APM 2663 Sample Test 2
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Important:

- Recall that the word *if* in a definition means *if and only if.*
- To receive full credit for a question, you should provide all logical steps.
- Recall that $\mathbb{N}$ is the set of positive integers.
- Recall that $\mathbb{Z}$ is the set of integers.
- Recall that $\mathbb{Q}$ is the set of rational numbers.
- Recall that $\mathbb{R}$ is the set of real numbers.
- This test is worth 110 marks. If you receive x marks, your grade will be $\min\{x, 100\}$%.

(1) Give a polynomial time algorithm, using the arithmetic model, to solve the following problem: Input n numbers and output their average. [10 marks]

(2) Find the gcd of 1575 and 231. Write the gcd as $1575x + 231y$ for some $x, y \in \mathbb{Z}$. [10 marks]

(3) Let a, b, c be positive integers. Let d be the gcd of a and b and g be the gcd of ca and cb. Prove that $g = cd$. [10 marks]

(4) Show that $\log_8 15$ is irrational. [10 marks]

(5) Use mathematical induction to prove that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} \ \forall n \geq 1. \text{ [10 marks]}$$

(6) What is wrong with the following "proof" that any order for $n \geq 10$ pounds of fish can be filled with only five-pound fish?

> We use the strong form of mathematical induction. Here $n_0 = 10$. Since an order of ten pounds of fish can be filled with two five-pound fish, the assertion is true for $n = 10$. Now let $k > 10$ be an integer and suppose that any order of l pounds of fish, $10 \leq l < k$, can be filled with only five-pound fish. We must prove that an order for k pounds can be similarly filled. By by the induction hypothesis, we can fill an order for $k - 5$ pounds of fish, so, adding one more five-pounder, we can fill the order for k pounds. By induction, the assertion is true for every integer $n \geq 10$.

[10 marks]

(7) How many integers between 1 and 600 (inclusive) are divisible by at least one of 5, 7 and 17? [10 marks]

(8) Let $k, l, n \in \mathbb{N}$ such that $k + l \leq n$. Use a combinatorial argument to prove that

$$\binom{n}{k}\binom{n-k}{l} = \binom{n}{l}\binom{n-l}{k}.$$ [15 marks]

(If you are having trouble with a combinatorial proof, you may use a non-combinatorial proof for 10 marks.)

(9) Evaluate

$$\sum_{k=1}^{n} k\binom{n}{k}.$$

(*Hint: Use the Binomial Theorem and Calculus.*) [15 marks]

(10) Suppose you draw 7 cards from a standard deck of cards with 52 cards. How many hands are there of the following type: A pair of X's, a pair of Y's, one of U, one of V, and one of W, where U, V, W, X, Y are all distinct "numbers" from $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$? For example, 2 of clubs, 2 of spades, 3 of hearts, 3 of diamonds, 5 of clubs, 6 of spades, and J of clubs. [10 marks]

Solutions

(1) One way is the following:

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for i = 1 to n
  read a[i];
sum := 0;
for i = 1 to n
  sum := sum + a[i];
a := sum/n;
write a;
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This takes n read statements, n additions, 1 division and 1 write statement. So it the running time is $O(n)$.

(2) Using the Euclidean Algorithm gives the following sequence: $1575 = 6(231) + 189$, $231 = 1(189) + 42$, $189 = 4(42) + 21$ and $42 = 2(21)$. So the gcd is 21. Now using backward substitutions, we get

$$\begin{aligned} 21 &= 189 - 4(42) &&= 189 - 4(231 - 1(189) &&= 5(189) - 4(231) \\ &= 5(1575 - 6(231)) - 4(231) &&= 5(1575) + (-34)(231) \end{aligned}$$

(3) Since d is the gcd of a and b, there exist integers x and y such that $d = ax + by$. So $cd = cax + cby$. Since d is the gcd of a and b, $d|a$ and $d|b$; hence $(cd)|(ca)$ and $(cd)|(cb)$. So cd is a common divisor of ca and cb. Since g is the gcd of ca and cb, $cd \leq g$. Now, since $cd = cax + cby$, $g|(ca)$ and $g|(cb)$, $g|(cd)$. Hence $g \leq cd$. Therefore $g = cd$.

(4) Suppose not. Then $\log_8 15 = \frac{m}{n}$ where m and n are positive integers. So $15 = 8^{m/n}$ which implies $15^n = 8^m$. This gives $3^n 5^n = 2^{3m}$. Each side is a prime factorization of some integer. Since the two factorizations are different, it follows from the Fundamental Theorem of Arithmetics that $3^n 5^n \neq 2^{3m}$, which is a contradiction.

(5) The case $n = 1$ is true since LHS=RHS=1. Assume the statement is true for $n = k$, that is,

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$

We want to prove the statement for $n = k + 1$, that is, we want to prove

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}.$$

Now

$$\begin{aligned} & 1^3+2^3+3^3+\cdots+k^3+(k+1)^3 \\ = \; & (1^3+2^3+3^3+\cdots+k^3)+(k+1)^3 \\ = \; & \tfrac{k^2(k+1)^2}{4}+(k+1)^3 \\ = \; & \tfrac{k^2(k+1)^2}{4}+\tfrac{4(k+1)^3}{4} \\ = \; & \tfrac{(k+1)^2(k^2+4k+4)}{4} \\ = \; & \tfrac{(k+1)^2(k+2)^2}{4} \end{aligned}$$

as required. Hence the statement is true for all $n \geq 1$ by the principle of mathematical induction.

(6) The "proof" is wrong because it has only checked the initial case $n = 10$. When $n = 11$, the induction step will assume the result is true for $11-5$ which is not correct and hence flawed.

(7) Let a_i=number of integers between 1 and 600 (inclusive) that are divisible by i. Then $a_5 = \lfloor 600/5 \rfloor = 120$, $a_7 = \lfloor 600/7 \rfloor = 85$, $a_{17} = \lfloor 600/17 \rfloor = 35$, $a_{5\cdot 7} = \lfloor 600/35 \rfloor = 17$, $a_{5\cdot 17} = \lfloor 600/85 \rfloor = 7$, $a_{7\cdot 17} = \lfloor 600/119 \rfloor = 5$ and $a_{5\cdot 7\cdot 17} = \lfloor 600/595 \rfloor = 1$ So by the principle of inclusion and exclusion, the desired number is

$$120+85+35-17-7-5+1 = 212.$$

(8) Suppose we have n people and we want to pick two mutually exclusive groups, one with k members and the other with l members. One way to count the number of different combinations is to pick the k-member committee first and we have $\binom{n}{k}$ ways to accomplish this; we can now pick l members to form the second committee and we have $\binom{n-k}{l}$ to accomplish this (the two committees are mutually exclusive), and hence the total number is $\binom{n}{k}\binom{n-k}{l}$. This is the LHS. Another way is to pick the l-member committee first, then the k-member committee; this will give the RHS and the result follows.

(9) Recall the Binomial Theorem, $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Differentiating both sides gives $n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}$. Now let $x = 1$ gives

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}.$$

(10) Out of 13 "numbers," we pick one for X in $\binom{13}{1}$ ways. There are 4 suits for X and we choose 2 in $\binom{4}{2}$ ways. Now there are 12 remaining "numbers" as U, V, W, X, Y are all distinct. Out of these 12 "numbers," we pick one for Y in $\binom{12}{1}$ ways. There are 4 suits for Y and we choose 2 in $\binom{4}{2}$ ways. Now there are 11 remaining "numbers" as U, V, W, X, Y are all distinct. Out of these 11 "numbers," we pick one for W in

$\binom{11}{1}$ ways. There are 4 suits for W and we choose 1 in $\binom{4}{1}$ ways. Now there are 10 remaining "numbers" as U, V, W, X, Y are all distinct. Out of these 10 "numbers," we pick one for U in $\binom{10}{1}$ ways. There are 4 suits for U and we choose 1 in $\binom{4}{1}$ ways. Now there are 9 remaining "numbers" as U, V, W, X, Y are all distinct. Out of these 9 "numbers," we pick one for V in $\binom{9}{1}$ ways. There are 4 suits for V and we choose 1 in $\binom{4}{1}$ ways. So it would *seem* that the solution is

$$\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}\binom{10}{1}\binom{4}{1}\binom{9}{1}\binom{4}{1}.$$

However, we have overcounted. Since we cannot distinguish X and Y, we have overcounted by a factor of $2! = 2$. Since we cannot distinguish U, V, W, we have overcounted by a factor of $3! = 6$. Thus we have overcounted by a factor of $(2!)(3!) = 12$. Hence, the answer is

$$\frac{1}{12}\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}\binom{10}{1}\binom{4}{1}\binom{9}{1}\binom{4}{1}.$$