Lista 9 - Exercícios de entrega

Kevin Yukio Futema 11221539

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1)

a)

$\vdash (X, Y, Z | X + Y + Z + W = N) \sim Multinomial(N, \alpha_1, \alpha_2, \alpha_3)$ e determinar os $\alpha_1, \alpha_2, \alpha_3$.

$$\mathbb{P}(X = x, Y = y, Z = z | X + Y + Z + W = N) = \frac{\mathbb{P}(X = x, Y = y, Z = z, X + Y + Z + W = N)}{\mathbb{P}(X + Y + Z + W = N)}$$

Para as variáveis X, Y, Z, W, temos as funções geradoras de momentos $M_X(t)$, $M_Y(t)$, $M_Z(t)$, $M_W(t)$, dadas por

$$M_X(t) = e^{\lambda_1(e^t-1)} \quad M_Y(t) = e^{\lambda_2(e^t-1)} \quad M_Z(t) = e^{\lambda_3(e^t-1)} \quad M_W(t) = e^{\lambda_4(e^t-1)}$$

Seja $S = X + Y + Z + W$,como X, Y, Z, W são independentes, então podemos afimar que

$$M_S(t) = M_X(t)M_Y(t)M_Z(t)M_W(t) = e^{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)(e^t-1)}$$

E portanto,pela unicidade da função geradora de momentos,podemos afirmar que

$$S \sim Poisson(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$$

$$\mathbb{P}(X = x, Y = y, Z = z | X + Y + Z + W = N) = \frac{\mathbb{P}(X = x, Y = y, Z = z, W = N - x - y - z)}{\mathbb{P}(S = N)}$$

Pela independência de X, Y, Z, W,

$$\mathbb{P}(X = x, Y = y, Z = z | X + Y + Z + W = N) = \frac{\mathbb{P}(X = x)\mathbb{P}(Y = y)\mathbb{P}(Z = z)\mathbb{P}(W = N - x - y - z)}{\mathbb{P}(S = N)}$$

$$\mathbb{P}(X = x, Y = y, Z = z | X + Y + Z + W = N) = \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^y}{y!}\frac{e^{-\lambda_3}\lambda_3^z}{z!}\frac{e^{-\lambda_4}\lambda_1^{(N-x-y-z)}}{(N-x-y-z)!}\frac{N!}{e^{-(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^N}$$

Apenas para tornar mais clara a resolução considere

$$w = N - x - y - z \quad e \quad \theta = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$

$$\mathbb{P}(X = x, Y = y, Z = z | X + Y + Z + W = N) = \frac{N!}{x!y!z!w!}\frac{e^{-\theta}\lambda_1^x\lambda_2^y\lambda_3^z\lambda_4^w}{e^{-\theta}\theta^N} = \frac{N!}{x!y!z!w!}\left(\frac{\lambda_1}{\theta}\right)^x\left(\frac{\lambda_2}{\theta}\right)^y\left(\frac{\lambda_3}{\theta}\right)^z\left(\frac{\lambda_4}{\theta}\right)^w$$

onde $x + y + z + w = N$.

Logo podemos afirmar que $(X, Y, Z | X + Y + Z + W = N) \sim Multinomial\left(N, \frac{\lambda_1}{\theta}, \frac{\lambda_2}{\theta}, \frac{\lambda_3}{\theta}, \frac{\lambda_4}{\theta}\right)$, ou simplesmente

$$(X, Y, Z | X + Y + Z + W = N) \sim Multinomial\left(N, \frac{\lambda_1}{\theta}, \frac{\lambda_2}{\theta}, \frac{\lambda_3}{\theta}\right)$$

assim como queríamos mostrar, e

$$\alpha_1 = \frac{\lambda_1}{\theta} \quad \alpha_2 = \frac{\lambda_2}{\theta} \quad \alpha_3 = \frac{\lambda_3}{\theta}$$

b)

Quero encontrar a distribuição de $(X, Y|X + Z = n, X + Y + Z + W = N)$.

$$\mathbb{P}(X = x, Y = y|X + Z = n, X + Y + Z + W = N) = \frac{\mathbb{P}(X = x, Y = y, X + Z = n, X + Y + Z + W = N)}{\mathbb{P}(X + Z = n, X + Y + Z + W = N)}$$

$$\mathbb{P}(X = x, Y = y|X + Z = n, X + Y + Z + W = N) = \frac{\mathbb{P}(X = x)\mathbb{P}(Y = y)\mathbb{P}(Z = n - x)\mathbb{P}(W = N - n - y)}{\mathbb{P}(X + Z = n, Y + W = N - n)}$$

Sabemos ainda, pela tabela de contingência, que $X + Z$ e $Y + W$ são independentes.

$$\mathbb{P}(X = x, Y = y|X + Z = n, X + Y + Z + W = N) = \frac{\mathbb{P}(X = x)\mathbb{P}(Y = y)\mathbb{P}(Z = n - x)\mathbb{P}(W = N - n - y)}{\mathbb{P}(X + Z = n)\mathbb{P}(Y + W = N - n)}$$

De forma análoga ao que foi feito no item **a)** (com funções geradoras de momentos), obtemos

$$X + Z \sim Poisson(\lambda_1 + \lambda_3) \quad e \quad Y + W \sim Poisson(\lambda_2 + \lambda_4)$$

Para simplificar a notação considere

$$\theta_1 = \lambda_1 + \lambda_3 \quad e \quad \theta_2 = \lambda_2 + \lambda_4$$

$$\mathbb{P}(X = x, Y = y|X+Z = n, X+Y+Z+W = N) = \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^y}{y!}\frac{e^{-\lambda_3}\lambda_3^{(n-x)}}{(n-x)!}\frac{e^{-\lambda_4}\lambda_4^{(N-n-y)}}{(N-n-y)!}\frac{(N-n)!n!}{e^{-(\lambda_1+\lambda_3)}(\lambda_1+\lambda_3)^n e^{-(\lambda_2+\lambda_3)}(\lambda_2+\lambda_3)^{(N-n)}}$$

$$\mathbb{P}(X = x, Y = y|X + Z = n, X + Y + Z + W = N) = \binom{n}{x}\left(\frac{\lambda_1}{\theta_1}\right)^x\left(\frac{\lambda_3}{\theta_1}\right)^{(n-x)}\binom{N-n}{y}\left(\frac{\lambda_2}{\theta_2}\right)^y\left(\frac{\lambda_4}{\theta_2}\right)^{(N-n-y)}$$

Sejam T e U duas variaveis aleatórias independentes, tais que

$$T \sim Binomial\left(n, \frac{\lambda_1}{\theta_1}\right) \quad e \quad U \sim Binomial\left(N - n, \frac{\lambda_2}{\theta_2}\right)$$

Então podemos dizer que

$$\mathbb{P}(X = x, Y = y|X + Z = n, X + Y + Z + W = N) = \mathbb{P}(T = x)\mathbb{P}(U = y)$$

$$\beta_1 = \frac{\lambda_1}{\theta_1} \quad \beta_2 = \frac{\lambda_2}{\theta_2}$$

c)

$\vdash (X|X + Z = n, X + Y = r, X + Y + Z + W = N) \sim Hipergeometrica(N, n, r)$

$$\mathbb{P}(X = x|X + Z = n, X + Y = r, X + Y + Z + W = N) = \frac{\mathbb{P}(X = x, X + Z = n, X + Y = r, X + Y + Z + W = N)}{\mathbb{P}(X + Z = n, X + Y = r, X + Y + Z + W = N)}$$

Pela independência de X, Y, Z, W,

$$\mathbb{P}(X = x|X + Z = n, X + Y = r, X + Y + Z + W = N) = \frac{\mathbb{P}(X = x)\mathbb{P}(Z = n - x)\mathbb{P}(Y = r - x)\mathbb{P}(W = N - r - n + x)}{\mathbb{P}(X + Z = n, X + Y = r, X + Y + Z + W = N)}$$

Considere o conjunto

$$D = \{k \in \mathbb{N} : max\{0, N - n - r\} \leq k \leq min\{n, r\}\}$$

Pelo Teorema da probabilidade Total, obtemos

$$\mathbb{P}(X = x|X + Z = n, X + Y = r, X + Y + Z + W = N) = \frac{\mathbb{P}(X = x)\mathbb{P}(Z = n - x)\mathbb{P}(Y = r - x)\mathbb{P}(W = N - r - n + x)}{\sum_{k\in D}[\mathbb{P}(X = k)\mathbb{P}(Z = n - k)\mathbb{P}(Y = r - k)\mathbb{P}(W = N - r - n + k)]}$$

Iremos resolver separadamente o numerador e o denominador. Para o numerador:

$$\mathbb{P}(X = x)\mathbb{P}(Z = n - x)\mathbb{P}(Y = r - x)\mathbb{P}(W = N - r - n + x) = \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^{(r-x)}}{(r-x)!}\frac{e^{-\lambda_3}\lambda_3^{(n-x)}}{(n-x)!}\frac{e^{-\lambda_4}\lambda_4^{(N-n-r+x)}}{(N-n-r+x)!}$$

Note que $\lambda_1^x\lambda_2^{-x}\lambda_3^{-x}\lambda_4^x = 1$, pois $\frac{\lambda_1\lambda_4}{\lambda_2\lambda_3} = 1$.

Considere ainda $\theta = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$,

$$\mathbb{P}(X = x)\mathbb{P}(Z = n - x)\mathbb{P}(Y = r - x)\mathbb{P}(W = N - r - n + x) = e^{-\theta}\frac{\binom{r}{x}\binom{N-n}{r-x}}{\binom{N}{r}}\frac{N!}{r!(N-n)!(N-r)!n!}$$

$$\sum_{k\in D}[\mathbb{P}(X = k)\mathbb{P}(Z = n - k)\mathbb{P}(Y = r - k)\mathbb{P}(W = N - r - n + k)] = \sum_{k\in D} e^{-\theta}\frac{\binom{r}{k}\binom{N-n}{r-k}}{\binom{N}{r}}\frac{N!}{r!(N-n)!(N-r)!n!}$$

Para simplificar a notação, considere também

$$C = \frac{N!}{r!(N-n)!(N-r)!n!}$$

Desta forma, podemos escrever

$$\sum_{k\in D}[\mathbb{P}(X = k)\mathbb{P}(Z = n - k)\mathbb{P}(Y = r - k)\mathbb{P}(W = N - r - n + k)] = e^{-\theta}C\sum_{k\in D}\frac{\binom{r}{k}\binom{N-n}{r-k}}{\binom{N}{r}}$$

Note que $\sum_{k\in D}\frac{\binom{r}{k}\binom{N-n}{r-k}}{\binom{N}{r}} = 1$, pois é a soma para todo k do domínio da função de distribuição de probabilidade de uma variavel aleatória $Hipergeométrica(N, n, r)$, então

$$\sum_{k\in D}[\mathbb{P}(X = k)\mathbb{P}(Z = n - k)\mathbb{P}(Y = r - k)\mathbb{P}(W = N - r - n + k)] = e^{-\theta}C$$

Podemos então escrever,

$$\mathbb{P}(X = x|X + Z = n, X + Y = r, X + Y + Z + W = N) = \frac{e^{-\theta}\frac{\binom{r}{x}\binom{N-n}{r-x}}{\binom{N}{r}}C}{e^{-\theta}C} = \frac{\binom{r}{x}\binom{N-n}{r-x}}{\binom{N}{r}}1_{x\in D}(x)$$

Logo $(X|X + Z = n, X + Y = r, X + Y + Z + W = N) \sim Hipergeometrica(N, n, r)$

4) $X \sim Gama(\alpha, \lambda)$ e $Y \sim Gama(\beta, \lambda)$, X e Y independentes

$T = \frac{X}{X+Y}$, $U = X+Y$ $\quad 0 < T < 1 \quad U > 0$

$J_h(t,u) = \begin{vmatrix} \frac{\partial h_1}{\partial t} & \frac{\partial h_1}{\partial u} \\ \frac{\partial h_2}{\partial t} & \frac{\partial h_2}{\partial u} \end{vmatrix} = \begin{vmatrix} U & T \\ -U & 1-T \end{vmatrix} =$

$X = TU$ e $Y = U - TU$ $\quad X = h_1(T,U)$ e $Y = h_2(T;U)$

$= U - UT + UT = U$

$|J_h| = \frac{1}{|J_g(h_1(T,U), h_2(T,U))|} = \frac{U^2}{U - TU + 1}$

$f_{T,U}(t,u) = f_{X,Y}(h_1(t,u), h_2(t,u)) \cdot |J_h(t,u)| = f_X(h_1(t,u)) . f_Y(h_2(t,u)) \cdot |J_h(t,u)| =$

$= \frac{\lambda^\alpha \cdot e^{-\lambda t u}}{\Gamma(\alpha)} \cdot (tu)^{\alpha-1} \cdot \frac{\lambda^\beta}{\Gamma(\beta)} \cdot e^{-\lambda(u - tu)} \cdot (u-tu)^{\beta-1} \cdot U \cdot \mathbb{1}_{(0,1)}(t) \, \mathbb{1}_{(0,\infty)}(u)$

$f_{T,U}(t,u) = \frac{\lambda^{\alpha+\beta} e^{-\lambda u}}{\Gamma(\alpha)\Gamma(\beta)} \cdot (tu)^{\alpha-1} (u-tu)^{\beta-1} \cdot U \, \mathbb{1}_{(0,1)}(t) \, \mathbb{1}_{(0,\infty)}(u) \rightarrow$ $\boxed{\text{verificaremos no proximo item que } \iint_R f_{T,U}(t,u) = 1}$

b)

$f_U(u) = \int_0^1 f_{T,U}(t,u) dt = \frac{\lambda^{\alpha+\beta} e^{-\lambda u}}{\Gamma(\alpha+\beta)} \cdot u^{\alpha-1} u^{\beta-1} \cdot u \underbrace{\int_0^1 \frac{1}{B(\alpha,\beta)} \cdot t^{\alpha-1} \cdot (1-t)^{\beta-1} dt}_{1\text{, pois distribuição distribuição } Beta(\alpha,\beta)} =$

$f_U(u) = \frac{\lambda^{\alpha+\beta} e^{-\lambda u}}{\Gamma(\alpha+\beta)} u^{\alpha+\beta-1} \mathbb{1}_{(0,\infty)}(u)$ logo

$U \sim Gama(\alpha+\beta, \lambda)$, além de achar a distribuição marginal de U podemos verificar bem facilmente que $\iint_R f_{T,U}(t,u)\, dt\, du = 1$, pois $\int_0^\infty \int_0^1 f_{T,U}(t,u)\, dt\, du = \underbrace{\int_0^\infty f_U(u) du}_{Gama(\alpha+\beta,\lambda)} = 1$

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$$f_T(t) = \int_0^{\infty} f_{T,U}(t,u)\,du = \int_0^{\infty} \frac{\lambda^{\alpha+\beta} e^{-\lambda u}}{\Gamma(\alpha)\Gamma(\beta)} \cdot u \cdot (tu)^{\alpha-1} (u(1-t))^{\beta-1}\,du = \frac{t^{\alpha-1}(1-t)^{\beta-1}}{\beta(\alpha,\beta)} \underbrace{\int_0^{\infty} \frac{\lambda^{\alpha+\beta} e^{-\lambda u}}{\Gamma(\alpha+\beta)} \cdot u^{\alpha+\beta-1}\,du}_{\sim \text{Gama}(\alpha+\beta,\lambda)}$$

$$1$$

$$f_T(t) = \frac{t^{\alpha-1}(1-t)^{\beta-1}}{\beta(\alpha,\beta)} \mathbb{1}_{(0,1)}(t) \text{ logo } T \sim \text{Beta}(\alpha,\beta)$$

c) Sim, podemos afirmar que T e U são independentes pois

$$\boxed{f_{T,U}(t,u) = f_T(t) \cdot f_U(u)}$$

$$f_{T,U}(t,u) = \frac{\lambda^{\alpha+\beta} e^{-\lambda u}}{\Gamma(\alpha)\Gamma(\beta)} \cdot u \cdot (tu)^{\alpha-1} \cdot (u(1-t))^{\beta-1} \mathbb{1}_{(0,1)}(t) \mathbb{1}_{(0,\infty)}(u) =$$

$$= \underbrace{\frac{\lambda^{\alpha+\beta} e^{-\lambda u}}{\Gamma(\alpha+\beta)} \cdot u^{\alpha+\beta-1} \mathbb{1}_{(0,\infty)}(u)}_{f_U(u)} \underbrace{\frac{t^{\alpha-1}(1-t)^{\beta-1}}{\beta(\alpha,\beta)} \mathbb{1}_{(0,1)}(t)}_{f_T(t)} = f_U(u) \cdot f_T(t)$$

6) $X_1, X_2, \dots, X_n$ independentes

a) Quero encontrar a distribuição de $Y_1 = \min\{X_1, X_2, \dots, X_n\}$

$$F_{Y_1}(u) = P(\min\{X_1, X_2, \dots, X_n\} \leq u) = 1 - P(\min\{X_1, X_2, \dots, X_n\} > u) = 1 - P(X_1 > u, \dots, X_n > u)$$

independência

$$\stackrel{\uparrow}{=} 1 - \prod_{i=1}^{n} P(X_i > u) = 1 - \prod_{i=1}^{n} e^{-\lambda_i u} = 1 - e^{-\left(\sum_{i=1}^{n} \lambda_i\right) u}$$

$$f_{Y_1}(u) = \frac{F_{Y_1}(u)}{\partial u} = \left(\sum_{i=1}^{n} \lambda_i\right) e^{-\left(\sum_{i=1}^{n} \lambda_i\right) u} \mathbb{1}_{(0,\infty)}(u) \text{ logo } Y_1 \sim Exp\left(\sum_{i=1}^{n} \lambda_i\right)$$

b) mostrar que $P(Y_1 = X_k) = \dfrac{\lambda_k}{\lambda_1 + \dots + \lambda_n}$

$$P(Y_1 = X_k) = P(X_k < \underbrace{X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_n}_{A}) = E(P(X_k < A \mid X_k = u)) =$$

$$= \int_0^\infty P(A > u) \cdot \lambda_k e^{-\lambda_k u} \, du = \int_0^\infty \lambda_k P(Y_1 > u) \, du = \frac{\lambda_k}{\sum_{i=1}^{n} \lambda_i}$$

$$P(A > u) = P(X_1 > u) \cdot \ldots \cdot P(X_{k-1} > u) \cdot P(X_{k+1} > u) \cdot \ldots \cdot P(X_n > u)$$

$$P(X_k > u) = e^{-\lambda_k u}$$

$$P(A > u) \cdot e^{-\lambda_k u} = P(Y_1 > u) = e^{-\left(\sum_{i=1}^{n} \lambda_i\right) u}$$

9) X e Y contínuas com distribuição uniforme(0,1)

$T = X+Y$ e $U = \frac{X}{X+Y}$ $\rightarrow$ $\underset{(h_1)}{X = UT}$ e $\underset{(h_2)}{Y = T - TU = T(1-U)}$

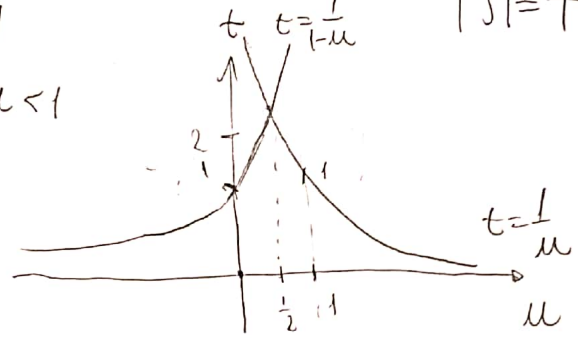
$J_h = \begin{vmatrix} U & T \\ 1-U & -T \end{vmatrix} = -UT - T + TU = -T$

$|J| = T$

$f_{T,U}(t,u) = f_{X,Y}(tu, t-tu)\,|J_h| = t\,\mathbb{1}_{(0,1)}(tu)\,\mathbb{1}_{(0,1)}(t-tu)$

$0 < tu < 1$ e $0 < t(1-u) < 1$

$t > 0$, $t < \frac{1}{u}$, $t < \frac{1}{1-u}$, $u < 1$



T

$\begin{cases} 0 < t < 1 \\ 0 < u < 1 \end{cases}$

$\begin{cases} 1 \le t < 2 \\ 1 - \frac{1}{t} < u < \frac{1}{t} \end{cases}$

U

$\begin{cases} 0 < u < \frac{1}{2} \\ 0 < t < \frac{1}{1-u} \end{cases}$

$\begin{cases} \frac{1}{2} \le u < 1 \\ 0 < t < \frac{1}{u} \end{cases}$

$f_T(t) = \begin{cases} f_{T_1}(t), & 0 < t < 1 \\ f_{T_2}(t), & 1 \le t < 2 \end{cases}$

$f_{T_1}(t) = \int_0^1 t\,du = t\,\mathbb{1}_{(0,1)}(t)$

$f_{T_2}(t) = \int_{1-\frac{1}{t}}^{\frac{1}{t}} t\,du = t \cdot u \Big|_{1-\frac{1}{t}}^{\frac{1}{t}} = t\left(\frac{1}{t} - 1 + \frac{1}{t}\right) = (-t+2)\,\mathbb{1}_{(1,2)}(t)$

$f_T(t) = \begin{cases} t, & 0 < t < 1 \\ 2 - t, & 1 \le t < 2 \end{cases}$

$f_U(u) = \begin{cases} f_{U_1}(u), & 0 < u < \frac{1}{2} \\ f_{U_2}(u), & \frac{1}{2} \le u < 1 \end{cases}$

$f_{U_1}(u) = \int_0^{\frac{1}{1-u}} t\,dt = \frac{t^2}{2}\Big|_0^{\frac{1}{1-u}} = \frac{1}{2(1-u)^2}\,\mathbb{1}_{(0,\frac{1}{2})}(u)$

$f_{U_2}(u) = \int_0^{\frac{1}{u}} t\,dt = \frac{t^2}{2}\Big|_0^{\frac{1}{u}} = \frac{1}{2u^2}\,\mathbb{1}_{(\frac{1}{2},1)}(u)$

$\vdash \int_{-\infty}^{\infty} f_T(t)\,dt = 1 \rightarrow \int_0^1 t\,dt + \int_1^2 2-t\,dt = \frac{t^2}{2}\Big|_0^1 + \left(2t - \frac{t^2}{2}\right)\Big|_1^2 = \frac{1}{2} + 2 - \frac{3}{2} = 1$ ✓

$\vdash \int_{-\infty}^{\infty} f_U(u)\,du = 1 \rightarrow \int_0^{\frac{1}{2}} \frac{1}{2(1-u)^2}\,du + \int_{\frac{1}{2}}^1 \frac{1}{2u^2}\,du = \frac{1}{2(1-u)}\Big|_0^{\frac{1}{2}} + \left(-\frac{1}{2u}\Big|_{\frac{1}{2}}^1\right) = 1 - \frac{1}{2} + \left(-\frac{1}{2} + 1\right) = \frac{1}{2} + \frac{1}{2} = 1$ ✓

Como $f_{T,U}(t,u) \neq f_T(t) \cdot f_U(u)$ T e U não são independentes