Theory Question 2

In polynomial least-squares approximation, we define the function f to be:

$$f(x) = b(x)^T c(x)$$

with $b(x)$ the basis vector and $c(x)$ the coefficient vector. The optimization objective is given by:

$$\min_c \sum_i w(x, p_i)(f(x) - f_i)^2 = \min_c \sum_i w(x, p_i)(b(x)^T c(x) - f_i)^2$$

This can be solved for c using a linear system:

$$\begin{bmatrix} w(x,p_1) & & \\ & \ddots & \\ & & w(x,p_N) \end{bmatrix} \begin{bmatrix} b(p_1)^T \\ \vdots \\ b(p_N)^T \end{bmatrix} c(x) = \begin{bmatrix} w(x,p_1) & & \\ & \ddots & \\ & & w(x,p_N) \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$$

which can be rewritten more compactly as:

$$W(x) \cdot B \cdot c(x) = W(x) \cdot \phi$$

Left-multiplying by $B^T \cdot W(x)$ we get the normal equations:

$$B^T \cdot W^2(x) \cdot B \cdot c(x) = B^T \cdot W^2(x) \cdot \phi$$

So the coefficients are given by:

$$c(x) = (B^T \cdot W^2(x) \cdot B)^{-1} B^T \cdot W^2(x) \cdot \phi$$

And the solution is obtained using the definition of f:

$$f(x) = b(x)^T c(x) = b(x)^T (B^T \cdot W^2(x) \cdot B)^{-1} B^T \cdot W^2(x) \cdot \phi$$

To simplify notation we rewrite:

$$f(x) = b(x)^T c(x) = b(x)^T H(x)^{-1} B^T \cdot W^2(x) \cdot \phi$$

where we defined $H(x) = B^T \cdot W^2(x) \cdot B$. Now we compute the derivative using the product rule:

$$\begin{aligned} f'(x) = {} & b'(x)^T H(x)^{-1} B^T \cdot W^2(x) \cdot \phi + \\ & b(x)^T (-H(x)^{-1} H'(x) H(x)^{-1}) B^T \cdot W^2(x) \cdot \phi + \\ & b(x)^T H(x)^{-1} B^T \cdot W^2(x)' \cdot \phi \\ = {} & b'(x)^T H(x)^{-1} B^T \cdot W^2(x) \cdot \phi + \\ & b(x)^T (-H(x)^{-1} (B^T \cdot (2W(x) \cdot W'(x)) \cdot B) H(x)^{-1}) B^T \cdot W^2(x) \cdot \phi + \\ & b(x)^T H(x)^{-1} B^T \cdot (2W(x) \cdot W'(x)) \cdot \phi \end{aligned}$$

which is the required result.