The Lambert W function

Michael Rogers, 2023

1 The Lambert W function.

One of the simpler, useful functions that can be defined implicitly is the Lambert W function, which is the implicit function $y = W(x)$ defined by the equation

$$ye^y = x. \tag{1}$$

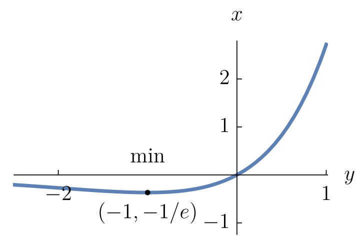

Figure 1 $f(y) = ye^y$.

$W(x) =$ `ProductLog[x]` *in Mathematica*

If $f(y) = ye^y$, then we may also define $W(x)$ to be the inverse function of f:

$$W(x) \overset{\text{def}}{=} f^{-1}(x), \quad x \geq -1/e,\ W(x) \geq -1$$

where we take the principal branch to be the solution to (1) for which $y \geq -1$. (There is another solution for which $y \leq -1$, which is sometimes denoted $W_{-1}(x)$.) It has an interesting history, having been rediscovered many times since the 1700s, and many applications.[1]

$W_{-1}(x) =$ `ProductLog[-1,x]` *in Mathematica*

The following identities are left as exercises (see Exercises 2, 3):

$$W^{-1}(y) = ye^y \tag{2}$$

$$e^{W(x)} = x/W(x) \tag{3}$$

$$\ln W(x) = \ln x - W(x) \tag{4}$$

Learning goal remarks. The Lambert W function is a robust source of examples and exercises that connect basic algebra, differentiation including implicit differentiation, and integration of logarithms and exponential functions. Its usefulness in specialized areas of a wide range of disciplines in the sciences and engineering make it interesting (and important within these specializations). However, Lambert W is at best of secondary importance in first-year calculus, and the facts presented here are not worth studying in depth at this time. Learning to apply formulas and being familiar with $\ln x$ and e^x are the important goals here.

1.1 Solving equations.

Many of the applications arise because the Lambert W function can solve equations that arise in many settings that cannot be solved in terms of the usual collection of functions encountered in high school and college mathematics. Since its addition to the mathematical software system Maple around 1992 and subsequently to Wolfram Mathematica in 1996,[2] interest in the W has increased in both mathematical and scientific research. It shows how new mathematics can sprout and grow from an old seed.

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[1] See this survey article, from which some of the examples used in these notes are taken: R.M. Corless et al., On the Lambert W Function, *Adv. Comp. Math.*, **5** (1996) 329-359 (Emory link).

[2] MapleSoft, *Maple*, https://maplesoft.com. Wolfram Research, Inc., *Mathematica*, https://wolfram.com.

Examples. A general strategy is to find a substitution or other manipulation to get the equation in the form $ue^u = z$; then $u = W(z)$ and we go from there.

1. Solve $xe^{2x} = a$. *Let $u =$ exponent*

$$xe^{2x} = a \Rightarrow 2xe^{2x} = 2a \overset{u=2x}{\Rightarrow} ue^u = 2a \Rightarrow u = 2x = W(2a) \Rightarrow x = \tfrac{1}{2}W(2a).$$

2. Solve $x^3e^x = a$. *Reduce power* $x^3 \xrightarrow{\sqrt[3]{\bullet}} x$

$$x^3e^x = a \overset{\sqrt[3]{\bullet}}{\Rightarrow} xe^{x/3} = a^{1/3} \overset{x=3u}{\Rightarrow} 3ue^u = a^{1/3} \Rightarrow u = x/3 = W(a^{1/3}/3)$$
$$\Rightarrow x = 3W(a^{1/3}/3).$$

3. Solve $x2^x = a$. *Change base*

$$x2^x = a \Rightarrow xe^{x\ln 2} = a \Rightarrow (x\ln 2)e^{x\ln 2} = a\ln 2 \overset{u=x\ln 2}{\Rightarrow} ue^u = a\ln 2$$
$$\Rightarrow u = x\ln 2 = W(a\ln 2) \Rightarrow x = W(a\ln 2)/\ln 2.$$

4. Solve $x + \ln x = a$. *Exponentiate*

$$x = W(e^a) \quad \text{(exercise)}.$$

5. Solve $2^x + x = a$.

$$x = a - \frac{W(2^a\ln 2)}{\ln 2} \quad \text{(exercise)}.$$

1.2 Derivatives and integrals.

PREREQ. CALCULUS

The following derivatives may be found with implicit differentiation and the identity 3:

$$W'(x) = \frac{1}{x + e^{W(x)}} = \frac{W(x)}{x(1 + W(x))} \tag{5}$$

$$W''(x) = -\frac{W(x)^2(W(x)+2)}{x^2(W(x)+1)^3} \tag{6}$$

See Exercises 4, 5.

In section ??, we developed a formula for the integral of the inverse of a function. If $f(y) = ye^y$, then $f^{-1}(x) = W(x)$, and the formula for the integral becomes

The ?? here and below are links to a part of the calculus text not included; ignore

$$\int_a^b W(x)\,dx = bW(b) - aW(a) - \int_{W(a)}^{W(b)} ye^y\,dy. \tag{7}$$

In exercise ?? in section ??, the integral of ye^y was shown to be

$$\int ye^y\,dy = (y-1)e^y + C. \tag{8}$$

Plugging $W(b)$ and $W(a)$ into (8) and subtracting in (7) leads to

$$\begin{aligned}
\int_a^b W(x)\,dx &= bW(b) - aW(a) - (y-1)e^y\Big|_{W(a)}^{W(b)} \\
&= bW(b) - aW(a) - \left[(W(b)-1)e^{W(b)} - (W(a)-1)e^{W(a)}\right] \\
&\overset{(3)}{=} bW(b) - aW(a) - \left[(W(b)-1)\frac{b}{W(b)} - (W(a)-1)\frac{a}{W(a)}\right] \\
&= \left(bW(b) - b + \frac{b}{W(b)}\right) - \left(aW(a) - a + \frac{a}{W(a)}\right) \\
&= x\left(W(x) + \frac{1}{W(x)} - 1\right)\Bigg|_a^b .
\end{aligned}$$

Another approach, which is good to try on integrals involving W, is the substitution:

$$w = W(x) \Leftrightarrow x = we^w, \qquad dx = (w+1)e^w\,dw. \tag{9}$$

Applying the substitution (9) to the integral of $W(x)$, we get

$$\begin{aligned}
\int W(x)\,dx &\overset{(9)}{=} \int w\cdot(w+1)e^w\,dw \\
&= \int we^w\,dw + \int w^2e^w\,dw
\end{aligned}$$

The first integral can be done with (8) and the second can be shown to be

$$\int w^2e^w\,dw = (w^2-2w+2)e^w + C \tag{10}$$

with the methods of lesson ??; see exercise 8 for more. Applying formulas (8) and (10), the integrals becomes

$$\begin{aligned}
\int W(x)\,dx &= (w-1)e^w + (w^2-2w+2)e^w + C = e^w(w^2-w+1) + C \\
&\overset{(3)}{=} \frac{x}{w}(w^2-w+1) + C \\
&\overset{w=W(x)}{=} x\left(W(x) + \frac{1}{W(x)} - 1\right) + C
\end{aligned}$$

Since these methods are developed further in second semester calculus, we won't go further into them. They may be used to find integrals of $x^nW(x)$ and even $\sin(W(x))$.

Example.

Here is an example that falls within the scope of our methods:

$$\int \frac{dx}{1+W(x)} \overset{(9)}{\underset{x=we^w}{=}} \int \frac{(w+1)e^w\,dw}{1+w}$$
$$= \int e^w\,dw = e^w + C = e^{W(x)} + C.$$

1.3 Towers of powers (an iterated sequence).

For a positive number z, define a sequence x_t, $t = 1, 2, 3, \ldots$ as iterations of $h(x) = z^x$:

$$x_1 = z, \quad x_{t+1} = z^{x_t}.$$

Then the first few terms are

$$x_1 = z,\ x_2 = z^z,\ x_3 = z^{z^z}, \ldots, x_n = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}, \ldots.$$

Let's solve for the fixed point:

$$x = h(x) \Rightarrow x = z^x \Rightarrow xe^{-x\ln z} = 1 \Rightarrow (-x\ln z)e^{-x\ln z} = -\ln z$$
$$\overset{u=-x\ln z}{\Rightarrow} ue^u = -\ln z \Rightarrow u = -x\ln z = W(-\ln z)$$
$$\Rightarrow x = -W(-\ln z)/\ln z.$$

For $e^{-e} < z < e^{1/e}$ or approximately $0.06599 < z < 1.4447$, the sequence x_t converges to $-W(-\ln z)/\ln z$. Follow the link to an interactive webpage illustrating the sequence. See Exercise 7 for more.

1.4 Solution to the Lotka-Volterra differential equation. *PREREQ. CALCULUS*

Volterra presented the following equations in the form of two autonomous rates:

$$\frac{dN_1}{dt} = (\varepsilon_1 - \gamma_1 N_2)N_1$$
$$\frac{dN_2}{dt} = (-\varepsilon_2 + \gamma_2 N_1)N_2$$

(Coincidentally, he was studying, like Robert May with his logistic map, oscillations in fish populations.) Lotka derived a similar system of differential equations studying chemical reactions. Today, it is called the Lotka-Volterra system.[3]

Here is a simplified form of the Lotka-Volterra system, in which we will assume the quantities a, b, x and y are positive, and a and b are constants:

$$\frac{dx}{dt} = ax(1-y), \quad \frac{dy}{dt} = -by(1-x). \tag{11}$$

[3] The details of these two examples are unimportant here, but for more see (1) V. Volterra, Variations and Fluctuations of the Number of Individuals in Animal Species living together, *ICES Journal of Marine Science* **3** (1928) 3–51 and A.J. Lotka, Contribution to the Theory of Periodic Reactions, *J. Phys. Chem.*, **14** (1910) 271–274; and (2) T.C. Scott *et al.*, The calculation of exchange forces: General results and specific models, *J. Chem. Phys.*, **99** (1993) 2841–2854 (Emory link).

From this, we derive the following separable differential equation we will solve:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{by(1-x)}{ax(1-y)} \Rightarrow$$

$$\frac{-b(1-x)}{x}\,dx = \frac{a(1-y)}{y}\,dy. \tag{12}$$

Integrating each side we obtain the following

$$\int \frac{-b(1-x)}{x}\,dx = -b\int\left(\frac{1}{x}-1\right)\,dx = -b\left(\ln|x|-x\right)+C_1.$$

$$\int \frac{a(1-y)}{y}\,dy = a\int\left(\frac{1}{y}-1\right)\,dy = a\left(\ln|y|-y\right)+C_2.$$

Setting these equal and recalling that the quantities x and y are positive, we get

$$\ln y - y = (-b/a)\left(\ln x - x\right) + C = \ln x^{-b/a} + bx/a + C.$$

We solve (by exponentiating and then letting $K = e^C$):

$$ye^{-y} = x^{-b/a}e^{bx/a}e^{C}$$

$$-\,ye^{-y} = -Kx^{-b/a}e^{bx/a}$$

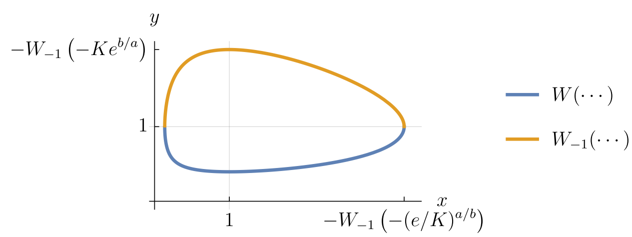

Figure 2 The solution curve in terms of two branches of the W function.

This has the form of the Lambert W equation (1), with $-y$ instead of y and a complicated expression on the right-hand side instead of x. In the case of the differential equation (12), both the principal branch W (where $W(x) \geq -1$) and the other branch W_{-1} (where $W_{-1}(x) \leq -1$) are valid solutions:

$$y = -W(-Kx^{-b/a}e^{bx/a}),\; -W_{-1}(-Kx^{-b/a}e^{bx/a}).$$

The two branches are shown in Figure 2. A point (x, y) on the graph represents the quantities $x(t)$ and $y(t)$ at some time t. As t increases, $x(t)$ and $y(t)$ change. Consider the signs of dx/dt and dy/dt in the system of equations (11). *Does the point (x, y) move clockwise or counterclockwise as t increases?*⋆[4] ⋆Q

[4] Spoiler: ccw.

2 Exercises

Exercises 1, 2, 3, and 6 do not require calculus; the others do.

1. Evaluate the Lambert W function:
(a) $W(3e^3)$. **(b)** $W(2e^2)$. **(c)** $W(e)$. **(d)** $W(0)$. **(e)** $W(2\ln 2)$.
(f) $W(e^2(4+\ln 4))$ (hard puzzle).

2. Prove the identity (3), $e^{W(x)} = x/W(x)$.

3. Prove the identity (4), $\ln W(x) = \ln x - W(x)$. Hint: $y = W(x)$ in equation (1).

4. Use implicit differentiation to prove identity (5):

$$W'(x) = \frac{1}{x + e^{W(x)}} = \frac{W(x)}{x(1+W(x))}.$$

5. Find and simplify the second derivative of the Lambert W function.

6. Use the Lambert W function to solve the following equations:
(a) $W(x) = 11$. **(b)** $x5^x = 2$. **(c)** $x^4e^x = 16$. **(d)** $x + \ln x = a$.
(e) A bit more involved: $2^x + x = a$. (Raise 2 to the power of each side and don't give up; you might need example 3 from section 1.1. Or subtract x then divide by 2^x.)

7. Let z be a positive constant, and let $h(x) = z^x$ be the tower-of-powers function from section 1.3, and let $x_* = -W(-\ln z)/\ln z$ be the fixed point of the iteration $x_{t+1} = h(x_t)$. (You will need to use properties from section 1.)
(a) Evaluate $h'(x)$ at the fixed point $x_* = -W(-\ln z)/\ln z$ and show it is equal to $-W(-\ln z)$.
(b) Solve $|h'(x_*)| = 1$ for z (which yields the critical boundary values for convergence).

8. Differentiate w^2e^w and deduce that $\int w^2e^w\,dx = w^2e^w - 2\int we^w\,dw$.

9. Use the substitution (9) to find the following integrals:
(a) $\displaystyle\int \frac{W(x)}{x}\,dx$.
(b) $\displaystyle\int \frac{\ln W(x)}{x}\,dx$, given $\int \ln x\,dx = x\ln x - x + C$ (from section ??).
(c) $\displaystyle\int_1^{2+\ln 2} W(e^t)\,dt$. Hint: Sub. $e^t = we^w$ (or $t = \ln x$ & follow with $x = we^w$).

10. Use the substitution (9) and formula (8) to find the following:

(a) $\int we^{2w}\,dw$. (Hint: First substitute $y = 2w$ before using (8).)

(b) $\int \frac{x}{W(x)}\,dx$. **(c)** $\int e^{W(x)}\,dx$.

11. Consider numerically solving the Lambert W equation $ye^y = x$. Then the function we want to find the root of is $f(y) = ye^y - x_0$ for a fixed value $x = x_0$. Let y_k denote the current estimate of the root. Note that in this setup, the roles of x and y are switched, and y is the independent variable. Thus the equation of a line looks like $x - x_1 = m(y - y_1)$, where m is the slope.

(a) (Newton's method.) Solve for the y intercept of the tangent line to $x = f(y)$ at $y = y_k$. Simplify. The result is the value of the updated estimate y_{k+1} of the root to the equation.

(b) (Halley's method.) Let $g(y) = f(y)/\sqrt{f'(y)}$. Solve for the y intercept of the tangent line to $x = g(y)$ at $y = y_k$. Simplify. The result is the value of the updated estimate y_{k+1} of the root to the equation.

2.1 *Answers and hints.*

1. To find $W(x_0)$ in these cases, set up $ye^y = x_0$ and guess y. **(a)** 3. **(b)** 2. **(c)** 1. **(d)** 0. **(e)** $\ln 2$. **(f)** $2 + \ln 2$. **2.** Solve the implicit equation (1) for $e^y = e^{W(x)}$. **3.** Take the logarithm of both sides of the implicit equation (1): $\ln(ye^y) = \ln x$. Since $\ln(ye^y) = W(x) + \ln W(x)$, we can solve for $\ln W(x)$ to obtain the identity. **4.** Implicit differentiation yields $\frac{dy}{dx}e^y + ye^y\frac{dy}{dx} = 1$. Now substitute $ye^y = x$ and solve for dy/dx; then apply identity (3). **5.** $W''(x) = -\left[W(x)^2(W(x)+2)\right]\big/\left[x^2(W(x)+1)^3\right]$. **6:** **(a)** $x = 11e^{11}$ by def. **(b)** $x = W(2\ln 5)/\ln 5$. **(c)** $x = 4W(1/2), 4W(-1/2)$. **(d)** Exponentiate: $x + \ln x = a \overset{e^\bullet}{\Rightarrow} xe^x = e^a \Rightarrow x = W(e^a)$. **(e)** $2^x + x = a \overset{2^\bullet}{\Rightarrow} 2^{2^x}2^x = 2^a \overset{u=2^x}{\Rightarrow} u2^u = 2^a \overset{\text{Ex. 3}}{\Rightarrow} u = 2^x = W(2^a\ln 2)/\ln 2 \Rightarrow x = \frac{\ln(W(2^a\ln 2)/\ln 2)}{\ln 2} = \frac{\ln(W(2^a\ln 2)) - \ln\ln 2}{\ln 2} \overset{(4)}{=} \frac{\ln(2^a\ln 2) - W(2^a\ln 2) - \ln\ln 2}{\ln 2} = \frac{a\ln 2 + \ln\ln 2 - W(2^a\ln 2) - \ln\ln 2}{\ln 2} = a - \frac{W(2^a\ln 2)}{\ln 2}$. **7:** **(a)** $h'(x) = \frac{d}{dx}z^x = \frac{d}{dx}e^{x\ln z} = e^{x\ln z}\ln z$. $h'(x_*) = e^{[-W(-\ln z)/\ln z]\ln z}\ln z = \left(e^{W(-\ln z)}\right)^{-1}\ln z \overset{(3)}{=} \left(-\ln z/W(-\ln z)\right)^{-1}\ln z = -W(-\ln z)$. **(b)** Solve $h'(x_*) = \pm 1$ separately: $-W(-\ln z) = -1 \Rightarrow W(-\ln z) = 1 \overset{\text{def.}}{\Rightarrow} 1e^1 = -\ln z \Rightarrow z = e^{-e}$. $-W(-\ln z) = 1 \Rightarrow W(-\ln z) = -1 \overset{\text{def.}}{\Rightarrow} -1e^{-1} = -\ln z \Rightarrow z = e^{1/e}$. **8.** Integrating both sides of $\frac{d}{dw}w^2e^w = w^2e^w + 2we^w$ yields $w^2e^w = \int w^2e^w\,dw + \int 2we^w\,dw$; then subtract $\int 2we^w\,dw$ from both sides. **9:** **(a)** $\frac{1}{2}W(x)(W(x)+2)+C$. **(b)** $\frac{1}{2}\left(\ln W(x)\right)^2 + W(x)(\ln(W(x))-1)+C$. **(c)** $5/2$. (The sub. $t = \ln x$ turns the indefinite form of this integral into the integral in part (a).) **10:** **(a)** $\frac{1}{4}e^{2w}(2w-1)+C$. **(b)** $x^2(2W(x)+1)/(4W(x)^2)+C$. **(c)** Same as part (b). **11.** Hint: The equation of the tangent has the form $x - f(y_k) = f'(y_k)(y - y_k)$ for Newton's method and with g replacing f in Halley's method. **(a)** $y_{k+1} = (x_0e^{-y_k} + y_k^2)/(y_k+1)$. **(b)** $\left[x_0\left(y_k^2+4y_k+2\right) + e^{y_k}y_k^3\right] \big/ \left[x_0\left(y_k+2\right) + e^{y_k}\left(y_k^2+2y_k+2\right)\right]$.