Multivariable Calculus Unit 4 Study Guide

1 Double Integrals

1.1 Definition:

$$\iint\limits_R f(x,y)\,dA$$

The area of the region R under the surface formed by $f(x,y)$ is computed by summing the tiny areas dA.

$$dA = dxdy \quad \text{or} \quad dA = dydx$$

1.2 Switching Order of Integration:

$$\int_0^2 \int_0^{\sqrt{x}} f(x,y)\,dydx$$

Given that x ranges from 0 to 2, y varies from 0 to $\sqrt{x}$. Thus the bound is $y = \sqrt{x} \to x = y^2$, $x = 0$, and $x = 2$. Plugging in $x = 0$ and $x = 2$ gives that y ranges from 0 to $\sqrt{2}$. Given the bounds, x varies from y^2 to 2.

$$\int_0^{\sqrt{2}} \int_{y^2}^2 f(x,y)\,dxdy$$

2 Volume Given Bounds

2.1 Given integer bounds:

Example: $\iint\limits_R xcos(xy)cos^2(\pi x)\,dA$, given $R = [0, \frac{1}{2}] \times [0, \pi]$

$$\int_0^{\frac{1}{2}} \int_0^{\pi} xcos(xy)cos^2(\pi x)\,dydx$$

$$= \int_0^{\frac{1}{2}} \left[cos^2(\pi x)sin(xy)\right]_0^{\pi}\,dx$$

$$= \int_0^{\frac{1}{2}} \left[cos^2(\pi x)sin(\pi x)\right]\,dx$$

$$= \left[-\frac{1}{3\pi} cos^3(\pi x) \right]_0^{\frac{1}{2}}$$

$$= \frac{1}{3\pi}$$

2.2 Given equation bounds for $z = f(x, y)$:

Example: $x^2 + y^2 = 9$, $z = 0$, and $z = 3 - x$

$$V = \iint\limits_R (3 - x)\, dA$$

Considering that x ranges from -3 to 3, then y must range from $-\sqrt{9 - x^2}$ to $\sqrt{9 - x^2}$.

$$V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (3 - x)\, dydx$$

$$= \int_{-3}^{3} (6\sqrt{9 - x^2} - 2x\sqrt{9 - x^2})\, dx$$

$$= 27\pi$$

2.3 Switching to Polar:

Switch x to $r\cos(\theta)$ and y to $r\sin(\theta)$. Switch dA to $rdrd\theta$. Switch bounds from rectangular coordinates to polar coordinates.

Example: Find the volume of a sphere given the equation $x^2 + y^2 + z^2 = 9$ which is ouside the cylinder $x^2 + y^2 = 1$.

$$z = \sqrt{9 - x^2 - y^2}$$

$$V = \iint\limits_R \sqrt{9 - x^2 - y^2}\, dA = \iint\limits_R \left(\sqrt{9 - r^2} \right) r\, drd\theta$$

Given the bounds, r ranges from 1 to 3, and θ ranges from 0 to 2π.

$$V = 2 * 4 \int_0^{\frac{\pi}{2}} \int_1^3 \left(\sqrt{9 - r^2} \right) r\, drd\theta$$

$$= 8 \int_0^{\frac{\pi}{2}} \frac{16\sqrt{2}}{3}\, d\theta$$

$$= \frac{64\sqrt{2}}{3}\pi$$

3 Area Given Bounds

3.1 Given equation bounds for $z = f(x, y)$:

Example: Area under the surface $\frac{x}{\sqrt{1+y^2}}$ enclosed by $y = x^2$, $y = 4$, and $x = 0$.

$$\iint\limits_R \frac{x}{\sqrt{1+y^2}}\, dA$$

Considering that y ranges from 0 to 4, then x must range from 0 to $\sqrt{y}$.

$$\int_0^4 \int_0^{\sqrt{y}} \frac{x}{\sqrt{1+y^2}}\, dxdy$$

$$= \int_0^4 \left[\frac{x^2}{2\sqrt{1+y^2}}\right]_0^{\sqrt{y}} dy$$

$$= \int_0^4 \frac{y}{2\sqrt{1+y^2}}\, dy$$

$$= \frac{\sqrt{17}-1}{2}$$

3.2 Given polar bounds:

Example: The region enclosed by the cardioid $r = 1 - cos(\theta)$.

$$\iint\limits_R 1\, dA = \iint\limits_R r\, drd\theta$$

Given bounds, θ ranges from 0 to 2π, and r ranges from 0 to $1 - cos(\theta)$.

$$A = \int_0^{2\pi} \int_0^{1-cos(\theta)} r\, drd\theta$$

$$= \int_0^{2\pi} \frac{(1-cos(\theta))^2}{2}\, d\theta$$

$$= \frac{3\pi}{2}$$

4 Surface Area

Using Cross Product: $dS = |\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}|\, dudv$
Given $f(u, v)$, find $\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$.

$$\frac{\partial \vec{r}}{\partial u} = \langle 1, 0, \frac{\partial f}{\partial u}\rangle$$

$$\frac{\partial \vec{r}}{\partial v} = \langle 0, 1, \frac{\partial f}{\partial v} \rangle$$

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = \left| \left\langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, 1 \right\rangle \right| = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1}$$

Using Jacobian: $dS = |J|\, dudv$, given $\vec{r}(u, v) = \langle \vec{r}_1(u, v), \vec{r}_2(u, v) \rangle$

$$|J| = \begin{vmatrix} \frac{\partial \vec{r}_1}{\partial u} & \frac{\partial \vec{r}_2}{\partial u} \\ \frac{\partial \vec{r}_1}{\partial v} & \frac{\partial \vec{r}_2}{\partial v} \end{vmatrix}$$

4.1 Given integer bounds:

Example: $y^2 + z^2 = 9$, given $R = [0, 2] \times [-3, 3]$

$$z = \sqrt{9 - y^2}$$

$$\vec{r}(x, y) = \langle x, y, \sqrt{9 - y^2} \rangle$$

$$\frac{\partial \vec{r}}{\partial x} = \langle 1, 0, 0 \rangle$$

$$\frac{\partial \vec{r}}{\partial y} = \langle 0, 1, -\frac{y}{\sqrt{9 - y^2}} \rangle$$

$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \sqrt{1 + \left(-\frac{y}{\sqrt{9 - y^2}}\right)^2} = \sqrt{\frac{9 - y^2 + y^2}{9 - y^2}} = \frac{3}{\sqrt{9 - y^2}}$$

$$S = \int_0^2 \int_{-3}^3 \frac{3}{\sqrt{9 - y^2}}\, dydx$$

$$= \int_0^2 3\pi\, dx$$

$$= 6\pi$$

4.2 Given equation bounds:

Example: $z^2 = 4x^2 + 4y^2$ bounded by $y = x$ and $y = x^2$

$$\vec{r}(x, y) = \langle x, y, \sqrt{4x^2 + 4y^2} \rangle$$

$$2z\frac{\partial z}{\partial x} = 8x \qquad 2z\frac{\partial z}{\partial y} = 8y$$

$$\frac{\partial z}{\partial x} = \frac{4x}{z} \qquad \frac{\partial z}{\partial y} = \frac{4y}{z}$$

$$\frac{\partial \vec{r}}{\partial x} = \langle 1, 0, \frac{4x}{z} \rangle$$

$$\frac{\partial \vec{r}}{\partial y} = \langle 0, 1, \frac{4y}{z} \rangle$$

$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \sqrt{\left(\frac{4x}{z}\right)^2 + \left(\frac{4y}{z}\right)^2 + 1} = \sqrt{\frac{4(4x^2 + 4y^2)}{z^2} + 1} = \sqrt{4+1} = \sqrt{5}$$

Given the bounds, x ranges from 0 to 1, and y ranges from x^2 to x.

$$S = \int_0^1 \int_{x^2}^x \sqrt{5}\, dydx$$

$$= \int_0^1 \sqrt{5}(x - x^2)\, dx$$

$$= \frac{\sqrt{5}}{6}$$

4.3 Switching to Polar:

Example: $z = xy$ bounded by $y = \frac{x}{\sqrt{3}}$, $y = 0$, and $x^2 + y^2 = 9$

$$\vec{r}(x, y) = \langle x, y, xy \rangle$$

$$\frac{\partial \vec{r}}{\partial x} = \langle 1, 0, y \rangle$$

$$\frac{\partial \vec{r}}{\partial y} = \langle 0, 1, x \rangle$$

$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \sqrt{x^2 + y^2 + 1} = \sqrt{r^2 + 1}$$

Given the bounds, r ranges from 0 to 3 because of the circle equation, and θ ranges from 0 to $\frac{\pi}{6}$ because of the line equation and that $tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$.

$$S = \int_0^{\frac{\pi}{6}} \int_0^3 (\sqrt{r^2 + 1}) r\, drd\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{3}(10\sqrt{10} - 1)\, d\theta$$

$$= \frac{\pi}{18}(10\sqrt{10} - 1)$$