Teaching Notes: Geometry of Linear Programming Problems

Dr. Arijit Das

Abstract

These notes present the geometric foundations of linear programming: Euclidean space, convex sets, half-spaces, convex polyhedra, extreme points (vertices), and the fundamental theorems that justify graphical and simplex-based solution methods. Proofs and classroom hints are provided.

1 Introduction

Linear programming problems (LPP) optimize a linear objective subject to linear constraints. Geometric understanding relies on elementary Euclidean geometry and convex analysis. The key geometric facts—feasible sets are convex polyhedra and optimal values of a linear objective occur at extreme points—are presented and proved here.

2 Euclidean space and vector operations

Definition 2.1 (Euclidean space)**.** *The n-dimensional Euclidean space is $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$. For $x, y \in \mathbb{R}^n$ the Euclidean inner product is $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ and the Euclidean norm is $\|x\| = \sqrt{\langle x, x\rangle}$.*

Vector operations used throughout:

$$x + y = (x_1 + y_1, \dots, x_n + y_n), \qquad \alpha x = (\alpha x_1, \dots, \alpha x_n).$$

3 Convex sets and basic properties

Definition 3.1 (Line in $\mathbb{R}^n$)**.** *Given two distinct points $u, v \in \mathbb{R}^n$, the* line *through u and v is*

$$L(u, v) = \{\, (1 - \lambda)u + \lambda v \;:\; \lambda \in \mathbb{R}\,\}.$$

It consists of all points obtained by extending indefinitely in both directions from u toward v.

Definition 3.2 (Line segment in $\mathbb{R}^n$)**.** *The* line segment *joining u and v is*

$$[u, v] = \{\, (1 - \lambda)u + \lambda v \;:\; \lambda \in [0, 1] \,\}.$$

It is the portion of the line between u and v, including the endpoints u and v.

Definition 3.3 (Linear combination)**.** *Let $x^{(1)}, x^{(2)}, \ldots, x^{(m)} \in \mathbb{R}^n$ be points (viewed as vectors) and let $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ be scalars. A* linear combination *of the points $x^{(1)}, \ldots, x^{(m)}$ is any point of the form*

$$y = \alpha_1 x^{(1)} + \alpha_2 x^{(2)} + \cdots + \alpha_m x^{(m)}.$$

Here multiplication by a scalar and addition are performed componentwise in $\mathbb{R}^n$.

Definition 3.4 (Convex set)**.** *A set $C \subseteq \mathbb{R}^n$ is* convex *if for all $u, v \in C$ and every $\lambda \in [0, 1]$,*

$$\lambda u + (1 - \lambda)v \in C.$$

Remark 3.5. Geometrically: the line segment joining any two points of C lies entirely in C.

Definition 3.6 (Convex linear combination of two points)**.** *Let $u, v \in \mathbb{R}^n$. A* convex linear combination *of u and v is any point of the form*

$$w = \lambda u + (1 - \lambda)v, \quad \text{where } \lambda \in [0, 1].$$

The set of all such points is exactly the line segment $[u, v]$ joining u and v.

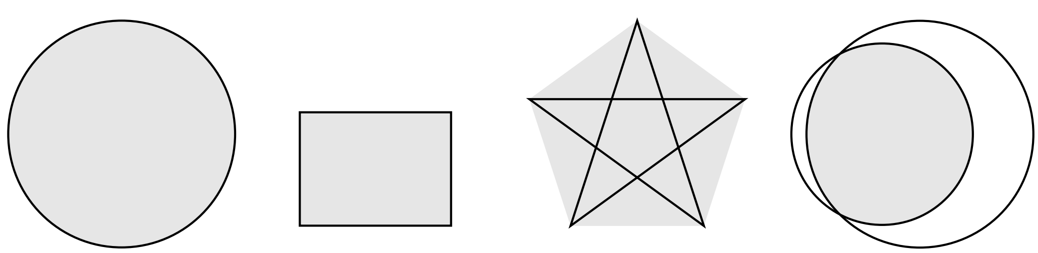
Definition 3.7 (Convex combination)**.** *Let $x^{(1)}, x^{(2)}, \ldots, x^{(m)} \in \mathbb{R}^n$. A* convex combination *of these points is any point of the form*

$$y = \lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots + \lambda_m x^{(m)},$$

where each $\lambda_i \geq 0$ and

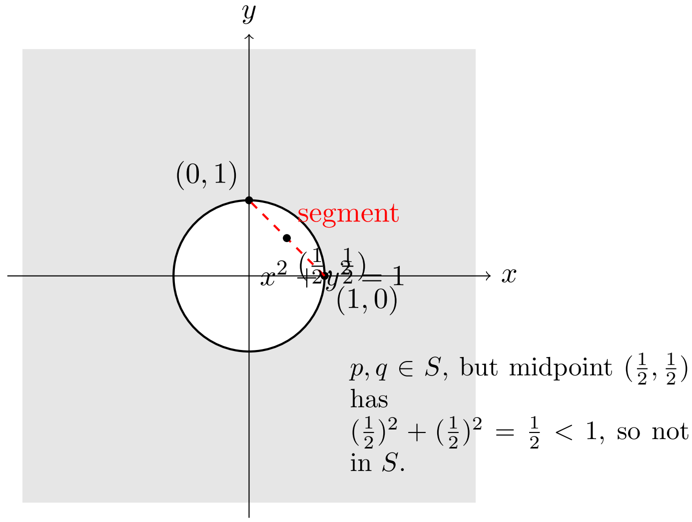
$$\sum_{i=1}^{m} \lambda_i = 1.$$

The coefficients λ_i are called the weights *of the convex combination.*

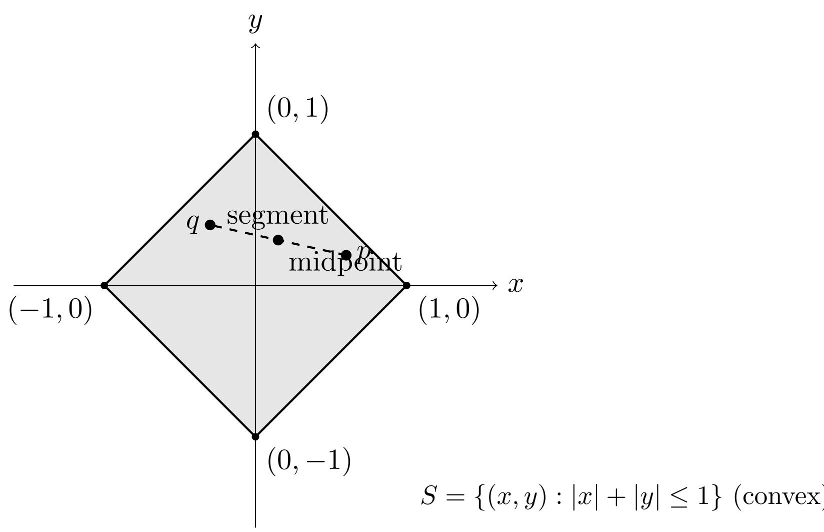


Circle: Convex Rectangle: Convex Star: Not Convex Crescent: Not Convex

Problem 3.8. Check the convexity of the set $S := \{(x, y) : x^2 + y^2 \geq 1\}$.



Problem 3.9. Check the convexity of the set $S := \{(x, y) : |x| + |y| \leq 1\}$.



Problem 3.10. Find the set of all convex combination of the points $(2, 5)$ and $(8, 10)$.

Solution: The set of all convex linear combination is given by:

$$S := \{\lambda(2, 5) + (1 - \lambda)(8, 10) : 0 \leq \lambda \leq 1\} = \{(5 - 3\lambda, 10 - 5\lambda) : 0 \leq \lambda \leq 1\}$$

Problem 3.11. Check the convexity of the set $S := \{(x, y) : xy \leq 2\}$.

Remark 3.12. Every empty and singleton set is convex.

Theorem 3.13. *Let A and B be convex subsets of $\mathbb{R}^n$. Then $A \cap B$ is convex.*

Proof. Take arbitrary points $u, v \in A \cap B$ and let $\lambda \in [0, 1]$. Since $u, v \in A$ and A is convex, the convex combination

$$w = \lambda u + (1 - \lambda)v$$

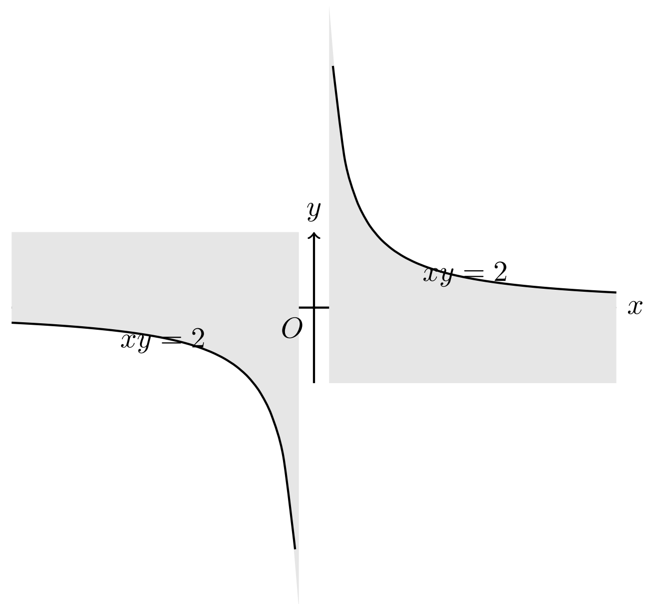


Figure 1: The set $S = \{(x, y) : xy \leq 2\}$ showing its non-convex region.

belongs to A. Similarly, since $u, v \in B$ and B is convex, the same point w belongs to B. Hence $w \in A \cap B$.

Because u, v and $\lambda \in [0, 1]$ were arbitrary, every convex combination of any two points of $A \cap B$ lies in $A \cap B$, so $A \cap B$ is convex. $\square$

Theorem 3.14. *Let $C_1, C_2, \ldots, C_k$ be convex subsets of $\mathbb{R}^n$ for some finite $k \geq 1$. Then the intersection $C = \bigcap_{i=1}^{k} C_i$ is convex.*

Proof. We proceed by induction on k.

Base case ($k = 1$): Trivial, since C_1 is convex by assumption.

Induction step: Assume the statement holds for $k - 1$ sets. Let

$$C' = \bigcap_{i=1}^{k-1} C_i,$$

which is convex by the induction hypothesis.

Now consider $C = C' \cap C_k$. Take arbitrary $u, v \in C$ and $\lambda \in [0, 1]$. Since $u, v \in C'$ and C' is convex, the convex combination $w = \lambda u + (1 - \lambda)v$ belongs to C'. Similarly, $u, v \in C_k$ and C_k is convex, so $w \in C_k$. Therefore $w \in C' \cap C_k = C$. Because u, v and λ were arbitrary, C is convex.

By induction, the intersection of any finite number k of convex sets is convex. $\square$

Theorem 3.15. *Intersections of (any collection of) convex sets are convex.*

Proof. Let $\{C_i\}_{i\in I}$ be convex and let $C = \bigcap_{i\in I} C_i$. Take $u, v \in C$. Then for each i, $u, v \in C_i$ hence $\lambda u + (1-\lambda)v \in C_i$ for every $\lambda \in [0,1]$. This holds for all i, so $\lambda u + (1-\lambda)v \in C$. Thus C is convex. $\square$

Remark 3.16. The union of two convex sets need not be convex in general.

Example 3.17 (Counterexample). Let

$$A = \{(x,y) \in \mathbb{R}^2 : (x+2)^2 + y^2 \le 1\}, \qquad B = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 \le 1\}.$$

Each of A and B is a closed disk of radius 1, hence convex. Consider

$$u = (-2,0) \in A, \qquad v = (2,0) \in B.$$

Their midpoint is

$$m = \frac{u+v}{2} = (0,0).$$

But $m \notin A$ because $\|(0,0) - (-2,0)\| = 2 > 1$, and $m \notin B$ because $\|(0,0) - (2,0)\| = 2 > 1$. Thus $m \notin A \cup B$, even though $u, v \in A \cup B$. Therefore $A \cup B$ is not convex.

Remark 3.18. The union of two convex sets is convex only in special cases (for instance, if one set contains the other). Equivalently, $A \cup B$ is convex iff $\operatorname{conv}(A \cup B) = A \cup B$, which usually forces one set to lie inside the convex hull of the other.

Problem 3.19. Show algebraically that the set

$$S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 4\}$$

is convex.

Proof. Let $u = (x_1, y_1), v = (x_2, y_2) \in S$, so that

$$x_1^2 + y_1^2 \le 4 \quad \text{and} \quad x_2^2 + y_2^2 \le 4.$$

Fix $\lambda \in [0,1]$ and set

$$w = \lambda u + (1-\lambda)v.$$

Then

$$\|w\|^2 = \|\lambda u + (1-\lambda)v\|^2 = \lambda^2\|u\|^2 + (1-\lambda)^2\|v\|^2 + 2\lambda(1-\lambda)\langle u, v\rangle.$$

By the Cauchy–Schwarz inequality,

$$\langle u, v\rangle \le \|u\|\,\|v\| \le 2 \cdot 2 = 4.$$

Hence

$$\|w\|^2 \le 4\lambda^2 + 4(1-\lambda)^2 + 2\lambda(1-\lambda)\cdot 4.$$

Simplifying,

$$\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) = (\lambda + (1-\lambda))^2 = 1,$$

so

$$\|w\|^2 \le 4.$$

Thus $w \in S$. Since $u, v \in S$ and $\lambda \in [0,1]$ were arbitrary, S is convex. $\square$

4 Hyperplanes and half-spaces

Definition 4.1 (Hyperplane)**.** *Let $a_1, a_2, \ldots, a_n$ and b be real numbers, with $\mathbf{a} = (a_1, a_2, \ldots, a_n) \neq \mathbf{0}$. A* hyperplane *in the n-dimensional Euclidean space $\mathbb{R}^n$ is the set of all points $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ satisfying the linear equation*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad \textit{or equivalently} \quad \mathbf{a}^{\mathsf{T}}\mathbf{x} = b.$$

Geometrically, a hyperplane is a flat, $(n-1)$-dimensional subset of $\mathbb{R}^n$ that divides the space into two disjoint open half-spaces*:*

$$\mathbf{a}^{\mathsf{T}}\mathbf{x} < b \quad \textit{and} \quad \mathbf{a}^{\mathsf{T}}\mathbf{x} > b.$$

In Linear Programming, each linear constraint of the form $\mathbf{a}^{\mathsf{T}}\mathbf{x} \le b$ (or $\ge b$) defines a closed half-space bounded by such a hyperplane.

Theorem 4.2. *Let $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $b \in \mathbb{R}$. The set*

$$H := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^{\mathsf{T}}\mathbf{x} = b\}$$

is convex.

Proof. Take any $\mathbf{u}, \mathbf{v} \in H$ and let $\lambda \in [0,1]$. By definition of H we have

$$\mathbf{a}^{\mathsf{T}}\mathbf{u} = b \quad \text{and} \quad \mathbf{a}^{\mathsf{T}}\mathbf{v} = b.$$

Consider the convex combination $\mathbf{w} = \lambda\mathbf{u} + (1-\lambda)\mathbf{v}$. Then

$$\mathbf{a}^{\mathsf{T}}\mathbf{w} = \mathbf{a}^{\mathsf{T}}\big(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}\big) = \lambda\,\mathbf{a}^{\mathsf{T}}\mathbf{u} + (1-\lambda)\,\mathbf{a}^{\mathsf{T}}\mathbf{v} = \lambda b + (1-\lambda)b = b.$$

Thus $\mathbf{w} \in H$. Because $\mathbf{u}, \mathbf{v}$ and λ were arbitrary, every convex combination of points of H lies in H, so H is convex. □

Corollary 4.3. *The closed half-spaces*

$$H^- := \{\mathbf{x} : \mathbf{a}^\mathsf{T}\mathbf{x} \le b\}, \qquad H^+ := \{\mathbf{x} : \mathbf{a}^\mathsf{T}\mathbf{x} \ge b\}$$

are convex.

Proof. Let H^- (the proof for H^+ is analogous). Take $\mathbf{u}, \mathbf{v} \in H^-$ and $\lambda \in [0,1]$. Then

$$\mathbf{a}^\mathsf{T}\big(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}\big) = \lambda\,\mathbf{a}^\mathsf{T}\mathbf{u} + (1-\lambda)\,\mathbf{a}^\mathsf{T}\mathbf{v} \le \lambda b + (1-\lambda)b = b,$$

so the convex combination belongs to H^-. Hence H^- is convex. □

5 Convex polyhedra and feasible region

Definition 5.1 (Convex polyhedron). *A* convex polyhedron *is an intersection of finitely many closed half-spaces:*

$$P = \{x \in \mathbb{R}^n : Ax \le b\}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. If P is bounded it is called a **polytope***.*

Remark 5.2. In an LPP, each linear inequality constraint defines a half-space; the feasible set is a convex polyhedron (often restricted further by nonnegativity $x \ge 0$).

Definition 5.3 (Extreme point or vertices). *A point $x \in C$ (convex set) is an* extreme point *(or* vertex*) of C if whenever*

$$x = \lambda u + (1-\lambda)v, \quad u, v \in C,\ \lambda \in (0,1),$$

it follows that $u = x$ and $v = x$ (i.e., x cannot be written as a nontrivial convex combination of two distinct points).

Remark 5.4. For polyhedra in $\mathbb{R}^n$ extreme points coincide with the geometric idea of **corner points**.

Definition 5.5 (Convex Hull). *Let $S \subseteq \mathbb{R}^n$ be any set of points. The* convex hull *of S, denoted by* $\mathrm{conv}(S)$*, is the smallest convex set containing S.*

Equivalently,

$$\operatorname{conv}(S) = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}_i \;:\; \mathbf{x}_i \in S,\ \lambda_i \ge 0,\ \sum_{i=1}^{k} \lambda_i = 1,\ k \in \mathbb{N} \right\}.$$

That is, the convex hull of S is the set of all convex combinations of finitely many points of S.

Theorem 5.6 (Feasible region is convex). *Let $P_F = \{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}$ be a feasible set given by linear inequalities. Then P_F is convex.*

Proof. Let $x, y \in P_F$. Then

$$Ax \le b \quad x \ge 0$$

$$Ay \le b \quad y \ge 0.$$

Consider the convex combination of x and y. For $0 \le \lambda \le 1$,

$$A\left(\lambda x + (1-\lambda)y\right) = \lambda Ax + (1-\lambda)Ay \le \lambda b + (1-\lambda)b = b.$$

Moreover, $\lambda x + (1-\lambda)y \ge 0$ as $x, y \ge 0$ and $\lambda \in [0,1]$. This implies $\lambda x + (1-\lambda)y$ is also belongs to the set P_F. So, P_F is convex. □

Theorem 5.7 (Finite extreme-point representation of a polytope). *If P is a bounded polyhedron (a polytope) then $P = \operatorname{conv}\{v_1, \dots, v_k\}$ for some finite set of extreme points $\{v_i\}$. (Every polytope is the convex hull of its extreme points.)*

Remark 5.8. This is a standard fact in convex geometry; it provides the representation used in the proof of the corner-point optimality theorem below.

Theorem 5.9 (Corner-point optimality / Extreme point theorem). *Let $P \subset \mathbb{R}^n$ be a nonempty bounded polyhedron (polytope) and let $c \in \mathbb{R}^n$. Consider the LP:*

$$\textit{maximize} \;\; c^\top x \quad \textit{s.t.} \quad x \in P.$$

Then there is an optimal solution attained at an extreme point of P.

Proof. Because P is a polytope, by the previous theorem $P = \operatorname{conv}\{v_1, \dots, v_k\}$ for finitely many extreme points v_i. Let $x^* \in P$ be an optimal solution (exists because P is nonempty and bounded and the objective is continuous). Express x^* as a convex combination

$$x^* = \sum_{i=1}^{k} \lambda_i v_i, \qquad \lambda_i \ge 0,\ \sum_i \lambda_i = 1.$$

Then

$$c^\top x^* = \sum_{i=1}^{k} \lambda_i (c^\top v_i).$$

Hence $c^\top x^*$ is a convex combination of the finite numbers $\{c^\top v_i\}$. Therefore at least one index j satisfies $c^\top v_j \geq c^\top x^*$. But x^* is optimal, so $c^\top v_j = c^\top x^*$, i.e., v_j is also optimal. Thus an extreme point v_j attains the optimum. □

Corollary 5.10. *If the objective $c^\top x$ has multiple optimal points on P, then the set of optimal solutions forms a face of P (possibly an edge or higher-dimensional face); in particular, if the objective plane is parallel to a face, every point on that face is optimal.*

Theorem 5.11 (**Fundamental Theorem of Linear Programming**). *Let*

$$\textit{maximize (or minimize) } \mathbf{c}^{\mathsf{T}}\mathbf{x} \quad \textit{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0$$

be a linear programming problem with feasible set

$$P_F = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0\}.$$

If P_F is nonempty and bounded, then:

1. *P_F is a convex polyhedron;*
2. *the objective function attains its optimum value at some point in P_F;*
3. *there exists at least one optimal solution that is an extreme point (vertex or corner point) of P_F.*

Proof. **Step 1: F is convex.** Each inequality $\mathbf{a}_i^{\mathsf{T}}\mathbf{x} \leq b_i$ defines a half-space, which is convex. The set $x_j \geq 0$ is also convex. Since

$$P_F = \bigcap_{i=1}^{m} \{\mathbf{x} : \mathbf{a}_i^{\mathsf{T}}\mathbf{x} \leq b_i\} \ \cap \ \bigcap_{j=1}^{n} \{\mathbf{x} : x_j \geq 0\},$$

So, P_F is convex. Again, P_F is the intersection of finitely many closed half spaces, so P_F is a convex polyhedron.

Step 2: Optimum exists. Since P_F is given as the intersection of closed half-spaces, it is closed. By assumption P_F is bounded, hence it is compact. The objective function $\mathbf{c}^{\mathsf{T}}\mathbf{x}$ is continuous, so by the Extreme Value Theorem it must attain a maximum and minimum on P_F.

Step 3: Optimum at an extreme point. Let us assume that P_F has finite number of vertices $\{X_1, X_2, ..., X_n\}$ and the optimum value of $\mathbf{c}^T\mathbf{x}$ attains at $\mathbf{x} = X_0 \in P_F$ (Which may not be a corner point).

Therefore,

$$\mathbf{c}^T X_0 \geq \mathbf{c}^T X_i \qquad \text{for all} \qquad 1 \leq i \leq n. \tag{1}$$

Since, $X_0 \in P_F$ and P_F is a convex polyhedron, by Theorem 5.7 X_0 can be written in the convex combination of the corner points $\{X_1, X_2, ..., X_n\}$. That is

$$X_0 = \lambda_1 X_1 + \lambda_2 X_2 + ... + \lambda_n X_n \tag{2}$$

Where $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^{n} \lambda_i = 1$. From equation (2) it follow

$$\mathbf{c}^T X_0 = \lambda_1 \mathbf{c}^T X_1 + \lambda_2 \mathbf{c}^T X_2 + ... + \lambda_n \mathbf{c}^T X_n \tag{3}$$

Let, $\mathbf{c}^T X_k := \max\{\mathbf{c}^T X_1, \mathbf{c}^T X_2, ..., \mathbf{c}^T X_n\}$ for some $1 \leq k \leq n$. Using this relation on equation (3)

$$\mathbf{c}^T X_0 \leq \mathbf{c}^T X_k \left(\lambda_1 + \lambda_2 + ... + \lambda_n\right) \implies \mathbf{c}^T X_0 \leq \mathbf{c}^T X_k \tag{4}$$

By combining (1) and (4), we can write

$$\mathbf{c}^T X_0 = \mathbf{c}^T X_k \qquad \text{for some} \qquad 1 \leq k \leq n.$$

Which indicate the optimal value of $\mathbf{c}^T\mathbf{x}$ is attained at at least one vertex of P_F. The proof is completed. □

Remark 5.12 (Alternative vertex argument without faces). If $\mathbf{x}^\star \in F$ is optimal and not an extreme point, then there exist distinct $\mathbf{u}, \mathbf{v} \in F$ and $\lambda \in (0, 1)$ with $\mathbf{x}^\star = \lambda\mathbf{u} + (1 - \lambda)\mathbf{v}$. Linearity gives $\mathbf{c}^\mathsf{T}\mathbf{x}^\star = \lambda\,\mathbf{c}^\mathsf{T}\mathbf{u} + (1 - \lambda)\,\mathbf{c}^\mathsf{T}\mathbf{v}$, so at least one of $\mathbf{u}, \mathbf{v}$ is also optimal. Repeating this decomposition and using that a polytope has finitely many vertices, one reaches a vertex that is optimal.

Remark 5.13 (Unbounded or empty feasible sets). If $F = \varnothing$, the LP is infeasible. If F is unbounded, the optimum may still exist (and is then attained at a vertex/face), but it may also be $+\infty$ (unbounded objective). The theorem above focuses on the bounded, nonempty case to guarantee existence and vertex attainment.

****

For query related to this material please contact: arijit.das@thapar.edu