AM5530 Programming Assignment

Veerendra Harshal
ME17B124

April 2020

Contents

1 Introduction

1.1 Problem Definition

The objective is to obtain the Blasius solution for laminar boundary layer over a flat plate in zero pressure gradient numerically using Runge-Kutta method and Newton-Raphson methods.

1.2 Governing Equation

$$f''' + \frac{1}{2}ff'' = 0 \tag{1}$$

$$Where, \quad f = f(\eta) \quad \eta = \frac{y}{\delta} \quad and \quad \delta = \sqrt{\frac{x\nu}{U_\infty}}$$

1.3 Boundary Conditions

$$For \quad \eta = 0, \quad f(0) = 0 \quad f'(0) = 0 \quad f''(0) = s$$

$$For \quad \eta \to \infty, \quad f'(\infty) = 1 \quad f'''(\infty) = 0$$

2 Numerical Formulation

We convert the third order ordinary differential equation into a system of 3 ordinary differential equations.

2.1 Variables

For the same, we introduce 3 functions of η that even though are dependant on each other , can be used to generate separate differential equations.

$$f = f(\eta) \quad u = f'(\eta) \quad v = f''(\eta)$$

2.2 Governing Equation(s)

To convert the present equation

$$f''' + \frac{1}{2}ff'' = 0$$

We shall use the previous two new variable introductions.

$$u = f'(\eta) \tag{2}$$

$$v = f''(\eta) \tag{3}$$

And substitute in the given governing equation to generate the final equation.

$$v' = -\frac{1}{2}fv \tag{4}$$

2.3 Boundary Conditions

For the new variables, new Boundary Conditions have to be defined.

$$f(0) = 0 \quad u(0) = \quad v(0) = s$$

3 Numerical Methods Used

3.1 Runge-Kutta Method

3.1.1 Definition

The most widely known member of the Runge–Kutta family is generally referred to as "RK4", the "classic Runge–Kutta method" or simply as "the Runge–Kutta method".
Let an initial value problem be specified as follows:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

Here y is an unknown function (scalar or vector) of time t, which we would like to approximate; we are told that $\frac{dy}{dt}$, the rate at which y changes, is a function of t and of y itself. At the initial time t_0 the corresponding y value is y_0. The function f and the initial conditions t_0, y_0 are given.
Now pick a step-size h > 0 and define

$$y_{n+1} = y_n + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right) t_{n+1} = t_n + h$$

for n = 0, 1, 2, 3, ..., using

$$k_1 = \ h \times f(t_n, y_n)$$

$$k_2 = \ h \times f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = \ h \times f\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = \ h \times f\left(t_n + h, y_n + k_3\right)$$

3.1.2 Runge Kutta Analysis

The Runge Kutta Method solves equations in the form of:

$$y' = g(x, y) \quad y(x_o) = y_o$$

Hence we rewrite our 3rd order Governing Equation that is now a Trivariate system of Simultaneous Differential Equations in this form.

$$y_i' = g_i(x, y_1, y_2, y_3) \quad y_i(x_o) = y_o$$

Which gives us

$$f' = u, \quad f(0) = 0 \tag{5}$$

$$u' = v, \quad u(0) = 0 \tag{6}$$

$$v' = -\frac{1}{2} f v, \quad v(0) = s \tag{7}$$

Now this system can be solved simultaneously for $f(\eta), u(\eta), v(\eta)$ for a given value of s.

3.2 Newton-Raphson Method

3.2.1 Definition

If the function satisfies sufficient assumptions and the initial guess is close, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

is a better approximation of the root than x_o. Geometrically, $(x_1, 0)$ is the intersection of the x-axis and the tangent of the graph of f at $(x_0, f(x_0))$: that is, the improved guess is the unique root of the linear approximation at the initial point. The process is repeated as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until a sufficiently precise value is reached.

4 Numerical Analysis

4.1 Flowchart

The flowchart describes what variables are created and used for the programs.

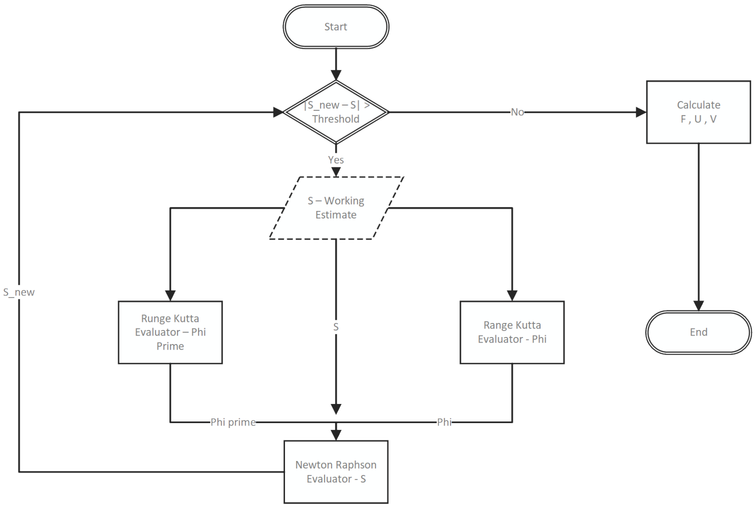

Figure 1: Flow Chart detailing the process.

4.2 Main File

The array eta is defined that covers the interval with equi-distant spacing. Then a loop is defined that shall run until convergence. Furthermore, the results are processed.

4.3 Accessory Function Files

4.3.1 Runge Kutta Evaluator - Coefficients

This function takes in the parameter s and returns a K-matrix that consists for a given f, u and v, the Runge-Kutta Coefficients each for f , u and v.

Input Takes in a 3×1 array which represents [f,u,v](η) and the step size h.

Output Returns a 3×4 array which gives row-wise the Runge-Kutta Coefficients k_i for f,u and v respectively.

4.3.2 Runge Kutta Evaluator - Φ

This function is for running Runge Kutta Method over eta. This takes in K-Matrix for each eta generated by the *Runge Kutta Evaluator - Coefficients* function by giving it the $[f, u, v](\eta)$.

Input Takes in the working estimate of the parameter s.

Output Returns the Error Function Φ that represents the error in the boundary condition $f'(\infty) - 1 = 0$.

4.3.3 Runge Kutta Evaluator - $\frac{\partial\Phi}{\partial s}$

This function approximates $\frac{\partial\Phi}{\partial s}$ by

$$\frac{\partial\Phi}{\partial s} = \frac{\Phi(s + \delta s) - \Phi(s)}{\delta s}$$

Input Takes in the working estimate of the parameter s.

Output Returns the derivative of the Error Function $\frac{\partial\Phi}{\partial s}$

4.3.4 Newton Raphson Evaluator - s

Function performs a Newton Raphson iteration.

Input Takes in the working estimate of the parameter s , Φ and $\frac{\partial\Phi}{\partial s}$.

Output Returns a better estimate of s , s_{new}

5 Results

5.1 Velocities

We now have numerical approximation of U and V as a function of η. To find them as functions of (x, y),

$$[U, V](\eta) = [U, V](\eta(x, y)) = [U, V](\frac{y\sqrt{U_\infty}}{\sqrt{x\nu}})$$

Eta	F	U	V
0	0	0	0.332057
1	0.165572	0.32978	0.323007
2	0.650024	0.629766	0.266752
3	1.396808	0.846044	0.16136
4	2.305747	0.955518	0.064234
5	3.283274	0.991542	0.015907
6	4.279621	0.998973	0.002402
7	5.279239	0.999922	0.00022
8	6.279214	0.999996	1.22E-05
9	7.279213	1	4.13E-07
10	8.279212	1	8.46E-09

Table 1: [F,U,V](η) for $\eta \in I$

5.2 Graphical Interpretation

5.2.1 Plot

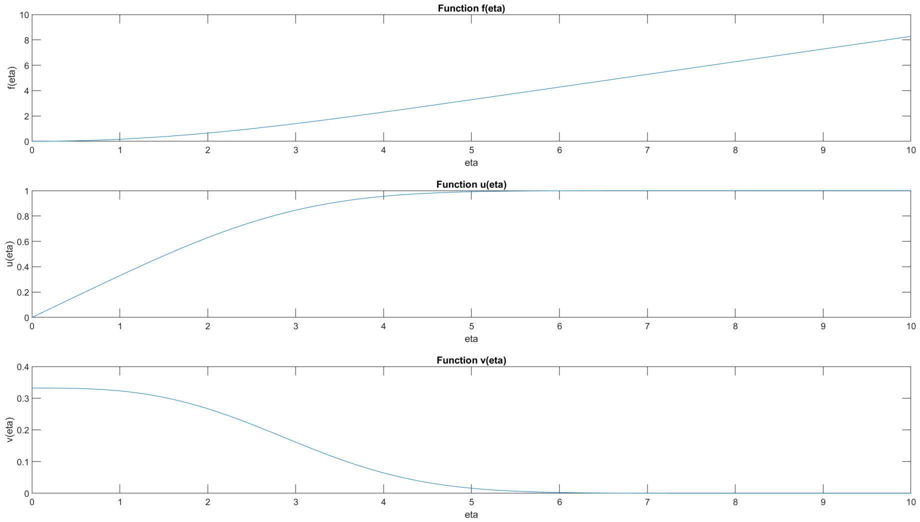

Figure 2: [F,U,V](η) .

5.2.2 Inference

$$\eta = y\sqrt{\frac{U_\infty}{\sqrt{x\nu}}}$$

Horizontal Component - U The Value of U Saturates after a while indicating that at $y \to \delta$ or $x \to 0$, $U \to U_\infty$

Vertical Component - V The Value of V decays to zero at higher η which indicates that at $y \to \delta$ or $x \to 0$, $V \to 0$

These agree with practical observations of the given physical system.