Theory Question 1

Informally, S defines an implicit function defined by $f(x) = 0$ i.e. an isosurface. This means that moving along the surface S does not change the value of the function. Any vector $v \in \mathbb{R}^3$ can be decomposed into a component tangent to the surface $v_{parallel}$ and a component normal to the surface v_{perp}. Evidently, the tangential component does not contribute to changing the value of the function as it moves along the isosurface. Now interpreting the gradient as the rate of change, the only component that remains is the one perpendicular to the surface and hence the gradient must be proportional to the normal to the surface.

More formually, consider any point P on the surface and any curve $C \in S$ that passes through P. C can be parametrized in cartesian coordinates as $c(t) = (x(t), y(t), z(t))^T$. Restraining the domain of f to that of C, we have f(c(t)) = 0. By basic differentiation rule we get:

$$\nabla f \cdot c'(t) = 0$$

Without loss of generality, let $P = c(T) = (x(T), y(T), z(T))$, then we can rewrite:

$$\nabla f(P) \cdot c'(T) = 0$$

We know $c'(t)$ is tangent to the surface and hence the dot product being null implies that ∇f is perpendicular to to the curve. Since this is the case for any curve passing through P, it is perpendicular to the tangent plane of the surface. In other words, the normal of the surface which is by definition the normal of the tangent plane to the surface is proportional to the gradient. Strict equality does not hold since the **null** dot product only characterizes angles and does not give information about norms.