Math 415 - Lecture 31

Markov matrices and Google

Textbook reading: Chapter 5.3

Suggested practice exercises: Chapter 5.3: 8, 9, 12, 14, 10.

Khan Academy video: Finding Eigenvectors and Eigenspaces example

Strang lecture: Lecture 21: Eigenvalues and eigenvectors Lecture 24: Markov Matrices and Fourier Series.

1 Review

1.1 Properties of eigenvectors and eigenvalues

- If $A\mathbf{x} = \lambda\mathbf{x}$ then $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ.All eigenvectors (plus $\mathbf{0}$) with eigenvalue λ form **eigenspace** of λ.
- λ is an eigenvalue of A $\iff$ $\underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0$.Why? Because $A\mathbf{x} = \lambda\mathbf{x} \iff (A-\lambda I)\mathbf{x} = \mathbf{0}$.By the way: this means that the eigenspace of λ is just $\mathrm{Nul}(A - \lambda I)$.
- E.g. if $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$ then $\det(A - \lambda I) = (3 - \lambda)(6 - \lambda)(2 - \lambda)$.

Example 1. Find the eigenvalues of A as well as a basis for the corresponding eigenspaces, where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

Solution.

- The characteristic polynomial is:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & 3-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)[(3-\lambda)^2 - 1] \\ &= (2-\lambda)(\lambda-2)(\lambda-4) \end{aligned}$$

- A has eigenvalues $2, 2, 4$ $\quad \left(A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \right)$

  Since $\lambda = 2$ is a double root, it has **(algebraic) multiplicity** 2.

- $\lambda_1 = 2$:

$$(A - \lambda_1 I)\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

  Two independent solutions: $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

  In other words, the eigenspace for $\lambda_1 = 2$ is span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

- $\lambda_2 = 4$: $\quad \left(A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \right)$

$$(A - \lambda_2 I)\mathbf{x} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- In summary, A has eigenvalues 2 and 4:
- eigenspace for $\lambda_1 = 2$ has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$,
- eigenspace for $\lambda_2 = 4$ has basis $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

2 Markov matrices

Definition 2. An $n \times n$ matrices A is **Markov matrix** if has non negative entries, and the entries in each column add to 1.

Theorem 1. *Let A be a Markov matrix. Then*

(i) 1 is an eigenvalue of A and any other eigenvalue λ satisfies $|\lambda| \leq 1$.

(ii) If A has only positive entries, then any other eigenvalue satisfies $|\lambda| < 1$.

Example 3. Let A be

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}.$$

Is A a Markov matrix?

Theorem 2. *Let A be an $n \times n$-Markov matrix with only positive entries and let* $\boldsymbol{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{R}^n$ *such that $z_1 + z_2 + \cdots + z_n = 1$. Then*

$$\boldsymbol{z}_\infty := \lim_{k\to\infty} A^k \boldsymbol{z} \text{ exists,}$$

*and $A\boldsymbol{z}_\infty = \boldsymbol{z}_\infty$. In this case $\boldsymbol{z}_\infty$ is often called the **steady state**.*

Proof. For simplicity, assume that A has n different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since A is a Markov matrix and has only positive entries, we can assume that $\lambda_1 = 1$ and $|\lambda_i| < 1$ for all $i = 2, \ldots, n$. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in \mathbb{R}^n$ such that $A\mathbf{v}_i = \lambda_i \mathbf{v_i}$. Since eigenvectors to different eigenvalues are linear independent, the above eigenvectors form a basis of $\mathbb{R}^n$. Thus there are scalar $c_1, \ldots, c_n$ such that

$$\mathbf{z} = c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n},$$

then

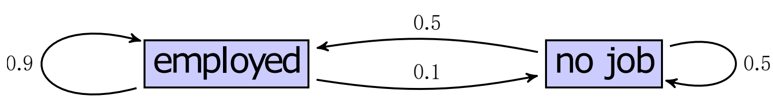
$$A^k\mathbf{z} = c_1\lambda_1^k\mathbf{v_1} + \cdots + c_n\lambda_n^k\mathbf{v_n} \to c_1\mathbf{v_1},$$

because $|\lambda_i| < 1$ for each $i = 2, \ldots, n$. (Exercise: Why is $c_1 \neq 0$?)

$\square$

Example 4. Consider a fixed population of people with or without job. Suppose that each year, 50% of those unemployed find a job while 10% of those employed lose their job.What is the unemployment rate in the long term equilibrium?

Solution.



x_t: proportion of population employed at time t (in years)
y_t: proportion of population unemployed at time t

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.9x_t + 0.5y_t \\ 0.1x_t + 0.5y_t \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

The matrix $\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$ is a **Markov matrix**. Its columns add to 1 and it has no negative entries.

$\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an equilibrium if $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$.

In other words, $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an eigenvector with eigenvalue 1.

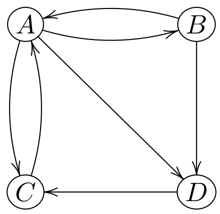
Eigenspace of $\lambda = 1$: Nul $\left(\begin{bmatrix} -0.1 & 0.5 \\ 0.1 & -0.5 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 5 \\ 1 \end{bmatrix}\right\}$

Since $x_\infty + y_\infty = 1$, we conclude that $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix}$.

Hence, the unemployment rate in the long term equilibrium is 1/6

3 Page rank (or: the 25000000000 \$ eigenvector)

Google's success is based on an algorithm to rank webpages, the **Page rank**, named after Google founder Larry Page. The idea is to determine how likely it is that a web user randomly gets to a given webpage. The webpages are ranked by these probabilities.
Suppose the internet consisted of the only four webpages A, B, C, D linked as in the following graph.



Imagine a surfer following these links at random. For the probability $PR_n(A)$ that she is at A (after n steps), we need to think about how she could have reached A. We add:

- the probability that she was at B (at exactly one step before), and left for A,(that's $PR_{n-1}(B) \cdot \frac{1}{2}$)
- the probability that she was at C, and left for A,
- the probability that she was at D, and left for A.

Hence: $PR_n(A) = PR_{n-1}(B) \cdot \frac{1}{2} + PR_{n-1}(C) \cdot \frac{1}{1} + PR_{n-1}(D) \cdot \frac{0}{1}$.
Encode the probabilties at step n in a state vector with four entries.

$$\begin{bmatrix} PR_n(A) \\ PR_n(B) \\ PR_n(C) \\ PR_n(D) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} PR_{n-1}(A) \\ PR_{n-1}(B) \\ PR_{n-1}(C) \\ PR_{n-1}(D) \end{bmatrix}$$

Definition 5. The **PageRank vector** is the long-term equilibrium. It is an eigenvector of the Markov matrix with eigenvalue 1.

Let's call the Markov matrix with the probabilities T:

- $T - 1I = \begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

  $\implies$ eigenspace of $\lambda = 1$ is spanned by $\begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$.

- Now we need to make the entries add up to 1.

$$\begin{bmatrix} PR(A) \\ PR(B) \\ PR(C) \\ PR(D) \end{bmatrix} = \frac{3}{16} \begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix}.$$

  This is the PageRank vector.

- The corresponding ranking of the webpages is A, C, D, B.

Remark. In practical situations the system might be too large for finding the eigenvalues by row operations.

- Google reports having met 60 trillion webpages. Google's search index is over $100,000,000$ gigabytes. Number of Google's servers is secret: about $2,500,000$ More than 1,000,000,000 websites (i.e. hostnames; about 75% not active)
- Thus we have a gigantic but very sparse matrix.

An alternative to row operations is the **power method** (see Theorem 2):

Power method
If T is an (acyclic and irreducible) Markov matrix, then for any $\mathbf{v}_0$ the vectors $T^n\mathbf{v}_0$ converge to an eigenvector with eigenvalue 1

Here:

$$T = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Start with an arbitrary state vector, hit it with powers of T.

$$\left(\begin{bmatrix} PR(A) \\ PR(B) \\ PR(C) \\ PR(D) \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix} \right), T \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.083 \\ 0.333 \\ 0.208 \end{bmatrix}$$

. Note that the ranking of the webpages is already A, C, D, B if we stop here.

$$T \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.083 \\ 0.333 \\ 0.208 \end{bmatrix}, \quad T^2 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.333 \\ 0.167 \end{bmatrix}, \quad T^3 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.396 \\ 0.125 \\ 0.292 \\ 0.188 \end{bmatrix}$$

Remark.

- If all entries of T are positive (no zero entries!), then the power method is guaranteed to work.
- In the context of PageRank, we can make sure that this is the case by replacing T with

$$(1-p) \cdot \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} + p \cdot \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Just to make sure: still a Markov matrix, now with positive entries Google used to use $p = 0.15$.