Title

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1 Introduction

The purpose of this document is to display the solution for an integral using Feynman integration. This is basically just me typesetting a problem that was worked out by Flammable Maths on his video "One Sick Integral." Also, I'm too lazy to say all of the technical stuff, so just assume it converges.

2 Derivation

$$I = \int_0^\infty \frac{dx}{1+x^n} \tag{1}$$

In order to tackle this, we first have to parameterize the function. We will soon use Feynman integration in order to get rid of the denominator. We define a function $I(t)$; notice that I is $I(t)$ at $t = 0$.

$$I(t) := \int_0^\infty \frac{e^{-t(1+x^n)}}{1+x^n} dx \tag{2}$$

$$I(0) = I$$

We now use Feynman integration and find $I'(t)$. Then eliminate the denominator by differentiating under the integral.

$$
\begin{aligned}
I'(t) &= \frac{d}{dt}\int_0^\infty \frac{e^{-t(1+x^n)}}{1+x^n}dx \\
\rightarrow I'(t) &= \int_0^\infty \frac{1}{1+x^n}\frac{\partial}{\partial t}e^{-t(1+x^n)}dx \\
&= -\int_0^\infty e^{-t(1+x^n)}dx \\
&= e^{-t}\int_0^\infty e^{-tx^n}dx \\
\text{Let } \xi &= tx^n \\
d\xi &= ntx^{n-1}dx \\
\rightarrow^{(t>0)}\ dx &= \frac{d\xi}{nt}x^{1-n} \\
\text{Notice } x^{1-n} &= x\cdot x^{-n}\text{, thus } x = \sqrt[n]{\frac{\xi}{t}} \text{ and } x^{-n} = \frac{t}{\xi} \\
\rightarrow dx &= \frac{1}{n\xi}\sqrt[n]{\frac{\xi}{t}}d\xi
\end{aligned}
$$

$$
\rightarrow I'(t) = -\frac{e^{-t}}{n\sqrt[n]{t}}\int_0^\infty \xi^{\frac{1}{n}-1}e^{-\xi}d\xi \tag{3}
$$

Here, we can use Bernoulli's definition of the gamma function.

$$
\Gamma(z) = \int_0^\infty x^{x-1}e^{-x}dx, \quad \Re(z) > 0 \tag{4}
$$

From equation (4), we can rewrite the improper integral.

$$
I'(t) = -\frac{e^{-t}}{n\sqrt[n]{t}}\Gamma\left(\frac{1}{n}\right) \tag{5}
$$

Recall $I = I(0)$ and note that $I \rightarrow \infty = 0$. Thus,

$$
\int_0^\infty I'(t)dt = -\int_0^\infty \frac{e^{-t}}{n\sqrt[n]{t}}\Gamma\left(\frac{1}{n}\right)dt = I(t\rightarrow\infty) - I(0) = -I \tag{6}
$$

Rewrite the improper integral in (6) and cancel the negatives.

$$
I = \frac{\Gamma(\frac{1}{n})}{n}\int_0^\infty t^{-\frac{1}{n}}e^{-t}dt \tag{7}
$$

If we rewrite the exponent to:

$$
t^{-\frac{1}{n}} = t^{1-\frac{1}{n}-1} \tag{8}
$$

Using equation (4), we can redefine (8) to be $\Gamma(1-\frac{1}{n})$, thus,

$$I = \frac{\Gamma(\frac{1}{n})\Gamma(\frac{1}{n} - 1)}{n} \tag{9}$$

Finally, we simplify the answer using Euler's reflection formula.

$$\Gamma(z)\Gamma(z - 1) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z} \tag{10}$$

Thus,

$$I = \frac{\pi}{n} \csc\left(\frac{\pi}{n}\right) \tag{11}$$

Again, credit to Papa Flammy for the work shown. Link to his YouTube channel: https://www.youtube.com/watch?v=m1x-xsvdYCQ