COMP9319 Web Data Compression and Search

Space Efficient Linear Time Construction of Suffix Arrays

Suffix Array

- Sorted order of suffixes of a string *T*.
- Represented by the starting position of the suffix.

Text	M	I	S	S	I	S	S	I	P	P	I	$
Index	1	2	3	4	5	6	7	8	9	10	11	12
Suffix Array	12	11	8	5	2	1	10	9	7	4	6	3

Brief History

- Introduced by Manber and Myers in 1989.
- Takes O($n \log n$) time, and 8n bytes.
- Many other non-linear time algorithms.

Authors	Time	Space (bytes)
Manber & Myers	$n \log n$	$8n$
Sadakane	$n \log n$	$9n$
String-sorting	$n^2 \log n$	$5n$
Radix-sorting	n^2	$5n$

Our Result

1. Among the first linear-time direct suffix array construction algorithms. Solves an important open problem.
2. For constant size alphabet, only uses $8n$ bytes.
3. Easily implementable.
4. Can also be used as a space efficient suffix tree construction algorithm.

Notation

- String $T = t_1 \ldots t_n$.
- Over the alphabet $\Sigma = \{1 \ldots n\}$.
- t_n = '\$', '\$' is a unique character.
- $T_i = t_i \ldots t_n$, denotes the i-th suffix of T.
- For strings α and β, $\alpha < \beta$ denotes α is lexicographically smaller than β.

Overview

- Divide all suffixes of *T* into two types.
  - Type S suffixes = $\{T_i \mid T_i < T_{i+1}\}$
  - Type L suffixes = $\{T_j \mid T_j > T_{j+1}\}$
  - The last suffix is both type *S* and *L*.
- Sort all suffixes of one of the types.
- Obtain lexicographical order of all suffixes from the sorted ones.

Identify Suffix Types

Type	L	S	L	L	S	L	L	S	L	L	L	L/S
Text	M	I	S	S	I	S	S	I	P	P	I	$

The type of each suffix in *T* can be determined in one scan of the string.

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Notation

Type		S			S			S				S
Text	M	I	S	S	I	S	S	I	P	P	I	$

Type *S* suffixes

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Notation

Type		S			S			S				S
Text	M	I	S	S	I	S	S	I	P	P	I	$

Type *S* positions/ characters

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Notation

Type		S			S			S				S
Text	M	I	S	S	I	S	S	I	P	P	I	$

Type *S* substrings

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Sorting Type *S* Suffixes

- Sort all type *S* substrings.
- Replace each type *S* substrings by its bucket number.
- New string is the sequence of bucket numbers.
- Sorting all type *S* suffixes = Sorting all suffixes of the new string.

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Sorting Type *S* Substrings

Type		S			S			S				S
Text	M	I	S	S	I	S	S	I	P	P	I	$

Bucket Sort	$	I	I	I
		P	S	S
		P	S	S
		I	I	I
		$		

Sort each substring until the next type *S* character

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Sorting Type *S* Substrings

Type

Text	3	3	2	1

Bucket Sort

$	I	I	I
	P	S	S
	P	S	S
	I	I	I
	$		

Substitute the substrings with the bucket numbers to obtain a new string. Apply sorting recursively to the new string.

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Sorting Type *S* Substrings

Type

Text	3	3	2	1

Bucket Sort

$	I	I	I
	P	S	S
	P	S	S
	I	I	I
	$		

Bucket Sort takes potentially O(n^2) time

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Solution

- Observation: Each character participates in the bucket sort at most twice.
  - Type *L* characters only participate in the bucket sort once.
- Solution:
  - Sort all the characters once.
  - Construct m lists according the distance to the closest type *S* character to the left

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Illustration

Type		S			S			S				S
Index	1	2	3	4	5	6	7	8	9	10	11	12
Text	M	I	S	S	I	S	S	I	P	P	I	$
Distance	0	0	1	2	3	1	2	3	1	2	3	4

Sorted Order of characters	12	2	5	8	11	1	9	10	3	4	6	7

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Illustration

Type		S			S			S				S
Index	1	2	3	4	5	6	7	8	9	10	11	12
Text	M	I	S	S	I	S	S	I	P	P	I	$
Distance	0	0	1	2	3	1	2	3	1	2	3	4

The Lists

9	3	6
10	4	7
5	8	11
12		

Sort the type *S* substrings using the lists

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T	M	I	S	S	I	S	S	I	P	P	I	$
Type		S			S			S				S
Pos	1	2	3	4	5	6	7	8	9	10	11	12
A	12	2	5	8	11	1	9	10	3	4	6	7

Step 1. Record the S−distances

Pos	1	2	3	4	5	6	7	8	9	10	11	12
Dist	0	0	1	2	3	1	2	3	1	2	3	4

Step 2. Construct S−distance Lists

1	9	3	6
2	10	4	7
3	5	8	11
4	12		

Step 3. Sort all type S substrings

Original

12	2	5	8

Sort according to list 1

12	8	5	2

Sort according to list 2

12	8	5	2

Sort according to list 3

12	8	5	2

Sort according to list 4

12	8	5	2

Fig. 3. Illustration of the sorting of type *S* substrings of the string MISSISSIPPI$.

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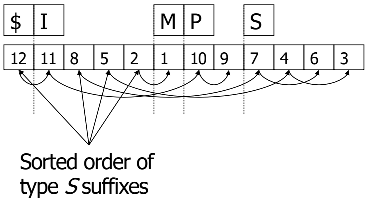
Construct Suffix Array for all Suffixes

- The first suffix in the suffix array is a type S suffix.
- For $1 \leq i \leq n$, if $T_{SA[i]-1}$ is type L, move it to the current front of its bucket



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Run-Time Analysis (Sketch)

- Identify types of suffixes -- O(n) time.
- Bucket sort type S (or L) substrings -- O(n) time.
- Construct suffix array from sorted type S (or L) suffixes -- O(n) time.

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Conclusion

- Among the first suffix array construction algorithm takes O(n) time.
- The algorithm can be easily implemented in 8n bytes (plus a few Boolean arrays).
- Equal or less space than most non-linear time algorithm.

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Exercise

- Consider the popular example string S:
- **bananainpajamas$**

1. Construct the suffix array of S using the linear time algorithm
2. Then compute the BWT(S)
3. What's the relationship between the suffix array and BWT ?

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Step – Identify the type of each suffix

- **LSLSLSSSLSLSLSL**L/S
- **bananainpajamas$**
- **1**
- **1234567890123456**

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Step – Compute the distance from S

- **LSLSLSSSLSLSLSL**L/S
- **bananainpajamas$**
- **1111111**
- **1234567890123456**
- **0012121112121212**

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Step – Sort order of chars

- LSLSLSSSLSLSLSL L/S
- bananainpajamas$
- 1111111
- 1234567890123456
- 0012121112121212
- $a bijmn ps
- 1 111 11 1
- 6246024171335895

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Step – Construct m-Lists

- LSLSLSSSLSLSLSL L/S
- bananainpajamas$
- 1111111
- 1234567890123456
- 0012121112121212
- $a bijmn ps
- 1 111 11 1
- 6246024171335895

Scan this once and bucket it according to dist.

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Step – Generate m-Lists

- List 1
- [7],[11],[13],[3,5,8],[9],[15]
- List 2
- [16],[4,6,10,12,14]

- 2022222011111111
- $a bijmn ps
- 1 111 11 1
- 6246024171335895

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Step – Sort S substrings

Bucket the S substrings

[16],[2,4,6,10,12,14],[7],[8]

- 1111111
- 1234567890123456
- 0012121112121212
- $a bijmn ps
- 1 111 11 1
- 6246024171335895

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Step – Sort S substrings

Bucket the S substrings
[16],[2,4,6,10,12,14],[7],[8]
After using List 1:
[16],[6],[10],[12],[2,4],[14],[7],[8]
List 2 useless. Then?

- List 1
- [7],[11],[13],[3,5,8],[9],[15]
- List 2
- [16],[4,6,10,12,14]

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Step – Sort S substrings

Bucket the S substrings
[16],[2,4,6,10,12,14],[7],[8]
After using List 1:
[16],[6],[10],[12],[2,4],[14],[7],[8]
List 2 useless. Consider 6 before 4:
[16],[6],[10],[12],[4],[2],[14],[7],[8]

- List 1
- [7],[11],[13],[3,5,8],[9],[15]
- List 2
- [16],[4,6,10,12,14]

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Step – Generate the Suffix Array

[16],[6],[10],[12],[4],[2],[14],[7],[8]

```
▪ $a     bijmn  ps
▪ 1   111  11    1
▪ 6246024171335895

▪ $a     ins
▪ 1 11   1  1
▪ 6602424785
```

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Step – Generate the Suffix Array

```
▪ $a     bijmn  ps
▪ 1   111  11    1
▪ 6246024171335895

▪ $a     in s
▪ 1 11   1   1
▪ 66024247585
```

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Step – Generate the Suffix Array

```
▪ $a     bijmn  ps
▪ 1   111  11    1
▪ 6246024171335895

▪ $a     in ps
▪ 1 11   1    1
▪ 660242475895
```

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Step – Generate the Suffix Array

```
▪ $a     bijmn  ps
▪ 1   111  11    1
▪ 6246024171335895

▪ $a     ijn ps
▪ 1 11   1 1   1
▪ 6602424715895
```

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Step – Generate the Suffix Array

```
▪ $a     bijmn  ps
▪ 1   111  11    1
▪ 6246024171335895

▪ $a     ijn  ps
▪ 1 11   1 1    1
▪ 66024247153895
```

type S

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Step – Generate the Suffix Array

```
▪ $a     bijmn  ps
▪ 1   111  11    1
▪ 6246024171335895

▪ $a     bijn  ps
▪ 1 11   1  1    1
▪ 660242417153895
```

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Step – Generate the Suffix Array

```
$a     bijmn  ps
1   111  11    1
6246024171335895

$a     bijmn  ps
1 11  1  11    1
6602424171353895
```

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Final answer

```
bananainpajamas$
         1111111
1234567890123456

Suffix Array:
1 11  1  11    1
6602424171353895
```

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Final answer

```
bananainpajamas$
         1111111
1234567890123456

Suffix Array:
1 11  1  11    1
6602424171353895
```

What is the BWT(S) ?

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BWT is easy!

```
bananainpajamas$
         1111111
1234567890123456

Suffix Array:
1 11  1  11    1
6602424171353895
BWT:
1  1  11 11    1
5591313660242784
```

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BWT construction in linear time

```
bananainpajamas$
         1111111
1234567890123456

BWT:
1  1  11 11    1
5591313660242784
snpjnbm$aaaaaina
```

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