Simplex Method

Solve the following LPP by using Simplex Method:

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 9 \\ 2x_1 + 3x_2 + 5x_3 &\leq 30 \\ 2x_1 - x_2 - x_3 &\leq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution:
We write the given LPP in the standard form:

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 + s_1 &= 9 \\ 2x_1 + 3x_2 + 5x_3 + s_2 &= 30 \\ 2x_1 - x_2 - x_3 + s_3 &= 8 \\ x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

We assume that the initial non-basic variables are x_1, x_2, x_3 that is we set $x_1 = x_2 = x_3 = 0$. Therefore the initial basic feasible solution is given by $s_1 = 9, s_2 = 30, s_3 = 8$.

We prepare the simplex table as follows:

	c_j	3	2	5	0	0	0		
e_i	CSV	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
0	s_1	1	1	1	1	0	0	9	9
0	s_2	2	3	**5**	0	1	0	30	6 $\rightarrow$
0	s_3	2	-1	-1	0	0	1	8	-8
$E_j = \sum_{i=1}^{3} a_{ij} e_i$		0	0	0	0	0	0		
$c_j - E_j$		3	2	5 $\uparrow$	0	0	0		

The first row c_j represents coefficients of decision variables in the objective function. Column CSV represents *Current Solution Variables* or the *Basis Variables.* Column of e_i contains coefficients of Basis Variables in the objective function. We then write the equality constraints in the matrix form $AX = B$. Column b contains the R.H.S. constants. Row E_j is calculated using formula mentioned in the table. These entries are dot products of column e_i with respective columns of decision variables. Next we calculate $c_j - E_j$ row.

Optimality Conditions for Maximization Type Problem:

(1) If all $c_j - E_j$ entries are ≤ 0, solution is optimal.

(2) If there exists j such that $c_j - E_j > 0$, we can improve upon the solution as follows:

Select maximum among positive $c_j - E_j$. In our case, it is 5 belonging to the column of variable x_3. This column is called *Key Column* or *Pivotal Column.* The variable; namely x_3, in this column is an *incoming variable* for the next iteration. As the number of variables in the basis remains the same; so one variable has to go out of the basis. We do this by computing the column of θ.

θ column entries are ratios of b column entries to the *key column* entries. Next we select the *least positive* (> 0) θ entry. In our case, it is 6; that belongs to the row of s_2. This row is called *Key Row* or *Pivotal Row;* and the variable in this row is an *incoming variable* for the next iteration.

Intersection of key row and key column is called *Pivotal element* or simply *Pivot.* This number is highlighted.

Change of Basis:

Make the pivotal entry as 1 and the other entries in that column (pivotal column) as zeros by using the following elementary row operations:

$$R_2 \to R_2/5, R_1 \to R_1 - R_2, R_3 \to R_3 + R_2$$

	c_j	3	2	5	0	0	0		
e_i	CSV	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
0	s_1	**3/5**	2/5	0	1	-1/5	0	3	5 $\rightarrow$
5	x_3	2/5	3/5	1	0	1/5	0	6	15
0	s_3	12/5	-2/5	0	0	1/5	1	14	35/6
$E_j = \sum_{i=1}^{3} a_{ij} e_i$		2	3	5	0	1	0		
$c_j - E_j$		1 $\uparrow$	-1	0	0	-1	0		

From the optimality condition, it is clear that we can further improve upon the existing solution. We change the basis as follows:

$$R_1 \to \frac{5}{3}R_1, R_2 \to R_2 - \frac{2}{5}R_1, R_3 \to R_3 - \frac{12}{5}R_1$$

	c_j	3	2	5	0	0	0		
e_i	CSV	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
3	x_1	1	2/3	0	5/3	-1/3	0	5	
5	x_3	0	1/3	1	-2/3	1/3	0	4	
0	s_3	0	-2	0	-4	1	1	2	
$E_j = \sum_{i=1}^{3} a_{ij} e_i$		3	11/3	5	5/3	2/3	0		
$c_j - E_j$		0	-5/3	0	-5/3	-2/3	0		

Since all $c_j - E_j$ are ≤ 0, optimal solution is

$$x_1 = 5, x_2 = 0, x_3 = 4, Z_{max} = 35$$

Note that x_2 is non-basic variable, therefore $x_2 = 0$. Also note that the matrix of basis variables is identity matrix in each simplex table.

Optimality Conditions for Minimization Type Problem:

(1) If all $c_j - E_j$ entries are ≥ 0, solution is optimal.

(2) If there exists j such that $c_j - E_j < 0$, we can improve upon the solution as follows:
Select the minimum among negative $c_j - E_j$. The corresponding column is the pivotal column.