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PHYS 512 – Q1

The analytical solution to $\frac{dy}{dx} = \frac{y}{1+x^2}$ is

$$y = c_0 \cdot \exp(\arctan(x)) \tag{1}$$

Given $y(-20) = 1$, we can derive $c_0 = y(0)$:

$$y(-20) = 1 \tag{2}$$

$$c_0 \cdot exp\big(arctan(-20)\big) = 1$$

$$c_0 = \frac{1}{\exp\big(arctan(-20)\big)}$$

And from this, we can derive $y(x)$

$$y(x) = \frac{\exp(\arctan(x))}{\exp(\arctan(-20))} \tag{3}$$

$$y(x) = \frac{\exp\,(\arctan(x))}{\exp\,(-\arctan(20))}$$

$$y(x) = \exp(\arctan(20)) * \exp(\arctan(x))$$

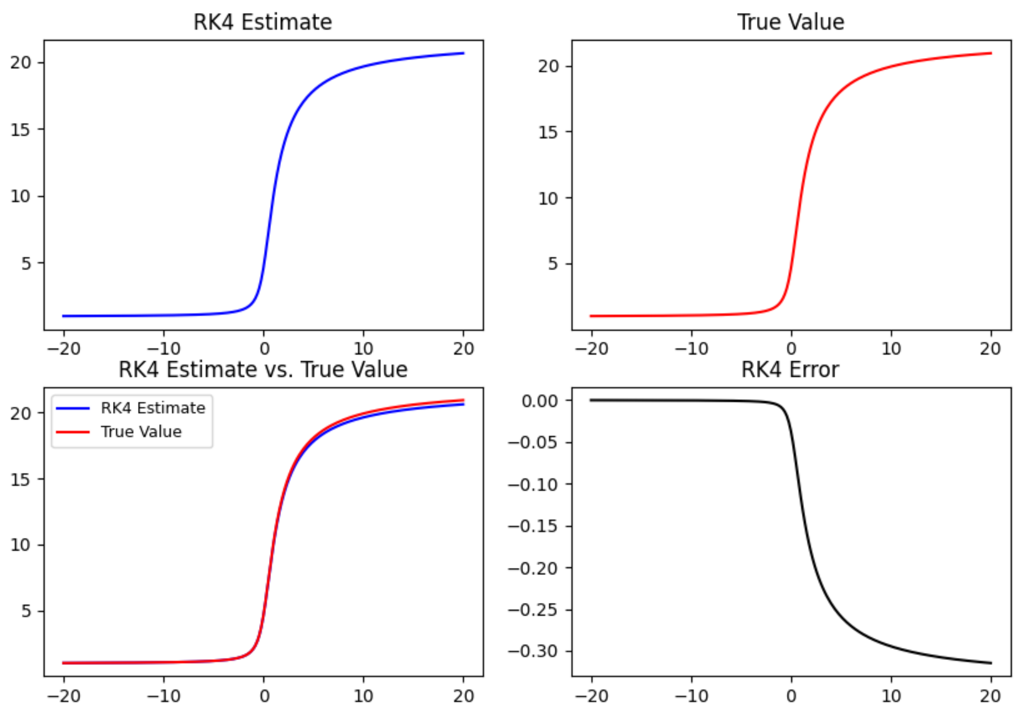

Figure 1: Fourth Order Runge-Kutta Estimate of $\frac{dy}{dx} = \frac{y}{1+x^2}$

As seen in Figure 1, the RK4 estimate of $\frac{dy}{dx} = \frac{y}{1+x^2}$ is quite accurate.

Now, for the second part of this question, we will define a function in python that uses two steps of $\frac{h}{2}$ to cancel out the leading-order error term in the RK4 estimate.

As derived in Numerical Recipes, the error of an RK4 with step h is $O(h^5)$. Thus, the error of an RK4 with step $\frac{h}{2}$ must be

$$O\left(\left(\frac{h}{2}\right)^5\right) = \frac{1}{32}O(h^5) \tag{4}$$

However, $\frac{h}{2}$ only represents a half-step. For one full step, we would have an error of

$$2 \cdot \frac{1}{32} O(h^5) = \frac{1}{16} O(h^5) \tag{5}$$

Now, we want to eliminate any terms with $O(h^5)$. To do so, we can find a linear combination of $O(h^5)$ and $\frac{1}{16} O(h^5)$ such that

$$c_1 \cdot O(h^5) + c_2 \cdot \frac{1}{16} O(h^5) = 0 \tag{6}$$

$$c_1 + \frac{c_2}{16} = 0$$

And of course, we must normalize the linear combination:

$$c_1 + c_2 = 1 \tag{7}$$

Then, equations (6) and (7) yield $c_1 = -\frac{1}{15}$ and $c_2 = \frac{16}{15}$. Let y_h and $y_{\frac{h}{2}}$ be the RK4 estimates of one step h and two steps $\frac{h}{2}$, respectively. Then, the following linear combination of y_h and $y_{\frac{h}{2}}$ is an RK4 estimate with its leading-order error term canceled out:

$$y_{n+1} = -\frac{1}{15} \cdot y_h + \frac{16}{15} \cdot y_{\frac{h}{2}} \tag{8}$$

Now, we want rk4_step() and rk4_stepd() to use the same number of function calls for a given estimate. Notice that for one step, rk4_step() uses 4 function evaluations, and rk4_stepd() uses 12 function evaluations . This means that for a 200-step estimate of an ODE, rk4_step() uses 800 function calls. So, for rk4_stepd() to use 800 function calls, we need to use $\frac{800}{12} \sim 67$ steps.

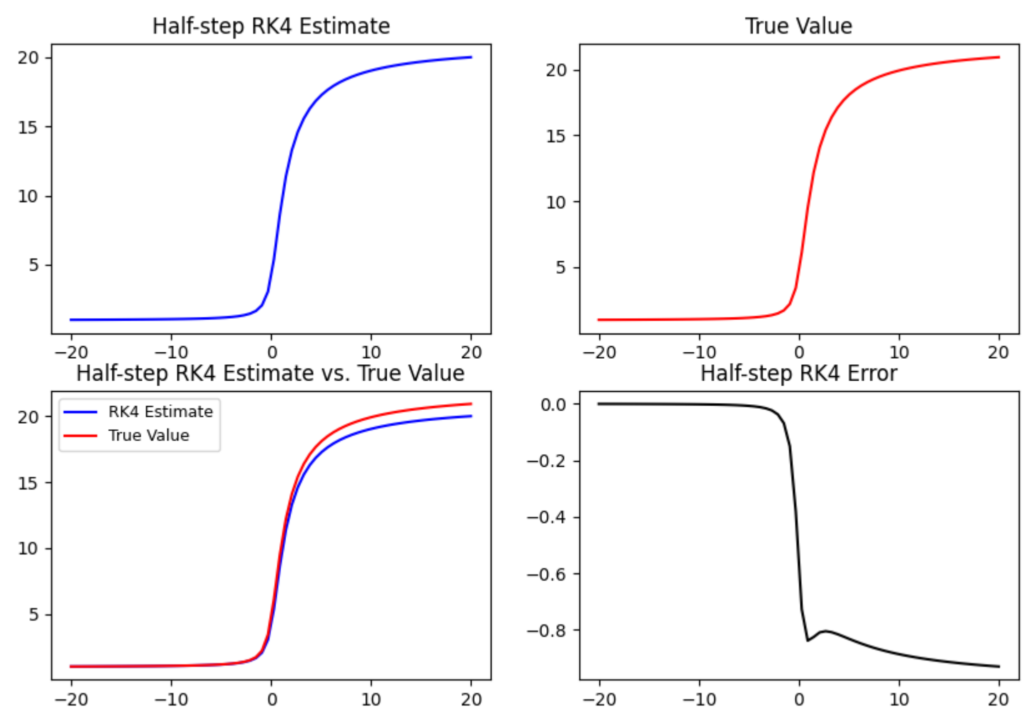

Figure 2: Half-step RK4 Estimate of $\frac{dy}{dx} = \frac{y}{1+x^2}$

As seen in Figure 2, the half-step RK4 estimate is decent as well. However, for 800 function calls, the full-step RK4 estimate is better. Indeed, the standard deviations of the full-step and half-step RK4 estimates are 0.14069699516880915 and 0.43038041858856557, respectively. However, if we were to compare the two estimates for the same number of steps, we would get very similar standard deviations.