Forward/Backward Spatial Smoothing Techniques for Coherent Signal Identification

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***Abstract*—In the context of coherent signal classification, a spatial smoothing scheme first suggested by Evans *et al.*, and subsequently studied by Shan *et al.*, is further investigated. It is proved here that by making use of a set of forward and complex conjugated backward subarrays simultaneously, it is always possible to estimate any K directions of arrival using at most $3K/2$ sensor elements. This is achieved by creating a smoothed array output covariance matrix that is structurally identical to a covariance matrix in some noncoherent situation. By incorporating the eigenstructure-based techniques on this smoothed covariance matrix, it then becomes possible to correctly identify all directions of arrival irrespective of their correlation.**

I. Introduction

In recent years, considerable effort has been spent in developing high resolution techniques for estimating the directions of arrival of multiple signals using multiple sensors. These methods [1]–[4], in general, exploit specific eigenstructure properties of the sensor array output covariance matrix and are known to yield high resolution even when the signal sources are partially correlated. However, when some of the signals are perfectly correlated (coherent), as happens, for example, in multipath propagation, these techniques encounter serious difficulties. Several alternatives have been proposed [5]–[11] to take care of this situation, of which the spatial smoothing scheme first suggested by Evans *et al.* [9], [10] and extensively studied by Shan *et al.* [11], [12] is specially noteworthy. Their solution is based on a preprocessing scheme that partitions the total array of sensors into subarrays and then generates the average of the subarray output covariance matrices. Shan *et al.* have shown that when this average of subarray covariance matrices is used in conjunction with the eigenstructure-based multiple signal classification technique developed by Schmidt [3], in the case of independent and identical sensor noise, it is possible to estimate all directions of arrival irrespective of their degree of correlation. However, this forward-only smoothing scheme makes use of a larger number of sensor elements than the conventional ones, and in particular requires $2K$ sensor elements to estimate any K directions of arrival.

In this paper, we analyze an improved spatial smoothing scheme—called the forward/backward smoothing scheme—and prove that at most $[3K/2]$[1] elements are enough to estimate any K directions of arrival. In addition to the forward subarrays, this scheme makes use of complex conjugated backward subarrays of the original array to achieve superior performance. In this context, it is instructive to note the observations of Evans *et al.* [10], "The combined effect of spatial smoothing and forward/backward averaging cannot increase an array's direction finding capability beyond $[2M/3]$ coherent signals (with M representing the number of sensor elements)." While this statement is correct and coincides with the bounds in [13], Evans *et al.* do not provide a proof for it. A special case of the general situation, where the multipath coefficients are treated to be real, is proved in [13]. However, this is an unrealistic assumption, as in practice all multipath coefficients will be invariably complex numbers and in that case it is necessary to reason differently.

For clarity of presentation, Section II deals with a completely coherent situation and proves that to estimate any K coherent directions of arrival, it is sufficient to have an array of $[3K/2]$ sensors. The proof for the general source scene is sketched in the Appendix.

II. Direction Finding in a Coherent Environment

Consider a uniform linear array consisting of M identical sensors and receiving signals from K narrow-band coherent signals that arrive at the array from directions $\theta_1, \theta_2, \cdots, \theta_K$. At any instant, these K signals $u_1(t), u_2(t), \cdots, u_K(t)$ are phase-delayed amplitude-weighted replicas of one of them—say, the first—and hence,

$$u_k(t) = \alpha_k u_1(t), \qquad k = 1, 2, \cdots, K \tag{1}$$

where α_k represents the complex attenuation of the kth signal with respect to the first signal $u_1(t)$. Using complex signal representation, the received signal $x_i(t)$ at the ith sensor can be expressed as

$$x_i(t) = \sum_{k=1}^{K} u_k(t) \exp\left(-j\pi(i-1)\cos\theta_k\right) + n_i(t). \tag{2}$$

Here the interelement distance is taken to be half wavelength and $n_i(t)$ represents the additive noise at the ith

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[1]The symbol $[\times]$ stands for the integer part of $\times$.

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sensor. It is assumed that the signals and noises are stationary, zero mean uncorrelated random processes, and further, the noises are assumed to be uncorrelated and identical between themselves with common variance σ^2. Rewriting (2) in common vector notation and with $\omega_k = \pi \cos \theta_k$; $k = 1, 2, \cdots, K$, we have

$$x(t) \triangleq [x_1(t), x_2(t), \cdots, x_M(t)]^T = Au(t) + n(t) \tag{3}$$

where $x(t)$ is the $M \times 1$ array output data vector and

$$u(t) = [u_1(t), u_2(t), \cdots, u_K(t)]^T, \tag{4}$$

$$n(t) = [n_1(t), n_2(t), \cdots, n_M(t)]^T, \tag{5}$$

$$A = \sqrt{M}\,[a(\omega_1), a(\omega_2), \cdots, a(\omega_K)] \tag{6}$$

with $a(\omega_k)$ representing the direction vector associated with the arrival angle θ_k; i.e.,

$$a(\omega_k) = \frac{1}{\sqrt{M}}\left[1, \exp(-j\omega_k), \exp(-j2\omega_k), \cdots, \exp(-j(M-1)\omega_k)\right]^T. \tag{7}$$

Here A is an $M \times K$ matrix with Vandermonde-structured distinct columns ($M > K$), and hence, is of rank K. From our assumptions, it now follows that the array output covariance matrix $R \triangleq E[x(t)\,x^\dagger(t)]$ has the form[2]

$$R = AR_uA^\dagger + \sigma^2 I \tag{8}$$

where $R_u = E[u(t)\,u^\dagger(t)]$ represents the source covariance matrix that remains as nonsingular so long as the sources are at most partially correlated. In that case, $AR_uA^\dagger$ is also of rank K and hence, if $\{\lambda_1 \ge \lambda_2 > \cdots \ge \lambda_M\}$ and $\{\beta_1, \beta_2, \cdots, \beta_M\}$ are the eigenvalues and the corresponding eigenvectors of R, then the above rank property implies that $\lambda_i = \sigma^2$, $i \ge K+1$ and further [3]

$$\beta_i^\dagger a(\omega_k) = 0, \quad i = K+1, K+2, \cdots, M, \quad k = 1, 2, \cdots, K. \tag{9}$$

The high resolution eigenstructure-based techniques make use of (9) (these relationships are true only when R_u is of full rank) to estimate the actual directions of arrival $\theta_1, \theta_2, \cdots, \theta_K$, respectively. However, when the signals are coherent as in (1), the above conclusion is no longer true and different relations hold. In that case, using (1) in (4) and with $E[|u_1(t)|^2] = 1$, it is easy to see that

$$R_u = \alpha\alpha^\dagger; \quad \alpha = [\alpha_1, \alpha_2, \cdots, \alpha_K]^T \tag{10}$$

and from (8), the array output covariance matrix reduces to

$$R = A\alpha\alpha^\dagger A^\dagger + \sigma^2 I \triangleq bb^\dagger + \sigma^2 I. \tag{11}$$

Here $b = A\alpha$, and again reasoning as before, it follows that $\lambda_2 = \lambda_3 = \cdots = \lambda_M = \sigma^2$ and hence,

$$\beta_i^\dagger b = 0, \quad i = 2, 3, \cdots, M. \tag{12}$$

[2]From here on, † denotes the complex conjugate transpose.

Because of their Vandermonde structure, no linear combination of direction vectors can result in another direction vector. Consequently, b is no longer a legitimate direction vector and hence (12) will not be able to estimate any true arrival angles. The crucial role played by the nonsingularity of R_u in this discussion has prompted Evans *et al.* and subsequently Shan *et al.* to introduce a preprocessing scheme [9]–[11] which guarantees full rank for the equivalent R_u in (8) even when the signals are all coherent. This preprocessing spatial smoothing scheme starts by dividing a uniform linear array with M_o sensors into uniformly overlapping subarrays of size M (see Fig. 1). Let $x_l^f(t)$ stand for the output of the lth subarray for $l = 1, 2, \cdots, L \triangleq M_o - M + 1$, where L denotes the total number of these forward subarrays. Using (2)–(6), we have

$$x_l^f(t) \triangleq [x_l(t), x_{l+1}(t), \cdots, x_{l+M-1}(t)]^T = AB^{l-1}u(t) + n_l(t), \quad 1 \le l \le L \tag{13}$$

where B^{l-1} denotes the $(l-1)$th power of the $K \times K$ diagonal matrix

$$B = \mathrm{diag}\,[\nu_1, \nu_2, \cdots, \nu_K]; \quad \nu_i = \exp(-j\omega_i), \quad i = 1, 2, \cdots, K. \tag{14}$$

Then, the covariance matrix of the lth subarray is given by

$$R_l^f = E\left[x_l^f(t)\left(x_l^f(t)\right)^\dagger\right] = AB^{l-1}R_u(B^{l-1})^\dagger A^\dagger + \sigma^2 I. \tag{15}$$

Following [9]–[11], define the forward spatially smoothed covariance matrix R^f as the mean of the forward subarray covariance matrices, and this gives

$$R^f = \frac{1}{L}\sum_{l=1}^{L} R_l^f = AR_u^fA^\dagger + \sigma^2 I. \tag{16}$$

In a completely coherent environment, using (10), the forward-smoothed source covariance matrix R_u^f takes the form

$$R_u^f \triangleq \frac{1}{L}\sum_{l=1}^{L} B^{l-1}R_u(B^{l-1})^\dagger = \frac{1}{L}CC^\dagger \tag{17}$$

where

$$C = [\alpha, B\alpha, B^2\alpha, \cdots, B^{L-1}\alpha]$$

$$= \begin{bmatrix} \alpha_1 & & & O \\ & \alpha_2 & & \\ & & \ddots & \\ O & & & \alpha_K \end{bmatrix} \begin{bmatrix} 1 & \nu_1 & \nu_1^2 & \cdots & \nu_1^{L-1} \\ 1 & \nu_2 & \nu_2^2 & \cdots & \nu_2^{L-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \nu_K & \nu_K^2 & \cdots & \nu_K^{L-1} \end{bmatrix}$$

$$\triangleq DV. \tag{18}$$

Clearly the rank of R_u^f is equal to the rank of C. Since $C = DV$ and the square matrix D is of full rank, the rank

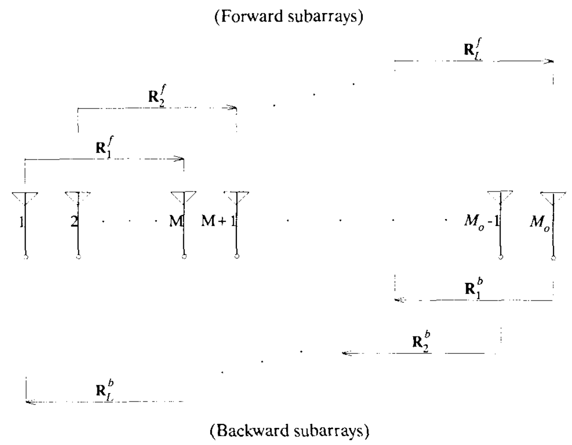

Fig. 1. The forward/backward spatial smoothing scheme.

of $\boldsymbol{C}$ is the same as that of $\boldsymbol{V}$. Now the rank of the $K \times L$ Vandermonde matrix $\boldsymbol{V}$ is $\rho(\boldsymbol{V}) = \min(K, L)$ and, hence, $\rho(\boldsymbol{V}) = \boldsymbol{K}$ iff $L \geq K$. Thus, if $L = M_o - M + 1 \geq K$ or equivalently $M_o \geq M + K - 1$, the smoothed source covariance matrix $\boldsymbol{R}_u^f$ is nonsingular and $\boldsymbol{R}^f$ has exactly the same form as the covariance matrix for a noncoherent case. Therefore, the conclusions in (9) will hold for $\boldsymbol{R}^f$ in (16) and, as pointed out by Shan *et al.*, one can successfully apply the eigenstructure methods to this smoothed covariance matrix regardless of the coherence of the signals. However, in this case, the number of sensor elements M_o must be at least $(M + K - 1)$, and recalling from (9) that the size M of each subarray must also be at least $K + 1$, it follows that the minimum number of sensors needed is $2K$ compared to $K + 1$ for the conventional one. In what follows, we present the improved spatial smoothing scheme that makes use of the forward and appropriate backward subarrays to reduce the required number of sensor elements to $[3K/2]$.

Toward this purpose, additional L backward subarrays are generated from the same set of sensors by grouping elements at $\{M_o, M_o - 1, \cdots, M_o - M + 1\}$ to form the first backward subarray and elements at $\{M_o - 1, M_o - 2, \cdots, M_o - M\}$ to form the second one, etc. (see Fig. 1). Let $\boldsymbol{x}_l^b(t)$ denote the complex conjugate of the output of the lth backward subarray for $l = 1, 2, \cdots, L$, where L as before denotes the total number $(M_o - M + 1)$ of these subarrays. Thus,

$$\begin{aligned} \boldsymbol{x}_l^b(t) &= \left[x_{M_o-l+1}^*(t), x_{M_o-l}^*(t), \cdots, x_{L-l+1}^*(t)\right]^T \\ &= \boldsymbol{A}\boldsymbol{B}^{l-1}\left(\boldsymbol{B}^{M_o-1}\boldsymbol{u}(t)\right)^* + \tilde{\boldsymbol{n}}_l^*(t), \qquad 1 \leq l \leq L \end{aligned} \tag{19}$$

where $\boldsymbol{B}$ is as defined in (14). The covariance matrix of the lth backward subarray is given by

$$\boldsymbol{R}_l^b = E\left[\boldsymbol{x}_l^b(t)\left(\boldsymbol{x}_l^b(t)\right)^\dagger\right] = \boldsymbol{A}\boldsymbol{B}^{l-1}\boldsymbol{R}_{\tilde{u}}\left(\boldsymbol{B}^{l-1}\right)^\dagger\boldsymbol{A}^\dagger + \sigma^2\boldsymbol{I} \tag{20}$$

with

$$\begin{aligned} \boldsymbol{R}_{\tilde{u}} &\triangleq \boldsymbol{B}^{-(M_o-1)}E\left[\boldsymbol{u}^*(t)\,\boldsymbol{u}^T(t)\right]\left(\boldsymbol{B}^{-(M_o-1)}\right)^\dagger \\ &= \boldsymbol{B}^{-(M_o-1)}\boldsymbol{R}_u^*\left(\boldsymbol{B}^{-(M_o-1)}\right)^\dagger. \end{aligned} \tag{21}$$

As before, define the spatially smoothed backward subarray covariance matrix $\boldsymbol{R}^b$ as the mean of these subarray covariance matrices; i.e.,

$$\boldsymbol{R}^b = \frac{1}{L}\sum_{l=1}^{L}\boldsymbol{R}_l^b = \boldsymbol{A}\boldsymbol{R}_u^b\boldsymbol{A}^\dagger + \sigma^2\boldsymbol{I}. \tag{22}$$

In a completely coherent environment $\boldsymbol{R}_u$ is given by (10) and in that case using (10) in (21) $\boldsymbol{R}_{\tilde{u}}$ simplifies to

$$\boldsymbol{R}_{\tilde{u}} = \boldsymbol{\delta}\boldsymbol{\delta}^\dagger, \tag{23}$$

where

$$\begin{aligned} \boldsymbol{\delta} &= \left[\delta_1, \delta_2, \cdots, \delta_K\right]^T; \qquad \delta_k = \alpha_k^*\nu_k^{-(M_o-1)}, \\ k &= 1, 2, \cdots, K \end{aligned} \tag{24}$$

with ν_k, $k = 1, 2, \cdots, K$ as defined in (14). Finally, using (23) the backward-smoothed source covariance matrix $\boldsymbol{R}_u^b$ is given by

$$\boldsymbol{R}_u^b \triangleq \frac{1}{L}\sum_{l=1}^{L}\boldsymbol{B}^{l-1}\boldsymbol{R}_{\tilde{u}}\left(\boldsymbol{B}^{l-1}\right)^\dagger = \frac{1}{L}\boldsymbol{E}\boldsymbol{E}^\dagger \tag{25}$$

where

$$\boldsymbol{E} = \left[\boldsymbol{\delta}, \boldsymbol{B}\boldsymbol{\delta}, \boldsymbol{B}^2\boldsymbol{\delta}, \cdots, \boldsymbol{B}^{L-1}\boldsymbol{\delta}\right] = \boldsymbol{F}\boldsymbol{V} \tag{26}$$

with $\boldsymbol{V}$ as in (18) and

$$\boldsymbol{F} = \operatorname{diag}\left[\delta_1, \delta_2, \cdots, \delta_K\right]. \tag{27}$$

Reasoning as before, it is easy to see that the backward spatially smoothed covariance matrix $\boldsymbol{R}^b$ will be of full rank so long as $\boldsymbol{R}_u^b$ is nonsingular, and this is guaranteed whenever $L \geq K$. Again, it follows that the backward subarray averaging scheme also requires at most $2K$ sensor elements to estimate the directions of arrival of K sources irrespective of their coherence.

It remains to show that by simultaneous use of the forward and backward subarray averaging schemes, it is possible to further reduce the number of extra sensor elements. To see this, following Evans *et al.* [10], define the forward/backward smoothed covariance matrix $\tilde{\boldsymbol{R}}$ as the mean of $\boldsymbol{R}^f$ and $\boldsymbol{R}^b$; i.e.,

$$\tilde{\boldsymbol{R}} = \frac{\boldsymbol{R}^f + \boldsymbol{R}^b}{2}. \tag{28}$$

Using (16), (17), (22), and (25) in (28) we have

$$\tilde{\boldsymbol{R}} = \boldsymbol{A}\left[\frac{1}{2L}\left(\boldsymbol{C}\boldsymbol{C}^\dagger + \boldsymbol{E}\boldsymbol{E}^\dagger\right)\right]\boldsymbol{A}^\dagger + \sigma^2\boldsymbol{I} = \boldsymbol{A}\tilde{\boldsymbol{R}}_u\boldsymbol{A}^\dagger + \sigma^2\boldsymbol{I} \tag{29}$$

with

$$\tilde{\boldsymbol{R}}_u = \frac{1}{2L}\left[\boldsymbol{C}\boldsymbol{C}^\dagger + \boldsymbol{E}\boldsymbol{E}^\dagger\right] = \frac{1}{2L}\boldsymbol{G}\boldsymbol{G}^\dagger. \tag{30}$$

Here

$$G = [\boldsymbol{\alpha}, B\boldsymbol{\alpha}, B^2\boldsymbol{\alpha}, \cdots, B^{L-1}\boldsymbol{\alpha}, \boldsymbol{\delta}, B\boldsymbol{\delta}, B^2\boldsymbol{\delta}, \cdots, B^{L-1}\boldsymbol{\delta}] = [DV|FV] = D[V|HV] \triangleq DG_0, \quad (31)$$

with D, V as in (18) and

$$H = \operatorname{diag}[\epsilon_1, \epsilon_2, \cdots, \epsilon_K]; \quad \epsilon_k = \delta_k/\alpha_k, \quad k = 1, 2, \cdots, K. \quad (32)$$

We will now prove that the modified source covariance matrix $\tilde{R}_u$ given by (30) will be nonsingular regardless of the coherence of the K signal sources so long as $2L \geq K$, provided that whenever equality holds among some of the members of the set $\{\epsilon_k\}_{k=1}^{K}$ in (32), the largest subset with equal entries must at most be of size L.

To appreciate this restriction, first consider the case where all ϵ_k, $k = 1, 2, \cdots, K$ are equal. In that case, it is easy to see that G_0 and hence $\tilde{R}_u$ will be of rank min (L, K) irrespective of the backward smoothing. However, in practice, this equality condition almost never occurs. This is because α_k in (1), which represents the complex attenuation of the kth source with respect to the reference source, is a signal property, and δ_k in (24), which is a function of the interelement phase delay of the kth source with respect to the reference element, is mainly an array geometry property. Thus, in an actual situation, all ϵ_k, $k = 1, 2, \cdots, K$ will be distinct and the simultaneous equality condition for all of them makes it an almost never occurring event. From these arguments, it also follows that the above restrictions on the equality among some of the ϵ_ks will almost always be satisfied. To be specific with regard to these restrictions, we will assume that

$$\epsilon_i \neq \epsilon_j, \quad \text{for any } i = 1, 2, \cdots, L, \quad \text{and } j = L+1, L+2, \cdots, K. \quad (33)$$

A special case of the general situation, where all α_k, $k = 1, 2, \cdots, K$, in (1) are real, is treated in [13]. In that case, using (24) and (32) in (31), it is easy to see that G_0 is a Vandermonde matrix with distinct columns and hence is of rank K so long as $2L \geq K$. This, however, is an unrealistic assumption as, in practice, all α_ks will be invariably complex numbers and in that case it is necessary to argue differently as follows.

From (30), $\tilde{R}_u$ will be nonsingular so long as G is of full row rank, and using (31) this is further equivalent to having full rank for G_0. Clearly, for G (or G_0) to have full row rank, it is necessary that $2L \geq K$ and with $L = M_o - M + 1$, this reduces to $2M_o \geq 2M + K - 2$. Again, recalling that in the presence of K signals the size M of each subarray must be at least $K + 1$, it follows that the number of sensors M_o needed must satisfy $2M_o \geq 3K$ or, equivalently, the minimum number of sensors must be at least $\lceil 3K/2 \rceil$. To see that this requirement is also sufficient, consider the quadratic product

$$y^\dagger G_0 G_0^\dagger y = y^\dagger V V^\dagger y + y^\dagger H V V^\dagger H^\dagger y \quad (34)$$

where y is any arbitrary $K \times 1$ vector. We will show that

$$y^\dagger G_0 G_0^\dagger y > 0 \quad (35)$$

for any $y \neq O$, thus proving the positive-definite property of $G_0 G_0^\dagger$ or $\tilde{R}_u$. Clearly, (35) needs to be demonstrated only for a typical $y_0 \in N(V^\dagger)$, the null space of $V^\dagger$. In that case, $V^\dagger y_0 = O$ and hence the first term in (34) reduces to zero. To prove our claim, it is enough to show that for such a typical y_0, $H^\dagger y_0$ does not belong to $N(V^\dagger)$. Since the Vandermonde structured matrix $V^\dagger$ is of full row rank L, the dimension of $N(V^\dagger)$ is $K - L$. Let v_{L+1}, $v_{L+2}, \cdots, v_K$ be a set of linearly independent basis vectors for $N(V^\dagger)$. With respect to the basis vectors for the K-dimensional space, these null space basis vectors can always be chosen such that [14]

$$v_l = [v_{1l}, v_{2l}, \cdots, v_{Ll}, 0, \cdots, 0, 1, 0, \cdots, 0]^T. \quad (36)$$

(In (36), the 1 is at the lth location.) These v_l, $l = L+1, L+2, \cdots, K$ are linearly independent and, moreover, for any $j \in \{L+1, L+2, \cdots, K\}$, using the diagonal nature of H, it is also easy to see that $H^\dagger v_j$ is linearly independent of the remaining v_l, $l = L+1, \cdots, K$, $l \neq j$. Further, the pair v_j and $H^\dagger v_j$, $j = L+1, L+2, \cdots, K$, is also linearly independent of each other. To see this, note that because of the full row rank property of $V^\dagger$, at least one of the v_{il}, $i = 1, 2, \cdots, L$ in (36) must be nonzero for every l. Let $v_{i_o j}$ be such an entry in v_j. Then the minor formed by the i_oth and jth rows of the matrix $[v_j | H^\dagger v_j]$ has the form

$$\begin{vmatrix} v_{i_o j} & \epsilon_{i_o}^* v_{i_o j} \\ 1 & \epsilon_j^* \end{vmatrix} = v_{i_o j}(\epsilon_j^* - \epsilon_{i_o}^*) \quad (37)$$

and is nonzero from (33). Thus, the matrix $[v_j | H^\dagger v_j]$ is of rank 2. This proves the linear independence of v_j and $H^\dagger v_j$. From the above discussion, it follows that $H^\dagger v_j$ is linearly independent of v_j, $j = L+1, L+2, \cdots, K$, and hence, $H^\dagger v_j \notin N(V^\dagger)$, $j = L+1, L+2, \cdots, K$. Now for any $y_0 \in N(V^\dagger)$, we have

$$y_0 = \sum_{j=L+1}^{K} k_j v_j, \quad (38)$$

which gives

$$H^\dagger y_0 = \sum_{j=L+1}^{K} k_j H^\dagger v_j. \quad (39)$$

Since all k_j cannot be zero in (39), it follows that $H^\dagger y_0 \notin N(V^\dagger)$, and hence, $V^\dagger H^\dagger y_0 \neq O$. This proves our claim and establishes that $\tilde{R}_u$ will be nonsingular under the mild restrictions in (33). In that case, the eigenvalues of $\tilde{R}$ satisfy $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_K > \tilde{\lambda}_{K+1} = \tilde{\lambda}_{K+2} = \cdots \tilde{\lambda}_M = \sigma^2$. Consequently, as in (9), the eigenvectors corresponding to equal eigenvalues are orthogonal to the direction vectors associated with the true directions of arrival; i.e.,

$$\tilde{\beta}_i^\dagger a(\omega_k) = 0, \quad i = K+1, K+2, \cdots, M, \quad k = 1, 2, \cdots, K. \tag{40}$$

Here $\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_M$ are the eigenvectors of $\tilde{R}$ corresponding to the eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_M$, respectively.

To summarize, we have proved that as long as the number of sensor elements is at least $\lceil 3K/2 \rceil$ (with K representing the number of signal sources present in the scene), it is almost always possible to estimate all arrival angles irrespective of the signal correlations by simultaneous use of the forward and backward subarray averaging scheme. Since the smoothed covariance matrix $\tilde{R}$ in (28) has exactly the same form as the covariance matrix for some noncoherent situation as in (8), the eigenstructure-based techniques can be applied to this smoothed covariance matrix, irrespective of the coherence of the signals, to successfully estimate their directions of arrival.

The Appendix extends the proof for the forward/backward smoothing scheme to a mixed source scene consisting of partially correlated signals with complete coherence among some of them.

III. Simulation Results

In this section, simulation results are presented to illustrate the performance of the forward/backward spatial smoothing scheme and to compare it to the conventional eigenstructure-based technique [3].

Fig. 2 represents a coherent source scene where the reference signal arriving from 70° undergoes multipath reflection, resulting in three additional coherent arrivals along 45°, 115°, and 127°. A six-element uniform array is used to receive these signals. The input signal-to-noise ratio (SNR) of the reference signal is 5 dB, and the attenuation coefficients of the three coherent sources are taken to be (0.4, 0.8), (−0.3, −0.7), and (0.5, −0.6), respectively. In the notation $\alpha = (a, b)$, here a and b represent the real and imaginary parts, respectively, of the complex attenuation coefficient α. Three-hundred data samples are used to estimate the array output covariance matrix using the standard maximum likelihood procedure. The application of the conventional eigenstructure method [3] to this covariance matrix resulted in Fig. 2(a). However, first applying the forward/backward spatial smoothing scheme with two forward and two backward ($L = 2$) subarrays of five ($M = 5$) sensors each, and then reapplying the eigenstructure technique on the smoothed covariance matrix $\tilde{R}$ resulted in Fig. 2(b). All four directions of arrival can be clearly identified, and the improvement in performance in terms of resolvability, irrespective of the signal coherence, is also visible in this case.

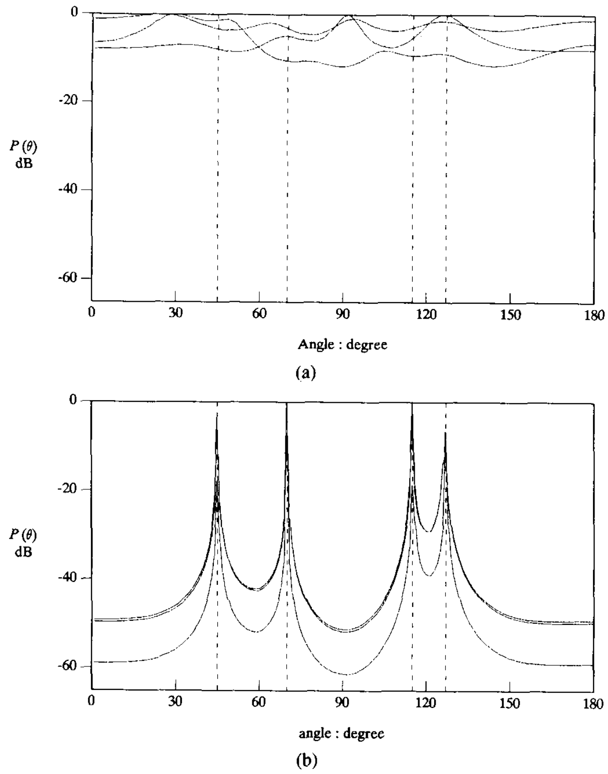

Fig. 2. Direction finding in a coherent scene. A six-element uniform array receives signals from four coherent sources with multipath coefficients (0.4, 0.8), (1., 0.), (−0.3, −0.7) and (0.5, −0.6). The arrival angles of the four coherent signals are 45°, 70°, 15°, and 127°. Input SNR of the reference signal is 5 dB. Three-hundred data samples are used to estimate the covariance matrix. (a) $P(\theta)$ using the conventional MUSIC scheme. (b) $P(\theta)$ using the forward/backward smoothing scheme. Here

$$P(\theta) = 1 \Big/ \sum_{i=K+1}^{M} \left| \beta_i^\dagger a(\omega) \right|^2, \qquad \omega = \pi \cos \theta.$$

IV. Conclusions

This paper reexamines the problem of locating the directions of arrival of coherent signals and, in that context, a spatial smoothing scheme, first introduced by Evans *et al.* and analyzed by Shan *et al.*, is further investigated. It is proved here that by simultaneous use of a set of forward and complex conjugated backward subarrays, it is always possible to estimate any K directions of arrival using at most $\lceil 3K/2 \rceil$ sensor elements. This is made possible by creating a smoothed array output covariance matrix that is structurally identical to a covariance matrix in some noncoherent situation, thus enabling one to correctly identify all directions of arrival by incorporating the eigenstructure-based techniques [3] on this smoothed matrix. This is a considerable saving compared to the forward-only smoothing scheme [11] that requires as many extra sensor elements as the total number of coherent signals present in the scene.

Appendix

Coherent and Correlated Signal Scene

We will demonstrate here that the forward/backward smoothing scheme discussed in Section II readily extends to the general situation where the source scene consists of $K + J$ signals $u_1(t), u_2(t), \cdots, u_K(t), u_{K+1}(t), \cdots,$

$u_{K+J}(t)$, of which the first K signals are completely coherent and the last $(J + 1)$ signals are partially correlated. Thus, the coherent signals are partially correlated with the remaining set of signals. Further, the respective arrival angles are assumed to be $\theta_1, \theta_2, \cdots, \theta_K, \theta_{K+1}, \cdots, \theta_{K+J}$. As before, the signals are taken to be uncorrelated with the noise, and the noise is assumed to be identical and uncorrelated from element to element. With symbols as defined in the text and using (2), the output $x_i(t)$ of the ith sensor element at time t in this case can be written as

$$x_i(t) = u_1(t) \sum_{k=1}^{K} \alpha_k \exp\left(-j(i-1)\omega_k\right) + \sum_{k=1}^{K+J} u_k(t) \cdot \exp\left(-j(i-1)\omega_k\right) + n_i(t), \quad i = 1, 2, \cdots, M. \tag{A.1}$$

With $\boldsymbol{x}(t)$ as in (3), this gives

$$\boldsymbol{x}(t) = \tilde{\boldsymbol{A}}\boldsymbol{v}(t) + \boldsymbol{n}(t), \tag{A.2}$$

where

$$\tilde{\boldsymbol{A}} = \sqrt{M}\left[\boldsymbol{a}(\omega_1), \boldsymbol{a}(\omega_2), \cdots, \boldsymbol{a}(\omega_K), \boldsymbol{a}(\omega_{K+1}), \cdots, \boldsymbol{a}(\omega_{K+J})\right] \tag{A.3}$$

with $\boldsymbol{a}(\omega_k)$; $k = 1, 2, \cdots, K + J$ as defined in (7) and

$$\boldsymbol{v}(t) \triangleq \begin{bmatrix} \boldsymbol{u}_1(t) \\ \boldsymbol{u}_2(t) \end{bmatrix}. \tag{A.4}$$

Here

$$\boldsymbol{u}_1(t) = \left[u_1(t), u_2(t), \cdots, u_K(t)\right]^T = u_1(t)\boldsymbol{\alpha} \tag{A.5}$$

with $\boldsymbol{\alpha}$ as in (10) and

$$\boldsymbol{u}_2(t) = \left[u_{K+1}(t), u_{K+2}(t), \cdots, u_{K+J}(t)\right]^T. \tag{A.6}$$

Following (13)–(17), (19)–(20), (25), and (28), the forward/backward smoothed covariance matrix $\tilde{\boldsymbol{R}}$ in this case can be written as

$$\tilde{\boldsymbol{R}} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{R}}_0\tilde{\boldsymbol{A}}^\dagger + \sigma^2\boldsymbol{I}, \tag{A.7}$$

where

$$\tilde{\boldsymbol{R}}_0 = \frac{1}{2L}\sum_{l=1}^{L} \tilde{\boldsymbol{B}}^{l-1}(\boldsymbol{R}_v + \boldsymbol{R}_{\tilde{v}})(\tilde{\boldsymbol{B}}^{l-1})^\dagger. \tag{A.8}$$

It remains to show that $\tilde{\boldsymbol{R}}_0$ is of full rank irrespective of the coherency among some of the arrivals. Here

$$\tilde{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{B}_2 \end{bmatrix} \tag{A.9}$$

where

$$\boldsymbol{B}_1 = \text{diag}\left[\nu_1, \nu_2, \cdots, \nu_K\right] \tag{A.10}$$

and

$$\boldsymbol{B}_2 = \text{diag}\left[\nu_{K+1}, \nu_{K+2}, \cdots, \nu_{K+J}\right] \tag{A.11}$$

with ν_k, $k = 1, \cdots, K + J$ as given by (14) and

$$\boldsymbol{R}_v \triangleq E\left[\boldsymbol{v}(t)\,\boldsymbol{v}^\dagger(t)\right] = \begin{bmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{12}^\dagger & \boldsymbol{R}_{22} \end{bmatrix}. \tag{A.12}$$

Using (A.4)–(A.6), it is easy to see that

$$\boldsymbol{R}_{11} = E\left[\boldsymbol{u}_1(t)\,\boldsymbol{u}_1^\dagger(t)\right] = \boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger \tag{A.13}$$

where $\boldsymbol{\alpha}$ is as before and $E[|u_1(t)|^2] = 1$. Similarly,

$$\boldsymbol{R}_{12} = E\left[\boldsymbol{u}_1(t)\,\boldsymbol{u}_2^\dagger(t)\right] = \boldsymbol{\alpha}\boldsymbol{\gamma}^\dagger \tag{A.14}$$

with

$$\boldsymbol{\gamma} = \left[\gamma_1, \gamma_2, \cdots, \gamma_J\right]^T, \tag{A.15}$$

where

$$\gamma_i \triangleq E\left[u_1(t)\,u_{K+i}^*(t)\right], \quad i = 1, 2, \cdots, J, \tag{A.16}$$

and

$$\boldsymbol{R}_{22} = E\left[\boldsymbol{u}_2(t)\,\boldsymbol{u}_2^\dagger(t)\right]. \tag{A.17}$$

From the partially correlated assumption among the later J signals, it follows that their correlation matrix $\boldsymbol{R}_{22}$ is of full rank and hence it has the representation

$$\boldsymbol{R}_{22} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\dagger \tag{A.18}$$

where $\boldsymbol{\Lambda}$ is again a full rank matrix of size $J \times J$. In a similar manner following (21), $\boldsymbol{R}_{\tilde{v}}$ can be written as

$$\boldsymbol{R}_{\tilde{v}} = \tilde{\boldsymbol{B}}^{-(M_o-1)}\boldsymbol{R}_v^*(\tilde{\boldsymbol{B}}^{-(M_o-1)})^\dagger = \begin{bmatrix} \tilde{\boldsymbol{R}}_{11} & \tilde{\boldsymbol{R}}_{12} \\ \tilde{\boldsymbol{R}}_{12}^\dagger & \tilde{\boldsymbol{R}}_{22} \end{bmatrix}, \tag{A.19}$$

and, proceeding as before,

$$\tilde{\boldsymbol{R}}_{11} = \boldsymbol{\delta}\boldsymbol{\delta}^\dagger \tag{A.20}$$

with $\boldsymbol{\delta}$ as in (24) and

$$\tilde{\boldsymbol{R}}_{12} = \boldsymbol{\delta}\tilde{\boldsymbol{\gamma}}^\dagger \tag{A.21}$$

with

$$\tilde{\boldsymbol{\gamma}} = \left[\tilde{\gamma}_1, \tilde{\gamma}_2, \cdots, \tilde{\gamma}_J\right]^T, \tag{A.22}$$

where

$$\tilde{\gamma}_i = \gamma_i^*(\nu_{K+i})^{-(M_o-1)}, \quad i = 1, 2, \cdots, J. \tag{A.23}$$

Here γ_i is as defined in (A.16) and ν_{K+i} is obtained by extending the definition in (14). Further,

$$\tilde{\boldsymbol{R}}_{22} = \boldsymbol{B}_2^{-(M_o-1)}\boldsymbol{R}_{22}(\boldsymbol{B}_2^{-(M_o-1)})^\dagger = \tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{\Lambda}}^\dagger \tag{A.24}$$

with

$$\tilde{\boldsymbol{\Lambda}} = \boldsymbol{B}_2^{-(M_o-1)}\boldsymbol{\Lambda}, \tag{A.25}$$

where $\tilde{\boldsymbol{\Lambda}}$ again is a full rank matrix of size $J \times J$. With (A.9)–(A.25) in (A.8), it simplifies to

$$\tilde{R}_0 = \frac{1}{2L} = \begin{bmatrix} \sum_{l=1}^{L} B_1^{l-1}(R_{11} + \tilde{R}_{11})(B_1^{l-1})^\dagger & \sum_{l=1}^{L} B_1^{l-1}(R_{12} + \tilde{R}_{12})(B_2^{l-1})^\dagger \\ \sum_{l=1}^{L} B_2^{l-1}(R_{12}^\dagger + \tilde{R}_{12}^\dagger)(B_1^{l-1})^\dagger & \sum_{l=1}^{L} B_2^{l-1}(R_{22} + \tilde{R}_{22})(B_2^{l-1})^\dagger \end{bmatrix}$$

$$= \frac{1}{2L}\begin{bmatrix} G_1 G_1^\dagger & G_1 G_2^\dagger \\ G_2^\dagger G_1 & G_3 G_3^\dagger \end{bmatrix} = \frac{1}{2L}\begin{bmatrix} G_1 & O \\ G_2 & G_4 \end{bmatrix}\begin{bmatrix} G_1^\dagger & G_2^\dagger \\ O & G_4^\dagger \end{bmatrix} \tag{A.26}$$

where

$$G_1 = [\alpha, B_1\alpha, \cdots, B_1^{L-1}\alpha, \delta, B_1\delta, \cdots, B_1^{L-1}\delta], \tag{A.27}$$

$$G_2 = [\gamma, B_2\gamma, \cdots, B_2^{L-1}\gamma, \tilde{\gamma}, B_2\tilde{\gamma}, \cdots, B_2^{L-1}\tilde{\gamma}], \tag{A.28}$$

$$G_3 = [\Lambda, B_2\Lambda, \cdots, B_2^{L-1}\Lambda, \tilde{\Lambda}, B_2\tilde{\Lambda}, \cdots, B_2^{L-1}\tilde{\Lambda}], \tag{A.29}$$

and G_4 satisfies

$$G_3 G_3^\dagger = G_2 G_2^\dagger + G_4 G_4^\dagger. \tag{A.30}$$

Define

$$\tilde{G} = \begin{bmatrix} G_1 & O \\ G_2 & G_4 \end{bmatrix}. \tag{A.31}$$

Then

$$\tilde{R}_0 = \frac{1}{2L}\tilde{G}\tilde{G}^\dagger. \tag{A.32}$$

Clearly, the rank of $\tilde{R}_0$ is the same as that of $\tilde{G}$. An examination of (A.27) shows that $G_1 G_1^\dagger$ is the average of the source covariance matrix corresponding to the completely coherent situation [see (31)] and, hence, from the result derived in Section II, it follows that $G_1 G_1^\dagger$ is of full rank K as long as $L \geq [K/2]$. Now it remains to show that G_4 is also of full row rank J, which together with (A.31) implies that $\tilde{G}$ and, hence, $\tilde{R}_0$ is of full rank $K + J$. From (A.28)–(A.30), we have

$$G_4 G_4^\dagger = G_3 G_3^\dagger - G_2 G_2^\dagger$$

$$= \left[\sum_{l=1}^{L} B_2^{l-1}(\Lambda\Lambda^\dagger - \gamma\gamma^\dagger)(B_2^{l-1})^\dagger + \sum_{l=1}^{L} B_2^{l-1}(\tilde{\Lambda}\tilde{\Lambda}^\dagger - \tilde{\gamma}\tilde{\gamma}^\dagger)(B_2^{l-1})^\dagger \right]. \tag{A.33}$$

In the first summation here, Λ and γ are matrices of ranks J and 1, respectively, and hence the matrix $(\Lambda\Lambda^\dagger - \tilde{\gamma}\tilde{\gamma}^\dagger)$ is at least of rank $J - 1$. Once again, resorting to the argument used in establishing (35) in Section II, it follows that each summation and hence G_4 is of full row rank J so long as $L > 1$. This establishes the nonsingularity of $\tilde{R}_0$ for $L \geq [K/2]$. As a result, the smoothed covariance matrix $\tilde{R}$ in (A.7) has structurally the same form as the covariance matrix for some noncoherent set of $K + J$ signals. Hence, the eigenstructure-based techniques can be applied to this smoothed matrix irrespective of the coherence of the original set of signals to successfully estimate their directions of arrival. This completes the proof.

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