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PHYS 512 - Assignment 7

Problem 1

Suppose t represents timesteps. Then, $t \pm dt = t \pm 1$. So we have:

$$\frac{f(x,t+1) - f(x,t-1)}{2dt} = -v\frac{f(x+dx,t) - f(x-dx,t)}{2dx}$$

Now, let $f(x,t) = \xi^t e^{ikx}$.

Then,

$$\frac{\xi^{t+1}e^{ikx} - \xi^{t-1}e^{ikx}}{2dt} = -v\frac{\xi^t e^{ikx}e^{ikdx} - \xi^t e^{ikx}e^{-ikdx}}{2dx}$$

$$\frac{\xi^t e^{ikx}[\xi - \xi^{-1}]}{2} = -\frac{v}{dx}dt\,\frac{\xi^t e^{ikx}[e^{ikdx} - e^{-ikdx}]}{2}$$

$$\xi - \xi^{-1} = -\alpha\,[e^{ikx} - e^{-ikdx}]$$

$$\longrightarrow \alpha = \frac{v}{dx}dt$$

$$\xi - \xi^{-1} = -2i\alpha\sin(kdx)$$

$$\xi^2 + \xi\,[2i\alpha\sin(kdx)] - 1 = 0$$

Solve for $\xi \longrightarrow$

$$\xi = \frac{-2i\alpha\sin(kdx) \pm \sqrt{-4\alpha^2\sin^2(kdx) + 4}}{2}$$

$$\xi = -i\alpha\sin(kdx) \pm \sqrt{1 - \alpha^2\sin^2(kdx)}$$

Now, given the CFL condition is satisfied ($|\alpha| \leq 1$), we get

$$|\xi|^2 = \alpha^2 \sin^2(kdx) + 1 - \alpha^2 \sin^2(kdx)$$

$$|\xi|^2 = 1$$

Thus, energy is preserved if the CFL condition is satisfied. ✓
(There is no amplitude dissipation using the leapfrog method.)