Analysis of Dependent Risk Data Based on α-Mixture Model

A Project Report submitted by

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in partial fulfilment of the requirements for the award of the degree
M.Sc. Mathematics

Indian Institute of Technology Jodhpur
Department of Mathematics

Nov 2023

Declaration

I hereby declare that the work presented in this Project Report titled **Analysis of Dependent Risk Data Based on α-Mixture Model**, submitted to the *Indian Institute of Technology Jodhpur* in partial fulfilment of the requirements for the award of the degree of M.Sc, is a true and accurate record of the research work conducted under the supervision of *Dr. Nil Kamal Hazra*. The contents of this project report, in full or in part, have not been submitted to, and will not be submitted by me to, any other institute or university in India or abroad for the award of any degree, diploma, or certificate.

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Acknowledgements

I would like to express my sincere gratitude to my supervisor, **Dr. NIL Kamal Hazra**, Department of Mathematics, Indian Institute of Technology Jodhpur, for allowing me to embark on this field of research. His constant support, vision, sincerity, and determination during my project have given me a new way to work. Without his support, I would not be able to learn in such depth. I would also like to thank **Mr. Tanmay Sahoo** who is working under Nil Kamal Sir, for his constant help and support, and all my batch mates and seniors for all those moments that we experienced in IIT Jodhpur. I am also thankful to **my family** for giving me the opportunity to study and motivating me to chase my dreams. In the end, I would like to thank **Department of Mathematics and Indian Institute of Technology Jodhpur** for providing a good environment and facilities.

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Certificate

This is to certify that the project report titled **Analysis of Dependent Risk Data Based on α-Mixture Model**, submitted by **Veeresh Chaudhary (M22MA208)** to the Indian Institute of Technology Jodhpur for the award of the degree of M.Sc., is a bonafide record of the research work done by him under my supervision. The part of this thesis is based on the paper entitled "Mixture model for dependent competing risks data and application for diabetic retinopathy treatment" by Pal and Prajapati (2023). To the best of my knowledge, the contents of this report, in full or in part, have not been submitted to any other institute or university for the award of any degree or diploma.

Nil Kamal Hazra.

Dr. Nil Kamal Hazra

Abstract

In this thesis, we do a brief review of the paper entitled "Mixture model for dependent competing risks data and application for diabetic retinopathy treatment" by Pal and Prajapati (2023). First, we do a brief literature review of the existing models available for dependent competing data and one of the censoring schemes, which is a progressive censoring scheme. All the results, I discussed are available in the literature. This article examines the PH (Proportional Hazards) family of distributions and presents a methodology based on mixture models to address dependent multiple competing risk data within the framework of the PFFCS (Progressive First Failure Censoring Scheme). Specifically, the mixing weights serve as a representation of the prior probabilities of a unit experiencing failure from various causes. These weights play a pivotal role in elucidating the dependency, or in other words, they are essential for explaining the interdependence among different failure modes. The article assumes that the failure time distributions associated with various failure causes belong to the proportional hazard family of distributions. This family includes significant members such as the Weibull, Burr XII, and Gompertz Distributions. The estimation of model parameters is carried out using the maximum likelihood principle.

The second part of this thesis in our research, we derived the normal equation for any baseline distribution within the Progressive First Failure Censoring Scheme. This equation is formulated under the assumption of using an α-mixture for the survival function of the unit. By applying this equation, we were able to estimate the values of the necessary parameters for our model and investigated the use of an α - mixture model for estimating parameters of the exponential distribution in the context of the Progressive First Failure Censoring Scheme. We estimated the required parameters and compare the performance of the α - mixture model with that of a arithmetic mixture model using the Akaike Information Criterion(AIC) and the Bayesian Information Criterion(BIC) for the Exponential distribution. Our analysis demonstrates the efficacy of the α - mixture model in handling complex dependencies and provides insights into its practical application in modeling competing risks data.

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Chapter 1

Competing Risk Data

1.1 Introduction

Competing Risk can be defined as when the occurrence of one type of event will prevent the occurrence of others. In reliability and survival analysis (it deals with analysing the expected duration of time until an event of interest occurs, e.g death of a patient or the failure of a system) event of interest-sensitiveness to more than one cause. e.g A patient having cancer can die by accident or heart attack In statistical literature, these types of problems are known as competing risk problems and are used in many areas like medical science, economics, and engineering. Failures in real-world devices or systems typically arise from the influence of multiple causes, each vying for prominence and interacting with one another. Nowadays, there are several ways to model competing risk data e.g latent failure times model (in this model, observed time to failure is calculated as the minimum of two or more failure time random). variable associated with each other. The drawback of this model is that it requires that risks be independently distributed, but in the real world, most of the causes of failure are dependent on each other. e.g If an individual with an eye disease loses sight in one eye, the remaining eye may face an increased risk of blindness. Therefore, it appears more reasonable to acknowledge interdependence among the factors contributing to the failure. so the finite mixture model is used to analyse competing risk data under the progressive first failure censoring scheme. A Finite Mixture Model is put forward in this study to address the interdependence among the failure time distributions linked to risk factors under the Progressive First Failure Censoring Scheme (PFFCS). It is assumed that these failure time distributions belong to the flexible proportional hazard (PH) family.

In the second part of the thesis,we derived the normal equation applicable to any baseline distribution within the Progressive First Failure Censoring Scheme,by using α-mixture for the unit's survival function. Utilizing this equation, we successfully estimated the necessary parameters for our model. We then explored the use of an α-mixture model for parameter estimation in the context of the Progressive First Failure Censoring Scheme. We estimated the required parameters and compared the performance of the α-mixture model with that of an arithmetic mixture model using statistical tools such as the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) for the Exponential distribution. Our analysis showcases the effectiveness of the α-mixture model in handling intricate dependencies, offering valuable insights into its practical application for modeling competing risks data.

1.2 Censoring Scheme

Censoring is a condition in which the value of measurement or observation is only partially known.

1.2.1 Type-I Censoring Scheme

The duration of the experiment is fixed, but the count of observed failures is subject to random variation.
For example :
1. Product Testing:
Consider a scenario where a company is testing the lifespan of a light bulb. A sample of bulbs is continuously monitored until the first bulb fails. The experiment is then terminated for that specific bulb, and the time to failure is recorded. This process is repeated for each bulb in the sample.
2. Clinical Trials:
In a clinical trial, patients might be monitored for a fixed period to observe the occurrence of a particular event. Once the event of interest happens to a patient, their participation in the study ends, and the time until the event is noted.
In both examples, the focus is on a fixed period of observation, and the study is concluded for each subject when a specific event occurs. This is characteristic of a Type I censoring scheme.

1.2.2 Type-II Censoring Scheme

The count of observed failures remains fixed, but the duration of the experiment is subject to random variability.
For example:
1. Reliability Testing in Electronics:
Suppose a manufacturer wants to assess the reliability of a new electronic component. They may decide in advance to test a batch of components until a certain number fails.
2. Drug Efficacy Trials:
In a clinical trial testing the effectiveness of a new drug, researchers might decide to enroll a fixed number of participants and continue the study until a specified number of patients experience a particular outcome, such as improvement or remission. Once the predetermined number of favorable outcomes is reached, the trial concludes.
In both scenarios, the emphasis is on a fixed number of observed events, and the study terminates when that specific number is reached, regardless of the time it takes. This is characteristic of a Type II censoring scheme.

1.2.3 Right Censoring Scheme

It happens when a participant withdraws from the study prior to the event taking place, or the study concludes before the event has occurred.
For example:
1. In a clinical trial examining the impact of treatments on heart attack occurrence, the study concludes after a 5-year duration. Participants who have not experienced a heart attack by the end of the year are subject to censoring. If a patient exits the study at the time t the event will occur

in the interval (t, ∞)
2. A patient diagnosed with lung cancer is enrolled in a clinical trial to assess the impact of a drug on their survival. However, the patient experiences a fatal car accident after a duration denoted as T years, but the specific timing of this event is unknown.

1.2.4 Left Censoring Scheme

This situation arises when the event of interest has occurred before the time of observation, but the precise timing of the event remains unknown.
For example :
1. Manufacturing Defects:
Suppose a company starts monitoring a batch of newly produced items for defects. However, the monitoring process begins a week after the items have been manufactured. Any defects that occurred during the production week, before the monitoring started, represent left-censored data. The company knows that defects may have occurred, but the exact timing and nature of those defects are not observed.
2. In a clinical trial:
A patient enrolled who already had cancer. The event of interest which is the presence of cancer, has occurred before the patient's enrollment.

1.3 Order Statistics

Order Statistics are the outcomes derived from arranging a sample either in ascending or descending order.
Consider a random sample taken from a population with a size of n. The order statistics are the sorted values of the sample.
Let $Z_1, Z_2, ..., Z_n$ are the random variable then the order statistics

$$Z_1 < Z_2 <, ..., < Z_n$$

are also random variables which are defined by arranging the values of $Z_1, Z_2, ..., Z_n$ in increasing order.
Let $Z_1, Z_2, ..., Z_n$ represent a random sample drawn from a distribution with a sample size of n. The order statistics are denoted by

$$\begin{aligned}
Z_{(1)} &= \min(Z_1, Z_2, \ldots, Z_n) \\
Z_{(2)} &= \text{the 2nd smallest of } Z_1, Z_2, \ldots, Z_n \\
&\vdots \\
Z_{(n)} &= \max(Z_1, Z_2, \ldots, Z_n)
\end{aligned}$$

Example: Suppose we have a random sample of ages from a population: $30, 25, 40, 35, 28$.
Ascending Order: Arrange the ages in ascending order: $25, 28, 30, 35, 40$. The order statistics are $Z_{(1)} = 25$, $Z_{(2)} = 28$, $Z_{(3)} = 30$, $Z_{(4)} = 35$, $Z_{(5)} = 40$.

1.3.1 Distribution Order Statistics

Let $Z_1, Z_2, ..., Z_n$ represent a random sample drawn from a distribution with a sample size of n. We will find the minimum- and maximum-order statistics.

Minimum Order Statistics

let's derive the probability density function (pdf) for $Z_{(1)}$, which represents the minimum value in the sample.The cumulative distribution function (CDF) for the minimum value in the sample is:

$$\begin{aligned}
H_{Z_{(1)}}(z) &= P(Z_{(1)} \leq z) = 1 - P(Z_{(1)} > z) \\
&= 1 - P(Z_1 > z, Z_2 > z, \ldots, Z_n > z) \\
&= P(Z_1 > z)P(Z_2 > z) \ldots P(Z_n > z) \text{ by independence} \\
&= 1 - [P(Z_1 > z)]^n \text{ because the } Z_i \text{ are identically distributed} \\
&= 1 - [1 - H(z)]^n
\end{aligned}$$

Therefore, the probability density function (pdf) for the minimum value in the sample is:

$$\begin{aligned}
h_{Z_{(1)}}(z) &= \frac{d}{dz} H_{Z_{(1)}}(z) = \frac{d}{dz} \left\{1 - [1 - H(z)]^n\right\} \\
&= n[1 - H(z)]^{n-1} h(z)
\end{aligned}$$

1.3.2 Maximum Order Statistics

let's derive the probability density function (pdf) for $Z_{(n)}$, which represents the maximum value in the sample.The cumulative distribution function (CDF) for the maximum value in the sample is:

$$\begin{aligned}
H_{Z_{(n)}}(z) &= P(Z_{(n)} \leq z) = P(Z_1 \leq z, Z_2 \leq z, \ldots, Z_n \leq z) \\
&= P(Z_1 \leq z)P(Z_2 \leq z) \ldots P(Z_n \leq z) \text{ by independence} \\
&= [P(Z_1 \leq z)]^n \text{ because the } Z_i \text{ are identically distributed} \\
&= [H(z)]^n
\end{aligned}$$

Upon taking the derivative, we obtain the probability density function (pdf) for the maximum value to be:

$$\begin{aligned}
h_{Z_{(n)}}(z) &= \frac{d}{dz} H_{Z_{(n)}}(z) = \frac{d}{dz} [H(z)]^n \\
&= n[H(z)]^{n-1} h(z)
\end{aligned}$$

1.4 Continuous Distribution

Consider a random variable X, which is defined on the probability space (S, F, P).Let $F_X(.)$ be the CDF then X is said to be continuous if $F_X(.)$ is absolutely continuous.

1.4.1 Absolutely Continuous Function

$F_X(.)$ is said to be absolutely continuous function if there exists a non-negative function

$$f_X : R \to (0, \infty)$$

s.t

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du$$

where
$f_X(x)$ is the pdf which satisfies the following properties:

1. $f_X(x) > 0$

2. $\int_{-\infty}^{-\infty} dx = 1$

1.5 Progressive First Failure Censoring Scheme

Consider n sets of independent groups, with each set comprising m units, undergoing a life test. Data is under observation till rth failure from n independent group. Upon the occurrence of the first failure, denoted as $T^R_{1:r::n:m}$, the group experiencing the observed failure, along with R_1 groups selected randomly from the rest of the (n-1) groups, is systematically excluded from the trial (study).In the same way, upon the second failure, denoted as $T^R_{2:r::n:m}$, the set linked to this second observed failure, along with R_2 sets chosen randomly from the rest of the $(n - R_1 - 2)$ groups, is systematically removed from the trial. In the end, At the rth failure, represented as $T^R_{r:r::n:m}$, the group experiencing this observed failure, along with R_r groups chosen randomly from the rest of $(n - R_{r-1} - r)$ groups, is systematically excluded from the test. We have fixed R_r and $R = (R_1, R_2, ..., R_r)$ before starting experiment .So

$$n = \sum_{i=1}^{r} R_i + r$$

Let,
$t^R_{1:r:r:m} < t^R_{2:r:n:m} < .... < t^R_{r:r:n:m}$ are the progressive first failure censored order statistics with progressive censoring scheme R.
Now, considering that the times of failure for all mn units under examination adhere to an entirely continuous distribution cumulative distribution function (CDF) denoted as H(·), with the corresponding probability density function (PDF) denoted as h(·), the joint probability density function (pdf) of $T^R_{1:r:n:m}, T^R_{2:r::n:m}, ..., T^R_{r:r::n:m}$ Can be articulated in the following manner:
$h_{1,2,...,r}(t^R_{1:r:n:m}, t^R_{2:r:n:m}, ..., t^R_{r:r:n:m}) =$

$$Cm^r \prod_{i=1}^{r} h(t_{i:r:n:m})[1 - H(t_{i:r:n:m})]^{m(R_i+1)-1}$$

where

$$C = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots (n - \sum_{k=1}^{r-1} R_k - r + 1)$$

Note: Only r out of mn items have failed in PFFCS. It offers notable benefits in reducing both the cost and duration of testing.

1.6 Description of the Model and Likelihood Function.

Let T represent the stochastic variable indicating the lifespan of an individual unit. A distribution is considered a Proportional Hazard (PH) family if the cumulative distribution function (CDF) of T is expressed as follows:

$$H(t;\alpha,\lambda) = \begin{cases} 1-[1-H_0(t;\alpha)]^{\lambda} & \text{if } t>0 \\ 0 & \text{otherwise} \end{cases}$$

In this context, $\alpha > 0, \lambda > 0$ are unspecified parameters, and $H_0(t;\alpha)$ represents a completely continuous cumulative distribution function (CDF). The corresponding probability density function (PDF) is defined as
follows:

$$h(t;\alpha,\lambda) = \begin{cases} \lambda h_0(t;\alpha)[1-H_0(t;\alpha)]^{\lambda-1} & \text{if } t>0 \\ 0 & \text{otherwise} \end{cases}$$

Here, $h_0(t;\alpha)$ denotes the probability density function (PDF) of the baseline random variable.
Now, we introduce a flexible finite mixture model formulation to deal with dependent risk data with multiple risk factors under PFFCS. Let $T^R_{i:r:n:m}, i = 1,2,3,...,r,$ 2, 3,..., r, represent the order statistics of progressively first failure-censored observations from a Proportional Hazard (PH) family of distribution characterized by parameters α and λ, under the progressive censoring scheme denoted as R. From this point forward, t_i will be utilized in place of $t^R_{i:r:n:m}$ for the sake of notation simplicity.
Assuming $t = (t_1, t_2, ..., t_r)$, for given r observation , The likelihood function for α and λ is expressed as follows :

$$L(\alpha,\lambda|t) = h_{T1,T2,...,T_r}(t_1,t_2,...,t_r)$$

$$= Cm^r \prod_{i=1}^{r} h(t_i)[1-H(t_i)]^{m(R_i+1)-1}$$

$$= Cm^r \prod_{i=1}^{r} \lambda h_0(t_i,\alpha)[1-H_0(t_i,\alpha)]^{\lambda-1}\left[1-(1-H_0(t_i,\alpha))^{\lambda})\right]^{m(R_i+1)-1}$$

$$= Cm^r\lambda^r \prod_{i=1}^{r}[1-H_0(t_i,\alpha)]^{\lambda-1}h_0(t_i,\alpha)\Big[[1-H_0(t_i,\alpha)]^{\lambda(m(R_i+1)-1)}\Big]$$

Here

$$C = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots (n - \sum_{k=1}^{r-1} R_k - r + 1)$$

We presume that each element within a specific target group experiences failure due to a variety of P competing causes. Denoting the likelihood of failure attributed to cause j as π_j, where j takes values from 1 to P. The probability density function (PDF) describing the lifespan of the unit is subsequently defined by $\sum_{j=1}^{P} \pi_j h(t; \alpha, \lambda_j)$. Consider the values $\pi_1, \pi_2, \cdots, \pi_P$ as the mixing coefficients in a finite mixture model, where each π_i represents the weight assigned to the ith component in the mixture with $\sum_{j=1}^{P} \pi_j = 1$ and $h(t; \alpha, \lambda_j)$ represents the probability density function (PDF) of the failure time for the unit due to cause j. Consider Δ to be a random variable defined by

$$\Delta = \begin{cases} 1, & \text{If the item experiences failure attributable to cause 1} \\ 2, & \text{If the item experiences failure attributable to cause 2} \\ \vdots & \\ P, & \text{If the item experiences failure attributable to cause P} \end{cases}$$

The configuration of the data set involving competing risks can be outlined as $(t, \delta) = (t_1, \delta_1), (t_2, \delta_2), \ldots, (t_r, \delta_r)$ where $(t_i, \delta_i), i = 1, 2, ..., r$ are the realization of the random vector (T, δ).The joint probability density function (PDF) is expressed as:
$g(t_i, \delta_i = j) \propto \pi_j h(t_i; \alpha, \lambda_j)[1 - H(t_i; \alpha, \lambda_j)]^{m(R_i+1)-1} \quad \forall i, j.$
For $i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, P,$ define

$$I_j(i) = \begin{cases} 1, & \text{if } \delta_i = j \\ 0, & \text{otherwise.} \end{cases}$$

The Likelihood Function for these r observations is as follows:

$$L(\theta|t, \delta) \propto \prod_{j=1}^{P-1} {\pi_i}^{r_j} (1 - \sum_{i=1}^{P-1} \pi_i)^{r_P} \prod_{j=1}^{P} {\lambda_j}^{r_j}$$

$$\times \prod_{i=1}^{r} \prod_{j=1}^{P} \left[[1 - H_0(t_i; \alpha)]^{\lambda_j - 1} h_0(t_i; \alpha) [1 - H_0(t_i; \alpha)]^{\lambda_j (m(R_i+1)-1)} \right]^{I_j(i)}$$

Here,$\theta = (\pi_1, \pi_2, \ldots, \pi_{P-1}, \alpha, \lambda_1, \lambda_2, \ldots, \lambda_P)$ and $r_j = \sum_{i-1}^{r} I_j(i)$ is the count of failures attributed to the jth cause, where j takes values from 1 to P. The log-likelihood function encompasses terms solely related to the parameters θ, which can be expressed as:

$$l(\theta|t, \delta) \propto \sum_{j=1}^{P-1} r_j + r_P ln(1 - \sum_{i=1}^{P-1} \pi_i) + \sum_{j=1}^{P} r_j ln_{\lambda_j}$$

$$+ \sum_{i=1}^{r} \sum_{j=1}^{P} \left[I_j(i) ln f_0(t_i; \alpha) + \lambda_j I_j(i) m(R_i + 1) ln(1 - H_0(t_i; \alpha)) \right]$$

1.7 Estimating the parameters by Maximizing the Likelihood Function.

To determine the Maximum Likelihood Estimate (MLE) of θ, we calculate the derivative of the log-likelihood function $l(\theta|t,\delta)$ concerning the model parameters. This process involves obtaining the normal equations, which represent the system of equations resulting from setting the derivatives equal to zero:

$$\frac{\partial l(\theta|t,\delta)}{\partial \pi_d} = \frac{r_d}{\pi_d} - \frac{r_P}{1-\sum_{i=1}^{P-1}\pi_i} = 0, d = 1, 2, ..., P-1$$

$$\frac{\partial l(\theta|t,\delta)}{\partial \lambda d} = \frac{r_d}{\lambda_d} + \sum_{i=1}^{r} I_d(i)m(R_i+1)ln(1-H_0(t_i;\alpha)) = 0$$

If the value of α is held constant, the Maximum Likelihood Estimates (MLEs) for the model parameters can be directly derived through the normal equations in the following manner:

$$\widehat{\pi}_d = \frac{r_d}{\sum_{j=1}^{P} r_j} = \frac{r_d}{r}, d = 1, 2, ..., P-1$$

$$\widehat{\lambda}_d = -\frac{r_d}{\sum_{i=1} rI_d(i)m(R_i+1)ln(1-H_0(t_i;\alpha))}$$

where $d = 1, 2, ..., P-1$.

1.8 Some Special cases

1.8.1 Weibull Distribution

By replacing $H_0(t;\alpha)$ with $1-e^{-t^{\alpha}}$, the Weibull Distribution characterized by a shape parameter α and a scale parameter λ would be obtained. Let $\theta = (\pi_1, \pi_2, \ldots, \pi_{P-1}, \alpha, \lambda_1, \lambda_2, \ldots, \lambda_P)$ be the parameters, The log-likelihood function is given by

$$l(\theta|t,\delta) = \sum_{i=1}^{P-1} r_i ln\pi_i + r_s ln(1-\sum_{i=1}^{P-1}\pi_i) + rln\alpha + \sum_{i=1}^{P} r_i ln_{\lambda_i}$$

$$+(\alpha-1)\sum_{i=1}^{P}\sum_{j=1}^{r} I_i(j)lnt_j - \sum_{i=1}^{P}\sum_{j=1}^{r}\lambda_j[k(R_j+1)]t_j^{\alpha}I_i(j)$$

Now , for the fixed α the MLE of π_d is given by

$$\frac{\partial l(\theta|t,\delta)}{\partial \pi_d} = \frac{r_d}{\pi_d} - \frac{r_P}{1-\sum_{i=1}^{P-1}\pi_i} = 0$$

therefore

$$\widehat{\pi}_d = \frac{r_d}{\sum_{j=1}^{P} r_j} = \frac{r_d}{r}$$

the MLE for the λ_d is given by

$$\frac{\partial l(\theta|t,\delta)}{\partial \lambda_d} = \frac{r_d}{\lambda_d} - \sum_{i=1}^{r} m[R+j+1] t_j^{\alpha} I_d(j) = 0$$

therefore,

$$\widehat{\lambda_d} = \frac{r_d}{\sum_{i=1}^{r} I_d(i) m(R_i+1) t_i^{\alpha}}$$

where $d = 1, 2, ..., P$.

1.8.2 Burr XII Distribution

By replacing $H_0(t;\alpha) = 1 - (1+t^{\alpha})^{-1}$, the Burr XII distribution with a shape parameter α and a scale parameter λ would be obtained. Let $\theta = (\pi_1, \pi_2, \ldots, \pi_{P-1}, \alpha, \lambda_1, \lambda_2, \ldots, \lambda_P)$ be the parameters, The log-likelihood function is given by

$$l(\theta|t,\delta) = \sum_{i=1}^{P-1} r_i ln\pi_i + r_P ln(1 - \sum_{i=1}^{P-1} \pi_i) + rln\alpha + \sum_{i=1}^{P} r_i ln\lambda_i$$

$$+(\alpha-1)\sum_{i=1}^{P}\sum_{j=1}^{r} I_i(j) lnt_j - \sum_{i=1}^{P}\sum_{j=1}^{r} \lambda_i [m(R_j+1)+1] ln(1+t^{\alpha}) I_i(j)$$

for the fixed value of α, the MLE of π_d is given by

$$\frac{\partial l(\theta|t,\delta)}{\partial \pi_d} = \frac{r_d}{\pi_d} - \frac{r_P}{1 - \sum_{i=1}^{P-1} \pi_i} = 0$$

therefore,

$$\widehat{\pi_d} = \frac{r_d}{\sum_{i=1}^{P} r_j} = \frac{r_d}{r}$$

the MLE for the λ_d is given by

$$\frac{\partial l(\theta|t,\delta)}{\partial \lambda_d} = \frac{r_d}{\lambda_d} - \sum_{i=1}^{r} [m(R_i+1)+1] ln(1+t_j^{\alpha} I_d(i)) = 0$$

therefore,

$$\widehat{\lambda_d} = \frac{r_d}{\sum_{i=1}^{r} [m(R_i+1)+1] ln(1+t_j^{\alpha}) I_d(i)}$$

.

where $d = 1, 2, ..., P$.

1.8.3 Gompertz Distribution

By replacing $H_0(t;\alpha) = 1 - e^{(e^{\alpha t}-1)}$, the Gompertz Distribution with a shape parameter α and a scale parameter λ would be obtained. Let $\theta = (\pi_1, \pi_2, \ldots, \pi_{P-1}, \alpha, \lambda_1, \lambda_2, \ldots, \lambda_P)$ be the parameters, The log-likelihood function is given by

$$l(\theta|t,\delta) = \sum_{i=1}^{P-1} r_i ln\pi_i + r_P ln(1 - \sum_{i=1}^{P-1} \pi_i) + rln\alpha + \sum_{i=1}^{P} r_i ln\lambda_i$$

$$+\alpha \sum_{i=1}^{P}\sum_{j=1}^{r} I_i(j)t_j - \sum_{i=1}^{P}\sum_{j=1}^{r} \lambda_i m(R_j+1)(e^{\alpha t_J}-1)I_i(j)$$

For the fixed value of α, the MLE of π_d is given by the equation:

$$\frac{\partial l(\theta|t,\delta)}{\partial \pi_d} = \frac{r_d}{\pi_d} - \frac{r_P}{1-\sum_{i=1}^{P-1}\pi_i} = 0$$

therefore,

$$\widehat{\pi_d} = \frac{r_d}{\sum_{i=1}^{P} r_j} = \frac{r_d}{r}$$

The MLE for the λ_d is given by the equation:

$$\frac{\partial l(\theta|t,\delta)}{\partial \lambda_d} = \frac{r_d}{\lambda_d} - \sum_{i=1}^{r} m(R_i+1)(e^{\alpha t_i}-1)I_d(i) = 0$$

therefore,

$$\widehat{\lambda_d} = \frac{r_d}{\sum_{i=1}^{r} I_d(i)m(R_i+1)(e^{\alpha t_i}-1)}$$

where $d = 1, 2, ..., P$.

Chapter 2

α- Mixture Model

2.1 α- Mixture Model

The finite alpha mixture of survival functions $\bar{H}_i, i = 1, 2, 3, \ldots, p$, is defined by their weighted αth power mean, given by

$$\bar{H}_\alpha(t) = \begin{cases} \left[\sum_{i=1}^{p} \left[\pi_i \bar{H}_i^\alpha(t) \right] \right]^{\frac{1}{\alpha}} & \text{when } \alpha \neq 0, \alpha \in \mathbb{R}, \\ \bar{H}_{\text{gm}}(t) & \text{when } \alpha = 0. \end{cases}$$

where $\pi_i > 0$, $\sum_{i=1}^{p} \pi_i = 1$, and

$$\bar{H}_{\text{gm}}(t) = \prod_{i=1}^{p} \bar{H}_i^{\pi_i}(t)$$

The α-mixture is a versatile family of mixture distributions that encompasses various models:
(a) When $\alpha = 1$, the α-mixture reduces to the standard arithmetic mixture.
(b) For $\alpha = 0$, it corresponds to the geometric mixture.
(c) When $\alpha = -1$, it represents the harmonic mixture of the baseline survival functions:

$$\bar{H}_{\text{hm}}(t) = \left[\sum_{i=1}^{p} \pi_i \bar{H}_i^{-1}(t) \right]^{-1}$$

(d) For $n = 2$ and $\alpha = 1$, the α-mixture can be understood as a binomial expansion mixture

$$\bar{H}_{1/m}(x) = \sum_{k=0}^{m} \binom{m}{k} \pi^k (1-\pi)^{m-k} \bar{H}^1_{1-k/m}(x) \bar{H}^2_{k/m}(x)$$

where $\binom{m}{k}$ is the binomial coefficient. In particular, for $\alpha = \frac{1}{2}$, the α-mixture gives

$$\bar{H}_{1/2}(x) = \pi^2 \bar{H}_1(x) + (1-\pi)^2 \bar{H}_2(x) + 2\pi(1-\pi)\sqrt{\bar{H}_1(x)\bar{H}_2(x)}$$

2.1.1 Survival Function

The survival function, denoted as $\bar{H}(t)$, is a fundamental concept in survival analysis and reliability theory. It represents the probability that a subject or system will survive beyond a certain time t. Mathematically, the survival function is defined as:

$$\bar{H}(t) = P(T > t)$$

where T is a random variable representing the time until an event occurs (e.g., failure, death), and t is a specific time point. The survival function provides important information about the survival characteristics of a population or system over time.

2.1.2 Failure Rate or Hazard Rate

It represents the instantaneous rate at which a system or component fails at time t, given that it has survived up to time t , often denoted by $\lambda(t)$

$$\begin{aligned}\lambda(t) &= \lim_{\Delta t \to 0} \frac{P(t < X \leq t + \Delta t \mid X > t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{H(t + \Delta t) - H(t)}{p(X > t) \cdot \Delta t} \\ &= \frac{h(t)}{\bar{H}(t)}\end{aligned}$$

where:
$h(t)$ is the pdf, representing the probability density of the time to failure at time t.
$\bar{H}(t)$ is the survival function, representing the probability that the system or component survives beyond time t.

2.1.3 AIC

The Akaike Information Criterion (AIC) is a measure used in statistics to assess the goodness of fit of a statistical model. It balances the trade-off between the goodness of fit of the model and the complexity of the model (number of parameters). AIC is calculated as:

$$\text{AIC} = 2k - 2\log(L)$$

where:
k is the number of estimated parameters in the model.
L is the maximum value of the likelihood function for the model.

A lower AIC value indicates a better-fitting model, with the best model being the one with the lowest AIC value. AIC is useful for model selection, where different models are compared to determine which one best explains the data while taking into account model complexity.

2.1.4 BIC

The Bayesian information criterion (BIC), also known as the Schwarz criterion (SBC), is used in statistics for selecting the best model from a set of models. It considers both the likelihood function and the number of parameters in a model. Unlike the Akaike information criterion (AIC), the BIC penalizes models more heavily for having more parameters, which helps prevent overfitting. This means that while adding parameters can increase the likelihood of the model, the BIC discourages overly complex models by imposing a larger penalty for additional parameters.
BIC is calculated as:

$$\mathrm{BIC} = klog(N) - 2\log(L)$$

where:
k is the number of estimated parameters in the model.
L is the maximum value of the likelihood function for the model.
N is the number of observations, or , the sample size,or ,the number of data points.

Given any two estimated models, the model with the lower value of BIC is the one to be preferred.

2.2 Description of the α -Mixture Model and Likelihood Function

Each unit in a specific target population is assumed to fail due to one of p competing failure causes. Let π_j denote the prior probability that a unit fails due to cause j, where $j = 1, 2, \ldots, p$.
The associated survival function of the unit in the case of a arithmetic-mixture (finite mixture model) is given by

$$\sum_{j=1}^{p} \pi_j \bar{H}(t; \beta, \lambda_j)$$

The associated survival function of the unit in the case of an alpha mixture model is given by

$$\left[\sum_{j=1}^{p} \pi_j \bar{H}^{\alpha}(t; \beta\lambda_j)\right]^{\frac{1}{\alpha}}$$

Consider Δ to be a random variable defined by

$$\Delta = \begin{cases} 1, & \text{If the item experiences failure attributable to cause 1} \\ 2, & \text{If the item experiences failure attributable to cause 2} \\ \vdots & \\ p, & \text{If the item experiences failure attributable to cause p} \end{cases}$$

The configuration of the data set involving competing risks can be outlined as $(t, \delta) = (t_1, \delta_1), (t_2, \delta_2), \ldots, (t_m, \delta_m$

where $(t_i, \delta_i), i = 1, 2, ..., r$ are the realization of the random vector (T, δ).The joint probability density function (PDF) is expressed as:

$$g(t_i; \beta, \lambda_j) \propto \pi_j h(t_i; \beta, \lambda_j) \Big[\sum_{j=1}^{p} \pi_j \bar{H}^{\alpha}(t_i; \beta, \lambda_j) \Big]^{\frac{(k(R_i+1)-1)}{\alpha}}$$

for i = 1,2,. . . ,m and for j = 1,2,. . . ,p.

Here, we take m sample size from n independent groups with each group containing k units be subjected to a life test.(This is obtained by apply PH- family of distribution under Progressive First Failure Censoring Scheme taking $\alpha-$mixture for survival function of the unit.)

Define

$$I_j(i) = \begin{cases} 1, & \text{if } \delta_i = j \\ 0, & \text{otherwise.} \end{cases}$$

The Likelihood Function for these m observations is as follows:

$$L(\theta|t, \delta) \propto \Big(\prod_{j=1}^{p-1} {\pi_j}^{m_j} \Big) \Big(1 - \sum_{i=1}^{p} \pi_i)^{m_p} \Big(\prod_{i=1}^{p} \lambda_i m_j \Big)$$

$$\times \prod_{i=1}^{m} \prod_{j=1}^{p} \Big[(1 - H(t_i; \beta))^{\lambda_j - 1)} h(t_i; \beta) \Big[\sum_{q=1}^{p} \pi_q (\bar{H}^{\alpha}(t_i; \beta))^{\lambda_q} \Big]^{\frac{(k(R_i+1)-1)}{\alpha}} \Big]^{I_j(i)}$$

where $\theta = (\pi_1, \pi_2, ..., \pi_{p-1}, \alpha, \beta, \lambda_1, \lambda_2, ..., \lambda_p)$ and $m_j = \sum_{i=1}^{m} I_j(i)$ is the number of failure due to jth cause; j=1,2,...,p.

The corresponding log-likelihood function is given by

$$l(\theta|t, \delta) \propto \sum_{j=1}^{p-1} m_j ln(\pi_j) + m_p ln(1 - \sum_{i=1}^{p-1} \pi_i) + \sum_{j=1}^{p} m_j ln(\lambda_j)$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{p} \Big[I_j(i) ln \Big[(\bar{H}(t_i; \beta))^{\lambda_j - 1} h(t_i; \beta) \Big[\sum_{q=1}^{p} \pi_q (\bar{H}^{\alpha}(t_i; \beta))^{\lambda_q} \Big]^{\frac{(k(R_i+1)-1)}{\alpha}} \Big] \Big]$$

$$\propto \sum_{j=1}^{p-1} m_j ln(\pi_j) + m_p ln(1 - \sum_{i=1}^{p-1} \pi_i) + \sum_{j=1}^{p} m_j ln(\lambda_j)$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{p} \Big[I_j(i)(\lambda_j - 1) ln(\bar{H}(t_i; \beta)) + I_j(i) ln(h(t_i; \beta)) + \frac{I_j(i)(k(R_i+1)-1)}{\alpha} ln \Big[\sum_{q=1}^{p} \pi_q (\bar{H}^{\alpha}(t_i; \beta))^{\lambda_q} \Big] \Big]$$

Let β=1,, we have $\theta = (\pi_1, \pi_2, ..., \pi_{p-1}, \alpha, \lambda_1, \lambda_2, ..., \lambda_p)$, When we differentiate the log-likelihood function with respect to the model parameter and set the derivative equal to zero, we obtain the normal equation. This equation represents the condition for finding the maximum likelihood estimate (MLE) of the parameter and these are :

$$\frac{\partial l}{\partial \lambda_d} = \frac{m_d}{\lambda_d} + \sum_{i=1}^{m} I_d(i) ln(\bar{H}(t_i)) + \sum_{i=1}^{m} \sum_{j=1}^{p} \Big[\frac{I_j(i)(k(R_i+1)-1)\pi_d (\bar{H}^{\alpha}(t_i))^{\lambda_d} ln(\bar{H}(t_i))}{\sum_{q=1}^{p} \pi_q (\bar{H}^{\alpha}(t_i))^{\lambda_q}} \Big] = 0$$

for d=1,2,...,p.

$$\frac{\partial l}{\partial \pi_d} = \frac{m_d}{\pi_d} - \frac{m_p}{(1-\sum_{i=1}^{p-1}\pi_i)} + \sum_{i=1}^{m}\sum_{j=1}^{p}\left[\frac{I_j(i)(k(R_i+1)-1)((\bar{H}^{\alpha}(t_i))^{\lambda_d} - (\bar{H}^{\alpha}(t_i))^{\lambda_{d+1}} - \ldots - (\bar{H}^{\alpha}(t_i))^{\lambda_{d+p-1}})}{\alpha\Big(\sum_{q=1}^{p}\pi_q(\bar{H}^{\alpha}(t_i))^{\lambda_j}\Big)}\right]$$

$$\frac{\partial l}{\partial \pi_d} = 0 \ \ ;\text{for } d = 1,2,\ldots,p-1$$

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{m}\sum_{j=1}^{p}\left[\frac{I_j(i)(k(R_i+1)-1)}{\alpha^2}\left[\frac{\alpha\sum_{q=1}^{p}\pi_q(\bar{H}^{\alpha}(t_i))^{\lambda_q}log(\bar{H}(t_i))^{\lambda_q}}{\sum_{q=1}^{p}\pi_q(\bar{H}^{\alpha}(t_i))^{\lambda_q}} - ln\Big(\sum_{q=1}^{p}\pi_q(\bar{H}^{\alpha}(t_i))^{\lambda_q}\Big)\right]\right] = 0$$

2.3 Exponential Distribution

If we take $\bar{H}(t_i) = e^{-t_i}$,we get survival function for the exponential distribution.
Let $\theta = (\pi_1, \pi_2, ..., \pi_{p-1}, \alpha, \lambda_1, \lambda_2, ..., \lambda_p)$ be the parameters to be estimated. The corresponding log-likelihood function is given by

$$l(\theta|t,\delta) \propto \sum_{j=1}^{p-1} m_j ln(\pi_j) + m_p ln(1-\sum_{i=1}^{p-1}\pi_i) + \sum_{j=1}^{p} m_j ln(\lambda_j) + \sum_{i=1}^{m}\sum_{j=1}^{p}\Bigg[I_j(i)(\lambda_j-1)(-t_i) + I_j(i)(-t_i)$$
$$+\frac{I_j(i)(k(R_i+1)-1)log(\sum_{q=1}^{p}\pi_q e^{-t_i\alpha\lambda_q})}{\alpha}\Bigg]$$

The log-likelihood function is differentiated with respect to parameters to get normal equations:

$$\frac{\partial l}{\partial \lambda_d} = \frac{m_d}{\lambda_d} - \sum_{i=1}^{m} I_d(i)(t_i) + \sum_{i=1}^{m}\sum_{j=1}^{p}\left[\frac{I_j(i)(k(R_i+1)-1)\pi_d e^{-t_i\alpha\lambda_d}(-t_i)}{\sum_{q=1}^{p}\pi_q e^{-t_i\alpha\lambda_q}}\right] = 0$$

for d=1,2,...,p

$$\frac{\partial l}{\partial \pi_d} = \frac{m_d}{\lambda_d} - \frac{m_p}{1-\sum_{i=1}^{p-1}\pi_i} + \sum_{i=1}^{m}\sum_{j=1}^{p}\left[\frac{I_j(i)(k(R_i+1)-1)(e^{-t_i\alpha\lambda_d} - e^{-t_i\alpha\lambda_{d+1}} - \ldots - e^{-t_i\alpha\lambda_{d+p-1}})}{\alpha\Big(\sum_{q=1}^{p}\pi_q e^{-t_i\lambda_q\alpha}\Big)}\right] = 0$$

for d=1,2,...,p-1

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{m}\sum_{j=1}^{p}\left[\frac{I_j(i)(k(R_i+1)-1)}{\alpha^2}\left[\frac{-\alpha\sum_{q=1}^{p}\pi_q e^{-t_i\alpha\lambda_q}t_i\lambda_q}{\sum_{q=1}^{p}\pi_q e^{-t_i\alpha\lambda_q}} - ln\Big(\sum_{q=1}^{p}\pi_q e^{-t_i\alpha\lambda_q}\Big)\right]\right] = 0$$

2.4 Simulation Study

The dataset contains information on two causes of blindness. Initially, there were 28 observations attributed to cause 1 ($\delta = 1$) and 10 observations attributed to cause 2 ($\delta = 0$). To apply the progressive first failure censoring scheme to this dependent risk dataset, these 38 observations are randomly grouped into 19 sets. Each set contains 2 elements, listed in ascending order.
Now, from the data with n=19,we choose m=15.

Table 2.1: Progressive first-failure censoring competing risks data

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Data	79	178	203	272	276	285	350	356	392	471	503	584	622	663	966
δ_i	1	1	0	0	1	0	1	1	1	0	1	1	0	0	0
R_i	1	0	0	0	0	0	1	0	0	0	0	0	1	1	0

It can be seen from the data set that:
$m_1 = 8, m_2 = 7, p = 2, k = 2$ and $\pi_1 = \pi, \pi_2 = 1 - \pi$,
$m_j = \sum_{j=1}^{m} I_j(i)$ is the number of failure due to jth cause j=1,2,...,p.
The analysis involves applying an α-mixture and an arithmetic mixture(AM) to the exponential distribution, with the resulting table displayed in Figure 1.

2.4.1 Simulation for Exponential Distribution

After substituting the values into the log-likelihood function, it is:

$$l(\theta|t,\delta) = 8ln(\pi) + 7ln(1-\pi) + 8ln(\lambda_1) + 7ln(\lambda_2)$$

$$+\sum_{i=1}^{15}\left[- 8t_i(\lambda_1 - 1) - 8t_i - 7t_i(\lambda_2 - 1) - 7t_i + \frac{15(2(R_i+1)-1)ln(\pi e^{-t_i\alpha\lambda_2} + (1-\pi)e^{-t_i\alpha\lambda_2})}{\alpha}\right]$$

Normal equations with respect to parameter $\theta = (\pi, \alpha, \lambda_1, \lambda_2)$ are :

$$\frac{\partial l}{\lambda_1} = \frac{8}{\lambda_1} - \sum_{i=1}^{15} 8t_i + \sum_{i=1}^{15}\left[\frac{15(2(R_i+1)-1)\pi e^{-t_i\alpha\lambda_1}(-t_i)}{(\pi e^{-t_i\alpha\lambda_1} + (1-\pi)e^{-t_i\alpha\lambda_2})}\right] = 0$$

$$\frac{\partial l}{\partial\lambda_2} = \frac{7}{\lambda_2} - \sum_{i=1}^{15} 7t_i + \sum_{i=1}^{15}\left[\frac{15(2(R_i+1)-1)(1-\pi)e^{-t_i\alpha\lambda_2}(-t_i)}{(\pi e^{-t_i\alpha\lambda_1} + (1-\pi)e^{-t_i\alpha\lambda_2})}\right] = 0$$

$$\frac{\partial l}{\partial\pi} = \frac{8}{\lambda_1} - \frac{7}{1-\pi} + \sum_{i=1}^{15}\left[\frac{15(2(R_i+1)-1)(e^{-t_i\alpha\lambda_1} - e^{-t_i\alpha\lambda_2})}{\alpha(\pi e^{-t_i\alpha\lambda_1} + (1-\pi)e^{-t_i\alpha\lambda_2})}\right] = 0$$

$$\frac{\partial l}{\partial\alpha} = \sum_{i=1}^{15}\left[\frac{15(2(R_i+1)-1)}{\alpha^2}\left(\frac{\alpha(\pi e^{-t_i\alpha\lambda_1}(-t_i\lambda_1) + (1-\pi)e^{-t_i\alpha\lambda_2}(-t_i\lambda_2))}{(\pi e^{-t_i\alpha\lambda_1} + (1-\pi)e^{-t_i\alpha\lambda_2})} - ln(\pi e^{-t_i\alpha\lambda_1} + (1-\pi)e^{-t_i\alpha\lambda_2})\right)\right] = 0$$

Normal equations with respect to parameter $\theta = (\pi, \lambda_1, \lambda_2)$ are :

$$\frac{\partial l}{\lambda_1} = \frac{8}{\lambda_1} - \sum_{i=1}^{15} 8t_i + \sum_{i=1}^{15} \left[\frac{15(2(R_i+1)-1)\pi e^{-t_i\lambda_1}(-t_i)}{(\pi e^{-t_i\lambda_1} + (1-\pi)e^{-t_i\lambda_2})} \right] = 0$$

$$\frac{\partial l}{\partial\lambda_2} = \frac{7}{\lambda_2} - \sum_{i=1}^{15} 7t_i + \sum_{i=1}^{15} \left[\frac{15(2(R_i+1)-1)(1-\pi)e^{-t_i\lambda_2}(-t_i)}{(\pi e^{-t_i\lambda_1} + (1-\pi)e^{-t_i\lambda_2})} \right] = 0$$

$$\frac{\partial l}{\partial\pi} = \frac{8}{\lambda_1} - \frac{7}{1-\pi} + \sum_{i=1}^{15} \left[\frac{15(2(R_i+1)-1)(e^{-t_i\lambda_1} - e^{-t_i\lambda_2})}{(\pi e^{-t_i\lambda_1} + (1-\pi)e^{-t_i\lambda_2})} \right] = 0$$

2.4.2 Figure 1

λ_1 ($\alpha-$ mixture)	λ_2 ($\alpha-$mix.)	π ($\alpha-$mix.)	λ_1(am)	λ_2 (am)
0.0001029356	0.001409701	0.994106125	0.06806040	0.07202681
0.0001317113	0.003875933	0.998331788	0.09932851	0.09974433
0.4000648896	0.399882865	0.454584487	0.39932752	0.39974382
0.4999326099	0.499276362	0.435778272	0.49958297	0.49993526
0.4992365234	0.500019404	0.424442462	0.49958297	0.49993526
0.2682346779	0.270702873	0.113456301	0.29932752	0.29974382
0.3494410241	0.349459684	0.47353227	0.34932752	0.34974382
0.042656215	0.044191994	0.007292904	0.06806040	0.07202681

α- initial	α($\alpha-$ mixture)	AIC ($\alpha-$mixture)	AIC (am)	π (am)
-5	-5.2921895	510.6737	33633.71	0.09654395
-2	-2.8159622	820.7852	47363.91	0.42298884
-1	-0.9775838	189979.5559	189773.86	0.42297467
-0.5	-0.4854964	237278.2452	237356.20	0.43131327
0.5	0.5164323	237299.7478	237356.20	0.43131327
1	1.4726184	128184.5507	142298.50	0.42297467
2	2.0149604	165993.5038	166035.87	0.42297467
5	5.7658037	20943.3018	33633.71	0.09654395

α initial	α($\alpha-$ mixture)	BIC ($\alpha-$mixture)	BIC(arithmetic -mixture)
-5	-5.2921895	513.5059	33635.83
-2	-2.8159622	823.6174	47366.03
-1	-0.9775838	189982.3881	189775.98
-0.5	-0.4854964	237281.0774	237358.32
0.5	0.5164323	237302.5800	237358.32
1	1.4726184	128187.3829	142300.62
2	2.0149604	165996.3360	166037.99
5	5.7658037	20946.1340	33635.83

2.5 Conclusion

This article explores the application of mixture model-based methodology to address dependent multiple competing risks data within the Proportional Hazards (PH) family of distributions, specifically under the Progressive First Failure Censoring Scheme (PFFCS). The significance of mixing weights is emphasized, as they represent prior probabilities of unit failure from various causes, playing a pivotal role in elucidating dependency. The assumption is made that latent failure times associated with individual competing risks adhere to different parameterizations within the PH family.

The estimation of model parameters follows the maximum likelihood principle, involving the solution of a one-dimensional optimization problem for obtaining parameter estimates.

While acknowledging the simplicity of our proposed model, it is recognized that exact dependency characterization among competing causes may pose challenges in practical applications. Future directions for research include exploring generalizations to other distribution families and incorporating model selection criteria based on the available data.

The second part of our study showcases the utility of the α-mixture model as a powerful tool for modeling dependent competing risks data.By estimating parameters of the exponential distribution using the Progressive First Failure Censoring Scheme, We have shown that the α mixture model gives us better results as compared to the arithmetic mixture model. Furthermore, our comparison with a arithmetic mixture model highlights the advantages of the alpha mixture model in handling complex data structures. Overall, our findings contribute to the advancement of statistical methods for analyzing competing risks data and underscore the importance of considering alternative modeling approaches in such contexts.

2.6 References

1. Progressive Censoring Theory, Methods, and Application N.Balakrishnan Rita Aggarwala .
2. A First Course in Order Statistics Barry C. Arnold N. Balakrishnan H. N. Nagaraja .
3. On estimation based on progressive first-failure-censored sampling Shuo-Jye Wu , Cokun Ku .
4. Pal, A. and Prajapati, D. (2023). Mixture model for dependent competing risks data and application for diabetic retinopathy treatment. Applied Mathematical Modelling, **113**, 514-527.