Math 415 - Lecture 32

Complex numbers and eigenvectors

Textbook reading: first part of Chapter 5.5

Suggested practice exercises: 5.5 1, 2, 3

Khan Academy video: Complex Numbers (part 1)

Strang lecture: Lecture 21: Eigenvalues and eigenvectors

1 Review

1.1 Properties of eigenvectors and eigenvalues

- If $A\mathbf{x} = \lambda\mathbf{x}$ then $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ. All eigenvectors (plus $\mathbf{0}$) with eigenvalue λ form **eigenspace** of λ.
- λ is an eigenvalue of $A \iff \underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0$. Why? Because $A\mathbf{x} = \lambda\mathbf{x} \iff (A - \lambda I)\mathbf{x} = \mathbf{0}$.
  By the way: this means that the eigenspace of λ is just $\text{Nul}(A - \lambda I)$.
- E.g. if $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$ then $\det(A - \lambda I) = (3 - \lambda)(6 - \lambda)(2 - \lambda)$.
- Eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ of A corresponding to different eigenvalues are independent.
- By the way:
  - product of eigenvalues = determinant
  - sum of eigenvalues = “trace” (sum of diagonal entries)

2 Eigenbasis?

2.1 Number of (independent) eigenvectors

An $n \times n$ matrix A has up to n different eigenvalues. Namely, the roots of degree n characteristic polynomial $\det(A - \lambda I)$.

- For each eigenvalue λ, A has at least one eigenvector. That is because $\text{Nul}(A - \lambda I)$ has dimension at least one.
- If λ has multiplicity m, then A has up to m (independent) eigenvectors for λ.

Ideally, we would like to find a total of n (independent) eigenvectors for A. This would give an **EIGENBASIS**. Why can there be no more than n independent eigenvectors?!

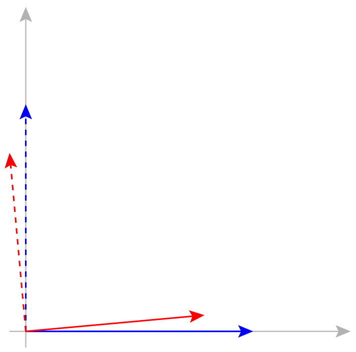
Two sources of trouble: eigenvalues can be

- complex numbers (that is, not enough real roots), or
- repeated roots of the characteristic polynomial.

2.2 Trouble I: complex eigenvalues

Example 1. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Geometrically, what is the trouble?

Solution.



Observe that $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$, i.e. multiplication by A is a rotation by $90°$ (counter-clockwise). There is no vector that parallel after rotation by $90°$. Thus A has no eigenvector!

3 Complex numbers review

Definition. $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

- $i = \sqrt{-1}$, or $i^2 = -1$.
- Any point in $\mathbb{R}^2$ can be viewed as a complex number:
  $\binom{x}{y} \leftrightarrow x + iy$

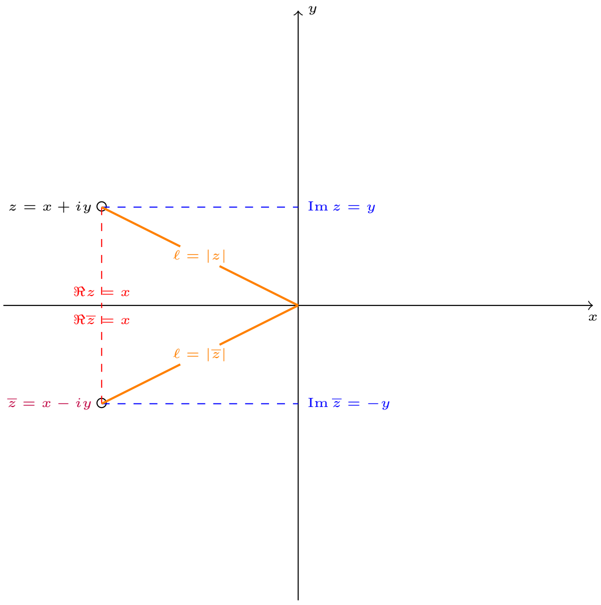
Definition. Let $z = x + iy$ be a complex number

Real part The real part of z, denoted $\Re(z)$ is defined by $\Re(z) = x$.

Imaginary part The imaginary part of z, denoted $\operatorname{Im}(z)$ is defined by $\operatorname{Im}(z) = y$.

Complex conjugate The complex conjugate of z, denoted $\overline{z}$, is defined by $\overline{z} = x - iy$.

Absolute value The absolute value, or magnitude of z, denoted $|z|$ or $\|z\|$, is given by $|z| = \sqrt{x^2 + y^2}$.



Adding complex numbers

Definition. Given $z = x + iy$, $w = u + iv$, we define

$$z + w = (x + u) + i(y + v)$$

Remark. This corresponds exactly to addition of vectors in $\mathbb{R}^2$.

Multiplying complex numbers

Definition. Given $z = x + iy$, $w = u + iv$, we define

$$\begin{aligned} zw &= (x + iy)(u + iv) \\ &= xu + x(iv) + (iy)u + (iy)(iv) \\ &= (xu - yv) + i(xv + yu) \end{aligned}$$

Absolute value and complex conjugate

Remark.

- $\overline{\overline{z}} = z$
- $|z|^2 = z\overline{z}$
- $|z| = |\overline{z}|$

Proof.

$$\boxed{\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - x(iy) + (iy)x - (iy)(iy) \\ &= x^2 + y^2 \end{aligned}}$$

$\square$

3.1 Complex Linear Algebra

Until now we took as our scalars the real numbers. In particular we used the vector space $\mathbb{R}^n$ of column vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

If c is a real number (a scalar) we defined

$$c\mathbf{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}.$$

Now we want to use **COMPLEX** scalars. We need a new context to make sense of this.

Definition. $\mathbb{C}^n$ is the (complex) vector space of *complex* column vectors $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$, where $z_1, z_2, \ldots . z_n$ are complex numbers.

- Now multiplication by a complex scalar makes sense.
- We can define subspaces, span, independence, basis, dimension for $\mathbb{C}^n$ in the usual way.
- We can multiply complex vectors by complex matrices. Column space and Null space still make sense.
- The only difference is the dot product, you need to use the complex conjugate to get a good notion of length. (Later more.)

4 Back to eigenvectors

Example 2. Find the complex eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Now, we can use complex numbers!

Solution (continued)**.**

- $\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$
  So the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$.
- $\lambda_1 = i : \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \mathbf{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$
  Let us check $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix}$
- $\lambda_2 = -i : \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \mathbf{x}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Summary: We had $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

- Eigenvalues: $i, -i$ These are conjugates!
- Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ These are also conjugates!

Theorem 1. *If A is a matrix with real entries and λ is a* ***complex eigenvalue****, then $\bar{\lambda}$ is also a complex eigenvalue. Furthermore, if $\mathbf{x}$ is an eigenvector with eigenvalue λ, then $\bar{\mathbf{x}}$ is an eigenvector with eigenvalue $\bar{\lambda}$.*

Proof. Observe that $\bar{A} = A$. Thus

$$A\bar{x} = \bar{A}\bar{x} = \bar{(}Ax) = \bar{(}\lambda x) = \bar{\lambda}\bar{x}.$$

Thus $\bar{x}$ is an eigenvector of A to the eigenvalue $\bar{\lambda}$. □

Remark. Note that we are using vectors in $\mathbb{C}^2$, instead of vectors in $\mathbb{R}^2$. Works pretty much the same!

Example 3. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. What is the trouble?

Solution.

- $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$
  So: $\lambda = 1$ is the only eigenvalue (it has multiplicity 2).
- $\lambda = 1 :$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  So the eigenspace is span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Only dimension 1!

- Trouble: We can not find an **Eigenbasis** for this matrix. This kind of problem cannot really be fixed. We have to lower our expectations and look for generalized eigenvectors. These are solutions to $(A - \lambda I)^2\mathbf{x} = \mathbf{0}, (A - \lambda I)^3\mathbf{x} = \mathbf{0}, \ldots$