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First-Order
Modal Logic

1.   Introduction

We now turn to the matter of combining first-order logic, including scoped-terms, with modal sentential logic. Syntactically, there is almost nothing to do; we simply add two symbols '□' and '◇', with the stipulation that they are both one-place connectives, written in prefix notation. Semantically and logically, however, every step of the way presents challenging philosophical issues not present in either first-order logic or modal SL, in isolation from each other.

2.   Quantificational Domain

The first major issue concerns the domain over which the variables range. How do we read '$\exists x$', '$\forall x$', and '$\imath x$'. As noted in the chapter on Modal Predicate Logic, we adopt a possibilist stance, according to which the variables (and their affiliated constants) range over the class of *all possible objects*. In order to produce "inner domains" or "actualist domains", we add a new logical predicate 'Æ', and affiliated actualist quantifiers, on the basis of which we have the following readings.

$\exists x Fx$		some *possible* thing is F
$\exists' x Fx$	$=_{\text{df}} \exists x(\text{Æ}x \,\&\, Fx)$	some *actual* thing is F
$\forall x Fx$		every *possible* thing is F
$\forall' x Fx$	$=_{\text{df}} \forall x(\text{Æ}x \rightarrow Fx)$	every *actual* thing is F

We also now add a further rule that requires actual existence to imply possible existence.[1]

Æ/E!
$\text{Æ}[\tau]$ —————— $\exists \nu[\nu = \tau]$
τ is any closed singular-term, and ν is any variable.

3.   Scoped-Terms in Modal Logic; *De Re* versus *De Dicto*

We have already seen how scoped-terms offer an analysis of Russell's theory of descriptions within a modern free logic system. We now see how scoped-terms enable us to formalize the important distinction between modalities *de re* and modalities *de dicto*.

By way of illustration, consider the following sentence.

Jay believes that Kay is the murderer

This has two plausible explications, which may be formulated as follows.

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[1] If we wish, we can also add an existence predicate 'E!' via the definition: $E!\tau =_{\text{df}} \exists \nu[\nu=\tau]$.

Jay believes *of Kay* that she is the murderer

Jay believes the proposition ⟨Kay is the murderer⟩[2]

In the first case, the belief is of-a-thing (*de re*); in the second case, the belief is of-a-proposition (*de dicto*).

We can formulate the first reading within our logical system as follows.

(ᴋ/x)	⟦jℬ⟧	[x = ᴍ]
it is true of Kay that	Jay believes that	she is the murderer

In this case, the singular-term 'Kay' has wide scope, which is syntactically conveyed in the same manner quantifier-scope is conveyed.

We can also reverse the scopes of the two operators to obtain the following.

⟦jℬ⟧	(ᴋ/x)	[x = ᴍ]
Jay believes that	it is true of Kay that	she is the murderer

Except for scoped-descriptions, scoped-terms are otiose in *non-modal* contexts. In this example, this amounts to the equivalence of the following pairs.

$$(ᴋ/x)[x = ᴍ] \qquad \equiv \qquad ᴋ = ᴍ$$
$$⟦jℬ⟧(ᴋ/x)[x = ᴍ] \qquad \equiv \qquad ⟦jℬ⟧[ᴋ = m]$$

In other words, the traditional distinction between *de re* and *de dicto* turns out to be a distinction of scope. If the singular-term has wide scope, then we are speaking *de re*; if the modal operator as wide scope, then we are speaking *de dicto*.

4.   Russell's Yacht

Bertrand Russell tells a joke, which we freely paraphrase as follows.[3]

A person, upon seeing his friend's yacht for the first time, comments,

"I expected your yacht to be bigger than it is"

to which the yacht owner replies

"How could that be? It must be exactly as big as it is!"

The problem raised in this example can be solved in at least two ways. One way, which is linguistically natural, and which we consider in a later chapter, uses the modifier 'actual'. In this section, we employ existing logical resources to solve this problem in a manner similar to Russell's solution.

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[2] We propose to enclose a sentence S in corner brackets to form the name of the proposition that S expresses. This is accomplished in ordinary English by prefixing the sentence with the subnective 'that'.

[3] Russell, 'On denoting', *Mind*, 14 (1905), 479-93.

First, for the sake of simplifying the modalities involved, let us reformulate the joke as follows, where we name the yacht 'Kiwi'.[4]

(Q?)   could Kiwi be bigger than it is?
(A!)   of course not; Kiwi must be exactly as big as it is!

The *contents* of (Q?) and (A!) can be expressed as follows.

(Q)   possibly, Kiwi is bigger than it is
(A)   necessarily, Kiwi is exactly as big as it is

We propose to analyze 'bigger than' in terms of the intermediate notion of *size*, so that

α is bigger than β

is semantically equivalent to:

α's size is (mathematically) greater than β's size $[s(\alpha) > s(\beta)]$

Similarly, we propose to analyze 'exactly as big as' so that

α is exactly as big as β

is semantically equivalent to:

α's size is (mathematically) identical to β's size $[s(\alpha) = s(\beta)]$

Thus, symbolizing 'Kiwi' by 'K', we *might* rephrase (Q) and (A) as follows.

(1)   possibly, $s(K) > s(K)$
        i.e., there is an accessible world in which K's size is greater than K's size.

(2)   necessarily, $s(K) = s(K)$
        i.e., in every accessible world, K's size is identical to K's size.

Clearly, (1) is false, and (2) is true [at least, supposing $s(K)$ exists at every accessible world.]

As with many jokes, Russell's joke is based on a simple misunderstanding. Specifically, the yacht owner hears the question (Q?) as (1) and retorts by saying (2). But of course, the questioner (presumably) does not intend the question that way.

According to Russell, the problem can be analyzed as follows. The sentence involves a description – 'the size of Kiwi' – which occurs twice. Furthermore, the scope of the first occurrence may or may not be different from the scope of the second occurrence. As you recall, the notion of scope is critical to Russell's analysis of definite descriptions; indeed, without it, his analysis is logically inconsistent.

Rather than dwell on Russell's own syntactic scheme, which is a bit awkward, we use the scoped-term scheme we have already developed as a substitute for Russell's scheme. As noted earlier in this chapter, this scheme enables us to distinguish *de re* (wide scope) uses of singular-terms from *de dicto* (narrow scope) uses of singular-terms.

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[4] Not to be confused with another famous yacht, from New Zealand, with a similar name.

Let us apply scoped-terms to the Kiwi-problem. On the one hand, (Q) can be formally rendered as follows,

$$(s(\kappa)/x) \diamondsuit [s(\kappa) > x]$$

which can be read as:

it is true of Kiwi's size (x) that: possibly Kiwi's size is bigger than it (x)

In this case, the term '$s(\kappa)$' occurs twice, once outside the scope of '$\diamondsuit$', and once inside the scope of '$\diamondsuit$'.

On the other hand, (A) can be formally rendered by any of the following equivalent formulas.

$$\square(s(\kappa)/x)(s(\kappa)/y)[x = y]$$
$$\square(s(\kappa)/x)[x = x]$$
$$\square[s(\kappa) = s(\kappa)]$$

In each formula, the term '$s(\kappa)$' also occurs twice, both inside the scope of '$\square$'.

5.   Leibniz's Law

A major problem in quantified modal logic is how to formulate, and understand, Leibniz's Law. Recall that, in standard first-order logic, Leibniz's Law is written in rule-form as follows.

LL	
$\Phi[\sigma/\nu]$ $\sigma = \tau$ ─────── $\Phi[\tau/\nu]$	$\Phi[\sigma/\nu]$ $\tau = \sigma$ ─────── $\Phi[\tau/\nu]$

Basically, this says the following.

suppose one has a formula Φ with one free variable ν;
suppose the result of replacing each free occurrence of ν by σ is true;
suppose $\sigma = \tau$;
then the result of replacing each free occurrence of ν by τ is true.

The following are examples of the sorts of sentences that pose counterexamples to this formulation of Leibniz's Law, which we will call the *substitutional formulation*. With the exception of the last example, each of these arguments involves an intensional operator, which is bold-faced.

9 is **necessarily** greater than 7;
9 <u>is</u> the number of planets;[5]
therefore, the number of planets is **necessarily** greater than 7.

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[5] There are two issues concerning how many planets there are. On the one hand, according to a recent discovery, there is a trans-plutonian body dubbed 'Quaoar', which would bring the total of planets to 10. On the other hand, it has long been debated whether Pluto is indeed a planet. A negative answer to the latter would bring the total to 8, since Quaoar is not a planet if Pluto is not a planet.

Coach Jones **is looking for** the tallest student (to play on the basketball team);
Kay <u>is</u> the tallest student;
therefore, Coach Jones **is looking for** Kay (to play on the basketball team).

the Babylonians **believed** that the morning star ≠ the evening star
the morning star <u>is</u> the evening star (both being Venus);
therefore, the Babylonians **believed** that the morning star ≠ the morning star.

Bush **dreams of** being the greatest U.S. president;
Lincoln <u>is</u> the greatest U.S. president;
therefore, Bush **dreams of** being Lincoln.

Bush was **once** a child;
Bush <u>is</u> the president of the U.S.;
therefore, the president of the U.S was **once** a child.

until recently, I did not **know** that Bodensee is Lake Constanz;[6]
Bodensee <u>is</u> Lake Constanz;
therefore, until recently, I did not **know** that Bodensee is Bodensee.

the temperature <u>is</u> 70;
the temperature is rising;
therefore, 70 is rising.

There are various accounts of what goes wrong in the above inferences. According to an account that traces to Frege,[7] some singular-terms *change reference* when they are placed inside certain syntactic contexts. In these contexts – which are said to be *referentially opaque* (not *transparent*) – some singular-terms are *referentially oblique* (not *direct*).

The chief problem with this account is that, although these arguments are all invalid according to their most salient readings, every one of these arguments (except perhaps the last one) admits a reading according to which the argument is valid. For example, it might be that Bush dreams of being Lincoln, and *this* is *precisely the sense* in which he dreams of being the greatest U.S. president.

It is crucial to understand that these sentences are not counter-examples to Leibniz's Law, but only to the substitutional formulation of Leibniz's Law. In order to see this, we recall the pre-formal rendering of Leibniz's Law.

Leibniz's Law – Original Formulation

If a is b (i.e., $a=b$),
then whatever is true of a is also true of b,
and whatever is true of b is also true of a.

With this in mind, let us consider the argument.

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[6] A true story! As a child, I saw a lake my dad referred to as 'Bodensee' (its German name). Much later, I heard about Lake Constanz, but I did not realize for quite some time that they are one and the same.
[7] Gottlob Frege, "Über Sinn und Bedeutung", *Zeitschrift für Philosophie und philosophische Kritik*, 100 (1892), 25-50.

> Bush **dreams of** being the greatest U.S. president;
> Lincoln i̲s̲ the greatest U.S. president;
> therefore, Bush **dreams of** being Lincoln.

The original formulation of Leibniz's Law requires a "true of" premise and a "true of" conclusion. So in order to apply Leibniz's Law to this argument, it must be rewritten – for example, as follows.

> **it is true of** the greatest U.S. president that Bush dreams of being **him**;
> Lincoln i̲s̲ the greatest U.S. president;
> therefore, **it is true of** Lincoln that Bush dreams of being **him**.

Notice that this argument is unproblematically valid, although it is unlikely to be sound, since its first premise is unlikely to be true.

A remaining question is how to formulate Leibniz's Law correctly. As it turns out, we have the technical machinery at hand, in the form of scoped-terms. In particular, the formal account of Leibniz's Law goes as follows, where σ and τ are closed singular-terms, and Φ is a formula.

Leibniz's Law – Scoped-Term Formulation	
$\sigma=\tau$ $(\sigma/v)\Phi$ ——— $(\tau/v)\Phi$	$\sigma=t$ $(\tau/v)\Phi$ ——— $(\sigma/v)\Phi$

These are *de re* principles, which can be read respectively as follows.

σ is τ; it is true of σ that Φ; therefore, it is true of τ that Φ.	σ is τ; it is true of τ that Φ; therefore, it is true of σ that Φ.

As it turns out, we do not propose the above principles as primitive rules of our system. Rather, we propose the following as our primitive rule, which is a *limited substitution* principle.

Leibniz's Law – Limited-Substitution Formulation	
$\Phi[\sigma/v]$ $\sigma = \tau$ ——— $\Phi[\tau/v]$	$\Phi[\sigma/v]$ $\tau = \sigma$ ——— $\Phi[\tau/v]$
Here, v must be ***modally free*** in Φ for both σ and τ.	
To say that v is ***modally free*** for ε in Φ is to say that: either: ε is a constant, or: no occurrence of v in Φ falls within the scope of a modal operator in Φ.	

To see that the scoped-term version follows from the latter, we offer the following derivation schema.

(1)	$(\sigma/\nu)\Phi$	Pr
(2)	$\sigma = \tau$	Pr
(3)	~~SHOW~~: $(\tau/\nu)\Phi$	Def (τ/ν)
(4)	~~SHOW~~: $\exists\nu\{\nu=\tau \,\&\, \Phi\}$	7,$\exists$I
(5)	$\exists\nu\{\nu=\sigma \,\&\, \Phi\}$	1,Def (τ/ν)
(6)	$a=\sigma \,\&\, \Phi$	5,$\exists$O
(7)	$a=\tau \,\&\, \Phi$	2,6,LL (adjusted)

Notice in lines (6) and (7) that the occurrences of σ and τ that participate in the substitution do not fall inside modal operators, so the adjusted form of LL can be applied.[8]

6.   Some Identities are Absolute; Some Aren't

Another curiosity of modal logic concerns how the modal operators interact with identity. We have already suggested that the fact that the present author believes that Bodensee is Bodensee does not entail that the present author believes that Bodensee is Lake Constanz, even if Bodensee is Lake Constanz. Similarly, we have suggested that, although it necessary that 9 is 9, it is not necessary that 9 is the number of planets, even if 9 is the number of planets.

According to one well-known view, due originally to Kripke,[9] all identities are *absolute* – which is to say that they are all true *everywhere* or *nowhere*. This is controversial, to say the least, and it has been hotly debated. By way of steering a middle ground in this dispute, we propose that identity statements generally admit two readings – a *Kripkean reading*, and an *ordinary reading*. In particular, according to the Kripkean reading, the following two arguments are valid, but according to the ordinary reading, these arguments are both invalid.

> the number of planets is 9;
> therefore, necessarily the number of planets is 9.

> Hesperus (the evening star) is Phosphorus (the morning star);
> therefore, necessarily Hesperus is Phosphorus.

The Kripkean readings, which render these arguments valid, are formalized as follows.

$$\text{N} = 9 \;/\; (\text{N}/x)\,(9/y)\,\Box\,[x=y]$$

$$\text{H} = \text{P} \;/\; (\text{H}/x)\,(\text{P}/y)\,\Box\,[x=y]$$

Both of these arguments are provable in our system, provided we add the unstated premises about the existence of all the entities in question, and we add one more rule of inference. For example, consider proving the last one.

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[8] Notice that σ and τ *may* also occur inside Φ, but these do not participate in the substitution.
[9] Saul Kripke, *Naming and Necessity*, Harvard University Press, 1980.

(1)	$\text{H} = \text{P}$	/0	Pr
(2)	$\exists x[x = \text{H}]$	/0	Pr
(3)	$\exists x[x = \text{P}]$	/0	Pr
(4)	~~SHOW~~: $(\text{H}/x)(\text{P}/y)\square[x=y]$	/0	Def (τ/v)
(5)	~~SHOW~~: $\exists x\{x=\text{H}\ \&\ (\text{P}/y)\square[x=y]\}$	/0	6,8,QL
(6)	$a = \text{H}$	/0	2,∃O
(7)	$b = \text{P}$	/0	4,∃O
(8)	~~SHOW~~: $(\text{P}/y)\square[a=y]$	/0	Def (τ/v)
(9)	~~SHOW~~: $\exists y\{y=\text{P}\ \&\ \square[a=y]\}$	/0	DD
(10)	~~SHOW~~: $\square[a=b]$	/0	ND
(11)	~~SHOW~~: $a=b$	/01	DD
(12)	$a=\text{P}$	/0	1,6,LL
(13)	$a=b$	/0	7,12,Eq(=)
(14)	$a=b$	/01	==Rep==
(15)	$b=\text{P}\ \&\ \square[a=b]$	/0	5,10,SL
(16)	$\exists y\{y=\text{P}\ \&\ \square[a=y]\}$	/0	15,∃I
(17)	$a=\text{H}\ \&\ (\text{P}/y)\square[a=y]$	/0	3,8,SL

What we have shown is — supposing Hesperus and Phosphorus are one and the same (and exist), the thing that is Hesperus is necessarily identical to the thing that is Phosphorus; in other words, that thing (presumably the planet Venus) is necessarily identical to itself.

Note the appearance of a new rule of derivation – *Identity-Repetition* – which is formally presented as follows.

=Rep	
$a = b$	/i
$a = b$	/j (old)
a and b are ***constants***.	

Identity-Repetition claims that identity assertions involving constants are ***absolute***, which is to say that constants are constant. Note carefully, however, that Identity-Repetition is restricted to *constants*, which formally encodes the following semantic principle.

constants *rigidly* denote their referents, but no other singular-terms do so.[10]

So, although we have shown the *de re* necessary identity of Hesperus and Phosphorus, we have not shown the *de dicto* necessity of the proposition ⟨Hesperus is Phosphorus⟩. In the proposed system, this is attempted in the following incomplete derivation.

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[10] We take variables to be non-referential, but to be purely anaphoric. But, insofar as they are referential, they are also rigidly referential.

(1)	$H = P$	/0	Pr
(2)	$\exists x[x = H]$	/0	Pr
(3)	$\exists x[x = P]$	/0	Pr
(4)	~~SHOW~~: $\Box[H = P]$	/0	Def (τ/v)
(5)	SHOW: $H = P$	/01	
(6)	$a = H$	/0	2,$\exists$O
(7)	$b = P$	/0	4,$\exists$O
(8)	$a = b$	/0	1,6,7,Eq(=)
(9)	$a = b$	/01	8,=Rep

Remember! The simple singular-terms 'H' and 'P' are proper nouns, not constants, so we are not entitled to apply =Rep to line (1).

So, it seems that the question whether Hesperus is necessarily Phosphorus is ambiguous between whether we are speaking of things – is there one thing, or two things? Or, we are speaking of propositions – is the proposition ⟨Hesperus=Phosphorus⟩ necessary?

By way of explaining how this is implemented in the semantics, we adopt a metaphor according to which we distinguish between "actors" and "roles". In particular, a proper noun can refer either to the actor or to the role. For example, in contexts in which we are actually talking about acting (playing a role in a performance), we can easily understand the following identity claims.

last night, Olivier was Hamlet, but tonight Gibson will be Hamlet

We describe this formally by saying that 'Olivier' and 'Gibson' refer to actors, and 'Hamlet' refers to a role.

The key in the proposed logic (semantics) is summarized in the following.

> variables and constants refer exclusively to actors,
> proper nouns *fundamentally* refer to roles,
> although they can be *scopally-manipulated* to refer to actors.

As a result of this semantic distinction, we have that

> some identities are necessary, and some identities are contingent.

As we pursue it, modal logic syntactically implements this difference by distinguishing between wide and narrow scope for the associated singular-terms. In particular, whenever singular-terms have wide scope, any true identity involving them is also necessary, and any false identity is impossible. On the other hand, whenever singular-terms have narrow scope, which is to say they appear in their natural grammatical position, some true identities are necessary, but some are not.

7.   *De Se* **Belief**

A number of prominent philosophers, including David Lewis[11] and Hector Castañeda[12], have discussed an odd problem about beliefs in reference to oneself, which are called *de se beliefs*.

By way of illustration, consider a situation in which Jay witnesses a murder. For the sake of argument, let us suppose Jay witnesses the murder in the same way that many of us witnessed the murder of Lee Harvey Oswald by Jack Ruby. Some of us, including the present author, saw it *live* on TV; others saw it on the evening news that same day; still others have seen it in documentaries. Given this sense of 'see', we have the following *de re* claim.

(1)   Jay believes of the person he saw commit the murder that **that person** is the murderer

$(\text{S}/x)\ [\![\text{J}\mathbb{B}]\!]\ [x = \text{M}]$

Here 'S' abbreviates 'the person Jay saw commit the murder', and 'M' abbreviates 'the murderer'. Their internal structure is unimportant here, so we simply treat them as atomic expressions.

Let us furthermore suppose that Jay <u>is</u> the person he saw commit the murder! It isn't often that we see ourselves on the evening news. Thus, we have the following additional claim.

(2)   Jay <u>is</u> the person Jay saw commit the murder

$\text{J} = \text{S}$

From (1) and (2), using the scoped-term version of Leibniz's law, we have the following.

(3)   Jay believes **of Jay** that **he** is the murderer

$(\text{J}/x)\ [\![\text{J}\mathbb{B}]\!]\ [\text{J} = \text{M}]$

Now things get interesting! Let us further suppose that Jay suffers from an odd malady in virtue of which he leads a double life of which he is unaware. In this case, we can suppose that Jay doesn't know about his other guise, and he doesn't recognize his other guise on TV to be himself. So, although Jay believes *of himself* that he is the murderer, he does not believe *of himself* that he-**himself** is the murderer! What belief is he missing? Well, it seems that he does not recognize the person on TV to be himself. In other words, it seems that Jay does not believe himself to be Jay!

This sounds like the yacht that might be bigger than itself. But, just as with the yacht example, we have to be careful with the scope locations of the various noun phrases. If we symbolize it properly, we get the following.

$\sim (\text{J}/x)\ [\![\text{J}\mathbb{B}]\!]\ [x=\text{J}]$

This has one *de re* occurrence of 'Jay' and one *de dicto* occurrence.[13]   This mixed usage of 'Jay' should not be confused with the following pure *de dicto* formula,

$\sim [\![\text{J}\mathbb{B}]\!]\ [\text{J} = \text{J}]$

or the following pure *de re* formulas.

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[11] David Lewis, "Attitudes de dicto and de se", Philosophical Review, 88: 513-543.
[12] H.N. Castañeda, (1966) "'He' - A study in the logic of self-consciousness", *Ratio*, 7: 130-157.
[13] In addition to the original occurrence in '$[\![j\mathbb{B}]\!]$', which is not important here.

$\sim$ (J/x) $[\![J\mathbb{B}]\!]$ [x=x]

$\sim$ (J/x) (J/y) $[\![j\mathbb{B}]\!]$ [x=y]

which are all logically false. For example, the following are the respective derivations of the opposites. We add the presupposed premise that Jay exists in the two *de re* examples.

(1)	~~SHOW~~: $[\![J\mathbb{B}]\!][J = J]$	/0	$[\![J\mathbb{B}]\!]$D
(2)	~~SHOW~~: J = J	/01	Ref=

(1)	$\exists x[x=J]$	/0	Pr
(2)	~~SHOW~~: (J/x)$[\![J\mathbb{B}]\!][x=x]$	/0	Def (τ/v)
(3)	~~SHOW~~: $\exists x\{x=J$ & $[\![J\mathbb{B}]\!][x=x]\}$	/0	4,5,QL
(4)	$\quad a=J$	/0	1,$\exists$O
(5)	$\quad$ ~~SHOW~~: $[\![J\mathbb{B}]\!][a=a]$	/0	$[\![J\mathbb{B}]\!]$D
(6)	$\quad$ ~~SHOW~~: $a=a$	/01	Ref=

(1)	$\exists x[x=J]$	/0	Pr
(2)	~~SHOW~~: (J/x)(J/y)$[\![J\mathbb{B}]\!][x=y]$	/0	Def (τ/v)
(3)	~~SHOW~~: $\exists x\{x=J$ & (J/y)$[\![J\mathbb{B}]\!][x=y]\}$	/0	4,5,QL
(4)	$\quad a=j$	/0	1,$\exists$O
(5)	$\quad$ ~~SHOW~~: $[\![(J/y)\mathbb{B}]\!][a=y]$	/0	Def (τ/v)
(6)	$\quad$ ~~SHOW~~: $\exists y\{y=J$ & $[\![J\mathbb{B}]\!][a=y]\}$	/0	4,7,QL
(7)	$\quad\quad$ ~~SHOW~~: $[\![J\mathbb{B}]\!][a=a]$	/0	$[\![J\mathbb{B}]\!]$D
(8)	$\quad\quad$ ~~SHOW~~: $a=a$	/01	Ref=

Recall that '$[\![J\mathbb{B}]\!]$' is a box-modality; so to show that Jay believes that P, we must show that P holds in every Jay-belief-accessible world.

Notice what happens when we try to prove the corresponding *de re* formula, where we include the obvious presupposition that Jay exists.

(1)	$\exists x[x=J]$	/0	Pr
(2)	SHOW: (J/x)$[\![J\mathbb{B}]\!][x=J]$	/0	Def (τ/v)
(3)	SHOW: $\exists x\{x=J$ & $[\![J\mathbb{B}]\!][x=J]\}$	/0	4,5,QL
(4)	$\quad a=J$	/0	1,$\exists$O
(5)	$\quad$ SHOW: $[\![J\mathbb{B}]\!][a=J]$	/0	$[\![J\mathbb{B}]\!]$D
(6)	$\quad$ SHOW: $a=J$	/01	??
(7)	$\quad$???		

Basically, what we need is to go from (4) to (6), but no modal inference allows this. Recall that whereas 'a' is a constant, introduced by $\exists$O, 'J' is a proper noun, so =Rep is impermissible here.

We have shown that it is not a matter of logic that Jay believes himself to be Jay (properly formulated). Perhaps it is a matter of fact. Well, given our premises, we can prove this is false, so Jay as a matter of fact does not believe himself to be Jay.

(1)	$(s/x)[\![J\mathbb{B}]\!][x = M]$	/0	Pr
(2)	$J = S$	/0	Pr
(3)	$\sim[\![J\mathbb{B}]\!][J = M]$	/0	Pr
(4)	SHOW: $\sim(J/x)[\![J\mathbb{B}]\!][x{=}J]$	/0	ID
(5)	$(J/x)[\![J\mathbb{B}]\!][x{=}J]$	/0	As
(6)	SHOW: ✕	/✶	DD
(7)	$\exists x\{x{=}J\ \&\ [\![J\mathbb{B}]\!][x{=}J]\}$	/0	5,Def (t/v)
(8)	$a{=}J\ \&\ [\![J\mathbb{B}]\!][a{=}J]$	/0	7,∃O
(9)	$(J/x)[\![J\mathbb{B}]\!][x = m]$	/0	1,2,LL(t/v)
(10)	$\exists x\{x{=}J\ \&\ [\![J\mathbb{B}]\!][x{=}M]\}$	/0	9,Def (t/v)
(11)	$b{=}J\ \&\ [\![J\mathbb{B}]\!][b{=}M]$	/0	10,∃O
(12)	$a{=}b$	/0	8a,11a,LL
(13)	$J{\neq}M$	/01	3,$\sim[\![J\mathbb{B}]\!]$O
(14)	$a{=}J$	/01	8b,$[\![J\mathbb{B}]\!]$O
(15)	$b{=}M$	/01	11b,$[\![J\mathbb{B}]\!]$O
(16)	$a{=}b$	/01	12,=Rep
(17)	$J{=}M$	/01	14-16,IL
(18)	✕	/✶	13,17,SL

We can actually prove something a bit stronger,

> Jay believes of himself that he is not Jay
>
> $(J/x)\,[\![J\mathbb{B}]\!]\,[x{\neq}J]$

using a correspondingly stronger premise:

> Jay believes that Jay is not the murderer
>
> $[\![J\mathbb{B}]\!]\,[J{\neq}M]$

The following is the derivation.

(1)	$(s/x)[\![J\mathbb{B}]\!][x = M]$	/0	Pr
(2)	$J = s$	/0	Pr
(3)	$[\![J\mathbb{B}]\!][J \neq m]$	/0	Pr
(4)	SHOW: $(J/x)[\![J\mathbb{B}]\!][x \neq J]$	/0	Def (τ/v)
(5)	SHOW: $\exists x\{x{=}J\ \&\ [\![J\mathbb{B}]\!][x{\neq}J]\}$	/0	8a,9,QL
(6)	$(J/x)[\![J\mathbb{B}]\!][x = m]$	/0	1,2,LL(τ/v)
(7)	$\exists x\{x{=}J\ \&\ [\![J\mathbb{B}]\!][x{=}m]\}$	/0	6,Def (τ/v)
(8)	$a{=}J\ \&\ [\![a\mathbb{B}]\!][a{=}m]$	/0	7,∃O
(9)	SHOW: $[\![J\mathbb{B}]\!][a{\neq}J]$	/0	$[\![j\mathbb{B}]\!]$D
(10)	SHOW: $a \neq J$	/01	ID
(11)	$J \neq m$	/01	3,$[\![J\mathbb{B}]\!]$O
(12)	$a = m$	/01	8b,$[\![J\mathbb{B}]\!]$O
(13)	$a \neq J$	/01	11,12,IL

At this point, we propose our official formal account of *de se* belief.

> α believes of himself that he/himself is F
>
> =: $(J/x)\,[\![J\mathbb{B}]\!]\,\{x{=}J\ \&\ Fx\}$
>
> =:   Jay believes of Jay that he is Jay and he is F

By way of concluding this section, we prove that every *de se* belief implies both the corresponding *de re* belief and the corresponding *de dicto* belief, and we show how the converse implication does not hold, by constructing an incomplete derivation.

(1)	$(J/x)⟦J𝔹⟧\{x=J \ \& \ Fx\}$	/0	Pr
(2)	~~SHOW~~: $(J/x)⟦J𝔹⟧Fx$	/0	Def (τ/v)
(3)	~~SHOW~~: $\exists x\{x=J \ \& \ ⟦J𝔹⟧Fx\}$	/0	5a,6,QL
(4)	$\exists x\{x=J \ \& \ ⟦J𝔹⟧[x=J \ \& \ Fx]\}$	/0	1, Def (τ/v)
(5)	$a=J \ \& \ ⟦J𝔹⟧[a=J \ \& \ Fa]$	/0	4,∃O
(6)	~~SHOW~~: $⟦J𝔹⟧Fa$	/0	⟦J𝔹⟧D
(7)	~~SHOW~~: Fa	/01	8,SL
(8)	$a=J \ \& \ Fa$	/01	5b,⟦J𝔹⟧O

(1)	$(J/x)⟦J𝔹⟧\{x=J \ \& \ Fx\}$	/0	Pr
(2)	~~SHOW~~: $⟦J𝔹⟧FJ$	/0	⟦J𝔹⟧D
(3)	~~SHOW~~: FJ	/01	
(4)	$\exists x\{x=J \ \& \ ⟦J𝔹⟧[x=J \ \& \ Fx]\}$	/0	1, Def (τ/v)
(5)	$a=J \ \& \ ⟦J𝔹⟧[a=J \ \& \ Fa]$	/0	4,∃O
(6)	$a=J \ \& \ Fa$	/01	5b,⟦J𝔹⟧O
(7)	FJ	/01	6a,6b,LL

(1)	$(J/x)⟦J𝔹⟧Fx$	/0	Pr
(2)	$⟦J𝔹⟧FJ$	/0	Pr
(3)	SHOW: $(J/x)⟦J𝔹⟧\{x=J \ \& \ Fx\}$	/0	Def (τ/v)
(4)	SHOW: $\exists x\{x=J \ \& ⟦J𝔹⟧[x=J \ \& \ Fx]\}$	/0	6a,7,QL
(5)	$\exists x\{x=J \ \& \ ⟦J𝔹⟧Fx\}$	/0	1,Def (τ/v)
(6)	$a=J \ \& \ ⟦J𝔹⟧Fa$	/0	5,∃O
(7)	SHOW: $⟦J𝔹⟧[a=J \ \& \ Fa]$	/0	⟦J𝔹⟧D
(8)	SHOW: $a=J \ \& \ Fa$	/01	
(9)	Fa	/01	6b,⟦J𝔹⟧O
(10)	$a=J$?????	/01	✗✗✗✗

> In other words, to construct a counter-model for this argument form, we need to make sure that the referent of 'J' is different in world 0 and world 01.

8.   Recognition

Using our actor/role metaphor, it seems that *de se* belief involves *de re* belief **plus** recognizing the identity between the actor (*res*) one is, and the role one is playing (say, on TV!), or to use Castañeda's term, one's *guise*. For example, one must recognize that the person on TV (one's *guise*) is oneself (one's *res*). The metaphysical/epistemological distinction between *res* and *guise* is reflected in the formal language in a fairly simple manner. On the one hand, there are *logical* noun phrases – i.e., constants and variables – which refer to actors both directly and rigidly. On the other hand, there are non-logical noun phrases – i.e., all other noun phrases – which refer to roles directly, and to actors indirectly and non-rigidly.

Now, the problem of *de se* belief is the problem of recognizing when a *guise* is a *res*. So the problem of *de se* belief is a special case of the general problem of recognition. For example, consider the following sentence.

I didn't recognize you

Presumably, the first person *sees* the second person, but does not *recognize* him/her. To recognize someone is to identify a *guise* and a *res* – to realize/recognize/believe that the person <u>is</u> the person. Oh

no – the yacht again! Clearly, the two instances of 'the person' are asymmetrical; otherwise, we have a triviality. The following are the pure *de re* readings [where I = 'I', and 'U' = 'you'].

$$(\text{U}/x) \sim [\![\text{I}\mathbb{B}]\!] \, [x=x]$$
$$(\text{U}/x) \, (\text{U}/\textsf{y}) \sim [\![\text{I}\mathbb{B}]\!] \, [x=\textsf{y}]$$

And the following is the pure *de dicto* reading.

$$\sim [\![\text{I}\mathbb{B}]\!] \, [\text{U}=\text{U}]$$

None of these is satisfactory because they are all logically false, granted that 'U' is referentially proper.

That leaves us with the mixed (*de re/de dicto*) reading.

$$(\text{U}/x) \sim [\![\text{I}\mathbb{B}]\!] \, [x=\text{U}]$$

According to this formulation, a recognition statement is a belief statements about the "real" identity of a guise – alternatively, what actor is playing what role – which is formulated using a combination of wide-scope and narrow-scope noun phrases.

Now, compare this formula with the "missing information" in the Jay/murderer example.

$$(\text{J}/x) \sim [\![\text{J}\mathbb{B}]\!] \, [x=\text{J}]$$

We see that the form of the latter is a special case of the form of the former. The difference merely concerns who is recognizing whom.

9.   Asymmetrical Identity

The examples above suggest that a given object can figure in the same sentence both as a *guise* and as a *res*. Formally speaking, a noun phrase can occur both as a wide-scope (*de re*) expression, denoting a *res*, and as a narrow-scope (*de dicto*) expression, denoting a *guise*.

The asymmetry between *guise* and *res* can be used in another kind of sentence that has perplexed a few philosophers and linguists. Consider the following, where Φ is any sentence phrase you wish.

if I were you, then Φ

First, we note that 'were' is the subjunctive form of 'be'. Since 'I' and 'you' are both singular-terms, this is transitive-'be', also known as the 'is' of identity.

Let us first consider the pure *de re* readings of 'I' and 'you'.

$$(\text{I}/x) \, (\text{U}/\textsf{y}) \, \Box_{\copyright} \, \{x=\textsf{y} \rightarrow \Phi\}$$

The '$\Box_{\copyright}$' here is a "contextual" necessity operator, appropriate to the particular sentences involved in the subjunctive conditional.[14] It is used here primarily to make the modal scope completely clear.

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[14] The basic idea traces to David Lewis (Counterfactuals, 1973 [Blackwell] and 1986 [Harvard University Press]), according to whom, in evaluating the truth-value of a subjunctive conditional 'if it were the case that $\mathcal{A}$, then it would be the case that C', one considers wider and wider possibility-neighborhoods surrounding the world of evaluation, until one finds a

What are some examples of Φ that might be true? The following seems plausible.

$(\text{I}/x)\ (\text{U}/y)\ \Box_{©}\ \{x{=}y \rightarrow y{=}x\}$

if I were you, then you would be me.

This is logically true, granting the propriety of 'I' and 'U'. So is the following, granting 'I≠U'.

$(\text{I}/x)\ (\text{U}/y)\ \Box_{©}\ \{x{=}y \rightarrow (\#(P)/x)[\#(P) = x{-}1]\}$

if I were you, then there would be one less person

What about the corresponding pure *de dicto* readings of 'I' and 'you'? The general form is given as follows.

$\Box_{©}\ \{\text{I}{=}\text{U} \rightarrow \Phi\}$

So the following are the corresponding pure *de dicto* formulations of our earlier instances.

$\Box_{©}\ \{\text{I}{=}\text{U} \rightarrow \text{U}{=}\text{I}\}$

$(\#(P)/x)\ \Box_{©}\ \{\ \text{I}{=}\text{U} \rightarrow [\#(P) = x{-}1]\ \}$

These are also logically true, granted the plausible presuppositions.

Now, the problem obviously is that neither of the pure readings,

$(\text{I}/x)\ (\text{U}/y)\ \Box_{©}\ \{x{=}y \rightarrow y{=}x\}$

$\Box_{©}\ \{\text{I}{=}\text{U} \rightarrow \Phi\}$

has anything to do with what we mean when we say 'if I were you...'. In particular, it seems fairly clear that

       if I were you, then Φ

and   if you were me, then Φ

are not logically equivalent. Yet the corresponding formulas,

$(\text{I}/x)\ (\text{U}/y)\ \Box_{©}\ \{x{=}y \rightarrow \Phi\}$	$(\text{U}/x)\ (\text{I}/y)\ \Box_{©}\ \{x{=}y \rightarrow \Phi\}$
$\Box_{©}\ \{\text{I}{=}\text{U} \rightarrow \Phi\}$	$\Box_{©}\ \{\text{U}{=}\text{I} \rightarrow \Phi\}$

are logically equivalent, respectively.

It seems clear that a non-symmetrical account reading of 'a were b' is needed in order to account for the difference in the imagined circumstances of these two sentences. This means we need to concoct some sort of asymmetrical identity predicate, or we need to provide a non-symmetrical account of the subject and predicate nominative.   Since fiddling with a logical symbol, such as identity, is foolhardy, we take the second route.

In fact, we propose formally analyzing these two sentences as follows.

---

neighborhood in which the antecedent $\mathcal{A}$ is possible (i.e., true somewhere) in the neighborhood). One then checks whether the ordinary conditional $\mathcal{A}{\rightarrow}C$ is necessary (i.e., true everywhere) in this neighborhood.

if I were you, then Φ; $(\text{I}/x) \; \Box_{\copyright} \; \{x = \text{U} \rightarrow \Phi\}$

if you were me, then Φ; $(\text{U}/x) \; \Box_{\copyright} \; \{x = \text{I} \rightarrow \Phi\}$

In other words, we provide a wide-scope reading of the subject, and we provide a narrow-scope reading of the predicate nominative. Or, to use our standing metaphor, we treat the subject as an actor/*res*, and we treat the predicate nominative as a role/guise.

Yet another way to describe the asymmetry between 'I were you' and 'you were me' is to say that, when we consider an alternative world, there are two possibilities worth considering.

(1)   me-the-actor playing you-the-role
(2)   you-the-actor playing me-the-role

10.   Derivation Exercises

Directions: for each of the following argument forms, construct a formal derivation of the conclusion (marked by '/') from the premises (if any) in system QML(k). In cases in which two formulas are separated by '//', construct a derivation of each formula from the other. In what follows, any atomic singular-term is a proper noun, and not a constant. Constants are intra-derivational devices, referred to extensively in the rules.

1. $/ \forall x \forall y (x=y \rightarrow \Box[x=y])$
2. $/ \forall x \forall y (x \neq y \rightarrow \Box[x \neq y])$
3. $/ \forall x \forall y \{x \neq y \rightarrow \sim\Diamond[x=y]\}$
4. $/ \forall x \forall y (\Diamond[x=y] \rightarrow x=y)$
5. $/ \forall x \forall y (\Diamond[x \neq y] \rightarrow [x \neq y])$
6. $/ \forall x \forall y (x=y \rightarrow \Box(Fx \leftrightarrow Fy))$
7. $/ \forall x \forall y (x=y \rightarrow (\Box Fx \leftrightarrow \Box Fy))$
8. $/ \forall x \Box[x=x]$
9. $/ \forall x \Box[f(x) = f(x)]$
10. $/ \forall x \Box \exists y[x=y]$
11. $/ \forall x \exists y \Box[x=y]$
12. $/ \forall x \forall y (x=y \rightarrow \forall z \Box(Rxz \leftrightarrow Ryz))$
13. $/ \forall x \forall y (x=y \rightarrow \Box\Box[x=y])$
14. $/ \forall x \forall y (\Diamond\Diamond[x=y] \rightarrow \Box\Box[x=y])$
15. $\exists x[x=N]$
       $/ \forall x \{[x \neq N] \rightarrow (N/y) \sim\Diamond[x=y]\}$
16. $/ \forall x (Fx \rightarrow \exists x Fx)$
17. $FN \; ; \; \exists x[x=N] / \exists x Fx$
18. $FN \; ; \; \exists'x[x=N] / \exists'x Fx$
19. $/ \forall x \{\Box Fx \rightarrow \exists x \Box Fx\}$

20. $\forall x Fx \; ; \; \exists x[x=N] / (N/x)Fx$
21. $\forall x \Diamond Fx \; ; \; \exists x[x=N] / (N/x)\Diamond Fx$
22. $(N/x)Fx / \exists x Fx$
23. $/ \forall x \{\Box Fx \rightarrow \exists x \Box Fx\}$
24. $\forall'x \Box Fx \; ; \; \exists'x[x=N] / (N/x)\Box Fx$
25. $\Box \exists'x[x=\imath xFx] / \Box \exists'x \forall y(Fy \leftrightarrow y=x)$
26. $\sim\Diamond \exists'x Fx / \Box \forall'x[x \neq \imath xFx]$
27. $/ \forall x \{\Box Fx \leftrightarrow (x/y)\Box Fy)$
28. $\Box FN \; ; \; \Box \exists x[x=N] / \Box(N/x)Fx$
29. $M=N / (M/x)\Box Fx \leftrightarrow (N/x)\Box Fx$
30. $\exists x[x=N] \; \& \; FN // (N/x)Fx$
31. $(\imath xFx/y)Gy // \exists x(\forall y(Fy \leftrightarrow y=x) \; \& \; Gx)$
32. $\Box FN \; ; \; (N/x)\Box[x=N] \; / \; (N/x)\Box Fx$
33. $(\imath xFx/y)\Box[y=\imath xGx] \; ; \; \sim\Box[\imath xFx=\imath xGx]$
       $/ (\imath xFx/y)\Diamond[y \neq \imath xFx]$
34. $\forall'x \Box(Fx \rightarrow \exists'y[x=y]) \; ; \; \exists'x \Diamond Fx$
       $/ \Diamond \exists'xFx$
35. $(\imath xFx/y)\Box Gy / \exists x \{Fx \; \& \; \Box Gx\}$
36. $\Diamond FN \; ; \; (N/x)\Box[x=N] / (N/x)\Diamond Fx$
37. $(\imath xFx/y)\Diamond[y=\imath xGx]$
       $\sim\Diamond[\imath xFx = \imath xGx]$
       $/ (\imath xFx/y)\Diamond[y \neq \imath xFx]$

11.   Counter-Model Exercises

For each of the following, construct a "counter-model" in the following sense – construct a full but incomplete derivation, and explain why it cannot be finished. By a full derivation is meant a derivation in which all relevant rules are applied to all relevant lines.

1. $FN / \exists x Fx$
2. $\Box FN / \exists x \Box Fx$
3. $\Box FN \; ; \; \exists x[x=N] / \exists x \Box Fx$
4. $\Box FN \; ; \; \exists'x[x=N] / \exists'x \Box Fx$
5. $\Box FN \; ; \; \exists'x[x=N] / \Box \exists'x Fx$
6. $\forall x \Diamond Fx \; ; \; \exists x[x=N] / \Diamond FN$
7. $\forall x Fx / (N/x)Fx$
8. $/ \Box FN \rightarrow (N/x)\Box Fx$
9. $/ (N/x)\Box Fx \rightarrow \Box FN$
10. $/ M=N \rightarrow \Box[M=N]$
11. $\Box FN \; ; \; \exists x[x=N] / \Box \exists x Fx$
12. $\Box FN \; ; \; \exists'x[x=N] \; / \; \Box \exists x Fx$

12.   Answers to Selected Exercises

1.   Derivations

#1:

(1)	SHOW: $\forall x\forall y(x=y \to \Box[x=y])$	/0	U2CD
(2)	$a=b$	/0	As
(3)	SHOW: $\Box[a=b]$	/0	ND
(4)	SHOW: $a=b$	/01	2,=Rep

#2:

(1)	SHOW: $\forall x\forall y(x\neq y \to \Box[x\neq y])$	/0	U2CD
(2)	$a\neq b$	/0	As
(3)	SHOW: $\Box[a\neq b]$	/0	ND
(4)	SHOW: $a\neq b$	/01	2,=Rep(−)

#3:

(1)	SHOW: $\forall x\forall y\{x\neq y \to \sim\Diamond[x=y]\}$	/0	U2CD
(2)	$a\neq b$	/0	As.
(3)	SHOW: $\sim\Diamond[a=b]$	/0	ID
(4)	$\Diamond[a=b]$	/0	As
(5)	SHOW: ✕	/✶	DD
(6)	$a=b$	/01	4,$\Diamond$O
(7)	$a=b$	/0	6,=Rep
(8)	✕	/✶	2,7,SL

#4:

(1)	SHOW: $\forall x\forall y\{\Diamond[x=y] \to x=y\}$	/0	U2CD
(2)	$\Diamond[a=b]$	/0	As
(3)	SHOW: $a=b$	/0	4,=Rep
(4)	$a=b$	/01	2,$\Diamond$O

#5:

(1)	SHOW: $\forall x\forall y\{\Diamond[x\neq y] \to x\neq y\}$	/0	U2CD
(2)	$\Diamond[a\neq b]$	/0	As
(3)	SHOW: $a\neq b$	/0	4,=Rep(−)
(4)	$a\neq b$	/01	2,$\Diamond$O

#6:

(1)	SHOW: $\forall x\forall y\{x=y \to \Box(Fx \leftrightarrow Fy)\}$	/0	U2CD
(2)	$a=b$	/0	As
(3)	SHOW: $\Box(Fa \leftrightarrow Fb)$	/0	ND
(4)	SHOW: $Fa \leftrightarrow Fb$	/01	DD
(5)	$a=b$	/01	2,=Rep
(6)	$Fa \leftrightarrow Fa$	/01	SL
(7)	$Fa \leftrightarrow Fb$	/01	5,6,LL

#7:

(1)	SHOW: $\forall x\forall y\{x=y \to (\Box Fx \leftrightarrow \Box Fy)\}$	/0	U2CD
(2)	$a=b$	/0	As
(3)	SHOW: $\Box Fa \leftrightarrow \Box Fb$	/0	2,4,LL
(4)	$\Box Fa \leftrightarrow \Box Fa$	/0	SL

#8:

(1)	SHOW: $\forall x\Box[x=x]$	/0	UD
(2)	SHOW: $\Box[a=a]$	/0	ND
(3)	SHOW: $a=a$	/01	Ref=

#9:

(1)	SHOW: $\forall x\Box[f(x) = f(x)]$	/0	UD
(2)	SHOW: $\Box[f(a) = f(a)]$	/0	ND
(3)	SHOW: $f(a) = f(a)$	/01	DD
(4)	$f(a) = f(a)$	/01	Ref=

#10:

(1)	SHOW: $\forall x\Box\exists y[x=y]$	/0	UD
(2)	SHOW: $\Box\exists y[a=y]$	/0	ND
(3)	SHOW: $\exists y[a=y]$	/01	4,$\exists$I
(4)	$a=a$	/01	Ref=

#11:

(1)	SHOW: $\forall x\exists y\Box[x=y]$	/0	UD
(2)	SHOW: $\exists y\Box[a=y]$	/0	ID
(3)	$\sim\exists y\Box[a=y]$	/0	As
(4)	SHOW: ✕	/✶	6,7,SL
(5)	$\sim\Box[a=a]$	/0	3,$\sim\exists$O
(6)	$a\neq a$	/01	5,$\sim\Box$O
(7)	$a=a$	/01	Ref=

#12:

(1)	SHOW: $\forall x\forall y\{x=y \to \forall z\Box(Rxz \leftrightarrow Ryz)\}$	/0	U2CD
(2)	$a=b$	/0	As
(3)	SHOW: $\forall z\Box(Raz \leftrightarrow Rbz)$	/0	UD
(4)	SHOW: $\Box(Rac \leftrightarrow Rbc)$	/0	ND
(5)	SHOW: $Rac \leftrightarrow Rbc$	/01	DD
(6)	$Rac \leftrightarrow Rac$	/01	SL
(7)	$a=b$	/01	2,=Rep
(8)	$Rac \leftrightarrow Rbc$	/01	6,7,LL

#13:

(1)	SHOW: $\forall x\forall y\{x=y \to \Box\Box[x=y]\}$	/0	U2CD
(2)	$a=b$	/0	As
(3)	SHOW: $\Box\Box[a=b]$	/0	ND2
(4)	SHOW: $a=b$	/012	2,=Rep

#14:

(1)	SHOW: $\forall x\forall y\{\Diamond\Diamond[x=y] \to \Box\Box[x=y]\}$	/0	U2CD
(2)	$\Diamond\Diamond[a=b]$	/0	As
(3)	SHOW: $\Box\Box[a=b]$	/0	ND2
(4)	SHOW: $a=b$	/012	5,=Rep
(5)	$a=b$	/034	2,$\Diamond$O2

#15:

(1)	$\exists x[x=N]$	/0	Pr
(2)	SHOW: $\forall x\{[x\neq N] \to (N/y)\sim\Diamond[x=y]\}$	/0	UCD
(3)	$a\neq N$	/0	As
(4)	SHOW: $(N/y)\sim\Diamond[a=y]$	/0	Def (τ/ν)
(5)	SHOW: $\exists y\{y=N \ \& \sim\Diamond[a=y]\}$	/0	6,7, QL
(6)	$b=N$	/0	1,$\exists$O
(7)	SHOW: $\sim\Diamond[a=b]$	/0	ID
(8)	$\Diamond[a=b]$	/0	As
(9)	SHOW: ✕	/✶	3,12,SL
(10)	$a=b$	/01	8,$\Diamond$O
(11)	$a=b$	/0	10,=Rep
(12)	$a=N$	/0	6,11,LL

#16:

(1)	SHOW: $\forall x(Fx \to \exists xFx)$	/0	UCD
(2)	Fa	/0	As
(3)	SHOW: $\exists xFx$	/0	2,$\exists$I

#17:

(1)	FN	/0	Pr
(2)	∃x[x=N]	/0	Pr
(3)	SHOW: ∃xFx	/0	DD
(4)	a=N	/0	2,∃O
(5)	Fa	/0	1,4,LL
(6)	∃xFx	/0	5,∃I

#18:

(1)	FN	/0	Pr
(2)	∃′x[x=N]	/0	Pr
(3)	SHOW: ∃′xFx	/0	Def ∃′
(4)	SHOW: ∃x(Æx & Fx)	/0	8,∃I
(5)	∃x(Æx & x=N)	/0	2,Def ∃′
(6)	Æa & a=N	/0	5,∃O
(7)	Fa	/0	1,6b,LL
(8)	Æa & Fa	/0	6a,7,SL

#19:

(1)	SHOW: ∀x{□Fx → ∃x□Fx}	/0	UCD
(2)	□Fa	/0	As
(3)	SHOW: ∃x□Fx	/0	2,∃I

#20:

(1)	∀xFx	/0	Pr
(2)	∃x[x=N]	/0	Pr
(3)	SHOW: (N/x)Fx	/0	Def (τ/ν)
(4)	SHOW: ∃x{x=N & Fx}	/0	DD
(5)	a=N	/0	2,∃O
(6)	Fa	/0	1,∀O
(7)	a=N & Fa	/0	5,6,SL
(8)	∃x{x=N & Fx}	/0	7,∃I

#21:

(1)	∀x◇Fx	/0	Pr
(2)	∃x[x=N]	/0	Pr
(3)	SHOW: (N/x)◇Fx	/0	Def (τ/ν)
(4)	SHOW: ∃x(x=N & ◇Fx)	/0	DD
(5)	a=N	/0	2,∃O
(6)	◇Fa	/0	1,∀O
(7)	a=N & ◇Fa	/0	5,6,SL
(8)	∃x(x=N & ◇Fx)	/0	7,∃I

#22:

(1)	(N/x)Fx	/0	Pr
(2)	SHOW: ∃xFx	/0	DD
(3)	∃x(x=N & Fx)	/0	1,Def (τ/ν)
(4)	a=N & Fa	/0	3,∃O
(5)	∃xFx	/0	4b,∃I

#23:

(1)	SHOW: ∀x{□Fx → ∃x□Fx}	/0	UCD
(2)	□Fa	/0	As
(3)	SHOW: ∃x□Fx	/0	2,∃I

#24:

(1)	∀′x□Fx	/0	Pr
(2)	∃′x[x=N]	/0	Pr
(3)	SHOW: (N/x)□Fx	/0	Def (τ/ν)
(4)	SHOW: ∃x(x=N & □Fx)	/0	9,∃I
(5)	∀x(Æx → □Fx)	/0	1,Def ∀′
(6)	∃x(Æx & x=N)	/0	2,Def ∃′
(7)	Æa & a=N	/0	6,∃O
(8)	□Fa	/0	5,7a,QL
(9)	a=N & □Fa	/0	7b,8,SL

#25:

(1)	□∃′x[x=ιxFx]	/0	Pr
(2)	SHOW: □∃′x∀y(Fy ↔ y=x)	/0	ND
(3)	SHOW: ∃′x∀y(Fy ↔ y=x)	/01	Def ∃′
(4)	SHOW: ∃x{Æx & ∀y(Fy ↔ y=x)}	/01	10,∃I
(5)	∃′x[x=ιxFx]	/01	1,□O
(6)	∃x{Æx & x=ιxFx}	/01	5,Def ∃′
(7)	Æa & a=ιxFx	/01	6,∃O
(8)	∀x(Fx ↔ x=a)	/01	7b,ιO
(9)	∀y(Fy ↔ y=a)	/01	8,AV
(10)	Æa & ∀y(Fy ↔ y=a)	/01	7a,9,SL

#26:

(1)	~◇∃′xFx	/0	Pr
(2)	SHOW: □∀′x[x≠ιxFx]	/0	ND
(3)	SHOW: ∀′x[x≠ιxFx]	/01	Def ∀′
(4)	SHOW: ∀x{Æx → x≠ιxFx}	/01	UCD
(5)	Æa	/01	As
(6)	SHOW: a≠ιxFx	/01	ID
(7)	a=ιxFx	/01	As
(8)	SHOW: ※	/✳	12,13,SL
(9)	~∃′xFx	/01	1,~◇O
(10)	~∃x(Æx & Fx)	/01	9,Def ∃′(−)
(11)	~(Æa & Fa)	/01	10,~∃O
(12)	~Fa	/01	5,11,SL
(13)	∀x{Fx ↔ x=a}	/01	7,ιO
(14)	Fa	/01	13,OO

#27:

(1)	SHOW: ∀x{□Fx ↔ (x/y)□Fy}	/0	UD
(2)	SHOW: □Fa ↔ (a/y)□Fy	/0	3,9,SL
(3)	SHOW: □Fa → (a/y)□Fy	/0	CD
(4)	□Fa	/0	As
(5)	SHOW: (a/y)□Fy	/0	Def (τ/ν)
(6)	SHOW: ∃y{y=a & □Fy}	/0	8,∃I
(7)	a=a	/0	Ref=
(8)	a=a & □Fa	/0	4,7,SL
(9)	SHOW: (a/y)□Fy → □Fa	/0	CD
(10)	(a/y)□Fy	/0	As
(11)	SHOW: □Fa	/0	DD
(12)	∃y{y=a & □Fy}	/010,Def (τ/ν)
(13)	b=a & □Fb	/0	12,∃O
(14)	□Fa	/013a,13b,LL

#28:

(1)	□FN	/0	Pr
(2)	□∃x[x=N]	/0	Pr
(3)	SHOW: □(N/x)Fx	/0	ND
(4)	SHOW: (N/x)Fx	/01	Def (τ/ν)
(5)	SHOW: ∃x{x=t & Fx}	/01	10,∃I
(6)	FN	/01	1,□O
(7)	∃x[x=t]	/01	2,□O
(8)	a=N	/01	7,∃O
(9)	Fa	/01	6,8,LL
(10)	a=N & Fa	/01	8,9,SL

#29:

(1)	M=N	/0	Pr
(2)	SHOW: (M/x)□Fx ↔ (N/x)□Fx	/0	3,11,SL
(3)	SHOW: (M/x)□Fx → (N/x)□Fx	/0	CD
(4)	(M/x)□Fx	/0	As
(5)	SHOW: (N/x)□Fx	/0	Def (τ/v)
(6)	SHOW: ∃x(x=N & □Fx)	/0	DD
(7)	∃x(x=M & □Fx)	/04, Def (τ/v)	
(8)	a=M & □Fa	/0	7,∃O
(9)	a=N	/0	1,8a,LL
(10)	a=N & □Fa	/0	8b,9,SL
(11)	∃x(x=N & □Fx)	/0	9,∃I
(12)	SHOW: (N/x)□Fx → (M/x)□Fx	/0	CD
(13)	obtained from (4)-(11) *mutatis mutandis*		

#30a:

(1)	∃x[x=N] & FN	/0	Pr
(2)	SHOW: (N/x)Fx	/0	Def (τ/v)
(3)	SHOW: ∃x{x=N & Fx}	/0	6,∃I
(4)	a=N	/0	1a,∃O
(5)	Fa	/0	1b,4,LL
(6)	a=N & Fa	/0	4,5,SL

#30b:

(1)	(N/x)Fx	/0	Pr
(2)	SHOW: ∃x[x=N] & FN	/0	5,6,SL
(3)	∃x{x=N & Fx}	/0 1,Def (τ/v)	
(4)	a=N & Fa	/0	3,∃O
(5)	∃x[x=N]	/0	4a,∃I
(6)	FN	/0	4a,4b,LL

#31a:

(1)	(ιxFx/y)Gy	/0	Pr.
(2)	SHOW: ∃x{∀y(Fy ↔ y=x) & Gx}	/0	DD
(3)	∃y(y=ιxFx & Gy)	/01, Def (τ/v)	
(4)	a=ιxFx & Ga	/0	3,∃O
(5)	∀x(Fx ↔ x=a)	/0	4a,ιO
(6)	∀y(Fy ↔ y=a)	/0	5,AV
(7)	∀y(Fy ↔ y=a) & Ga	/0	4b,6,SL
(8)	∃x{∀y(Fy ↔ y=x) & Gx}	/0	7,∃I

#31b:

(1)	∃x{∀y(Fy ↔ y=x} & Gx}	/0	Pr
(2)	SHOW: (ιxFx/y)Gy	/0	Def (τ/v)
(3)	SHOW: ∃y{y=ιxFx & Gy}	/0	7,∃I
(4)	∀y(Fy ↔ y=a) & Ga	/0	1,∃O
(5)	a=ιyFy	/0	4a,ιI
(6)	a=ιyFy & Ga	/0	4b,5,SL
(7)	a=ιxFx & Ga	/0	6,AV

#32:

(1)	□FN	/0	Pr
(2)	(N/x)□[x=N]	/0	Pr
(3)	SHOW: (N/x)□Fx	/0	Def (τ/v)
(4)	SHOW: ∃x(x=N & □Fx)	/0	12,∃I
(5)	∃x(x=N & □[x=N])	/02, Def (τ/v)	
(6)	a=N & □[a=N]	/0	5,∃O
(7)	SHOW: □Fa	/0	ND
(8)	SHOW: Fa	/01	DD
(9)	FN	/01	1,□O
(10)	a=N	/01	6b,□O
(11)	Fa	/01	9,10,LL
(12)	a=N & □Fa	/0	6a,7,SL

#33:

(1)	(ιxFx/y)□[y=ιxGx]	/0	Pr
(2)	~□[ιxFx = ιxGx]	/0	Pr
(3)	SHOW: (ιxFx/y)◇[y ≠ ιxFx]	/0	Def (τ/v)
(4)	SHOW: ∃y(y=ιxFx & ◇[y ≠ ιxFx])	/0	11,∃I
(5)	∃y(y=ιxFx & □[y=ιxGx])	/01, Def (τ/v)	
(6)	a=ιxFx & □[a=ιxGx])	/0	5,∃O
(7)	ιxFx ≠ ιxGx	/01	2,~□O
(8)	a = ιxGx	/01	6b,□O
(9)	a ≠ ιxFx	/01	7,8,LL(−)
(10)	◇[a ≠ ιxFx]	/0	9,◇I
(11)	a=ιxFx & ◇[a ≠ ιxFx]	/0	6a,10,SL

#34:

(1)	∀'x□(Fx → ∃'y[x=y])	/0	Pr
(2)	∃'x◇Fx	/0	Pr
(3)	SHOW: ◇∃'xFx	/0	15,◇I
(4)	∀x{Æx → □(Fx → ∃'y[x=y])}	/0	1,Def ∀'
(5)	∃x{Æx & ◇Fx}	/0	2,Def ∃'
(6)	Æb & ◇Fb	/0	5,∃O
(7)	□(Fb → ∃'y[b=y])	/0	4,6a,QL
(8)	Fb	/01	6b,◇O
(9)	Fb → ∃'y[b=y]	/01	7,□O
(10)	∃'y[b=y]	/01	8,9,SL
(11)	∃y{Æy & b=y}	/01	10,Def ∃'
(12)	Æc & b=c	/01	11,∃O
(13)	Æb	/0112a,12b,LL	
(14)	∃x{Æx & Fx}	/01	8,13,QL
(15)	∃'xFx	/01	14,Def ∃'

#35:

(1)	(ιxFx/y)□Gy	/0	Pr
(2)	SHOW: ∃x{Fx & □Gx}	/0 4b,5,6,QL	
(3)	∃y{y=ιxFx & □Gy}	/0 1,Def (τ/v)	
(4)	a=ιxFx & □Ga	/0	3,∃O
(5)	∀x(Fx ↔ x=a)	/0	4a,ιO
(6)	a=a	/0	Ref=

#36:

(1)	◇FN	/0	Pr
(2)	(N/x)□[x=N]	/0	Pr
(3)	SHOW: (N/x)◇Fx	/0 4,Def (τ/v)	
(4)	SHOW: ∃x{x=N & ◇Fx}	/0 6a,10,QL	
(5)	∃x{x=N & □[x=N]}	/0 2,Def (τ/v)	
(6)	a=N & □[a=N]	/0	5,∃O
(7)	FN	/01	1,◇O
(8)	a=N	/01	6b,□O
(9)	Fa	/01	7,8,LL
(10)	◇Fa	/0	9,◇I

#37:

(1)	(ιxFx/y)◇[y=ιxGx]	/0	Pr
(2)	~◇[ιxFx = ιxGx]	/0	Pr
(3)	SHOW: (ιxFx/y)◇[y ≠ ιxFx]	/04, Def (τ/v)	
(4)	SHOW: ∃y{y=ιxFx & ◇[y≠ιxFx]	/0 6a,11,QL	
(5)	∃y{y=ιxFx & ◇[y=ιxGx]}	/0 1,Def (τ/v)	
(6)	a=ιxFx & ◇[a=ιxGx]	/0	5,∃O
(7)	a=ιxGx	/01	6,◇O
(8)	ιxFx ≠ ιxGx	/01	2,~◇O
(9)	ιxFx ≠ a	/01	7,8,LL
(10)	a ≠ ιxFx	/019, Sym=(−)	
(11)	◇[a ≠ ιxFx]	/0	10,◇I

2.   Counter-Models

#1:

(1)	FN	/0	Pr
(2)	SHOW: $\exists x Fx$	/0	??

One cannot apply $\exists$I to (1), because 'N' is not a
constant.
'N' may not refer.
Keep in mind, being F at a world does not entail existing
at that world.

#2:

(1)	$\Box$FN	/0	Pr
(2)	SHOW: $\exists x \Box Fx$	/0	??

same problem as #1.

#3:

(1)	$\Box$FN	/0	Pr
(2)	$\exists x[x=\text{N}]$	/0	Pr
(3)	SHOW: $\exists x \Box Fx$	/0	??
(4)	$a=\text{N}$	/0	2,$\exists$O
(5)	FN	/01	1,$\Box$O

Although a is N <u>at 0</u>, a *need not* be N <u>at 01</u>,
so we cannot apply LL to 4, 5

#4:

(1)	$\Box$FN	/0	Pr
(2)	$\exists'x[x=\text{N}]$	/0	Pr
(3)	SHOW: $\exists'x \Box Fx$	/0	Def $\exists'$
(4)	SHOW: $\exists x(\text{Æ}x \,\&\, \Box Fx)$	/0	6a,7,QL
(5)	$\exists x(\text{Æ}x \,\&\, x=\text{N})$	/0	2,Def $\exists'$
(6)	$\text{Æ}a \,\&\, a=\text{N}$	/0	5,$\exists$O
(7)	SHOW: $\Box Fa$	/0	ND
(8)	SHOW: Fa	/01	
(9)	FN	/01	1,$\Box$O

a is N <u>at 0</u>, but a *need not* be N <u>at 01</u>, so we cannot apply
LL to 6b, 9.

#5:

(1)	$\Box$FN	/0	Pr
(2)	$\exists'x[x=\text{N}]$	/0	Pr
(3)	SHOW: $\Box\exists'x Fx$	/0	ND
(4)	SHOW: $\exists'x Fx$	/01	Def $\exists'$
(5)	SHOW: $\exists x(\text{Æ}x \,\&\, Fx)$	/01	
(6)	$\exists x(\text{Æ}x \,\&\, x=\text{N})$	/0	2, Def $\exists'$
(7)	$\text{Æ}a \,\&\, a=\text{N}$	/0	6,$\exists$O
(8)	FN	/01	1,$\Box$O

two problems: (1) we know that a exists <u>at 0</u>, but we
don't know that a exists <u>at 01</u>; (2) we know that a is N <u>at
0</u>, but we don't know that a is N <u>at 01</u>.

#6:

(1)	$\forall x \Diamond Fx$	/0	Pr
(2)	$\exists x[x=\text{N}]$	/0	Pr
(3)	SHOW: $\Diamond$FN	/0	
(4)	$a=\text{N}$	/0	2,$\exists$O
(5)	$\Diamond Fa$	/0	1,$\forall$O
(6)	Fa	/01	5,$\Diamond$O

We can't apply LL to 4, 6; they refer to different worlds.

#7:

(1)	$\forall x Fx$	/0	Pr
(2)	SHOW: $(\text{N}/x)Fx$	/0	Def (τ/v)
(3)	SHOW: $\exists x(x=\text{N} \,\&\, Fx)$	/0	

We don't know whether N exists anywhere;
if we did, we could finish.

#8:

(1)	SHOW: $\Box$FN $\to$ (N/x)$\Box$Fx	/0	CD
(2)	$\Box$FN	/0	As
(3)	SHOW: (N/x)$\Box$Fx	/0	Def (τ/v)
(4)	SHOW: $\exists x(x=\text{N} \,\&\, \Box Fx)$	/0	

2 is *de dicto*; 4 is *de re*;
for example, the fact that the president must submit a
budget to the congress does not entail that there is an
individual who must submit such a budget.

#9:

(1)	SHOW: (N/x)$\Box$Fx $\to$ $\Box$FN	/0	CD
(2)	(N/x)$\Box$Fx	/0	As
(3)	SHOW: $\Box$FN	/0	ND
(4)	SHOW: FN	/01	

2 is *de re*; 3 is *de dicto*;
for example; suppose that the particular individual who
is, in fact, president (e.g., from 1992 to 1999, Bill
Clinton) is required by custom to be faithful to Hillary
Clinton; it does not follow that the president is required
by custom to be faithful to Hillary Clinton.

#10:

(1)	SHOW: M=N $\to$ $\Box$[M=N]	/0	CD
(2)	M=N	/0	As
(3)	SHOW: $\Box$[M=N]	/0	ND
(4)	SHOW: M=N	/01	??

'M=N' may be a "contingent identity", in which case it
is not a necessary
(or believed, or obligatory) identity.
For example, let 'M' abbreviate 'the morning star', and
let 'N' abbreviate 'the evening star'.

#11:

(1)	$\Box$FN	/0	Pr
(2)	$\exists x[x=\text{N}]$	/0	Pr
(3)	SHOW: $\Box\exists x Fx$	/0	ND
(4)	SHOW: $\exists x Fx$	/01	DD
(5)	$a=\text{N}$	/0	2,$\exists$O
(6)	can't apply LL to line 1 [N is a proper noun]!		

#12:

(1)	$\Box$FN	/0	Pr
(2)	$\exists'x[x=\text{N}]$	/0	Pr
(3)	SHOW: $\Box\exists x Fx$	/0	ND
(4)	SHOW: $\exists x Fx$	/01	DD
(5)	$\exists x(\text{Æ}x \,\&\, x=\text{N})$	/0	2,Def $\exists'$
(6)	$\text{Æ}a \,\&\, a=\text{N}$	/0	5,$\exists$O
(7)	can't apply LL to line 1 [N is a proper noun]!		