Multivariable Calculus Unit 3 Study Guide

1 Continuity

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

Test the following:

$$\lim_{along\ x-axis} \frac{x^2*0}{x^4+0^2} = \lim_{x\to 0} \frac{0}{x^4} = 0$$

$$\lim_{along\ y-axis} \frac{0^2*y}{0^4+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

$$\lim_{along\ y=x} \frac{x^2*x}{x^4+x^2} = \lim_{x\to 0} \frac{x^3}{x^4+x^2} = 0$$

$$\lim_{along\ y=x^2} \frac{x^2*x^2}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

2 Total Differential and Linear Approximation

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

Let $f(x,y) = \sqrt{x^2+y^2}$, use the total differential to approximate $f(1.02, 1.99)$.

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}\frac{\partial}{\partial x}(x^2+y^2) = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}}\frac{\partial}{\partial y}(x^2+y^2) = \frac{y}{\sqrt{x^2+y^2}}$$

$$f(1,2) = \sqrt{5} \quad \Delta x = 0.02 \quad \Delta y = -0.01$$

$$\Delta f = \frac{1}{\sqrt{1^2+2^2}}(0.02) + \frac{2}{\sqrt{1^2+2^2}}(-0.01) = 0$$

$$f(1.02, 1.99) = f(1,2) + \Delta f = \sqrt{5}$$

3 Chain Rule

$$f(x(t), y(t))$$

$$\begin{array}{ccc} & f & \\ x & & y \\ | & & | \\ t & & t \end{array}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Let $z^2x - zy^2 + 2 = 0$ where $x = rcos\theta$ and $y = rsin\theta$. Find $\frac{\partial z}{\partial r}$ when $r = 2$ and $\theta = \frac{\pi}{2}$.

$$\begin{array}{ccccc} & & z & & \\ & x & & y & \\ r & \theta & & r & \theta \end{array}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

Let $f(x, y, z) = z^2x - zy^2 + 2$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{z^2}{2xz - y^2}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = -\frac{-2yz}{2xz - y^2}$$

$$\frac{\partial x}{\partial r} = cos\theta$$

$$\frac{\partial y}{\partial r} = sin\theta$$

$$\frac{\partial z}{\partial r} = \frac{-z^2}{2xz - y^2}cos\theta + \frac{2yz}{2xz - y^2}sin\theta$$

Calculate x, y, z when $r = 2$ and $\theta = \frac{\pi}{2}$

$$x = 2cos\frac{\pi}{2} = 0$$

$$y = 2sin\frac{\pi}{2} = 2$$

$$z^2 * 0 - z * 2^2 + 2 = 0, z = \frac{1}{2}$$

$$\frac{\partial z}{\partial r} = \frac{-(\frac{1}{2})^2}{(2)(0)(\frac{1}{2}) - (2)^2}cos(\frac{\pi}{2}) + \frac{(2)(2)(\frac{1}{2})}{(2)(0)(\frac{1}{2}) - (2)^2}sin(\frac{\pi}{2}) = -\frac{1}{2}$$

4 Directional Derivative

The directional derivative in the given direction $\vec{u}$ at point $P : (x_0, y_0, z_0)$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

$$\nabla f(x_0, y_0, z_0) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

ex. Find $D_{\vec{u}} f$ of $f(x, y) = x^2 + 2y^2$ in the direction $\langle 3, 4 \rangle$ at $(1, 2)$

$$\nabla f = \langle 2x, 4y \rangle \to \langle 2, 8 \rangle$$

$$\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$$

$$D_{\vec{u}} f = \langle 2, 8 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{38}{5}$$

The unit vector $\vec{u}$ in the direction of fastest increase or maximum $D_{\vec{u}} f$ given a point $P : (x_0, y_0, z_0)$

$$\vec{u} = \nabla f(x_0, y_0, z_0)$$

ex. Find a unit vector in the direction in which $f(x, y) = (x^2 + y^2)^{\frac{3}{2}}$ increases most rapidly at the point $(5, -13)$

$$\vec{u} = \langle \frac{3}{2}(x^2 + y^2)^{\frac{1}{2}}(2x), \frac{3}{2}(x^2 + y^2)^{\frac{1}{2}}(2y) \rangle \to \langle 15\sqrt{194}, -39\sqrt{194} \rangle$$

$$\vec{u} = \langle \frac{15\sqrt{194}}{582}, \frac{-39\sqrt{194}}{582} \rangle = \langle \frac{5\sqrt{194}}{194}, \frac{-13\sqrt{194}}{194} \rangle$$

5 Tangent Planes and Normal Vectors

Normal vector $\vec{n}$ to the tangent plane at point $P : (x_0, y_0, z_0)$

$$\vec{n} = \nabla f(x_0, y_0, z_0)$$

Tangent plane equation at point P

$$T = \nabla f(x_0, y_0, z_0) * \langle x - x_0, y - y_0, z - z_0 \rangle$$

Line normal to the surface at point P

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \nabla f(x_0, y_0, z_0)$$

Given surface $2z^2 cos(xy) + \frac{z}{x} + 2$ find the normal vector $\vec{n}$ and the tangent plane at the point $(\frac{1}{3}, \pi, -1)$.

$$w(x, y, z) = 2z^2 cos(xy) + \frac{z}{x} + 2$$

$$\nabla w = \langle -2yz^2 sin(xy) - \frac{z}{x^2}, -2xz^2 sin(xy), 4zcos(xy) + \frac{1}{x} \rangle$$

$$\vec{n} = \nabla w(\frac{1}{3}, \pi, -1) = \langle -\sqrt{3}\pi + 9, -\frac{\sqrt{3}}{3}, 1 \rangle$$

$$T = \langle -\sqrt{3}\pi + 9, -\frac{\sqrt{3}}{3}, 1 \rangle \cdot \langle x - \frac{1}{3}, y - \pi, z + 1 \rangle = (9 - \sqrt{3}\pi)x + \frac{2\pi - y}{\sqrt{3}} + z - 2$$

$$\vec{r}(t) = \langle \frac{1}{3}, \pi, -1 \rangle + t\langle -\sqrt{3}\pi + 9, -\frac{\sqrt{3}}{3}, 1 \rangle$$

6 Optimization and 2nd Derivative Test

Second derivative in the direction $\vec{u} = \langle u_1, u_2 \rangle$

$$D_{\vec{u}}^2 f = f_{xx}(u_1 + \frac{u_2 f_{xy}}{f_{xx}})^2 + \frac{u_2^2}{f_{xx}}(f_{xx}f_{yy} - {f_{xy}}^2)$$

Find critical points

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = 0$$

Second derivative test

$$D = f_{xx}f_{yy} - {f_{xy}}^2$$

Local minimum if $D > 0$ and $f_{xx} > 0$
Local maximum if $D > 0$ and $f_{xx} < 0$
Saddle point if $D < 0$

Optimization given constraints using Lagrange Multipliers

$$f(x,y) = xye^{\frac{-(x^2+y^2)}{2}} \qquad x^2 + y^2 \leq 4$$

$$g(x,y) = x^2 + y^2$$

$$\nabla f = \langle ye^{-\frac{(x^2+y^2)}{2}}(1 - x^2), xe^{-\frac{(x^2+y^2)}{2}}(1 - y^2) \rangle \qquad \nabla g = \langle 2x, 2y \rangle$$

$$\nabla f = \lambda \nabla g$$

$$ye^{-\frac{(x^2+y^2)}{2}}(1 - x^2) = 2x\lambda$$

$$xe^{-\frac{(x^2+y^2)}{2}}(1 - y^2) = 2y\lambda$$

$$x^2 + y^2 = 4$$

$$\frac{y(1 - x^2)}{x(1 - y^2)} = \frac{x}{y}$$

$$y^2(1 - x^2) = x^2(1 - y^2)$$

$$y^2 - x^2y^2 = x^2 - x^2y^2$$
$$y^2 = x^2$$
$$x^2 + x^2 = 4 \to (x, y) = (\pm\sqrt{2}, \pm\sqrt{2})$$
$$f(\sqrt{2}, \sqrt{2}) = f(-\sqrt{2}, -\sqrt{2}) = 2e^{-2}$$
$$f(\sqrt{2}, -\sqrt{2}) = f(-\sqrt{2}, \sqrt{2}) = -2e^{-2}$$

Absolute maximum at $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$
Absolute minimum at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$