August 2013

1.

Let $\{B_t\}$ be the Brownian motion, and let σ be the last visit to the level 0 before $t = 1$, i.e.,

$$\sigma = \sup\{t \leq 1 : B_t = 0\}.$$

1. Show that σ is not a stopping time
2. Show that
$$\sigma \overset{d}{=} \frac{X^2}{X^2 + Y^2}$$
where X and Y are independent unit normals.

Solution.

1. Since B_t is BM, $B_t^2 - t$ is a martingale. For the sake of contradiction, suppose σ is a stopping time. Then since σ is bounded, we may apply the Bounded optional sampling theorem,
$$\mathbb{E}[B_\sigma^2 - \sigma] = 0 \Rightarrow \mathbb{E}[\sigma] = 0.$$
This, implies $\mathbb{P}[\sigma = 0] = 1$. However, we know that σ has an arcsine distribution, so
$$\mathbb{P}[\sigma = 0] = \frac{2}{\pi} \arcsin(0) - 0,$$
a contradiction,
2. $\{\sigma \leq t\} = \{\sup_{t \leq s \leq 1} B_s < 0\} \cup \{\inf_{t \leq s \leq 1} B_s > 0\}$, so since $(\sup_{t \leq s \leq 1} B_s - B_t, B_t) \sim (|B_1 - B_t|, B_t)$ and $|B_1 - B_t|$ and B_t are independent.
$$\begin{aligned}\mathbb{P}[\sigma \leq t] &= 2\mathbb{P}[\sup_{t \leq s \leq 1} B_s < 0] = 2\mathbb{P}[\sup_{t \leq s \leq 1}(B_s - B_t) \leq -B_t] = 2\mathbb{P}[M_{1-t} < -B_t] = 2\mathbb{P}[|B_1 - B_t| < -B_t] \\ &= \mathbb{P}[|B_1 - B_t| < |B_t|] = \mathbb{P}[(B_1 - B_t)^2 < B_t^2]\end{aligned}$$
Since $B_1 - B_t \sim N(0, (1-t)) \sim \sqrt{1-t}X$, and $B_t \sim N(0, t) \sim \sqrt{t}Y$ where $X, Y \sim N(0, 1)$. Then
$$\mathbb{P}[\sigma \leq t] = \mathbb{P}[(1-t)X^2 < tY^2] = \mathbb{P}[\frac{X^2}{X^2 + Y^2} < t].$$
so $\sigma \sim \frac{X^2}{X^2+Y^2}$.

2.

Let M and N be two continuous local martingales. Show that $\mathcal{E}(M)\mathcal{E}(N)$ is a local martingale if and only if MN is a local martingale.

Solution. By the definition of quadratic covariation, MN is a local martingale if and only if $\langle M, N\rangle = 0$. Further, $\mathcal{E}(M)\mathcal{E}(N) = \mathcal{E}(M+N)\mathcal{E}(\langle M, N\rangle) = \mathcal{E}(M+N)\exp(\langle M, N\rangle)$. Since $M+N$ is a local martingale, $\mathcal{E}$ is a local martingale. Thus, if MN is a local martingale, $\langle M, N\rangle = 0$, so $\mathcal{E}(M)\mathcal{E}(N) = \mathcal{E}(M+N)e^0 = \mathcal{E}(M+N)$, so $\mathcal{E}(M)\mathcal{E}(N)$ is a local martingale.

$$\langle \mathcal{E}(M)\mathcal{E}(N)\rangle = \int_0^t \mathcal{E}(M)\mathcal{E}(N) d\langle M, N\rangle = 0$$

Since $\mathcal{E}(M)\mathcal{E}(N) > 0$, $\langle M, N\rangle$ must be constant (?). So, since $\langle M, N\rangle_0 = 0$, $\langle M, N\rangle = 0$, so MN is a local martingale.

3.

Let $\{B_t\}$ be a Brownian motion. For $c \in \mathbb{R}$, compute

$$\mathbb{P}[B_t + ct < 1, \forall\, t \geq 0].$$

Solution. Let $X_t = B_t + ct$. For $c > 0$, $\mathbb{P}[B_t + ct < 1, \forall\, t \geq 0] = 0$. Thus, assume $c \leq 0$. For Brownian motion B_t, we know that the exponential martingale $\exp(\lambda B_t - \frac{1}{2}\lambda^2 t)$ is a martingale for $\lambda \in \mathbb{R}$. Then, we can choose $\lambda = -2c$, so that $M_t = \exp(-2c(B_t + ct)) = \exp(-2cX_t)$. Since $c < 0$, M_t is a martingale such that $M_t \to 0$ and $t \to \infty$. Define $\tau = \inf\{t \geq 0 : M_t = \exp(-2c)\}$. Since M_t^τ is bounded and therefore uniformly integrable, by the optional sampling theorem

$$\mathbb{E}[M_\tau] = \mathbb{P}[\tau < \infty]\exp(-2c) + \mathbb{P}[\tau = \infty](0) = \mathbb{E}[M_0] = 1 \to \mathbb{P}[\tau < \infty] = \exp(2c)$$

$$\mathbb{P}[X_t < 1\ \forall\, t \geq 0] = \mathbb{P}[\sup_{t \geq 0} X_t < 1] = \mathbb{P}[\sup_{t \geq 0} M_t < \exp(-2c)] = 1 - \mathbb{P}[\tau < \infty] = 1 - \exp(2c).$$

January 2014

1.

Let $(M_t)_{0\le t\le T}$ be a submartingale and let $\lambda > 0$. Show that

$$\lambda \mathbb{P}(\max_{0\le t\le T} M_t \ge \lambda) \le \mathbb{E}[M_t \mathbf{1}_{\max_{0\le t\le T} M_t \ge \lambda}].$$

1. Consider a one-dimensional Brownian motion at $B_0 = 0$. Let $u, v : [0,\infty) \to \mathbb{R}$ be such that u is C^1, strictly increasing and $u(0) = 0$. Assume also that $v(t) \neq 0$ for each t and v has bounded variation. Show that the process
$$X_t = v(t)B_{u(t)}$$
is a semi-martingale and the martingale part is $\int_0^t v(s)dB_{u(s)}$.
2. Show that the martingale part is a Brownian motion if and only if $v^2(s)u'(s) = 1$ for each s
3. Find u, v such that X defined above is an Orstein-Uhlenbeck process with parameter β, $dX_t = \beta X_t dt + d\gamma_t$ for some Brownian motion γ.

Solution. Let $(M_t)_{0\le t\le T}$ be a submartingale. Then

$$\lambda \mathbb{P}[\max_{0\le t\le T} M_t \ge \lambda] = \mathbb{E}[\lambda \mathbf{1}_{\max_{0\le t\le T} M_t \ge \lambda}] \le \mathbb{E}[M_t \mathbf{1}_{\max_{0\le t\le T} M_t \ge \lambda}].$$

1. First, let's show that $M_t = \int_0^t v(s)dB_{u(s)}$ is a martingale. Let $r < t$. Let $\{\Delta_n\}$ be a sequence of partitions such that $\Delta_n \to$Id. Then let $M_t^n = \sum_{k=1}^n v(t_k)(B_{t_k\wedge t} - B_{t_{k-1}\wedge t})$. So, $\mathbb{E}[M_t^n|\mathcal{F}_r] = \sum_{k=1}^n v(t_k)\mathbb{E}[(B_{t_k\wedge t} - B_{t_{k-1}\wedge t})|\mathcal{F}_r] = \sum_{k=1}^n v(t_k)(B_{t_k\wedge r} - B_{t_{k-1}\wedge r}) = M_r^n$. Taking the limit as $n \to \infty$, M_t is a martingale. Now, we want to show that $X_t - \int_0^t v(s)dB_{u(s)}$ is of finite variation. To do this we look at the total variation,
$$\begin{aligned}\sup \sum_{i=1}^k &|v(t_i)B_{u(t_i)} - \int_0^{t_i} v(s)dB_{u(s)} - v(t_{i-1})B_{u(t_{i-1})} + \int_0^{t_{i-1}} v(s)dB_{u(s)}| \\ &= \sup \sum_{i=1}^k |(v(t_i)B_{u(t_i)} - v(t_{i-1})B_{u(t_{i-1})}) + \int_{t_{i-1}}^{t_i} v(s)dB_{u(s)}|\end{aligned}$$
2. By the Lévy characterization of a Brownian motion, we just need $\langle \int_0^t v(s)dB_{u(s)}\rangle = t$. Let $M_t = \int_0^t v(s)dB_{u(s)}$. Then $d\langle M_t\rangle = dM_t dM_t = v(t)^2 dB_{u(t)}dB_{u(t)} = v(t)^2 du(t)$, so $\langle M\rangle_t = \int_0^t v(s)^2 du(s) = \int_0^t v(s)^2 u'(s)ds = t$. Since $v^2(s), u'(s) \ge 0\ \forall\ s \in [0,\infty)$, $v(s)^2 u'(s) = 1\ \forall\ s$.
3. Since X_t is a semimartingale, we can apply Ito's formula,
$$dX_t = v(t)dB_{u(t)} + B_{u(t)}dv(t).$$
To be a OU process, need $\int_0^t v(s)dB_{u(s)}$ to be a Brownian motion, which implies $v^2u' = 1$ from above and $B_{u(t)}dv(t) = \beta v(t)B_{u(t)}dt$. This occurs when $dv(t) = \beta v(t)dt$, so $v(t) = Ce^{\beta t}$, $C \neq 0$. Then since $v^2u' = 1$, implies $u'(t) = C^{-2}e^{-2\beta t}$. Thus, $u(t) = -\frac{1}{2C^2\beta}e^{-2\beta t} + D$. Since $u(0) = 0$, $D = \frac{1}{2C^2\beta}$, so $u(t) = \frac{1}{2C^2\beta}(1 - e^{-2\beta t})$.

2.

(Range of Brownian Motion) Let B be a one-dimensional BM starting at zero. Define

$$S_t = \max_{s\le t} B_s, \quad I_t = \inf_{s\le t} B_s, \quad \theta_c = \inf\{t : S_t - I_t = c\},$$

for some $c > 0$.

1. Show that for each λ, the process
$$M_t = \cosh(\lambda(S_t - B_t))\exp(-\frac{\lambda^2 t}{2})$$
is a martingale.
2. Prove that $\mathbb{E}[\exp(-\frac{\lambda^2\theta_c}{2})] = \frac{2}{1+\cosh(\lambda c}$.

Solution

1. Let $0 \le s < t$.

$$\begin{aligned}
\mathbb{E}[\exp(\lambda(S_t - B_t))|\mathcal{F}_s] &= \mathbb{E}[\exp(\lambda((S_t - B_t) - (S_s - B_s) + (S_s - B_s))|\mathcal{F}_s] \\
&= \exp(\lambda(S_s - B_s))\mathbb{E}[\exp(\lambda((S_t - S_s) - (B_t - B_s))|\mathcal{F}_s] \\
\mathbb{E}[\exp(\lambda((S_t - S_s) - (B_t - B_s))|\mathcal{F}_s] &= \mathbb{E}[\exp(\lambda(|B_t - B_s|)] = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{\lambda x}e^{-\frac{x^2}{2(t-s)}dx} \\
&= \frac{2}{\sqrt{2\pi(t-s)}}e^{\frac{\lambda^2(t-s)}{2}}2\int_0^\infty e^{-\frac{x^2}{2(t-s)}+\lambda x-\frac{\lambda^2(t-s)}{2}}\,dx \\
&= \frac{2}{\sqrt{2\pi(t-s)}}e^{\frac{\lambda^2(t-s)}{2}}\int_0^\infty e^{-(\frac{x}{\sqrt{2(t-s)}}-\frac{\lambda\sqrt{t-s}}{\sqrt{2}})^2}\,dx \\
&= \frac{2}{\sqrt{2\pi(t-s)}}e^{\frac{\lambda^2(t-s)}{2}}\sqrt{2(t-s)}\int_0^\infty e^{-u^2}\,du \\
&= \frac{2}{\sqrt{\pi}}e^{\frac{\lambda^2(t-s)}{2}}\frac{\sqrt{\pi}}{2} = e^{\frac{\lambda^2(t-s)}{2}}.
\end{aligned}$$

Similarly, $\mathbb{E}[\exp(-\lambda(S_t - B_t))|\mathcal{F}_s] = \exp(\lambda(S_s - B_s))e^{\frac{\lambda^2(t-s)}{2}}$. Thus,

$$\mathbb{E}[\cosh(\lambda(S_t - B_t))|\mathcal{F}_s] = \cosh(\lambda(S_s - B_s))e^{\frac{\lambda^2(t-s)}{2}}$$

which implies

$$\mathbb{E}[M_t|\mathcal{F}_s] = \cosh(\lambda(S_s - B_s))e^{\frac{\lambda^2(t-s)}{2}}e^{\frac{-\lambda^2 t}{2}} = M_s.$$

So M_t is a martingale.

2. The stopped martingale $M_t^{\theta_c}$ is bounded and thus uniformly integrable. So, by the optional sampling theorem,

$$\begin{aligned}
\mathbb{E}[M_{\theta_c}] = \mathbb{E}[\cosh(\lambda(S_{\theta_c} - B_{\theta_c})\exp(-\frac{\lambda^2\theta_c}{2}] &= \mathbb{E}[M_0] = 1 \\
\Rightarrow \mathbb{E}[\exp(-\frac{\lambda^2\theta_c}{2}] &= \frac{1}{\mathbb{E}[\cosh(\lambda(S_{\theta_c} - B_{\theta_c}))]}
\end{aligned}$$

Since $B_{\theta_c} = S_{\theta_c}$ or $B_{\theta_c} = I_{\theta_c}$ with equal probability (by the symmetry of BM),

$$\mathbb{E}[\cosh(\lambda(S_{\theta_c} - B_{\theta_c})] = \frac{1}{2} + \frac{1}{2}\cosh(\lambda c).$$

So,

$$\mathbb{E}[\exp(-\frac{\lambda^2\theta_c}{2})] = \frac{2}{1+\cosh(\lambda c)}.$$

August 2014

2.

Let $(W_t)_{0\le t\le 1}$ a BM (defined only up to time one). Show that the two dimensional vector

$$\left(W_1, \int_0^1 sgn(W_s)dW_s\right)$$

has the following properties:

1. both marginals are normal $N(0,1)$
2. however, it is NOT a joint normal random vector

Solution.

1. From the definition of BM, $W_1 = W_1 - W_1 \sim N(0,1)$. Let Δ_n be a sequence of partitions of $[0,1]$ such that $\Delta_n \to Id$. Then $M_t^n = \sum_{i=1}^n sgn(W_{t_i})(W_{t_i} - W_{t_{i-1}})$. From the definition of BM, these increments $W_{t_i} - W_{t_{i-1}}$ are independent and $sgn(W_{t_i})(W_{t_i} - W_{t_{i-1}}) \sim N(0, t_i - t_{i-1}$. Thus, since Δ_n is a partition of $[0,1]$, $M_t^n \sim N(0,1)\ \forall\ n$. Thus, since $M_t^n \to M_t$ as $n\to\infty$, $M_t \sim N(0,1)$.
2.

3.

(Stratonovich integral and chain rule) For two continuous semi-martingales X and Y (on the same space and filtration), we define the Stratonovich integral

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y\rangle_t,$$

where $\int_0^t X_s dY_s$ represents the Itô integral. Show that if $f \in C^3$, and X is a continuous semimartingale, then we have the chain rule

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s.$$

Solution. Since $f \in C^3$ and X is a a continuous semi-martingale, we can apply Itô's formula,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s.$$

Since $\int_0^t f'(X_s) \circ dX_s = \int_0^t f'(X_s)dX_s + \frac{1}{2}\langle f'(X), X\rangle_t$, it is sufficient to show that

$$\int_0^t f''(X_s)d\langle X\rangle_s = \langle f'(X), X\rangle_t.$$

Since $f \in C^3, f' \in C^2$, so we can apply Itô again,

$$d\langle f'(X), X\rangle_t = df'(X_t)dX_t = (f''(X_t)dX_t + \frac{1}{2}f'''(X_t)d\langle X\rangle_t)dX_t = f''(X_t)d\langle X\rangle_t + \frac{1}{2}f'''(X_t)d\langle X\rangle_t dX_t = f''(X_t)d\langle X\rangle_t$$

So,

$$\int_0^t f''(X_s)d\langle X\rangle_s = \langle f'(X), X\rangle_t.$$

August 2015

1.

Consider two pairs of adapted continuous processes (H^i, X^i) defined on two filtered probability spaces, $(\Omega_i, \mathcal{F}_i, (\mathcal{F}_t^i)_{0\le t<\infty}, \mathbb{P}_i)$ for $i = 1, 2$. Assume that the two pairs have the same law (as two dimensional processes), and that X^1, X^2 are semi-martingalees. Show that the two stochastic integrals $I^i = \int H^i dX^i$ have the same law for $i = 1, 2$.

2.

Consider a binary random variable X such that $\mathbb{E}[X] = 0$. For a given Brownian motion B, construct a stopping time with property $\mathbb{E}[T] < \infty$ and such that B_T and X have the same distribution. Is the condition $\mathbb{E}[X] = 0$ necessary for the existence of such a stopping time?

Solution. Let X be a binary random variable. That is

$$\mathbb{P}[X = a] = 1 - \mathbb{P}[X = b]$$

for some $a \neq b \in \mathbb{R}$ such that

$$\mathbb{E}[X] = a\mathbb{P}[X = a] + b\mathbb{P}[X = b] = 0 \to a\mathbb{P}[X = a] + b(1 - \mathbb{P}[X = a]) \to \mathbb{P}[X = a] = \frac{b}{b-a}.$$

This implies one of a, b is negative. WLOG, assume $a > 0, b < 0$. So,

$$X = \begin{cases} a & p = \frac{b}{b-a} \\ b & p = \frac{a}{b-a}. \end{cases}$$

Let B_t be a Brownian motion. Consider the stopping time $\tau = T_a \wedge T_b$ where $T_a = \inf\{t \ge 0 : B_t = a\}$ and $T_b = \inf\{t \ge 0 : B_t = b\}$. Since B_t^τ is a bounded martingale and therefore uniformly integrable, we can apply the optional sampling theorem,

$$\mathbb{E}[B_\tau] = a\mathbb{P}[T_a < T_b] + b\mathbb{P}[T_b < T_a] = 0 \Rightarrow \mathbb{P}[T_a < T_b] = \frac{b}{b-a}.$$

So

$$B_\tau = \begin{cases} a & p = \frac{b}{b-a} \\ b & p = \frac{-a}{b-a}. \end{cases}$$

Since B_t is BM, $B_t^2 - t$ is a martingale. For fixed t, we can apply the bounded optional sampling theorem so that

$$\mathbb{E}[B_{t\wedge\tau}^2] = \mathbb{E}[t \wedge \tau].$$

Taking the limit as $t \to \infty$ we get

$$\mathbb{E}[B_\tau^2] = \mathbb{E}[\tau]$$

by the bounded convergence theorem for the LHS and the monotone convergence theorem for the RHS. Thus,

$$\mathbb{E}[\tau] = a^2\mathbb{P}[T_a < T_b] + b^2(1 - \mathbb{P}[T_a < T_b]) < a^2 + b^2 < \infty.$$

3.

Show that, for a continuous semimartingale M and continuous adapted process A of bounded variation, with $A_0 = 0$, we have the following equivalence:

1. M is actually a local martingale with $\langle M \rangle = A$
2. for each $f \in C_b^2$, we have that

$$f(M_t) - f(M_0) - \int_0^t f''(M_s)dA_s$$

is a martingale.

Solution. Let $f \in C_b^2$. By Itô's formula,

$$f(M_t) = f(M_0) + \int_0^t f'(M_s)dM_s + \frac{1}{2}\int_0^t f''(M_s)d\langle M \rangle_s$$

(1) $\to$ (2) Suppose M is a local martingale with $\langle M \rangle = A$. Then,

$$\int_0^t f'(M_s)dM_s = f(M_t) - f(M_0) - \frac{1}{2}\int_0^t f''(M_s)dA_s.$$

Since M_t is a local martingale and f' is bounded, $\int_0^t f'(M_s)dM_s$ is a local martingale. Further,

$$\langle \int_0^t f'(M_s)dM_s \rangle = \int_0^t f'(M_s)^2 dA_s \leq CA_t < \infty \ \forall t \geq 0$$

So, $f(M_t) - f(M_0) - \frac{1}{2}\int_0^t f''(M_s)dA_s$ is a martingale.
(2) $\to$ (1)

January & August 2016

1.

Let S be an exponential Brownian motion with drift,

$$S_t = 1 + \int_0^t \mu S_u du + \int_0^t S_u dB_u, \quad t \geq 0,$$

for some $\mu \in \mathbb{R}$, where B is the standard Brownian motion. Given $\epsilon \in (0,1)$, compute $\mathbb{E}[\tau_\epsilon]$, where $\tau_\epsilon = \inf\{t \geq 0 : S_t = \epsilon\}$.

Solution.

$$S_t = \exp((\mu - 1)t + B_t)$$

We can check this using Itô's formula,

$$dS_t = S_t d((\mu - \frac{1}{2})t + B_t) + \frac{1}{2}S_t d\langle(\mu - \frac{1}{2})t + B_t\rangle = S_t((\mu - \frac{1}{2})dt + dB_t + \frac{1}{2}dt = S_t(\mu dt + dB_t).$$

This is hitting time of Brownian motion with drift, Girsanov's theorem.

2.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let W be a Brownian motion of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, \mathbb{P})$. Show that, for each x, the process M^x defined by

$$M_t^x = f(x + W_t), \quad 0 \leq t < \infty$$

is a local sub-martingale if and only if f is convex.

Solution. $\Leftarrow$ Suppose f is convex. Let $0 \leq s < t < \infty$. Then

$$\mathbb{E}[M_t^x | \mathcal{F}_s] = \mathbb{E}[f(x + W_t)|\mathcal{F}_s] \geq f(\mathbb{E}[x + W_t|\mathcal{F}_s]) = f(x + W_s) = M_s^x.$$

So, M_t^x is a local submartingale.

$\Rightarrow$ Suppose $\{M_t^x\}$ is a local submartingale. Let $A, B \in \mathbb{R}$ and $\lambda > 0$. Choose $x, 0 \leq a < b$ such that such that $A = x + a$ and $B = x - b$ where and $\lambda = \frac{b}{a+b}$ and consider the stopping time $\tau = T_a \wedge T_{-b}$ where $T_a = \inf\{t \geq 0 : W_t = a\}$ and $T_{-b} = \inf\{t \geq 0 : W_t = -b\}$. Then, by the optional sampling theorem,

$$\mathbb{E}[M_\tau^x] \geq \mathbb{E}[M_0^x]$$

where

$$\mathbb{E}[M_\tau^x] = \frac{b}{a+b}f(x+a) + \frac{b}{a+b}f(x-b)$$

and

$$M_0^x = f(x) = f(\frac{b}{a+b}(x+a) + \frac{a}{a+b}(x-b)) = f(\frac{bx + ab + ax - ab}{a+b}) = f(x).$$

So, $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$, so f is convex.

3.

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ where the filtration satisfies the usual conditions. On this space, consider two standard, one dimensional Brownian motions W and B (BM wrt the same filtration). Assume that $\langle B, W\rangle_t = \rho t$, where $\langle B, W\rangle$ is the cross variation of B and W, and ρ is constant. Show that

1. $B = W$, if $\rho = 1$
2. B is independent of W if $\rho = 0$.

Solution.

1. Suppose $\langle B, W\rangle_t = t$. Consider the process $M_t = B_t - W_t$. Since B_t, W_t are martingales, M_t is a martingale. Then since $(B_t - W_t)^2 = B_t^2 - 2B_tW_t + W_t^2$ and we know $B_t^2 - t, W_t^2 - t$ and $B_tW_t - t$ are local martingales, $\langle B - W\rangle_t = 0$. Since $\langle B - W\rangle_t = \lim_{n\to\infty} \sum_{i=1}^{\infty} (B_{t_i^n \wedge t} - W_{t_i^n\wedge t} - B_{t_{i-1}^n\wedge t} + W_{t_{i-1}^n\wedge t})^2 = 0$, this implies $B - W$ has finite variation. Thus, since $B - W$ is a local martingale the finite variation, $B - W = 0$, so $B = W$.

2. Suppose $\langle B, W\rangle_t = 0$. By Lévy's characterization of Brownian motion, (B, W) is a 2-dimensional Brownian motion $\Leftrightarrow B_t^2 - t, W_t^2 - t, B_tW_t$ are local martingales. Since B, W are BM, $B_t^2 - t$ and $W_t^2 - t$ are local martingales. Since $\langle B, W\rangle_t = 0 = \frac{1}{2}(\langle B + w\rangle_t - \langle B\rangle_t - \langle W\rangle_t)$, $\langle B + W\rangle_t = \langle W\rangle_t + \langle B\rangle_t$. So, $B_t^2 + 2B_tW_t + W_t^2 - \langle B\rangle_t - \langle W\rangle_t$ is a local martingale, so $\{BW\}_t$ is a local martingale. Thus, (B, W) is a 2-dimensional BM, so B and W are independent by the definition of 2-dimensional brownian motion.

January 2019

1.

Let W be a one-dimensional BM. Let $\mu, \sigma \in \mathbb{R}$ and x an initial value. Solve in closed form the equation

$$\begin{cases} dX_t & = \mu X_t dt + \sigma X_t dW_t \\ X_0 = x. \end{cases}$$

Solution.

$$dX_t = X_t(\mu dt + \sigma dW_t)$$

From this form, we make a guess that $X_t = x\exp(\mu t + \sigma W_t)$ and check using Itô's formula.

$$dX_t = X_t d(\mu t + \sigma W_t) + \frac{1}{2}X_t d\langle \mu t + \sigma W_t\rangle_t = X_t(\mu dt + \sigma dW_t + \frac{1}{2}\sigma^2 dt) = X_t((\mu + \frac{1}{2}\sigma^2)dt + \sigma dW_t)$$

So, we need a correction of $-\frac{1}{2}\sigma^2 t$. Thus, $X_t = x\exp((\mu - \frac{1}{2}\sigma^2)t + W_t)$.

2.

Let W be a standard one-dimensional Brownian motion and M be its running maximum process, i.e.

$$M_t = \max_{0\le s\le t} W_s, \quad 0 \le t < \infty.$$

Consider a two-times continuously differentiable function $f : \{(x, m) : m \ge 0, -\infty < x \le m\} \to \mathbb{R}$. Find a necessary and sufficient condition so that the process Y defined by $Y_t = f(W_t, M_t), \quad 0 \le t < \infty$, is a local martingale.

Partial Solution. For almost all ω, the measure $dM(\omega)$ is singular with respect to the Lebesgue measure with support $\{t : W_t = M_t\}$. By Itô's formula,

$$\begin{aligned} dY_t &= f_x(W_t, M_t)dW_t + f_m(W_t, M_t)dM_t + \frac{1}{2}f_{xx}(W_t, M_t)dt + f_{xm}(W_t, M_t)dW_t dM_t + \frac{1}{2}f_{mm}(W_t, M_t)dM_t dM_t \\ &= f_x(W_t, M_t)dW_t + f_m(W_t, M_t)dM_t + \frac{1}{2}f_{xx}(W_t, M_t)dt \end{aligned}$$

For Y_t to be a local martingale, we need the finite variation parts to vanish, $f_{xx}(W_t, M_t) = 0$ everywhere and $f_m(W_t, M_t)$ on the support of dM, $f_m(W_t, M_t)$ on the diagonal $\{t : W_t = M_t\}$. These conditions are also sufficient.

3.

Consider a finite time horizon T and a RCLL sub-martingale $(M_t)_{0\le t\le T}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{0\le t\le T}$. Consider the optimization problem of finding the stopping time τ that maximizes the expected value M at the (random) time τ, namely the problem

$$\sup_{\tau \text{ a stopping time}} \mathbb{E}[M_\tau].$$

Find optimizer $\tau*$.

Solution. Since M_t is a RCLL sub-martingale, it admits a Doob-Meyer decomposition, $M_t = L_t + A_t$ where L_t is a martingale and A_t is a nondecreasing, predictable process. Since all viable stopping times are bounded by finite time horizon T, we can apply the bounded optional sampling theorem. For any stopping time τ, $\mathbb{E}[M_\tau] = \mathbb{E}[L_\tau] + \mathbb{E}[A_\tau] = \mathbb{E}[L_0] + \mathbb{E}[A_\tau]$. Since A_t is a nondecreasing process, this implies $\mathbb{E}[M_\tau]$ is maximized for $\tau = T$.

August 2009

4.

Let $(B_t)_{0\leq t<\infty}$ be a standard Brownian motion. Define the random time T_x by

$$T_x = \inf\{t \geq 0 : B_t = x\}, \; x \in \mathbb{R}.$$

Compute $\mathbb{P}[T_a < T_{-b}]$ and $\mathbb{E}[T_a \wedge T_{-b}]$ for $a, b > 0$.

Solution. Let $\tau = T_a \wedge T_{-b}$. Then, since $(B_t^\tau)_{0\leq t<\infty}$ is bounded, (B_t^τ) is a uniformly integrable martingale. Thus, by the optional sampling theorem,

$$\mathbb{E}[B_\tau] = a\mathbb{P}[T_a < T_{-b}] + -b(1 - \mathbb{P}[T_a < T_{-b}]) = (a+b)\mathbb{P}[T_a < T_{-b}] - b = \mathbb{E}[B_0] = 0$$

which implies

$$\mathbb{P}[T_a < T_{-b}] = \frac{b}{a+b}.$$

Since (B_t) is a Brownian motion, $B_t^2 - t$ is a martingale. Again, applying the bounded optional sampling theorem,

$$\mathbb{E}[B_{t\wedge\tau}^2] = \mathbb{E}[t \wedge \tau].$$

Then letting $t \to \infty$, by the bounded convergence theorem for the LHS and the monotone convergence theorem for the RHS,

$$\mathbb{E}[B_\tau^2] = \mathbb{E}[\tau]$$

$$\begin{aligned}\mathbb{E}[\tau] = \mathbb{E}[B_\tau^2] &= a^2\mathbb{P}[T_a < T_{-b}] + b^2(1 - \mathbb{P}[T_a < T_{-b}]) \\ &= (a^2 - b^2)\left(\frac{b}{a+b}\right) + b^2 = (a-b)(a+b)\frac{b}{a+b} + b^2 = (a-b)b + b^2 = ab.\end{aligned}$$

5.

Let (B_t) be a standard Brownian motion. Show that

$$\lim_{t\to\infty} \sqrt{t}\mathbb{P}[B_s \leq 1, \forall\, s \leq t] = \sqrt{\frac{2}{\pi}}.$$

Solution. Let M_t be the running maximum process of the Brownian motion. Then

$$\mathbb{P}[B_s \leq 1, \; \forall\, s \leq t] = \mathbb{P}[M_t \leq 1] = \mathbb{P}[|B_t| \leq 1]$$

since $M_t \sim |B_t|$. So,

$$\begin{aligned}\lim_{t\to\infty} \sqrt{t}\mathbb{P}[B_s \leq 1, \forall\, s \leq t] &= \lim_{t\to\infty} \sqrt{2}\mathbb{P}[|B_t| \leq 1] = \lim_{t\to\infty} \sqrt{t}\frac{2}{\sqrt{2\pi t}}\int_0^1 \exp(-x^2/2t)\; dx \\ &= \frac{2}{\sqrt{2\pi}} \lim_{t\to\infty} \int_0^1 \exp(-x^2/2t)\; dx = \sqrt{\frac{2}{\pi}}\int_0^1 \lim_{t\to\infty} \exp(-x^2/2t)\; dx = \sqrt{\frac{2}{\pi}}.\end{aligned}$$

where we can interchange the limit and integral due to the bounded convergence theorem.

6.

Let $(B_t)_{t\in[0,\infty)}$ be a B, and let $(X_t)_{t\geq 0}$ be its Lévy transform given

$$X_t = \int_0^t sgn(B_u) dB_u,$$

1. Show that X is a Brownian motion.
2. Show that the random variables B_t and X_t are uncorrelated.
3. Show that B_t and X_t are not independent.

Solution.

1. To show that X is a Brownian motion, we will show that X_t is a martingale and $\langle X\rangle_t = t$. Let $0 \leq s < t < \infty$. Consider a sequence of partitions of $[0,\infty)$ denoted $\Delta_n \to$Id. Then for $X_t^n = \sum_{k=1}^{n} sgn(B_{t_k} \wedge t)(B_{t_k\wedge t} - B_{t_{k-1}\wedge t})$, we consider the conditional expectation,

   $$\mathbb{E}[X_t^n|\mathcal{F}_s] = \sum_{k=1}^{\infty} sgn(B_{t_k\wedge t})\mathbb{E}[(B_{t_k\wedge t} - B_{t_{k-1}\wedge t})|\mathcal{F}_s] = \sum_{k=1}^{\infty} sgn(B_{t_k\wedge s}))(B_{t_k\wedge s} - B_{t_{k-1}\wedge s}) = X_s^n.$$

   So X^n is a martingale. Then $X_t^n \to X_t \; \forall\, t \in [0,\infty)$, so X is also a martingale.

   Or, if we have that X_t is a semi-martingale, since $X_t = \int_0^t sgn(B_s)dB_s$, $dX_t = sgn(B_t)dB_t$, so X_t is a local martingale.

   Now, we want $\langle X\rangle_t$.

   $$d\langle X\rangle_t = dX_t dX_t = (sgn(B_t)dB_t)(sgn(B_t)dB_t) = dB_t dB_t = d\langle B\rangle_t = dt$$

   So, $\langle X\rangle_t = t$. Thus, by Lévy's characterization of BM, X is a Brownian Motion.

2. To show that B_t and X_t are uncorrelated, $\text{Cov}(X_t, B_t) = 0$ First, consider

$$X_tB_t = \int_0^t X_u dB_u + \int_0^t B_u dX_u + \int_0^t sgn(B_u)du.$$

The first two terms are local martingales with quadratic variations in $\mathbb{L}^2$,

$$\langle \int_0^t X_u dB_u \rangle = \int_0^t X_u^2 du$$
$$\langle \int_0^t B_u dX_u \rangle = \int_0^t B_u^2 du$$

so they are true martingales. So,

$$\text{Cov}(X_tB_t) = \mathbb{E}[X_tB_t] = \mathbb{E}[\int_0^t X_u dB_u + \int_0^t B_u dX_u + \int_0^t sgn(B_u)du] = \mathbb{E}[\int_0^t sgn(B_u)du] = \int_0^t \mathbb{E}[sgn(B_u)]du = 0.$$

3. To show that B_t and X_t are not independent, we calculate $\mathbb{E}[X_tB_t^2]$. First, since $dB_t^2 = 2B_t dB_t + dt$ by Itô,

$$X_tB_t^2 = 2\int_0^t X_uB_u dB_u + \int_0^t X_u du + \int_0^t B_u^2 dX_u + 2\int_0^t B_u sgn(B_u)du.$$

$\int_0^t X_uB_u dB_u$ and $\int_0^t B_u^2 dX_u$ are true martingales, so

$$\mathbb{E}[X_tB_t^2] = \mathbb{E}[\int_0^t (X_u + 2B_u sgn(B_u))du] = \mathbb{E}[\int_0^t (X_u + 2|B_u|)du] = \int_0^t \mathbb{E}[X_u + 2|B_u|]du.$$

Since $\mathbb{E}[X_u] = 0$,

$$\mathbb{E}[X_tB_t^2] = 2\int_0^t \mathbb{E}[|B_u|]du > 0.$$

However, $\mathbb{E}[X_t]\mathbb{E}[B_t^2] = 0$, so $\text{Cov}(X_t, B_t^2) \neq 0$ which implies X_t and B_t are not independent.

January 2010

5.

Let $\{B_t\}_{t\in[0,\infty)}$ be a standard Brownian Motion and let $\{H_t\}_{t\in[0,\infty)}$ be a progressively measurable process such that

$$\forall\ t \geq 0, \quad \int_0^t H_u^2 du < \infty, \text{ and } \int_0^\infty H_u^2 du = \infty.$$

For $\sigma > 0$, show that $\int_0^\tau H_u dB_u \sim \mathcal{N}(0, \sigma^2)$, where $\tau = \inf\{t \geq 0 : \int_0^t H_u^2 du = \sigma^2\}$.

Solution. Let $M_t = \int_0^t H_u dB_u$. Since H_t is progressively measurable with the above properties, M_t is a local martingale. Then $\langle M \rangle_t = \int_0^t H_u^2 du$. Consider the exponential martingale

$$X_t = \exp(iuM_t + \frac{1}{2}u^2\langle M \rangle_t).$$

Then since M_t^τ is bounded and therefore uniformly integrable, we may apply the optional sampling theorem

$$\mathbb{E}[\exp(iuM_\tau)] = \frac{1}{\exp(\frac{1}{2}u^2\sigma^2)} = \exp(-\frac{1}{2}u^2\sigma^2)$$

Since the characteristic function of a normal random variable is $\varphi(t) = \exp(i\mu t - \frac{1}{2}\sigma^2 t^2)$, this implies $\int_0^\tau H_u\ dB_u \sim \mathcal{N}(0, \sigma^2)$.

August 2010

3.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{C}$ be a non-empty family of sub-σ-algebras of $\mathcal{F}$. For a random variable X in $\mathcal{L}^1$, prove that the family

$$\chi = \{\mathbb{E}[X|\mathcal{G}] : \mathcal{G} \in \mathcal{C}\}$$

is uniformly integrable.

Solution. The family $\{X\}$ is uniformly integrable, so there exists a convex, nondecreasing test function φ such that $\mathbb{E}[\varphi(X)] < \infty$. Then for $\mathcal{G} \in \mathcal{C}$. $\mathbb{E}[\varphi(\mathbb{E}[X|\mathcal{G}])] \leq \mathbb{E}[\mathbb{E}[\varphi(X)|\mathcal{G}]] = \mathbb{E}[\varphi(X)] < \infty$. Thus, χ is uniformly integrable.