FYS3150 - Project 1

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Abstract om du vil

I. THEORY

A. Matrix formulation of the discrete one-dimensional Poisson equation

The one-dimensional Poisson equation with Dirichlet boundary conditions is given by equation 1.

$$-\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0 \tag{1}$$

We defnine the discretized approximation to u to be v_i at points $x_i = ih$ evenly spaced between $x_0 = 0$ and $x_{n+1} = 1$. The step length between the points is $h = 1/(n+1)$. The boundary conditions from equation 1 then give $v_0 = v_{n+1} = 0$. An approximation to the second derivative of u, derived from the Taylor expansion, is

$$\frac{-v_{i-1} + 2v_i - v_{i+1}}{h^2} = f_i \quad for i = 1, 2, ..., n \tag{2}$$

where $f_i = f(x_i)$.

Written out for all i, equation 2 becomes

$$\begin{aligned}
-v_0 + 2v_1 - v_2 &= h^2 f_1 \\
-v_1 + 2v_2 - v_3 &= h^2 f_2 \\
&... \\
-v_{n-2} + 2v_{n-1} - v_n &= h^2 f_{n-1} \\
-v_{n-1} + 2v_n - v_{n+1} &= h^2 f_n
\end{aligned}$$

In general, this can be rearranged slightly so that

$$\begin{aligned}
2v_1 - v_2 &= h^2 f_1 + v_0 \\
-v_1 + 2v_2 - v_3 &= h^2 f_2 \\
&... \\
-v_{n-2} + 2v_{n-1} - v_n &= h^2 f_{n-1} \\
-v_{n-1} + 2v_n &= h^2 f_n + v_{n+1}
\end{aligned}$$

This system of equations can be written in matrix form as

$$\boldsymbol{A}\boldsymbol{v} = \tilde{\boldsymbol{b}}, \tag{3}$$

explicitly

$$\begin{bmatrix} 2 & -1 & 0 & 0 & ... & 0 \\ -1 & 2 & -1 & 0 & ... & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & ... & 0 & -1 & 2 & -1 \\ 0 & ... & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} h^2 f_1 + v_0 \\ h^2 f_2 \\ . \\ . \\ . \\ h^2 f_{n-1} \\ h^2 f_n + v_{n+1} \end{bmatrix}$$

With $v_0 = v_{n+1} = 0$, the right side reduces to $\tilde{b}_i = h^2 f_i$.

[1] Engeland, Hjorth-Jensen, Viefers, Raklev og Flekkøy, 2020, *Kompendium i FYS2140 Kvantefysikk, Versjon 3*, s. 81
[2] Griffiths, D. J, Schroeter, D. F., 2018, *Introduction to Quantum Mechanics, Third edition*, s. 44
[3] Griffiths, D. J, Schroeter, D. F., 2018, *Introduction to Quantum Mechanics, Third edition*, s. 75
[4] Griffiths, D. J, Schroeter, D. F., 2018, *Introduction to Quantum Mechanics, Third edition*, s. 133

II. APPENDIX

A. Utregninger til oppgave 3c

Likning

$$\mathbf{B}_0 \cdot \boldsymbol{\nabla} = \frac{B(\theta)}{r(\theta)}\frac{\partial}{\partial\theta}$$

blir

$$\mu_0\rho(\theta)\frac{\partial^2}{\partial t^2}\left(\frac{\xi(\theta,t)}{h_a(\theta)}\right) = \frac{1}{h_a^2(\theta)}\frac{B(\theta)}{r(\theta)}\frac{\partial}{\partial\theta}\left(h_a^2(\theta)\frac{B(\theta)}{r(\theta)}\frac{\partial}{\partial\theta}\left(\frac{\xi}{h_a(\theta)}\right)\right)$$

$$\mu_0\rho(\theta)\left(-\frac{\omega^2\xi(\theta,t)}{h_a(\theta)}\right) = \frac{1}{h_a^2(\theta)}\frac{B(\theta)}{r(\theta)}\frac{\partial}{\partial\theta}\left(h_a^2(\theta)\frac{B(\theta)}{r(\theta)}\frac{\partial}{\partial\theta}\left(\frac{\xi}{h_a(\theta)}\right)\right)$$

$$-\mu_0\rho(\theta)\frac{\omega^2\Xi(\theta)e^{i\omega t}}{h_a(\theta)} = \frac{1}{h_a^2(\theta)}\frac{B(\theta)}{r(\theta)}\frac{\partial}{\partial\theta}\left(h_a^2(\theta)\frac{B(\theta)}{r(\theta)}\frac{\partial}{\partial\theta}\left(\frac{\Xi(\theta)e^{i\omega t}}{h_a(\theta)}\right)\right)$$

Den tidsavhengige faktoren $e^{i\omega t}$ kan strykes på begge sider siden derivasjonen med hensyn på θ som vist ikke virker på den. Da får vi

$$-\mu_0\rho(\theta)\frac{\omega^2\Xi(\theta)}{h_a(\theta)} = \frac{1}{h_a^2(\theta)}\frac{B(\theta)}{r(\theta)}\frac{\partial}{\partial\theta}\left(h_a^2(\theta)\frac{B(\theta)}{r(\theta)}\frac{\partial}{\partial\theta}\left(\frac{\Xi(\theta)}{h_a(\theta)}\right)\right)$$

Deler på $B^2(\theta)$ på begge sider

$$-\frac{\mu_0\rho(\theta)}{B^2(\theta)}\omega^2\frac{\Xi(\theta)}{h_a(\theta)} = \frac{1}{B(\theta)h_a^2(\theta)}\frac{1}{r(\theta)}\frac{\partial}{\partial\theta}\left(B(\theta)h_a^2(\theta)\frac{1}{r(\theta)}\frac{\partial}{\partial\theta}\left(\frac{\Xi(\theta)}{h_a(\theta)}\right)\right)$$

$$v_A(\theta) = \frac{B(\theta)}{\sqrt{\mu_0\rho(\theta)}} \quad \text{gir}$$

$$-\frac{\omega^2}{v_A^2(\theta)}\frac{\Xi(\theta)}{h_a(\theta)} = \frac{1}{B(\theta)h_a^2(\theta)}\frac{1}{r(\theta)}\frac{\partial}{\partial\theta}\left(B(\theta)h_a^2(\theta)\frac{1}{r(\theta)}\frac{\partial}{\partial\theta}\left(\frac{\Xi(\theta)}{h_a(\theta)}\right)\right)$$

III. KODE