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PHYS 512 – Q3

For part a), we want to linearize the following equation

$$z - z_0 = a[(x - x_0)^2 + (y - y_0)^2] \tag{1}$$

$$\mathrm{z} = \mathrm{z}_0 + a[x^2 - 2xx_0 + x_0^2 + y^2 - 2yy_0 + y_0^2]$$

$$z = a(x^2 + y^2) - 2ax_0x - 2ay_0y + (z_0 + ax_0^2 + ay_0^2)$$

$$z = a(x^2 + y^2) + bx + cy + d$$

So, the linearized equation is $z = a(x^2 + y^2) + bx + cy + d$, where a, b, c and d are our new parameters. Note that

$$a = a$$

$$b = -2ax_0$$

$$c = -2ay_0$$

$$d = z_0 + ax_0^2 + ay_0^2$$

In terms of our new parameters, the old parameters are

$$a = a$$

$$x_0 = -\frac{b}{2a}$$

$$y_0 = -\frac{c}{2a}$$

$$z_0 = d - \frac{b^2}{4a} - \frac{c^2}{4a}$$

For part b), I solved the matrix equation $m = [A^T N^{-1} A]^{-1} A^T N^{-1} z$. This yielded the following parameters:

$$a = 1.66704455 \cdot 10^{-4}$$

$$b = 4.53599028 \cdot 10^{-4}$$

$$c = -1.94115589 \cdot 10^{-2}$$

$$d = -1.51231182 \cdot 10^{3}$$

In terms of the non-linear parameters, we get

$$a = 1.66704455 \cdot 10^{-4}$$

$$x_0 = -1.360488620415$$

$$y_0 = 58.221476144713$$

$$z_0 = -1512.877213364$$

Then, the following is an approximate equation for the given paraboloid:

$$z = 1.667 \cdot 10^{-4}(x^2 + y^2) + 4.536 \cdot 10^{-4}(x) - 1.941 \cdot 10^{-2}(y) - 1.512 \cdot 10^{3} \qquad (2)$$

And its nonlinear equivalent is:

$$z + 1512.877 = 1.667 \cdot 10^{-4}[(x + 1.360)^2 + (y - 58.221)^2] \qquad (3)$$

See below for a plot of this fit and the original data points. It can be difficult to compare the fit with the raw data in this image because it is 3-dimensional. For better visuals, one can run the code and rotate the plot that it produces.

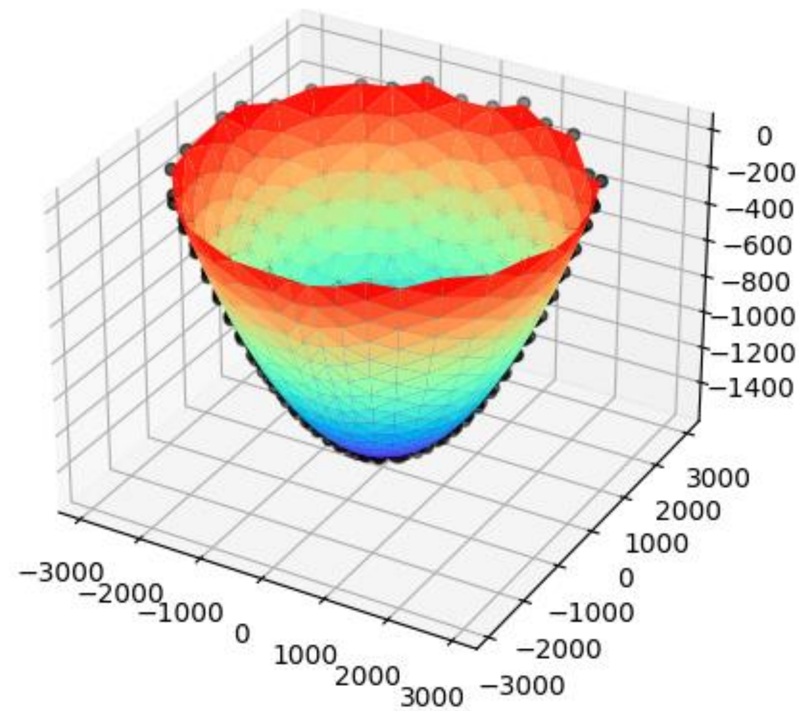

Figure 1: Least-squares Fit of Paraboloid Data

For part c), I considered the noise to be the standard deviation of $(z_{data} - z_{fit})$. Doing so, I obtained a noise of $\sigma \approx 3.7683$. Then, I calculated the error on parameters a, b, c and d using the equation derived in class: $err = (A^T N^{-1} A)^{-1}$. This yielded

$$\varepsilon_a = 6.45189976 \cdot 10^{-8}$$

$$\varepsilon_b = 1.25061100 \cdot 10^{-4}$$

$$\varepsilon_c = 1.19249564 \cdot 10^{-4}$$

$$\varepsilon_d = 3.12018436 \cdot 10^{-1}$$

Now, we want to find the focal length (f) of the paraboloid, which is given by

$$z - z_0 = \frac{(x - x_0)^2 + (y - y_0)^2}{4f} \tag{4}$$

$$f = \frac{(x - x_0)^2 + (y - y_0)^2}{4(z - z_0)}$$

Using this formula, I obtained $f = 1499.659984125217\ mm = 1.499659984125217\ cm$, with a standard deviation of $1.4819179194110214 \cdot 10^{-11}$. This is very close to the expected $f = 1.5\ cm$.

Now, we want to find the error bars of f. While computing f, there are four sources of uncertainties (x_0, y_0, z_0 and z). Thus, to find f_{min} and f_{max}, we can rewrite f as

$$f = \frac{\left(x - \left(x_0 \pm \varepsilon_{x_0}\right)\right)^2 + \left(y - \left(y_0 \pm \varepsilon_{y_0}\right)\right)^2}{4\left(z \pm \varepsilon_z - \left(z_0 \pm \varepsilon_{z_0}\right)\right)} \tag{5}$$

Where the error in x_0, y_0 and z_0 are given by

$$\varepsilon_{x_0} = \frac{|x_{0_{min}} - x_{0_{max}}|}{2}$$

$$\varepsilon_{y_0} = \frac{|y_{0_{min}} - y_{0_{max}}|}{2}$$

$$\varepsilon_{z_0} = \frac{|z_{0_{min}} - z_{0_{max}}|}{2}$$

Since x_0, y_0 and z_0 depend on a, b, c and d, the min's and max's are computed using the error in our parameters. Finally, the error in z is computed using the matrix equation $err_z = A \cdot err \cdot A^T$, where err is the matrix we used to obtain the error of our parameters.

I'm not sure if there is an error in my code, but Python has estimated my error bars to be zero.