Unit - 1 Physics of vibrations

Free oscillations

The oscillatory body oscillates with undiminished amplitude with its own natural frequency of vibrations for infinite length of time under the action of restoring force, until an external force affects its motion are called free oscillations.

Ex: * Oscillation of mass suspended to spring with negligible damping and small displacement.

* Oscillations of simple pendulum
* LC oscillations

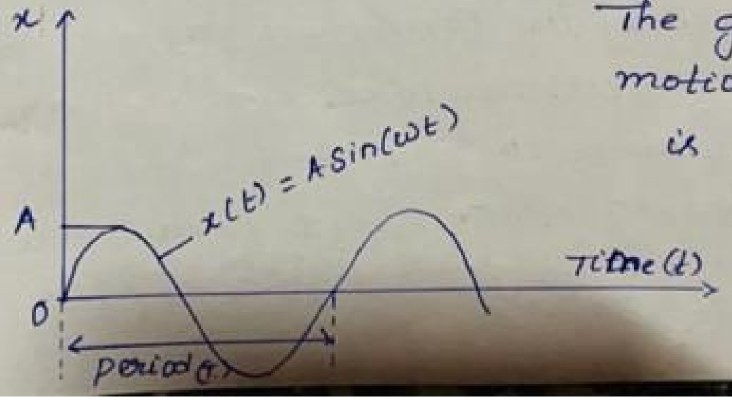
The free oscillation possess constant amplitude and period without any external force to set the oscillation. Ideally, free oscillation does not undergo damping. The frequency of free oscillation is also called as natural frequency since the frequency of free oscillation depends on the nature & the structure of the oscillating body.

Equation of Motion of Free oscillations

If 'm' is the mass of oscillating body with a force constant 'k' and 'x' as the displacement at the instant 't' of the oscillating body. then

$$\omega^2 x = 0$$

where $\omega \rightarrow$ Angular frequency



The general equation of motion for free oscillations is given by.

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

If the displacement $x(t)$ is known, then sequentially differentiating, we can find the velocity & accelaration of the body.

$$x(t) = A \sin(\omega t + \psi_0)$$

$$\therefore \quad x'(t) = -A\omega \sin(\omega t + \psi_0)$$

$$\text{ie } v(t) = -A\omega \sin(\omega t + \psi_0)$$

$$\text{ll}^{ly} \quad x''(t) = -A\omega^2 \cos(\omega t + \psi_0)$$

$$\text{ie } a(t) = v't = -A\omega^2 \cos(\omega t + \psi_0)$$

where $A \rightarrow$ Amplitude

$(\omega t + \psi_0) \rightarrow$ phase of oscillation

ψ_0 is the initial phase at time $t = 0$.

Damped vibrations or Damped oscillations

An oscillatory body oscillates such that its amplitude gradually decreases and comes to rest at equilibrium position in a finite interval of time due to the action of resistive force.

Ex: * Mechanical oscillations of simple pendulum
* Electrical oscillations of LC circuit
* A swing left free to oscillate after being pushed once.

In other words, it is a type of motion executed by a body subjected to the combined action of both the restoring force & resistive force and the motion always gets terminated with the body coming to a rest at the equilibrium position in a finite interval of time.

Differential Equation and Solution

Consider a body of mass 'm' executing vibration in resistive medium, the vibrations are damped due to the resistance offered by the medium.

Since resistive force is proportional to the velocity of the body & act opposite to its movement.

$$\text{Resistive force} = -r\frac{dx}{dt}$$

The vibrating body is constantly acted upon by restoring force whose magnitude is proportional to the displacement.

$$\text{Restoring force} = -Kx$$

The net force acting on the body is the resultant force ie sum of resistance force & restoring force.

$$F = -r\frac{dx}{dt} - Kx \longrightarrow (1)$$

According to Newton's II Law of motion.

$$F = m\frac{d^2x}{dt^2} \longrightarrow (2)$$

comparing eq^{ns} (1) & (2). we get

$$m\frac{d^2x}{dt^2} = -r\frac{dx}{dt} - Kx$$

$$\boxed{m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + Kx = 0} \longrightarrow (3)$$

Divide eq^n (3) by 'm'. we get

$$\frac{d^2x}{dt^2} + \left(\frac{r}{m}\right)\frac{dx}{dt} + \left(\frac{K}{m}\right)x = 0 \longrightarrow (4)$$

Since the natural frequency of vibration ω is given by,

$$\omega = \sqrt{k/m}$$

$$(k/m) = \omega^2$$

$$(r/m) = 2b$$

$\therefore$ eqⁿ (4) becomes,

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0$$

Eqⁿ (3) represents differential equation for damped vibrations.

The general equation for damped vibrations are given by,

$$\boxed{x = \frac{x_0}{2}\left\{\left[1+\frac{b}{\sqrt{b^2-\omega^2}}\right] e^{(-b+\sqrt{b^2-\omega^2})t} + \left[1-\frac{b}{\sqrt{b^2-\omega^2}}\right] e^{(-b-\sqrt{b^2-\omega^2})t}\right.}$$

As t varies, x also varies. But the nature of variati-on depends upon the term $\sqrt{b^2-\omega^2}$. The three possible domains of variations are,

$b^2 > \omega^2$, $b^2 = \omega^2$ and $b^2 < \omega^2$ which are dealt seperatly as follows.

Over, critical and underdamping

case (i) - overdamped / dead beat $[b^2 > \omega^2]$

It is a condition under which the restoring & the resistive forces acting on a body such that, the body is brought to a halt at the equilibrium position without oscillation.

when $b^2 > \omega^2$, $(b^2-\omega^2)$ is positive.

case (ii) critical Damping $[b^2 = \omega^2]$

It is a condition under which the restoring and resistive forces acting on a body are such that, the body is brought to a halt at the equilibrium position without oscillation, in the minimum time.

ie The body attains equilibrium position, rapidly is known as critical damping.

case (iii) under damping $[b^2 < \omega^2]$

It is a condition under which the restoring and the resistive forces acting on a body are such that the body vibrates with diminishing amplitude as the time progresses and ultimately comes to halt at the equilibrium position.

The body which is able to vibrate with diminishing amplitude, such vibrations are called under damped vibrations.

when $b^2 < \omega^2$, $(b^2-\omega^2)$ is negative.

Forced vibrations

It is a steady state of sustained vibrations of a body vibrating in a resistive medium under the action of an external periodic force which acts independently of the restoring force.

The forced vibrations survive as long as the external periodic force is present.

Ex :* Oscillations of a swing which is pushed periodica-
-lly by a person.

* The periodic variation of current in an LCR circuit driven by an AC source.

* The motion of hammer in a calling bell

Differential Equation for Force oscillations and General Solution

Consider a body of mass 'm' executing vibrations in a damping medium acted upon by an external periodic force $F \sin(pt)$, where p is the angular frequency of the external force. If 'x' is the displace-ment of the body at any instant of time 't' then the damping force which acts in a opposite direction to the movement of the body is equated to the term $\left[-r\,dx/dt\right]$, where 'r' is the damping Constant, and the restoring force is equated to the term $(-Kx)$, where 'K' is the force Constant.

The net force acting on the body is the resultant of all the three forces.

$$\therefore \text{Resultant force} = -r\frac{dx}{dt} - Kx + F \sin pt \rightarrow (1)$$

The body's motion due to the resultant force obeys the Newton's second law of motion on the basis of which we can write,

$$\text{Resultant force} = m\frac{d^2x}{dt} \rightarrow (2)$$

Equating eqⁿˢ 1 & 2

$$m\frac{d^2x}{dt^2} = -r\frac{dx}{dt} - Kx + F\sin(pt)$$

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + Kx = F\sin(pt)$$

The above eqⁿ represents the equation of motion for forced vibrations.

Dividing the above eqⁿ throughout by m, we get

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{K}{m}x = \frac{F}{m}\sin(pt) \longrightarrow (3)$$

Let $r/m = 2b$, $\omega = \sqrt{K/m}$

$\therefore$ eqⁿ (3) can be written as,

$$\boxed{\frac{d^2x}{dt^2} + 2b + \omega^2 x = \frac{F}{m}\sin(pt)} \longrightarrow (4)$$

The general solution for the above eqⁿ is given by,

$$\boxed{x = a\sin(pt - \alpha)} \longrightarrow (5)$$

where, x -> displacement

a -> Amplitude

α -> phase of the forced vibration

p -> Angular frequency of External force

t -> time at given instant

Resonance

Consider a body of mass 'm' vibrating in a resistive medium of damping Constant 'r' under the influence of an external force F Sin pt. If ω is the natural frequency of vibration for the body, then, the equation for the amplitude of vibrations is given by,

$$\boxed{a = \frac{F/m}{\sqrt{4b^2p^2 + (\omega^2 - p^2)^2}}}$$ → Amplitude for forced vibrations

The Condition at which $\omega = p$, the system will possess the ability to keep the same phase as that of the periodic force at all times and therefore the vibrating system will have the ability to receive completely the energy delivered by the periodic force.

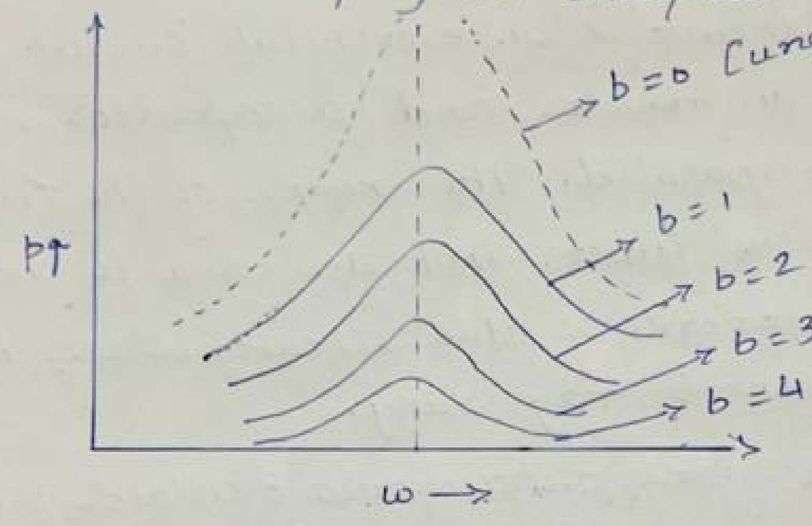
Thus resonance is defined as, " the condition when the frequency of a periodic force, acting on a vibrating body is equal to the natural frequency of vibrations of the body, the energy transfer from the periodic force to the body becomes maximum because of which the body is thrown into a state of wild oscillations.

Sharpness of Resonance

Sharpness of resonance is the rate at which the amplitude changes corresponding to a small change in the frequency of the applied External force at the stage of resonane.

$$\boxed{\text{Sharpness of resonance} = \frac{\Delta a}{\Delta t}}$$

Effect of damping on Sharpness of Resonance



The rate at which the change in amplitude occurs near resonance depends on damping. For small damping, the rate is high & the resonance is said to be sharp. For heavy damping it will be low & the resonance is said to be flat.

The above graph shows the response of amplitude to various degrees of damping.

Quality Factor

The amount of damping is described by the quantity called quality factor.

$$Q = \frac{\omega}{2b}$$

Quality factor is the number of cycles required for

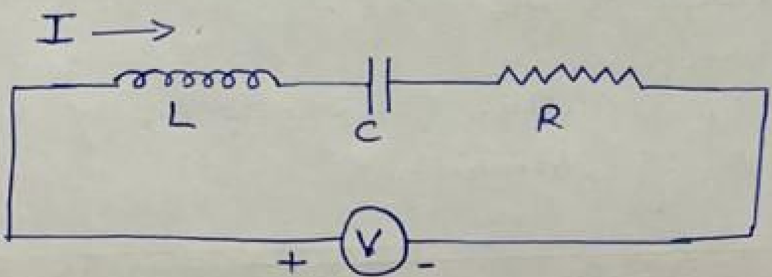
the energy to fall off by a factor $e^{2\pi}$ (≈ 535)
If Q value is more indicated the sustained oscillations overcoming the resistive forces.
Quality factor describes how much under damped is the oscillatory system.

LCR circuit and Resonance

An LCR circuit is an electrical circuit consisting of a resistor. an inductor and a capacitor, Connected in Series or in parallel. The name of the circuit is derived from the letters that are used to denote the constituent components of this circuit, where the sequence of the Components may vary.

The circuit forms a harmonic oscillator from current and resonates in a similar way as an LC circuit. Introducing the resistor increases the decay of these oscillations, which is also known as damping.

Series circuit

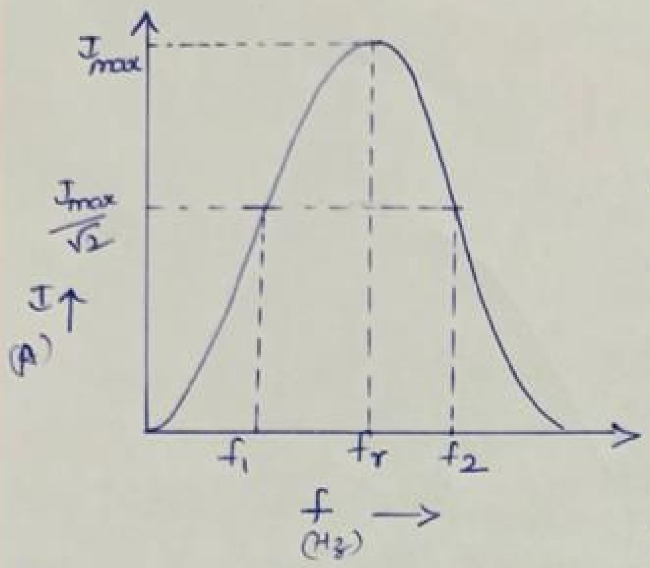


In this circuit, the three components are all in series with the voltage source.

$$V(t) = V_R + V_L + V_C$$

where V_R, V_L, V_C → Voltages across, R, L and C respectively

$v(t)$ → Time varying voltage from the source.



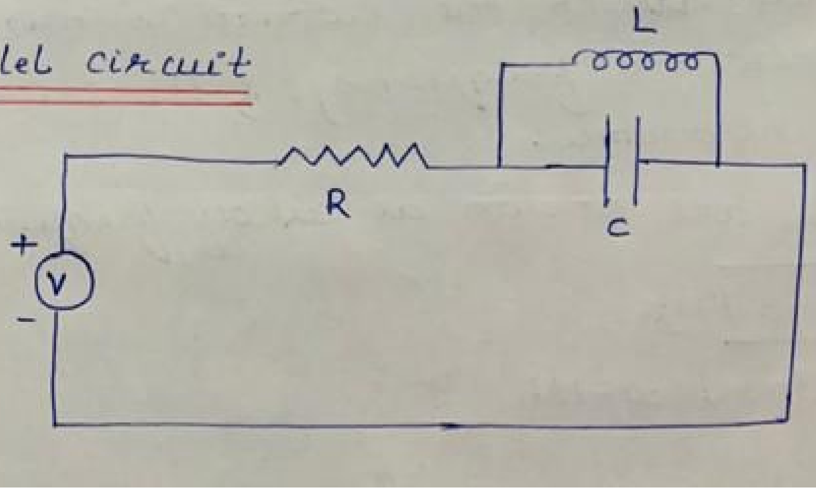
The graph shows the variation of current with respect to the frequency. As the frequency applied increases, the current also increases upto a certain point and reaches the saturation point. Then after increasing the frequency makes the current to decrease. The frequency at which the current is maximum, is known as resonant frequency and the condition is known as resonance. The condition at which applied frequency is same as natural frequency.

f_1 & f_2 are known as cut off frequencies.

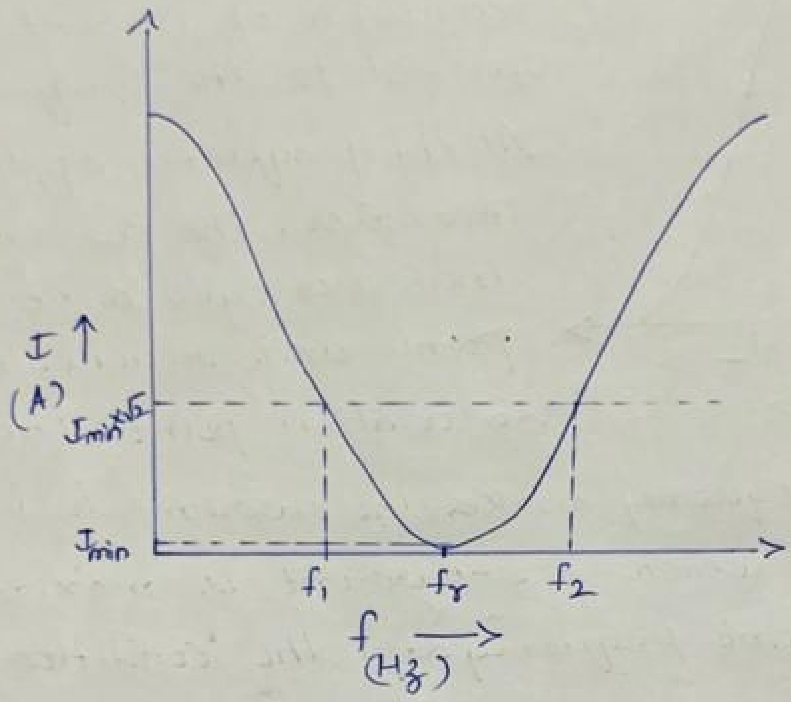
$$\boxed{BW = (f_2 - f_1)}\ Hz$$

BW → Band width

Parallel circuit



In this circuit, the resistor and capacitor are connected in series and inductor is connected parallel to capacitor.



The above graph represents the variation of current with respect to the applied frequency. As the applied frequency increases, the current decreases and reaches a minimum value. Then the further increase in frequency increases the current. The frequency at which the current is minimum is known as resonant frequency & the condition is known as resonance.

f_1 & f_2 are known as cut off frequencies.

$$\boxed{BW = (f_2 - f_1)}\ Hz$$

$BW \rightarrow$ Band width

Elasticity

Elasticity is one of the general properties of matter. Anybody which is not free to move when acted upon by a suitable force, undergoes a change in form. This Change in the form is called deformation. The change Could be in shape, size or both of them. If the Changes are within limit, the body regains its original shape or size when the system of deforming forces is withdrawn. This property of a substance is called elasticity.

Thus elasticity can be defined as, that property of matter by virtue of which a deformed body returns to its original state after the removal of the deforming force.

Stress

Force applied per unit area is Known as Stress.

$$\boxed{\text{Stress} = \frac{\text{Force}}{\text{Area}}}$$

Strain

The Ratio of change in dimension to original dimension is known as strain.

$$\boxed{\text{Strain} = \frac{\text{Change in Dimension}}{\text{Original Dimension}}}$$

Hooke's Law

Stress is proportional to Strain for any material within the elastic limit.

$$\text{Stress} \propto \text{Strain}$$

$$\boxed{\frac{\text{Stress}}{\text{Strain}} = \text{Constant / Modulus}}$$

Elastic Moduli

Young's Modulus (Y)

The ratio of Longitudinal stress to linear strain within the elastic limit is Known as Young's Modulus.

$$\text{Young's Modulus} = \frac{\text{Longitudinal Stress}}{\text{Linear strain}}$$

$$Y = \frac{F/a}{x/L}$$

$$\boxed{Y = \frac{FL}{xa} \; N/m^2}$$

Bulk Modulus (K)

The ratio of Compressive stress to the volume strain without change in shape of the body within the elastic limits is called the bulk modulus.

$$\text{Bulk Modulus} = \frac{\text{Compressive Stress}}{\text{volume Strain}}$$

$$K = \frac{F/a}{v/V}$$

$$\boxed{K = \frac{FV}{va} \; N/m^2}$$

Rigidity Modulus

The ratio of tangential stress to shear strain is Known as rigidity modulus.

$$\text{Rigidity Modulus} = \frac{\text{Tangential Stress}}{\text{Shear Strain}}$$

$$\eta = \frac{F/a}{x/L}$$

$$\boxed{\eta = \frac{FL}{xa} \ N/m^2}$$

poisson's Ratio

within the elastic limits of a body, the ratio of lateral strain to longitudinal strain is a constant called as poisson's ratio.

$$\text{poisson's Ratio} = \frac{\text{Lateral Strain}}{\text{Longitudinal strain}}$$

$$\sigma = \frac{d/D}{x/L}$$

$$\boxed{\sigma = \frac{dL}{xD}}$$

There are no units for poisson's ratio. It is a number & hence a dimensionless quantity.

Longitudinal Strain Coefficient

The longitudinal strain produced per unit stress is known as longitudinal strain coefficient.

$$\alpha = \frac{\text{Longitudinal Strain}}{\text{unit Stress}}$$

$$\alpha = \frac{x/L}{T}$$

$$\boxed{\alpha = \frac{x}{LT}} \quad \text{Where } T \rightarrow \text{unit Stress}$$

Lateral Strain Coefficient

The lateral strain produced per unit stress is called lateral strain coefficient.

$$\beta = \frac{d/D}{T}$$

$$\boxed{\beta = \frac{d}{DT}}$$

$d \rightarrow$ Change in dimension

$D \rightarrow$ original dimension

$T \rightarrow$ unit stress

Relation b/n α, β and σ

Consider the ratio,

$$\beta/\alpha = \frac{d/D\cancel{T}}{x/L\cancel{T}}$$

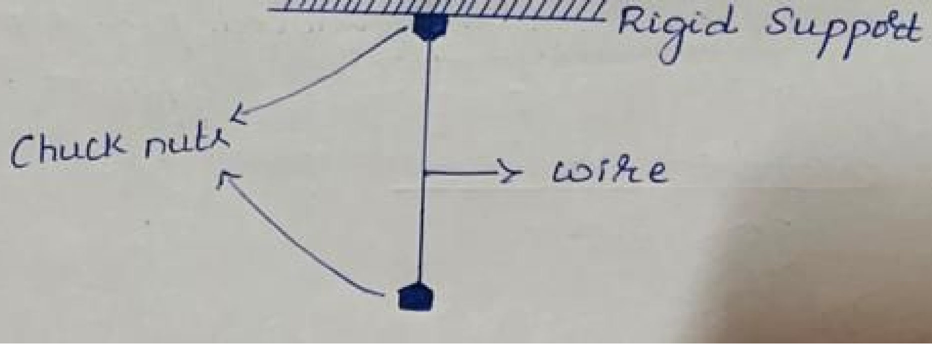
$$= \frac{d/D}{x/L}$$

$$\beta/\alpha = \frac{dL}{xD}$$

WKT $\sigma = \dfrac{dL}{xD}$

$$\therefore \boxed{\beta/\alpha = \sigma}$$

Torsional pendulum



consider a wire which is clamped vertically with the help of a chuck nut at the top and carrying a uniform body like disc, bar or cylinder at the other end. when the body is given a slight twist and let go, the system executes oscillation in the horizontal plane about the wire as axis. These oscillations are called torsional oscillations and the arrangement is called torsional pendulum.

The expression for time period of torsional oscillations is given by,

$$\boxed{T = 2\pi\sqrt{I/C}}$$

$C \rightarrow$ Couple per unit twist

$I \rightarrow$ Moment of Inertia

Applications

a) The working of torsion pendulum clocks is based on torsional oscillations.

b) To determine the frictional forces between solid surfaces & flowing liquid using forced torsion pendulums

c) Torsion springs are used in torsion pendulum clocks.

d) To determine the rigidity modulus of the material given.

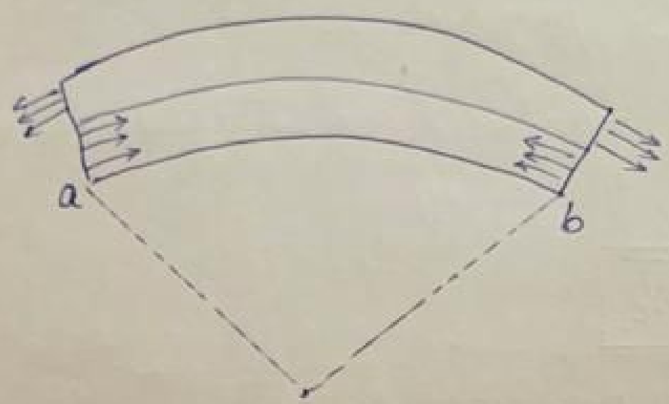
Bending of Beams

Beam

A beam is defined as a rod or bar of uniform cross section [circular or cylindrical] whose length is very much larger than its thickness so that the shearing stresses over any cross section are small and may be neglected.

Neutral Surface

when a straight bar is bent as shown in figure, the outer layers get elongated while the inner layers get contracted. However in between there will be a layer which is neither elongated nor contracted. This layer is known as neutral surface



Neutral axis

The line of intersection of the neutral surface and the plane of bending is known as neutral axis.

Types of Beams

There are four types of beams.

a) Simply supported beam: Ends of the beam are made to rest freely.

Ex Girders of railway over bridges

b) Fixed beam: Beam is fixed at both ends.

Ex: Bridges & other structures

c) cantilever beam: Beam is fixed at one end and free at another end.

Ex: Found in big buildings & appartments as balcony.

d) Continiously supported beam: More than two supports are provided to the beam.

Ex: Metro bridges

Bending Moment

The sum of the moments of all internal elastic forces acting over the whole cross section of the beam is known as bending moment.

The expression for bending moment is given by,

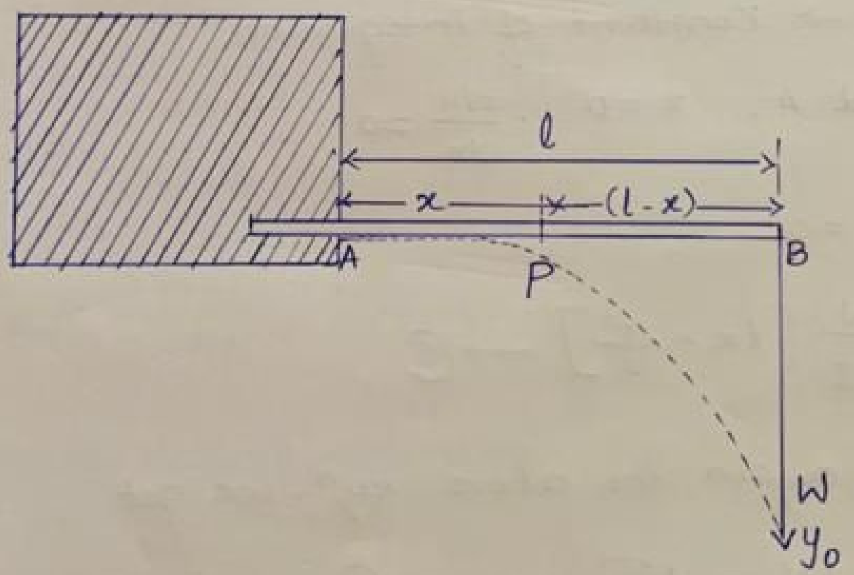
$$\boxed{BM = \frac{qI}{R}}$$

where $q \rightarrow$ young's modulus of the material

$I \rightarrow$ Moment of Inertia

$R \rightarrow$ Radius of curvature

Cantilever Experiment to determine young's Modulus



Let AB be a beam of length 'l' fixed at the end 'A' and loaded by a weight 'W' at B. The bent position is shown by the dotted line. Consider a section of the beam at 'P', distant 'x' from the fixed end 'A'.

Since the beam is in equilibrium in the bent position, the external bending couple at 'P' is balanced by the internal bending moment.

ie $w(l-x) = \frac{qI}{R} \rightarrow ①$

But according to differential calculus, 'R' is given by,

$\frac{1}{R} = \frac{d^2y}{dx} \rightarrow ②$ where $\frac{dy}{dx} \rightarrow$ Slope of the beam

$\therefore$ eqn ① can be written as,

$$W(l-x) = qI \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = \frac{W}{qI}(l-x) \rightarrow ③$$

Integrating above eqn, we get

$$\frac{dy}{dx} = \frac{W}{qI}\left[lx - \frac{x^2}{2}\right] + C_1 \rightarrow ④$$

where $C_1 \rightarrow$ Constant of integration

At the end A, $x=0$, $\frac{dy}{dx}=0$

$$\Rightarrow C_1 = 0$$

$$\therefore \frac{dy}{dx} = \frac{W}{qI}\left[lx - \frac{x^2}{2}\right] \rightarrow ⑤$$

Integrating again the above eqn. we get

$$y = \frac{W}{qI}\left[\frac{lx^2}{2} - \frac{x^3}{6}\right] + C_2 \rightarrow ⑥$$

where, $C_2 \rightarrow$ Constant of integration

Again at the end A, $x=0$ & $y=0$

$$\Rightarrow C_2 = 0$$

$$\therefore y = \frac{W}{qI}\left[\frac{lx^2}{2} - \frac{x^3}{6}\right] \rightarrow ⑦$$

If y_0 is the minimum depression at the end 'B' of the beam, ie at $x = l$.

$$y_0 = \frac{W}{qI}\left[\frac{l^3}{2} - \frac{l^3}{6}\right]$$

$$y_0 = \frac{Wl^3}{3qI} \longrightarrow (8)$$

For a beam of rectangular cross section,

$$I = \frac{bd^3}{12}$$

$$\therefore\ y_0 = \frac{Wl^3}{\cancel{3}_1 q}\left(\frac{\cancel{12}^4}{bd^3}\right)$$

$$y_0 = \frac{4Wl^3}{qbd^3}$$

$$W = mg$$

$$\therefore\ y_0 = \frac{4mgl^3}{qbd^3}$$

$$\boxed{q = \frac{4mgl^3}{bd^3 y_0}}$$

Using the above expression, the value of young's Modulus can be determined.