FYS4150
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Abstract

Introduction

Method

The implicit schemes are differential equations that can be rewritten as a set of linear equations. The Euler forward is explicit, the Euler backward is implicit, while the Crank-Nicolson scheme is a combination of the two preceding schemes. These systems can be rewritten as sets of linear equations.

The Euler backwards scheme is implicit, as it uses the current step i, and a later step $i+1$ to derive the previous step $i-1$.

$$u_t \approx \frac{u(x_i, t_j) - u(x_i, t_j - \Delta t)}{\Delta t} \tag{1}$$

$$u_{xx} \approx \frac{u(x_i + \Delta x, t_j) - 2u(x_i, t_j) + u(x_i - \Delta x, t_j)}{\Delta x^2} \tag{2}$$

It is possible to scale the above equation by $\alpha = \Delta t / \Delta x^2$, so the equation only depends on one scaled variable. This leads to:

$$u_{i,j-1} = -\alpha u_{i-1,j} + (1 + 2\alpha) u_{i,j} - \alpha u_{i+1,j} \tag{3}$$

Now the differential equation can be written as a set of linear equations with a matrix A times a vector V_j such that $AV_j = V_{j-1}$. Where A, defined from the above differential equations take the form:

$$A = \begin{bmatrix} 1+2\alpha & -\alpha & 0 & 0 & \cdots \\ -\alpha & 1+2\alpha & -\alpha & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\alpha & 1+2\alpha \end{bmatrix} \tag{4}$$

It is now possible to find the previous vector V_{j-1} when we already know what V_j, which we already know due to our initial conditions. A more generalized equation can be written as:

$$A^{-1}(AV_j) = A^{-1}(V_{j-1}) \tag{5}$$

and if we keep multiplying by A^{-1} we get the implicit scheme:

$$V_j = A^{-j} V_0 \tag{6}$$

A very similar process can be applied to the Euler forward method, but this scheme is explicit:

$$u_t = \frac{u(x_i, t_j + \Delta t) - u(x_i, t_j)}{\Delta t} \tag{7}$$

$$u_{xx} \approx \frac{u(x_i + \Delta x, t_j) - 2u(x_i, t_j) + u(x_i - \Delta x, t_j)}{\Delta x^2} \tag{8}$$

$$u_{i,j-1} = \alpha u_{i-1,j} + (1 - 2\alpha) u_{i,j} + \alpha u_{i+1,j} \tag{9}$$

$$A = \begin{bmatrix} 1-2\alpha & \alpha & 0 & 0 & \cdots \\ \alpha & 1-2\alpha & \alpha & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha & 1-2\alpha \end{bmatrix} \tag{10}$$

such that:

$$A^{-1}(AV_j) = A^{-1}(V_{j-1}) \tag{11}$$

We generalize again and get:

$$V_j = A^{-j}V_0 \tag{12}$$

The Crank-Nicolson scheme is is a combination of both implicit and explicit schemes, namely the Euler forward and Euler backward method.

$$\frac{\theta}{\Delta x^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + \frac{1-\theta}{\Delta x^2}(u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}) = \frac{1}{\Delta t}(u_{i,j} - u_{i,j-1}) \tag{13}$$

where θ determines whether the scheme is explicit when $\theta = 0$, or implicit when $\theta = 1$. However, it is when $\theta = 1/2$ where we have the actual Crank-Nicolson scheme which is stable for all Δx and Δt. To derive the Crank-Nicolson scheme we begin with the forward Euler method and Taylor expand $u(x, t+\delta t)$, $u(x+\delta x, t)$, $u(x-\delta x, t)$, $u(x+\delta x, t+\Delta t)$ and $u(x+\delta x, t+\Delta t)$ for $t+\Delta t/2$.

Again we scale the equation with $\alpha = \frac{\Delta t}{\Delta x^2}$ which results in the following equation:

$$-\alpha u_{i-1,j} + (2+2\alpha)u_{i,j} - \alpha u_{i+1,j} = \alpha u_{i-1,j-1} + (2-2\alpha)u_{i,j-1} + \alpha u_{i+1,j-1} \tag{14}$$

which can be rewritten as

$$(2I + \alpha B)V_j = (2I - \alpha B)V_{j-1} \tag{15}$$

$$V_j = (2I + \alpha B)^{-1}(2I - \alpha B)V_{j-1} \tag{16}$$

where I is the identity matrix and B is given by:

$$B = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & -1 & 2 \end{bmatrix} \tag{17}$$

Results

The truncation errors and stability is calculated in the Taylor expansion and these values are shown in the table below:

Method	Truncation error	Stability for
Euler Forward	Δx^2, Δt	Δx^2 and Δt^2
Euler Backward	Δx^2, Δt	Δx^2 and Δt^2
Crank-Nicolson	Δx^2, Δt^2	$\frac{1}{2}\Delta x^2 \geq \Delta t$

Table 1: Truncation errors and stability for the three methods

Discussion

From Table 1

Concluding remarks

Reference list