FYS4150
Project 5, deadline December 10.

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Abstract

Introduction

The aim of this project is to use three different finite difference schemes to simulate temperature variations in Earths crust and upper mantle. The three schemes are the implicit Euler backward, explicit Euler forward and the Crank-nicolson scheme. This is a combination of the Euler forward and backward schemes. In this report, there will be focus on testing the three algorithms in one dimension, before moving on to two dimensions. The first part will mostly revolve around idealized situations in one dimension, before moving on to an idealized two dimensional problem. This is followed by a three layered model of the earths crust, where each has different parameters. All the code and runs for the

Method

The three different schemes are differential equations that can be rewritten as a set of linear equations. The Euler forward is explicit, the Euler backward is implicit, while the Crank-Nicolson scheme is a combination of the two preceding schemes. These systems can be rewritten as sets of linear equations.

The Euler backwards scheme is implicit, as it uses the current step i, to derive the previous step $i-1$.

$$u_t \approx \frac{u(x_i, t_j) - u(x_i, t_j - \Delta t)}{\Delta t} \tag{1}$$

$$u_{xx} \approx \frac{u(x_i + \Delta x, t_j) - 2u(x_i, t_j) + u(x_i - \Delta x, t_j)}{\Delta x^2} \tag{2}$$

It is possible to scale the above equation by $\alpha = \Delta t/\Delta x^2$, so the equation only depends on one scaled variable. This leads to:

$$u_{i,j-1} = -\alpha u_{i-1,j} + (1 + 2\alpha)u_{i,j} - \alpha u_{i+1,j} \tag{3}$$

Now the differential equation can be written as a set of linear equations with a matrix A times a vector V_j such that $AV_j = V_{j-1}$. Where A, defined from the above differential equations take the form:

$$A = \begin{bmatrix} 1+2\alpha & -\alpha & 0 & 0 & \cdots \\ -\alpha & 1+2\alpha & -\alpha & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\alpha & 1+2\alpha \end{bmatrix} \tag{4}$$

It is now possible to find the previous vector V_{j-1} when V_j is known.This means that it is essential to have initial conditions to start the calculations. A more generalized equation can be written as:

$$A^{-1}(AV_j) = A^{-1}(V_{j-1}) \tag{5}$$

By continuing to multiply by A^{-1}, the implicit scheme takes the form:

$$V_j = A^{-j}V_0 \tag{6}$$

A very similar process can be applied to the Euler forward method, but this scheme is explicit:

$$u_t = \frac{u(x_i, t_j + \Delta t) - u(x_i, t_j)}{\Delta t} \tag{7}$$

$$u_{xx} \approx \frac{u(x_i + \Delta x, t_j) - 2u(x_i, t_j) + u(x_i - \Delta x, t_j)}{\Delta x^2} \tag{8}$$

$$u_{i,j-1} = \alpha u_{i-1,j} + (1 - 2\alpha)u_{i,j} + \alpha u_{i+1,j} \tag{9}$$

$$A = \begin{bmatrix} 1-2\alpha & \alpha & 0 & 0 & \cdots \\ \alpha & 1-2\alpha & \alpha & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha & 1-2\alpha \end{bmatrix} \tag{10}$$

such that:

$$V_{j+1} = AVj \tag{11}$$

We generalize again and get:

$$V_{j+1} = A^{j+1}V_0 \tag{12}$$

The Crank-Nicolson scheme is is a combination of both the implicit Euler backward and explicit Euler forward scheme.

$$\frac{\theta}{\Delta x^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + \frac{1-\theta}{\Delta x^2}(u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}) = \frac{1}{\Delta t}(u_{i,j} - u_{i,j-1}) \tag{13}$$

Where θ determines whether the scheme is explicit when $\theta = 0$, or implicit when $\theta = 1$. However, to get the actual Crank-Nicolson scheme, it is required to have $\theta = 1/2$. This is stable for all Δx and Δt. The Crank-Nicolson scheme is derived by Taylor expanding the forward Euler method $u(x, t+\Delta t)$, $u(x+\Delta x, t)$, $u(x-\Delta x, t)$, $u(x+\Delta x, t+\Delta t)$ and $u(x+\Delta x, t+\Delta t)$ for $t+\Delta t/2$.

Scaling the equation with $\alpha = \frac{\Delta t}{\Delta x^2}$ gives the following equation:

$$-\alpha u_{i-1,j} + (2+2\alpha)u_{i,j} - \alpha u_{i+1,j} = \alpha u_{i-1,j-1} + (2-2\alpha)u_{i,j-1} + \alpha u_{i+1,j-1} \tag{14}$$

which can be rewritten as

$$(2I + \alpha B)V_j = (2I - \alpha B)V_{j-1} \tag{15}$$

$$V_j = (2I + \alpha B)^{-1}(2I - \alpha B)V_{j-1} \tag{16}$$

where I is the identity matrix and B is given by:

$$B = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & -1 & 2 \end{bmatrix} \tag{17}$$

Results

The truncation errors and stability is calculated in the Taylor expansion and these values are shown in the table below:

Method	Truncation error	Stability for
Euler Forward	Δx^2, Δt	$\frac{1}{2}\Delta x^2 \geq \Delta t$
Euler Backward	Δx^2, Δt	Δx^2 and Δt^2
Crank-Nicolson	Δx^2, Δt^2	Δx^2 and Δt^2

Table 1: Truncation errors and stability for the three methods

Discussion

From Table 1

Concluding remarks

Reference list