

**FYS4150**  
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## **Abstract**

## Introduction

The aim of this project is to use three different finite difference schemes to simulate temperature variations in Earth's crust and upper mantle. The three schemes are the implicit Euler backward, explicit Euler forward and the Crank-nicolson scheme. This is a combination of the Euler forward and backward schemes. In this report, there will be focus on testing the three algorithms in one dimension, before moving on to two dimensions. The first part will mostly revolve around idealized situations in one dimension, before moving on to an idealized two dimensional problem. This is followed by a three layered model of the earth's crust, where each has different parameters. All the code, runs and figures for the project is accessible [here](#).

## Method

The three different schemes are differential equations that can be rewritten as a set of linear equations. The Euler forward is explicit, the Euler backward is implicit, while the Crank–Nicolson scheme is a combination of the two preceding schemes. These systems can be rewritten as sets of linear equations.

The Euler backwards scheme is implicit, as it uses the current step  $i$ , to derive the previous step  $i - 1$ .

$$u_t \approx \frac{u(x_i, t_j) - u(x_i, t_j - \Delta t)}{\Delta t} \quad (1)$$

$$u_{xx} \approx \frac{u(x_i + \Delta x, t_j) - 2u(x_i, t_j) + u(x_i - \Delta x, t_j)}{\Delta x^2} \quad (2)$$

It is possible to scale the above equation by  $\alpha = \Delta t / \Delta x^2$ , so the equation only depends on one scaled variable  $\alpha$ . With a Taylor expansion this leads to:

$$u_{i,j-1} = -\alpha u_{i-1,j} + (1 + 2\alpha)u_{i,j} - \alpha u_{i+1,j} \quad (3)$$

Now the differential equation can be written as a set of linear equations with a matrix  $A$  times a vector  $V_j$  such that  $AV_j = V_{j-1}$ . Where  $A$ , defined from the above differential equations take the form:

$$A = \begin{bmatrix} 1 + 2\alpha & -\alpha & 0 & \cdots & 0 \\ -\alpha & 1 + 2\alpha & -\alpha & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & -\alpha \\ 0 & \cdots & \cdots & -\alpha & 1 + 2\alpha \end{bmatrix} \quad (4)$$

It is now possible to find the previous vector  $V_{j-1}$  when  $V_j$  is known. This means that it is essential to have initial conditions to start the calculations. A more generalized equation can be written as:

$$A^{-1}(AV_j) = A^{-1}(V_{j-1}) \quad (5)$$

By continuing to multiply by  $A^{-1}$ , the implicit scheme takes the form:

$$V_j = A^{-j}V_0 \quad (6)$$

A very similar process can be applied to the Euler forward method, but this scheme is explicit:

$$u_t = \frac{u(x_i, t_j + \Delta t) - u(x_i, t_j)}{\Delta t} \quad (7)$$

$$u_{xx} \approx \frac{u(x_i + \Delta x, t_j) - 2u(x_i, t_j) + u(x_i - \Delta x, t_j)}{\Delta x^2} \quad (8)$$

$$u_{i,j-1} = \alpha u_{i-1,j} + (1 - 2\alpha)u_{i,j} + \alpha u_{i+1,j} \quad (9)$$

$$A = \begin{bmatrix} 1 - 2\alpha & \alpha & 0 & \cdots & 0 \\ \alpha & 1 - 2\alpha & \alpha & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \alpha \\ 0 & \cdots & \cdots & \alpha & 1 - 2\alpha \end{bmatrix} \quad (10)$$

such that:

$$V_{j+1} = AV_j \quad (11)$$

We generalize again and get:

$$V_{j+1} = A^{j+1}V_0 \quad (12)$$

The Crank-Nicolson scheme is a combination of both the implicit Euler backward and explicit Euler forward scheme.

$$\frac{\theta}{\Delta x^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + \frac{1-\theta}{\Delta x^2}(u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}) = \frac{1}{\Delta t}(u_{i,j} - u_{i,j-1}) \quad (13)$$

Where  $\theta$  determines whether the scheme is explicit when  $\theta = 0$ , or implicit when  $\theta = 1$ . However, to get the actual Crank-Nicolson scheme, it is required to have  $\theta = 1/2$ . This is stable for all  $\Delta x$  and  $\Delta t$ . The Crank-Nicolson scheme is derived by Taylor expanding the forward Euler method  $u(x, t+\Delta t)$ ,  $u(x+\Delta x, t)$ ,  $u(x-\Delta x, t)$ ,  $u(x+\Delta x, t+\Delta t)$  and  $u(x+\Delta x, t+2\Delta t)$  for  $t+\Delta t/2$ .

Scaling the equation with  $\alpha = \frac{\Delta t}{\Delta x^2}$  gives the following equation:

$$-\alpha u_{i-1,j} + (2 + 2\alpha)u_{i,j} - \alpha u_{i+1,j} = \alpha u_{i-1,j-1} + (2 - 2\alpha)u_{i,j-1} + \alpha u_{i+1,j-1} \quad (14)$$

which can be rewritten as

$$(2I + \alpha B)V_j = (2I - \alpha B)V_{j-1} \quad (15)$$

$$V_j = (2I + \alpha B)^{-1}(2I - \alpha B)V_{j-1} \quad (16)$$

where I is the identity matrix and B is given by:

$$B = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & -1 & 2 \end{bmatrix} \quad (17)$$

## Implementation

The explicit Euler scheme were first implemented due to its simplicity, not needing any tri-diagonal solvers. However, this simplicity comes on the cost of stability which can be seen in Table 1. Next, the implicit schemes were implemented, as well as the LU triangular solver which is the backbone for the scheme. Crank-Nicolson does not solely rely on an LU solver, but has an implicit component as well.

Only the explicit scheme were extended to an additional dimension. Again, this is for the sake of simplicity, but one has to be aware of the stability requirements at all times.

When the code is extended an additional dimension, the boundary conditions have to be extended accordingly. A problem arises at the edges of the two-dimensional grid where no boundary conditions exists. The problem is that the explicit scheme needs four adjacent points to calculate the current point, but at the edges there are only three available points. To resolve this issue, one has to use the point at the opposite edge.

To solve the geological case, the implicit scheme has to be adjusted with the  $1300C^\circ$  at the bottom of the lithosphere and  $8C^\circ$  at the top of the crust. These depths correspond to the first and last columns in the calculated matrix. Additionally, there is heat production at certain ranges which were also implemented between the given column numbers in out matrix.

## Results

The truncation errors and stability is calculated in the Taylor expansion and these values are shown in the table below:

Table 1: Truncation errors and stability for the three methods

Method	Truncation error	Stability for
Euler Forward	$\Delta x^2, \Delta t$	$\frac{1}{2}\Delta x^2 \geq \Delta t$
Euler Backward	$\Delta x^2, \Delta t$	$\Delta x^2$ and $\Delta t^2$
Crank-Nicolson	$\Delta x^2, \Delta t^2$	$\Delta x^2$ and $\Delta t^2$

Table 1 shows that the Euler backward and Crank-nicolson schemes have no limitations for  $\Delta x$  and  $\Delta t$ .

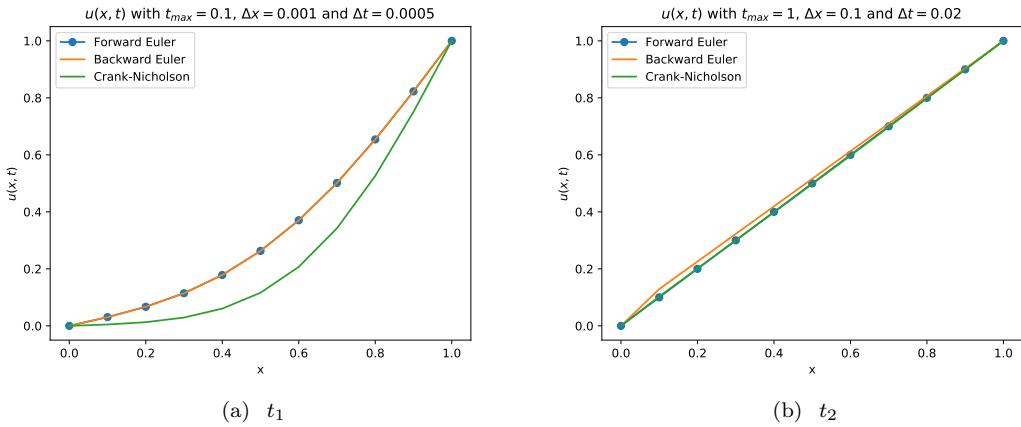


Figure 1: Comparison of  $t_1$  and  $t_2$ .

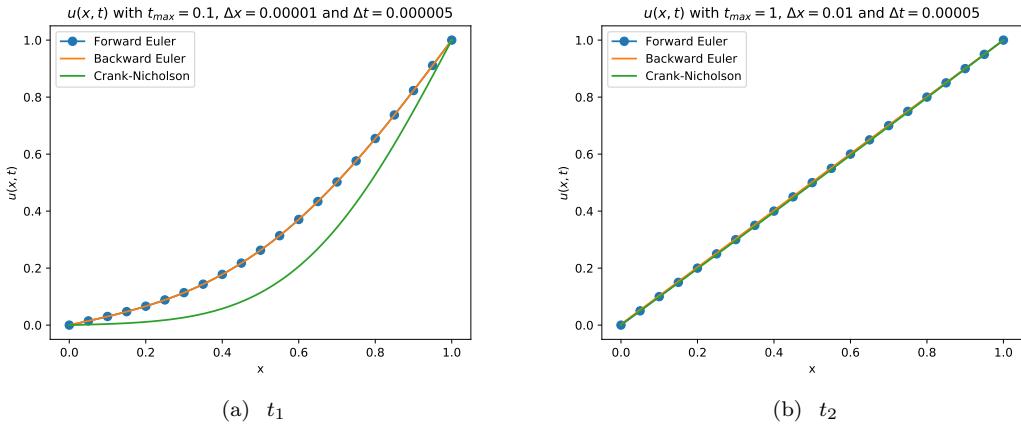


Figure 2: Comparison of  $t_1$  and  $t_2$ .

Figure ?? highlights the distribution of temperature at two different times. At  $t_1$  the different schemes produce different distributions. Forward and backward euler line up closely, in contrast to the Crank-nicolson scheme. At  $t_2$  the temperature distribution is linear and all the schemes line up closely.

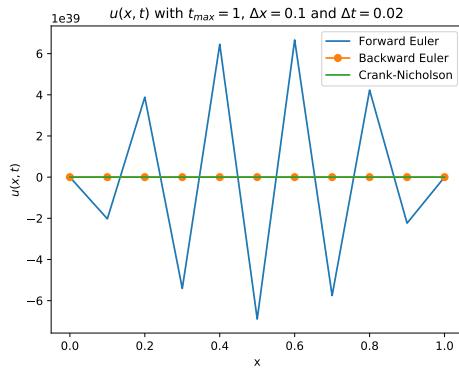


Figure 3: Plot showing the effect of choosing unfavorable  $\Delta x$  and  $\Delta t$

As can be read out of Table 1, the forward Euler scheme has a rigorous stability condition. Figure 3 shows a graphical representation of what happens when this condition is violated.

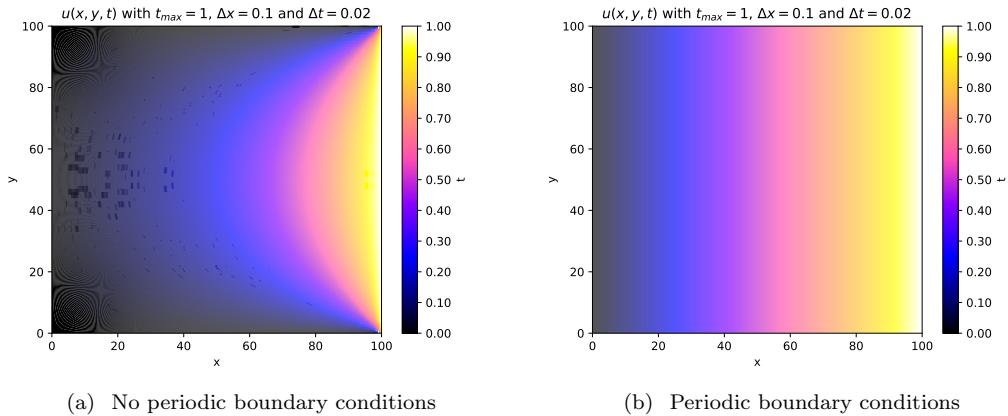


Figure 4: Two dimensional simulation with and without periodic boundary conditions

With periodic boundary conditions, the plot is giving the same result as in the one dimensional case (Figure ??A and Figure 4A). In other words, it is showing a uniform increase in temperature with increasing  $x$ . Without periodic boundary conditions, there is heat loss at all the edges except the edge with the heat source.

## Discussion

Table 1 shows that, even though the Euler forward method is easy to implement, there are serious limitations for the stability. The Euler backward and Crank-Nicolson schemes are stable for all  $\Delta x$  and  $\Delta t$ , but the implementation is more complicated. As the Euler Forward and Crank-Nicolson uses a tridiagonal solver in the computation process, it is expected to be slower than the Euler forward scheme, which uses a more direct approach. Figure 3 shows that the forward Euler fluctuates a lot when the stability condition is violated. The stable schemes actually produce correct results, but are drowned by the fluctuations of the forward Euler.

In the one dimensional case, the system converges to an equilibrium. This is shown graphical in figure ??A and ??B. Given the boundary conditions  $u(0, t) = 0$  and  $u(1, t) = 1$  for all  $t$ , this can be translated to a one dimensional rod with a heat source at  $x = 1$  and a heat sink at  $x = 0$ . When starting the computation, the only point with a nonzero temperature is at  $x = 1$ . The heat from this point then spreads through the rod. The heat rises quicker close to the heat source, as is shown in figure ??A, but will with time reach an equilibrium (figure ??B). It is important to note that the behavior of the system is entirely dependent on the boundary conditions. As an example, it would be possible to add another heat source at  $x = 0$  and the temperature would rise fast in the edges and slower in the middle.

The boundary conditions in the two dimensional case is depending on what kind of system being simulated. If the system in question is a two dimensional rod with a heat source at one end and heat loss at all the other edges, the temperature distribution would look like 4A. If the system is simulating the temperature distribution in the lithosphere, it is not expected to be much heat loss in the lateral direction. This can be simulated with periodic boundary conditions, which in essence removes all lateral heat loss.

The last case looked on is the three layered lithosphere model. The two bottom layers are enriched with a radioactive substance that alters the heat production in the layer. The enriched area is limited to 150 kilometers of lateral extension. As the enriched area is limited, there will be heat flow in horizontal direction.

## Errors

The errors that occur in this the project mainly stem from the instability of the explicit Euler in both one and two dimensions. By being cautious, the results stayed stable as can be observed in the result section of this paper. If the explicit Euler method became unstable, it would be vary obvious by looking at the output files. The values in the file would then mainly consist of NaN and Inf due to the calculation of extremely large variables.

## **Concluding remarks**

## References