FYS3150 - Project 3

Vegard Falmår and Sigurd Sørlie Rustad
Universitetet i Oslo[*]
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Abstract om du vil

I. INTRODUCTION

II. THEORY

A. Newton's laws and the motion of planets

The well-known Newtons second law reads

$$\sum \mathbf{F} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}, \tag{1}$$

where $\sum \mathbf{F}$ is the sum of forces acting on an object, m is its mass and $\mathbf{r}$ its position.

Newton's law of gravity says that the total gravitational force acting on an object i from all the other objects in a system is

$$F_{G,i} = -Gm_i \sum_{j\neq i} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \tag{2}$$

where $\mathbf{r}_i$ are the positions and m_i are the masses of the objects. G is the gravitational constant.

For the motions of planets, on which the gravitational force is the only force acting, the two equations 1 and 2 combined give us the differential equation that describes the motion of an object:

$$\frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2} = \frac{1}{m_i}F_{G,i} = -G\sum_{j\neq i} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \tag{3}$$

B. Velocity Verlet

Classical problem of Newtonian mechanics often involve solvinga set of two coupled first order differential equations, namely

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= v \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= a\end{aligned}$$

where x is position, v is velocity and a is acceleration. Doing a Taylor expansion around a point in time $t = t_0$ gives

$$\begin{aligned}x(t) &= x(t_0) + (t-t_0)\frac{\mathrm{d}x}{\mathrm{d}t}(t_0) + \frac{1}{2}(t-t_0)^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t_0) + O(h^3) \\ &= x(t_0) + (t-t_0)v(t_0) + \frac{1}{2}(t-t_0)^2 a + O(h^3)\end{aligned}$$

An expression can then be found for position at a time $t = t_0 \pm h$:

$$x(t\pm h) = x(t_0) \pm hv(t_0) + \frac{1}{2}h^2 a \pm O(h^3) \tag{4}$$

Discretizing equation 4 and letting $x_i = x(t)$, $x_{i+1} = x(t+h)$, $v_i = v(t)$ and $v_{i+1} = v(t+h)$, we get

$$\begin{aligned}x_{i+1} &\approx x_i + hv_i + \frac{h^2}{2}a_i \\ v_{i+1} &\approx v_i + ha_i + \frac{h^2}{2}\dot{a}_i,\end{aligned}$$

where $\dot{a}_i$ is the derivative of the acceleration with respect to time. Similarly to the case of Forward Euler, doing a Taylor expansion of a gives after some manipulation

$$\dot{a}_i \approx \frac{a_{i+1} - a_i}{h} \tag{5}$$

Insering this into the expressions we have for x_{i+1} and v_{i+1} gives us the equations that describe the Velocity Verlet method:

$$\begin{aligned}x_{i+1} &\approx x_i + hv_i + \frac{h^2}{2}a_i \\ v_{i+1} &\approx v_i + \frac{h}{2}(a_{i+1} + a_i)\end{aligned}$$

From equation 4 we see that the mathematical error in this approximation goes like $O(h^3)$.

C. Forward Euler

From equation 4 we see that by including only the two first terms in the Taylor expansion of $x(t)$ and $v(t)$ we get

$$\begin{aligned}x_{i+1} &\approx x_i + hv_i \\ v_{i+1} &\approx v_i + ha_i\end{aligned}$$

This is referred to as the Forward Euler method of solving differential equations. The mathematical error in this approximation goes like $O(h^2)$.

III. METHODS

A. Discretization of the equations of motion

Equation 3 is the equation of motion that describes the system of planets and stars in motion. We want to

[*] vegardfa@uio.no; sigurdsr@gmail.com

solve this equation numerically for all the planets in our solar system. In order to do this we must discretize the equations. We can rewrite the equation as a set of first order differential equations using the velocity:

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}$$
$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -G\sum_{j\neq i} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

The Verlet is not a self-starting algorithm. Mainly applicable when we are not interested in the velocity Velocity Verlet is energy conserving, Forward Euler is not

[1] Engeland, Hjorth-Jensen, Viefers, Raklev og Flekkøy, 2020, *Kompendium i FYS2140 Kvantefysikk, Versjon 3*, s. 81
[2] Griffiths, D. J, Schroeter, D. F., 2018, *Introduction to Quantum Mechanics, Third edition*, s. 44
[3] Griffiths, D. J, Schroeter, D. F., 2018, *Introduction to Quantum Mechanics, Third edition*, s. 75
[4] Griffiths, D. J, Schroeter, D. F., 2018, *Introduction to Quantum Mechanics, Third edition*, s. 133