Problem 7

a) Let X be a non-negative random variable with probability density function f, and let $a > 0$. We need to prove:

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Definition of expected value:

The expected value of X is given by:

$$E(X) = \int_0^\infty xf(x)dx$$

Since X is non-negative, the integral is from 0 to ∞.

Rewriting the integral:

We can split the integral into two parts:

$$E(X) = \int_0^a xf(x)dx + \int_a^\infty xf(x)dx$$

For $x \geq a$ in the second integral:

In the second integral, x takes on values greater than or equal to a. Therefore, we can write:

$$xf(x) \geq af(x) \text{ for } x \geq a$$

This implies:

$$\int_a^\infty xf(x)dx \geq \int_a^\infty af(x)dx$$

Relating with probability function

Pulling a out of the integral: $\int_a^\infty af(x)dx = a\int_a^\infty f(x)dx$ We know that $\int_a^\infty f(x)dx$ represents $P(X \geq a)$ (the probability that X is greater than or equal to a).

Therefore:

$$\int_a^\infty xf(x)dx \geq aP(X \geq a)$$

Combining the inequalities

Going back to our expression for $E(X)$:

$$
\begin{aligned}
E(X) &= \int_0^a xf(x)dx + \int_a^\infty xf(x)dx \\
E(X) &\geq \int_a^\infty xf(x)dx \text{ (since the first integral is non-negative)} \\
E(X) &\geq aP(X \geq a)
\end{aligned}
$$

Rearranging to get Markov's inequality

Divide both sides by a to obtain:

$$P(X \geq a) \leq \frac{E(X)}{a}$$

b) To prove:

$$P(X \leq a) \geq \frac{E(X \cdot I(X \leq a))}{a}$$

where I is the indicator function ($I(X \leq a) = 1$ if $X \leq a$, and 0 otherwise).

Using Definition of expected value:

$$E(X \cdot I(X \leq a)) = \int_0^\infty x \cdot I(X \leq a) \cdot f(x)dx$$

Simplifying using the indicator function:

$$E(X \cdot I(X \leq a)) = \int_0^a x \cdot I(X \leq a) \cdot f(x)dx + \int_a^\infty x \cdot I(X \leq a) \cdot f(x)dx$$

The indicator function makes the integrand zero for $x > a$ and the function is 1 for $x < a$:

$$E(X \cdot I(X \leq a)) = \int_0^a xf(x)dx$$

Fact that $x \leq a$ in the integral:

$$xf(x) \leq af(x) \text{ for } x \leq a$$

This implies:

$$\int_0^a xf(x)dx \leq \int_0^a af(x)dx$$

Relating to probability fucntion

Pull a out of the integral: $\int_0^a af(x)dx = a\int_0^a f(x)dx$ We know that $\int_0^a f(x)dx$ represents $P(X \leq a)$

Therefore:

$$\int_0^a xf(x)dx \leq aP(X \leq a)$$

Combining the inequalities:

$$E(X \cdot I(X \leq a)) = \int_0^a xf(x)dx$$
$$E(X \cdot I(X \leq a)) \leq aP(X \leq a)$$

Divide both sides by a to obtain:

$$P(X \leq a) \geq \frac{E(X \cdot I(X \leq a))}{a}$$

c) We need to prove that for any random variable X with expectation $\mu < \infty$ and variance $\sigma^2 > 0$:

$$P(|X - \mu| \geq n\sigma) \leq \frac{1}{n^2} \text{ for } n > 0.$$

Markov's inequality:

Markov's inequality (proved in part (a)): For a non-negative random variable Y and any $a > 0$,

$$P(Y \geq a) \leq \frac{E(Y)}{a}$$

Defining a new random variable:

Let $Y = (X - \mu)^2$. Note that Y is non-negative.

Applying Markov's inequality to Y:

Let $a = (n\sigma)^2 = n^2\sigma^2$. Applying Markov's inequality to Y and a, we get:

$$P(Y \geq n^2\sigma^2) \leq \frac{E(Y)}{n^2\sigma^2}$$

Substituting and simplifying

Substitute $Y = (X - \mu)^2$:

$$P((X-\mu)^2 \geq n^2\sigma^2) \leq \frac{E((X-\mu)^2)}{n^2\sigma^2}$$

$E((X-\mu)^2)$ is the variance, i.e σ^2:

$$P((X-\mu)^2 \geq n^2\sigma^2) \leq \frac{\sigma^2}{n^2\sigma^2}$$

Simplifying:

$$P((X-\mu)^2 \geq n^2\sigma^2) \leq \frac{1}{n^2}$$

The event $(X-\mu)^2 \geq n^2\sigma^2$ is equivalent to the event $|X-\mu| \geq n\sigma$. Therefore:

$$P(|X-\mu| \geq n\sigma) \leq \frac{1}{n^2}$$