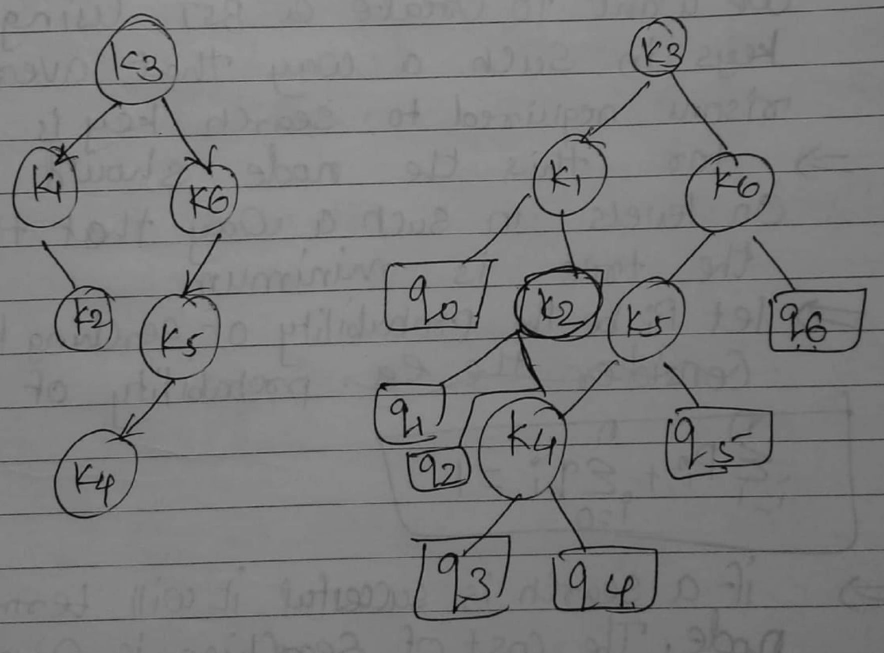
Optimal Binary search trees

⇒ An optimal binary search tree is a binary search tree for which the nodes are arranged on levels such that the tree cost is minimum

⇒ for better representation purpose of OBST we will consider "Extended binary search tree" which have the keys stored at their internal nodes. suppose n keys k1, k2, kn are stored at the internal nodes of a binary search tree.

⇒ It is assumed that keys are given in sorted order $k1 < k2 < \dots < kn$.

⇒ An extended binary search tree is obtained from the binary search tree by adding successor nodes to each of its internal nodes.



In the extended tree

- squares represents terminal nodes. These
- terminal nodes represent unsuccesful searches of the tree for key values. The searches did not end succesfully. because key

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value being searched are not stored in the tree

— round nodes represent internal nodes, these are actual keys stored in the tree

→ If a user searches a particular key in the tree 2 cases can occur

Case 1 – The key is found, so the corresponding weight p is incremented.

Case 2 – The key is not found so the corresponding q value is incremented.

p & q represents frequencies of successful searches & unsuccessful searches respectively.

⇒ n keys with their probabilities are given to us.

we want to create a BST using these n keys in such a way that average comparisons required to search key is minimum.

⇒ for this the node should be arranged on levels in such a way that the hight of the tree is minimum.

⇒ let P_i be the probability of searching key i, q_i be ~~Consider the eg.~~ probability of unsuccessful search

$$\boxed{\sum_{i=1}^{n} P_i + \sum_{i=0}^{n} q_i = 1}$$

⇒ if a search is successful it will terminate on internal node. The cost of searching is given by

Cost(k_i) = (level of k_i + 1) * P_i

⇒ if a search is unsuccessful, it terminates on external nodes (terminal nodes). The cost of searching is given by, level(e_i) * q_i

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The terminal node in the extended tree that is left successor of k_1 can be interpreted as representing all key value that are not stored and are less than k_1. Similarly, the terminal node in the extended tree that is the right successor of k_n, represents all key values not stored in the tree that are greater than k_n.

An obvious way to find an optimal binary search tree is to generate each possible binary search tree for the keys, calculate the weighted path length & keep the tree with minimum weighted path.

This search through all posible sol^n is not feasible since the no of search trees grows exponentially with "n".

An alternative is a recursive algo. a binary search tree has root, left subtree and right subtree. both subtrees must be the optimal binary search trees.

OBST(i,j) denotes the optimal binary search tree containing the keys k_i, k_{i+1}, k_j.

$w(i,j)$ - denotes the weight matrix for OBST(i,j)

$w(i,j)$ can be defined using formula.

$$w_{i,j} = \sum_{k=i+1}^{j} p_k + \sum_{k=i}^{j} q_k$$

$C(i,j)$ denote the cost matrix for OBST(i,j)

$$C_{i,i} = w_{i,j}$$

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$$C_{i,j} = w_{i,j} + \min_{i<k\leq j} \{ C_{i,k-1} + C_{k,j} \} \quad \text{--- (1)}$$

$R_{i,j}$ $0 \leq i \leq j \leq n$ denotes the root matrix for OBST (i, j)

Assign to $R_{i,j}$ the k value for which we obtain a minimum for eq^n (1)

Q: Find optimal binary search tree for keys do < if < int < while probability of succesful search p_1, p_2, p_3, p_4 and unsuccesful search

$$\begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 3 & 1 & 1 \end{array}$$

$$\begin{array}{ccccc} q_0 & q_1 & q_2 & q_3 & q_4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 & 1 & 1 \end{array}$$

OBST is given by OBST (0,4) — involving the given 4 keys.

q_0 q_1 q_2 q_3 q_4 ← q
2 3 1 1 1

w \ j		0	1	2	3	4
	q_0	2	8	12	14	16
p_1	3 → 1		3	7	9	11
p_2	3 → 2			1	3	5
p_3	1 → 3				1	3
p_4	1 → 4					1

$w(0,1) = p_1 + q_1 + w(0,0)$
$= 3+3+2$
$= 8$

$w(1,2) = p_2 + q_2 + w(1,1)$
$= 3+1+3$
$= 7$

$w(2,3) = p_3 + q_3 + w(2,2)$
$= 1+1+1$
$= 3$

$w(3,4) = p_4 + q_4 + w(3,3)$
$= 1+1+1$
$= 3$

$$\boxed{w(i,j) = p_j + q_j + w(i, j-1)}$$

$w(0,2) = p_2 + q_2 + w(0,1)$
$= 3 + 1 + 8 = 12$

$w(1,3) = p_3 + q_3 + w(1,2)$
$= 1 + 1 + 7 = 9$

$w(2,4) = p_4 + q_4 + w(2,3)$
$= 1 + 1 + 3$
$= 5$

$w(0,3) = p_3 + q_3 + w(0,2)$
$= 1 + 1 + 12 = 14$

$w(1,4) = p_4 + q_4 + w(1,3)$
$= 1 + 1 + 9 = 11$

$w(0,4) = p_4 + q_4 + w(0,3)$
$= 1 + 1 + 14$
$= 16$

Cost

~~Root~~ (write k values)

i \ j	0	1	2	3	4
0	0	8	19	25	32
1		0	7	12	19
2			0	10	8
3				0	3
4					0

~~Cost~~ Root

			2	3	1	1	1
			q_0	q_1	q_2	q_3	q_4
		i \ j	0	1	2	3	4
		0	0	1	1	2	2
3	P_1	1		0	2	2	2
3	P_2	2			0	2	3 or 4
1	P_3	3				0	4
1	P_4	4					0

$$\boxed{c(i,j) = \min_{i < k \le j} \{ c(i, k-1) + c(k, j) \} + w(i,j)}$$

k is $\underline{1}$

$Cost(0,1) = \cancel{\omega(i,j) + min}$

$= \omega(0,1) + min\{Cost(0,0) + Cost(1,1)\}$

$= 8 + 0 + 0$

$= 8$

$Cost(1,2) = \omega(1,2) + min\{Cost(1,1) + Cost(2,2)\}$ k = 2

$= 7 + 0 + 0$

$= 7$

$Cost(2,3) = \omega(2,3) + min\{Cost(1,2) + Cost(3,3), \cancel{C(1,2) + C(3,3)}\}$ k = 3

$\cancel{= 9 + min\{0+}$

$= 3 + min\{7 + 0\}$

$= 10$

$Cost(2,4) = \omega(2,4) + min\{Cost(2,2) + Cost(3,4), Cost(2,3) + Cost(4,4)\}$ k = 3, k = 4

$= 5 + min\{0 + 3, 10 + 0\}$ k = 3

$= 5 + 3$

$= 8$

$Cost(3,4) = \omega(3,4) + min\{Cost(3,3) + Cost(4,4)\}$

$= \cancel{3 + min(0)}$

$= 3$

$$\text{Cost}(0,2) = w(0,2) + \min \begin{Bmatrix} \text{Cost}(0,0) + \text{Cost}(1,2), & k=1 \\ \text{Cost}(0,1) + \text{Cost}(2,2) & k=2 \end{Bmatrix}$$

$$= 12 + \min \begin{Bmatrix} 0+7, \\ 8+0 \end{Bmatrix}$$

$$= 12+7$$

$$= 19$$

$$\text{Cost}(1,3) = w(1,3) + \min \begin{Bmatrix} \text{Cost}(1,1) + \text{Cost}(2,3), & k=2 \\ \text{Cost}(1,2) + \text{Cost}(3,3) & k=3 \end{Bmatrix}$$

$$= 9 + \min \begin{Bmatrix} 0+3, \\ 7+0 \end{Bmatrix}$$

$$= 12$$

$$\text{Cost}(0,3) = w(0,3) + \min \begin{Bmatrix} \text{Cost}(0,0) + \text{Cost}(1,3), & k=1 \\ \text{Cost}(0,1) + \text{Cost}(2,3) & k=2 \\ \text{Cost}(0,2) + \text{Cost}(3,3) & k=3 \end{Bmatrix}$$

$$= 14 + \min \begin{Bmatrix} 0+12, \\ 8+3, \\ 19+0 \end{Bmatrix}$$

$$= 14 + 11$$

$$= 25 \qquad k=2$$

$$\text{Cost}(1,4) = w(1,4) + \min \begin{Bmatrix} \text{Cost}(1,1) + \text{Cost}(2,4), & k=2 \\ \text{Cost}(1,2) + \text{Cost}(1,3), & k=3 \\ \text{Cost}(1,3) + \text{Cost}(4,4) & k=4 \end{Bmatrix}$$

$$= 11 + \min \begin{Bmatrix} 0+8, \\ 7+12, \\ 12+0 \end{Bmatrix}$$

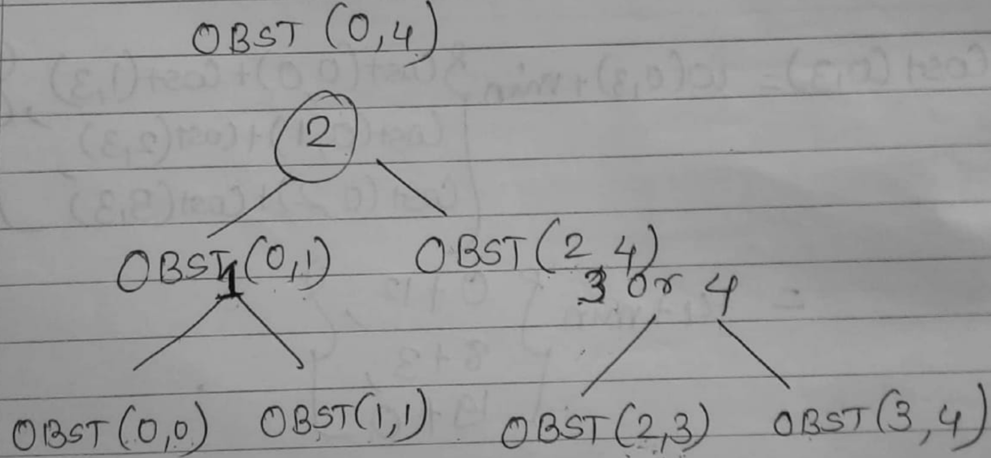
$$= 11+8 \qquad = 19$$

$$cost(0,4) = w(0,4) + \min \begin{cases} cost(0,0) + cost(1,4), & k=1 \\ cost(0,1) + cost(2,4), & k=2 \\ cost(0,2) + cost(3,4), & k=3 \\ cost(0,3) + cost(4,4) & k=4 \end{cases}$$

$$= 16 + \min \begin{cases} 0+19, \\ 8+8, \\ 19+3, \\ 25+0 \end{cases}$$

$$= 16 + 16$$

$$= \underline{\underline{32}}$$



for

Q.2] OBST

	k_1		k_2		k_3		k_4
	Cout	<	float	<	if	<	while

$p_1 \frac{1}{20}$ $p_2 \frac{1}{5}$ $p_3 \frac{1}{10}$ $p_4 \frac{1}{20}$

$q_0 \frac{1}{5}$ $q_1 \frac{1}{10}$ $q_2 \frac{1}{5}$ $q_3 \frac{1}{20}$ $q_4 \frac{1}{20}$

→ The ultimate cost is given by COST(0, 4)
Here in order to be convinient multiply prop by 20 and recomputed probabilities are

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$p_1 = 1$ $\qquad q_0 = 4$

$p_2 = 4$ $\qquad q_1 = 2$

$p_3 = 2$ $\qquad q_2 = 4$

$p_4 = 1$ $\qquad q_3 = 1$

$\qquad\qquad q_4 = 1$

W

	i \ j	q_0 (4) 0	q_1 (2) 1	q_2 (4) 2	q_3 (1) 3	q_4 (1) 4
	0	4	7	15	18	20
1 — p_1	1		2	10	13	15
4 — p_2	2			4	7	9
2 — p_3	3				1	3
1 — p_4	4					1

Root

i \ j	0	1	2	3	4
0	0	1	2	2	2
1		0	2	2	2
2			0	3	3
3				0	4
4					0

Cost

	i \ j	q_0 0	q_1 1	q_2 2	q_3 3	q_4 4
	0	0	7	22	32	$\boxed{39}$
p_1	1		0	10	20	27
p_2	2			0	7	12
p_3	3				0	3
p_4	4					0

$k=1$

$$Cost(0,1) = W(0,1) + C(0,0) + C(1,1) = 7+0+0 = 7$$

$$Cost(0,2) = W(0,2) + \min \begin{cases} C(0,0) + C(1,2) \\ C(0,1) + C(2,2) \end{cases} = 15 + 7 = 22$$

$k=2$ & $k=3$

$$Cost(1,3) = W(1,3) + \min \begin{cases} C(1,1) + C(2,3) & k=2 \\ C(1,2) + C(3,3) & k=3 \end{cases}$$

$$= 13 + 7 = 20$$

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k=3 or k=4.

$$Cost(2,4) = w(2,4) + \min\begin{cases} \overset{0}{c(2,2)} + \overset{3}{c(3,4)} & k=3 \\ \underset{7}{c(2,3)} + \underset{0}{c(4,4)} & k=4 \end{cases}$$

$$= 9 + 3 = 12$$

k=1 or k=2 or k=3

$$Cost(0,3) = w(0,3) + \min\begin{cases} \overset{0}{c(0,0)} + \overset{20}{c(1,3)} & k=1 \\ \underset{7}{c(0,1)} + \underset{7}{c(2,3)} & k=2 \\ \underset{22}{c(0,2)} + \underset{6}{c(3,3)} & k=3 \end{cases}$$

$$= 18 + 14$$

$$= 32$$

k=2,3,4

$$Cost(1,4) = w(1,4) + \min\begin{cases} \overset{0}{c(1,1)} + \overset{12}{c(2,4)} & k=2 \\ \overset{10}{c(1,2)} + \overset{3}{c(3,4)} & k=3 \\ \underset{20}{c(1,3)} + \underset{0}{c(4,4)} & k=4 \end{cases}$$

$$= 15 + 12$$

$$= 27$$

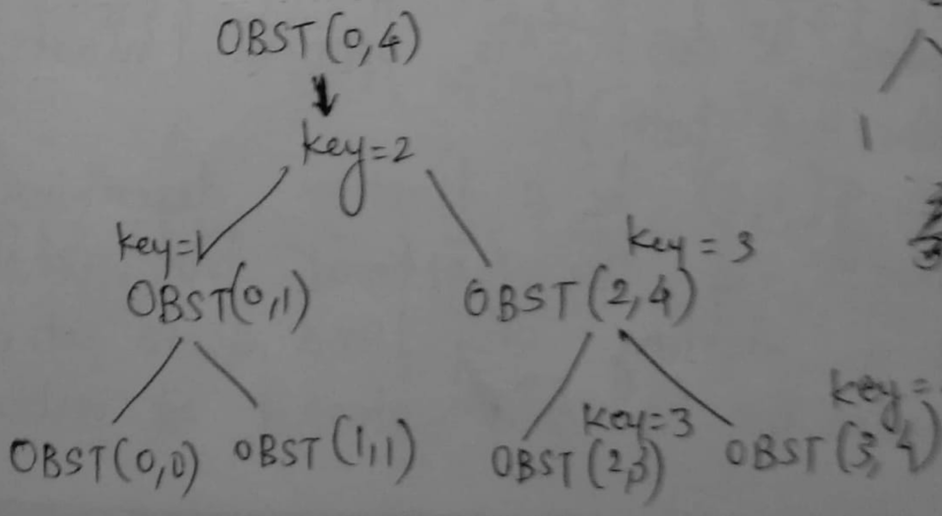
k=1,2,3,4

$$Cost(0,4) = w(0,4) + \min\begin{cases} \overset{0}{c(0,0)} + \overset{27}{c(1,4)} & k=1 \\ \overset{7}{c(0,1)} + \overset{12}{c(2,4)} & k=2 \\ \overset{22}{c(0,2)} + \overset{3}{c(3,4)} & k=3 \\ \overset{32}{c(0,3)} + \overset{0}{c(4,4)} & k=4 \end{cases}$$

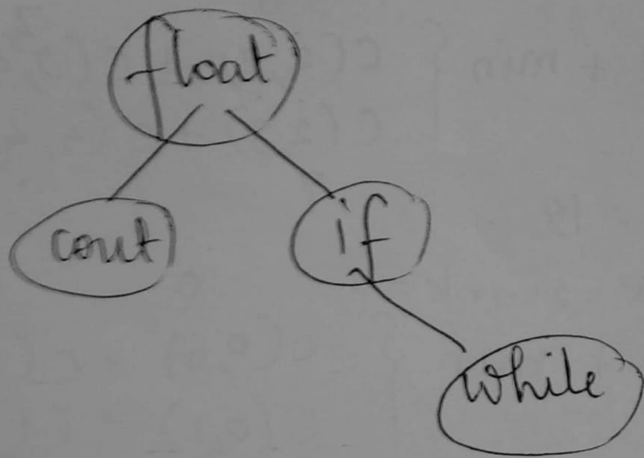
$$= 20 + 19$$

$$= \underline{39}$$

(*) OBST



Resultant OBST



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