}{t}$
4. $\dfrac{e^{-2t}\sin(2t)\cosh(t)}{t}$
5. $\dfrac{\sin t}{t}$
6. $\dfrac{\sin^2 t}{t}$
7. $\dfrac{\sin^3 t}{t}$
8. $\left[\dfrac{\sin(2t)}{\sqrt{t}}\right]^2$
9. $\dfrac{\sin^2 t}{t^2}$
10. $\dfrac{1-\cos t}{t^2}$

(h) Find $L[f(t)]$ & hence $L[f'(t)]$:-

1. $f(t) = \dfrac{\sin t}{t}$ Ans: $s\cot^{-1}(s) - 1$
2. $f(t) = \begin{cases} 3 & ; 0 \le t < 5 \\ 0 & ; t > 5 \end{cases}$ Ans: $L[f'(t)] = -3e^{-5s}$
3. $f(t) = e^{-5t}\sin t$ Ans: $L[f'(t)] = \dfrac{s}{s^2+10s+26}$
4. $f(t) = \left\{ \dfrac{1-\cos(2t)}{t} \right.$ Ans: $= s\log\left[\dfrac{\sqrt{s^2+4}}{s}\right]$

Use: $L[f'(t)] = s\,L[f(t)] - \underbrace{f(0)}_{\lim_{t\to 0} f(t)}$

(i)
1. $L\left[\int_0^t t e^{-4t}\sin(3t)\,dt\right]$
2. $L\left[e^{-4t}\int_0^t t\sin(3t)\,dt\right]$
3. $L\left[t\int_0^t e^{-4t}\sin(3t)\,dt\right]$
4. $L\left[\cosh(t)\int_0^t e^t\cosh(t)\,dt\right]$
5. $L\left[e^{-t}\int_0^t \dfrac{\sin t}{t}\,dt\right]$
6. $L\left[\int_0^t \dfrac{1-e^{-t}}{t}\,dt\right]$
7. $L\left[\int_0^t e^{-t}t^4\,dt\right]$
8. $L\left[\int_0^t t^2\sin t\,dt\right]$
9. $L\left[\int_0^t e^t\dfrac{\sin t}{t}\,dt\right]$
10. $L\left[\int_0^t\int_0^t\int_0^t t\sin t\,dt\,dt\,dt\right]$ Ans: $\dfrac{2}{s^2(s^2+1)^2}$

P.T.O.

(II) Evaluate the following integrals using L.T:-

(1) $\int_0^\infty e^{-2t} \sin^3 t \, dt$

(2) $\int_0^\infty e^{-4t} \cosh^3(t) \, dt$

(3) $\int_0^\infty e^{-3t} t^5 \, dt$

(4) $\int_0^\infty e^{-3t} t \sin t \, dt$

(5) $\int_0^\infty e^{-2t} t^2 \sin(3t) dt$

(6) $\int_0^\infty e^{-t} t^3 \sin t \, dt$ ✓

(7) $\int_0^\infty \left[\frac{\sin(2t) + \sin(3t)}{t e^t}\right] dt$

(8) $\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt$

(9) $\int_0^\infty e^{-\sqrt{2} t} \left(\frac{\sin t \sinh(t)}{t}\right) dt$ ✓

(10) $\int_0^\infty \frac{\cos(6t) - \cos(4t)}{t} dt$ ✓

(11) $\int_0^\infty e^{-t} \left[\int_0^t u^2 \sinh(u) \cosh(u) \, du\right] dt$ ✓

(12) $\int_0^\infty e^{-3t} t^2 \sinh(2t) \, dt$

(13) $\int_0^\infty e^{-t} \left\{\frac{1}{t} \int_0^t e^{-u} \sin u \, du\right\} dt$

(14) Find 'α' such that [$\alpha \to$ constant]

(a) $\int_0^\infty e^{-t} \frac{\sin(\alpha t)}{t} dt = \frac{\pi}{3}$

(Ans: $\alpha = \sqrt{3}$)

(b) $\int_0^\infty e^{-3t} t J_0(\alpha t) dt = \frac{3}{125}$

given $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ $(\alpha > 0)$

(Ans: $\alpha = 4$)

(c) $\int_0^\infty e^{-t} \, \mathrm{erf}(\alpha\sqrt{t}) dt = \frac{2}{\sqrt{5}}$

$(\alpha > 0)$

$L[\mathrm{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$

(Ans: $\alpha = 2$)

Answers (verify)

(1) $\frac{6}{65}$

(2) $\frac{12}{35}$

(3) $\frac{40}{243}$

(4) $\frac{3}{50}$

(5) $\frac{18}{2197}$

(6) zero

(7) $\frac{3\pi}{4}$ ✓

(8) $\frac{1}{4} \log 5$

(9) $\frac{\pi}{8}$

(10) $\log(2/3)$

(11) $-\frac{2}{125}$

(12) $\frac{124}{125}$

(13) $\frac{1}{4} \log 5 - \frac{1}{2} \cot^{-1}(2)$

$f(at)$

$\frac{a}{a} L\left[\frac{\sin(at)}{at}\right] = \frac{a}{a}\left[\frac{1}{a} F(\ldots\right.$ $= \ldots$ $= \frac{\pi}{2} - \ldots$

① Extra Problems: $= \frac{\pi}{2} - \tan^{-1}(s) = F(s)$

(i) Find $L\left[\frac{\sin t}{t}\right]$ & hence or otherwise find $L\left[\frac{\sin(at)}{t}\right]$

(ii) Find $L\left[t \int_0^t \frac{\sin u}{u} \, du\right]$ Ans: $(-1)\frac{d}{ds}\left[\frac{1}{s}\left\{\frac{\pi}{2} - \tan^{-1}(s)\right\}\right]$

(iii) Evaluate: $\int_0^\infty \frac{\sin t}{t} \, dt$; Ans: $\frac{\pi}{2}$

(iv) " $\int_0^\infty \left(\int_{u=0}^t e^{-t} \frac{\sin u}{u} \, du\right) dt$ Ans $= L\left[\int_0^t \frac{\sin u}{u} \, du\right]\Big|_{s=1} = \frac{\pi}{4}$ ✓

Note: e^{-t} is indept. of u ∴ treat it as const while $\int$ating w.r.t. u

(v) " $\int_0^\infty e^{-2t}\left(\int_0^t e^{-u} \frac{\sin u}{u} \, du\right) dt = L\left[\int_0^t e^{-u} \frac{\sin u}{u} \, du\right]\Big|_{s=2}$

Ans: $I = \frac{1}{2} \cot^{-1}(3)$

(vi) Find $L\left[e^{-t} \int_0^t \frac{\sin u}{u} \, du\right]$ Ans: $\frac{1}{s+1} \cot^{-1}(s+1)$ ✓

(vii) Find $L\left[\frac{1}{t} \int_0^t e^{-u} \sin u \, du\right]$ ∴ Ans: $\frac{1}{2}\left[\int_s^\infty \frac{1}{s(s^2+2s+2)} \, ds\right]$

↓ Assignment ①

$= \frac{1}{4} \log\left[\frac{s^2+2s+2}{s^2}\right] - \frac{1}{2}\cot^{-1}(s+1).$

(viii) Evaluate: $\int_0^\infty e^{-t} \frac{\sin t}{t} \, dt$ Ans : $L\left[\frac{\sin t}{t}\right]\Big|_{s=1} = \frac{\pi}{4}$

(ix) S.T. $\int_0^\infty e^{-\sqrt{2}\, t} \frac{1}{t} \sin t \sinh t \, dt = \frac{\pi}{8}$; Ans: $\frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right]\Big|_{s=\sqrt{2}}$ ✓

$= \frac{1}{2}\left[\tan^{-1}(2/s^2)\right]\Big|_{s=\sqrt{2}}$

$= \frac{\pi}{8}$

(x) S.T. $\int_0^\infty \frac{t^2 \sin(3t)}{e^{2t}} \, dt = \frac{18}{2197}$ | Ans. $\frac{18(s^2-3)}{(s^2+9)^3}\Big|_{s=2} = \frac{18}{2197}$

(2) Given: $L[J_0(t)] = \frac{1}{\sqrt{1+s^2}}$, find

(i) $L[t\, J_0(at)] =$ Ans: $\frac{s}{(s^2+a^2)^{3/2}}$ $\quad L[J_0(at)] = \frac{1}{a}\left[\frac{1}{\sqrt{1+s^2/a^2}}\right] = \frac{1}{\sqrt{s^2+a^2}}$

(ii) $L[e^{-bt} J_0(at)]$ Ans: $\frac{1}{\sqrt{(s+b)^2+a^2}}$ ✓

(iii) $\int_0^\infty J_0(t)\, dt$ Ans: 1

(iv) $\int_0^\infty e^{-t} J_0(t)\, dt$ Ans: $\frac{1}{\sqrt{2}}$

(v) $\int_0^\infty e^{-3t}\, t\, J_0(4t)\, dt$ Ans: $\frac{3}{125}$ ✓

(vi) $\int_0^\infty e^{-4t}\, t\, J_0(3t)\, dt$ Ans: $\frac{4}{125}$

(3) Given: $L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$ $\left[\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\, dx\right.$ expand & integrate w.r.t. x, sub the lts & then apply L.T$\left.\right]$

(i) Find: $\int_0^\infty e^{-t} \operatorname{erf}\sqrt{t}\, dt = L[\operatorname{erf}\sqrt{t}]\Big|_{s=1} = \frac{1}{\sqrt{2}}$

(ii) S.T $\int_0^\infty e^{-2t} \operatorname{erf} 2\sqrt{t}\, dt = \frac{1}{\sqrt{6}}$

$L(\operatorname{erf}\sqrt{4t})\Big|_{s=2}$ ✓

Ans

$L[\operatorname{erf}\sqrt{4t}]$ $(a=4)$ $= \frac{1}{4}\,\frac{1}{s/4\sqrt{1+s/4}}$

$= \frac{1}{4}\left[\frac{4}{s}\cdot\frac{2}{\sqrt{s+4}}\right]$

$I = \Big|_{s=2} = \pi/\sqrt{6}$

(iii) Find $L[e^{3t}\, t \operatorname{erf}\sqrt{t}]$ ✓

Ans: $\frac{3s-7}{2(s-3)^2(s-2)^{3/2}}$

(iv) Find $L[t \operatorname{erf} 2\sqrt{t}]$: ✓ Ans: $\frac{3s+8}{s^2(s+4)^{3/2}}$

(v) S.T. $\int_0^\infty t e^{-t^2} \operatorname{erf}(t)\, dt = \frac{1}{2\sqrt{2}}$

Let $t^2 = u$

$2t\,dt = du$

$t\,dt = \frac{1}{2} du$

u: 0 to ∞

$I = \int_0^\infty e^{-u} \operatorname{erf}(\sqrt{u}) \frac{du}{2}$

$= \frac{1}{2}\left[L\{\operatorname{erf}\sqrt{u}\}\Big|_{s=1}\right]$

$= \frac{1}{2}\left[\frac{1}{s\sqrt{s+1}}\Big|_{s=1}\right]$

$= \frac{1}{2\sqrt{2}}$

Note:

(vi) $L[\operatorname{erf}_c \sqrt{t}] = L[1 - \operatorname{erf}\sqrt{t}]$

Find $\int_0^\infty \operatorname{erf}_c \sqrt{t}\, dt = L[\operatorname{erf}_c \sqrt{t}]\Big|_{s=0}$

$= 1/2$