Double Cone Paradox Analysis

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Introduction

The double cone paradox has been known in the science community due to its curious behaviour and its importance in the science communication and classical mechanics. Its importance relates on the apparently upwards movement of the double cone in a tilted set of rails violating the laws of Newton, however, this curious event is totally consistent with theory owing to reasons explained in the document. This article is going to analyze and explain the phenomena basing in the paper *"The double cone: a mechanical paradox or a geometrical constraint?"*[1] making a first approach to the mathematics that describe the case, in order to take aside the physical explanation that domains the movement of the double cone. Finally, an experiment would be held with a focus on corroborate the relation among the theoretical and the experimental components previously exposed, comparing the data obtained both on this study and the basis paper study.

1 Mathematical analysis

Taking into account the mathematical process settled in the mentioned paper, it is possible to get a deeper comprehension throughout mincing the steps taken by the authors. In the same manner, two aspects of the phenomena would be studied in this section: A Geometrical Description and the Energy Conservation analysis.

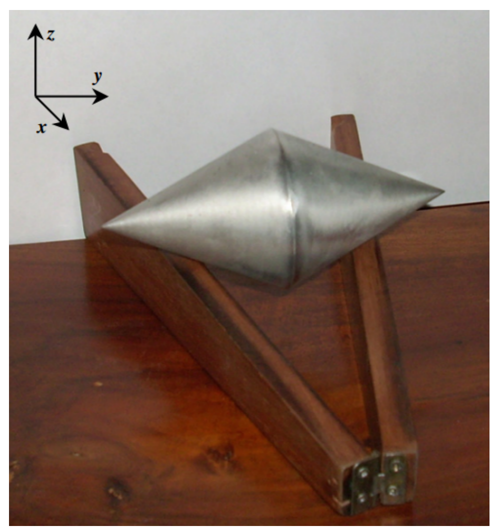


Figure 1: Experimental set up taken from the paper [1].

1.1 Geometrical Description

According to the schemes shown in the Figures 10 and 2, it is convenient to relate the variation of position (in the different axis) through time, furthermore, the most important relations to take into account are: how the distance between the x axis and the contact point of the cone in the rail $(y(x))$ varies when it moves (dx) (Equation 1), how the vertical distance between the object horizontal axis and the contact points with rails $(\zeta(y))$ varies when $y(x)$ varies (dy) (Equation 2) and how the contact points height $(z(x))$ varies when the object moves (dx) (Equation 3). Its importance lies in the trigonometrical relation they represent: each of them corresponds to the tangent of the angles Ψ, θ and φ respectively, making it easier to unify equations.

$$\frac{y(x+dx)-y(x)}{dx} = \frac{dy}{dx} = \tan\Psi, \tag{1}$$

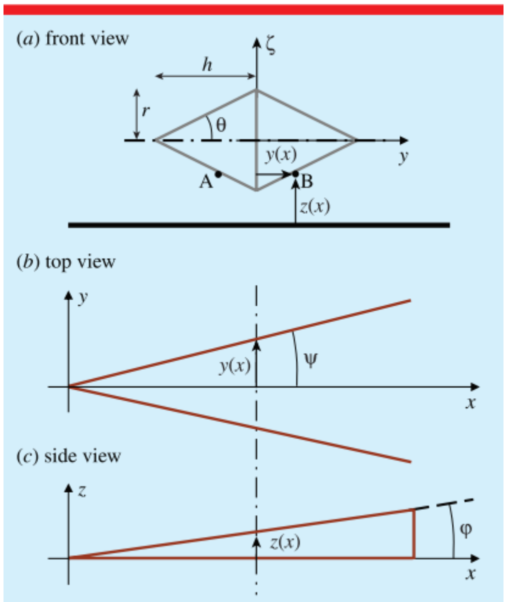


Figure 2: Perspectives of the double cone and the rails, taken from the paper [1]

$$\frac{\zeta(y+dy)-\zeta(y)}{dy}=\frac{d\zeta}{dy}=\tan\theta, \quad (2)$$

$$\frac{z(x+dx)-z(x)}{dx}=\frac{dz}{dx}=\tan\varphi. \quad (3)$$

Now, solving equations for the differentials dy, $d\zeta$ and dz, and also replacing (4) in (5) the following equations are obtained:

$$dy = \tan\Psi \ dx, \quad (4)$$

$$d\zeta = \ \tan\theta \ dy = \tan\theta \ \tan\Psi \ dx, \quad (5)$$

$$dz = \tan\varphi \ dx. \quad (6)$$

Here is noticeable that when dz increases, $d\zeta$ decreases. So on, both displacements are given in a vertical direction. With this in mind, the total vertical translation of the object horizontal axis with respect to the ground is obtained from the subtraction of (6) and (5):

$$dz - d\zeta = \tan\varphi \ dx - \tan\theta \ \tan\Psi \ dx. \quad (7)$$

For the phenomena to occur, this total translation should be negative (because of physical reasons that would be treated later). This condition is given by the following inequality:

$$dz - d\zeta < 0 \Rightarrow \tan\varphi < \tan\Psi \ \tan\theta, \quad (8)$$

knowing that the movement in x axis (dx) in this case is always positive so the condition is only required for the tangents.

In the basis paper, authors give an example of an experimental double cone system that satisfies the tangents condition, showing that Equation 8 is experimentally correct (in this study an own experimental process would be held to corroborate the geometrical condition). Moreover, authors display what they expected to be the "indifferent static equilibrium of the double cone on the rails". This angle ($\Psi \approx 13.8°$ in the paper) is gotten through this equation:

$$\Psi = \tan^{-1}(\frac{\tan\varphi}{\tan\theta}), \quad (9)$$

taking into account that angles φ and θ are known and constant.

1.2 Energy Conservation

Taking into account that the phenomenon described begins with an potential energy due to its initial height, it is pertinent to study its variation throughout the displacement of the double cone, which can be exposed in the equation (10), where the m variable corresponds to the mass of the cone, g consists on the gravitational acceleration of the earth and Δz shows the variation along the z axis or the height.

$$\Delta U = -mg\Delta z \quad (10)$$

It is important to consider that this potential energy mentioned before tents to decrease, being transformed into another, denominated kinetic energy, owing to the movement of the object studied. This behavior is consequent of the conservation of the mechanical energy principle, which can be synthesized is the equation (11).

$$\Delta U + \Delta k = 0, \quad (11)$$

The kinetic energy of a dot is expressed by its mass m_i and velocity v_i. Now, If the dot is rotating, v_i can be indicated by the turning radius and the angular velocity ω_i. (12)

$$K_i = \frac{1}{2}m_i v_i^2 \Rightarrow v_i = r_i\omega \tag{12}$$

The body analyzed has a geometry and volume, then, it is conformed by a set of n dots, and the kinetic energy would corresponds to the sum of all these elements.

$$\begin{aligned} K_r &= \sum_i^n K_i = \sum_i^n \frac{1}{2}m_i vi^2 \\ &= \sum_i^n \frac{1}{2}m_i r_i^2\omega^2 \\ &= \frac{1}{2}(\sum_i^n m_i r_i^2)\omega^2 \end{aligned} \tag{13}$$

Taking into account that the moment of inertia I equals the product of the mass and the square radius (14)

$$I = \sum_i^n m_i r_i^2, \tag{14}$$

the expression to the rotational kinetic energy is would finally be rewritten as (15), which corresponds to the mathematical expression of the kinetic energy for a rotational movement.

$$\Delta K = \frac{1}{2}I_{cm}\omega^2, \tag{15}$$

where ω consists on the angular velocity, and I_{cm} refers to the inertial momentum of the solid double cone (16).

$$I_{cm} = \frac{3}{5}mr^2, \tag{16}$$

Taking the principle (11), it is possible to execute an equalization of (10) and (15), in order to obtain an expression to ω (17).

$$\begin{aligned} mg\Delta z &= \frac{1}{2}I_{cm}\omega^2, \\ mg\Delta z &= \frac{1}{2}\cdot\frac{3}{5}mr^2\omega^2, \\ \omega^2 &= \frac{10mg\Delta z}{3mr^2}, \\ \omega &= \sqrt{\frac{10g\Delta z}{3r^2}}. \end{aligned} \tag{17}$$

This expression allows comparison between the theoretical result and the experimental one. It shows on an explicit manner the relation between the magnitudes considered in the analysis of the situation.

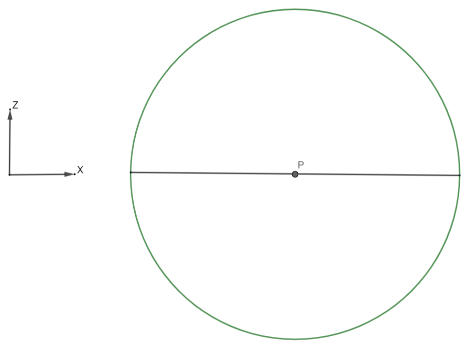
2 Physical analysis

Once mathematical analysis is stated, the physical description of the phenomena is needed to comprehend completely the behavior of the double cone and how this movement is not violating classical motion laws. Mathematical descriptions formed before will be used now to present their implications in the mechanical conditions that governs the double cone tendency and the demonstration of the accordance to what is known in physics.

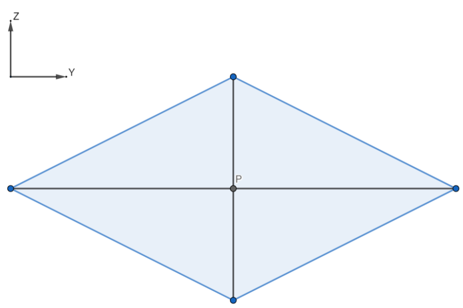
2.1 Geometrical Implications

According to the first analysis, it was shown that the only condition for the upward movement to happen is given by expression 8. Actually, this state indicates that double cone's CM is moving down. To explain this, first, it is necessary to establish the CM point.

It is known that the case analysed here has a perfect mass distribution along all its volume, this means, that the double cone here has a uniform mass density ρ, which is constant for every n particles of the solid. With this in mind and according to literature, it can be assured that the CM of the object is the geometrical symmetrical center [2]. So on, it is possible to find the geometrical center of the object in its two principal perspectives: in the XZ plane (Figure 3a) and in the YZ plane (Figure 3b).



(a) XZ plane



(b) YZ plane

Figure 3: Perspectives of the double cone showing the centers of mass (geometrical center) which describes the total volumetric center of mass of the rigid object.

After locating the center of mass, it can be analysed the net movement of the double cone according to the condition established (8) in this particular point. In other words, this shows how the CM height must change over time. So on, it must be always decreasing for the phenomenon to occur. Now, it is known that an object is always trying to get to its least potential energy state (equilibrium) [3], and that explains why the movement (that looks like is rolling up) is actually rolling down.

In fact, it can be appreciated the vertical displacement, due to the reasons previously presented, as in Figure 4.

Furthermore, for example, if the object analysed has a cylindrical shape, $d\zeta = 0$ because $\zeta(y)$ doesn't varies in time. So, the CM height of the cylinder

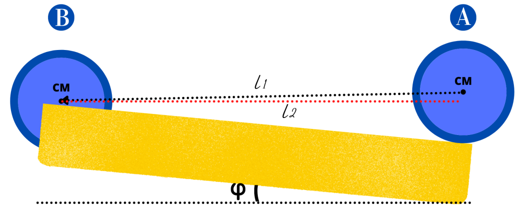


Figure 4: Net vertical movement of the CM. l_1 represents the CM movement from A to B, and l_2 is a horizontal line that touches the CM in moment B. Height difference between l_1 and l_2 in moment A shows the height displacement of CM from A to B.

corresponds only to dz, and, in order to accomplish the Eq 8, the cylinder must, instead of move up, move down the tilted rails, what is seen normally in nature [4].

2.2 Mechanical Energy Conservation

The physical analysis of the behaviour of a system can be explained through the variation of its energy. In this case, as the phenomenon studied consist on the movement of a double cone, the energy associated is the mechanical one.

Firstly, it is possible to assume that the double cone is at rest at the initial time of the motion. Thus, it is correct to affirm that its kinetic energy is equivalent to zero. However, according to the reference axis taken by this study, the body has potential energy at this moment, being this one, the total mechanical energy at that point. Throughout the displacement, the double cone acquires speed due to the transformation of the potential energy into kinetic energy, which can be rotational or translational, depending on the way it moves.

The model proposed in the paper consist on a conservative energy system (11), where the friction force between the surfaces contributes to the rotational movement of the double cone. Consequently, the total mechanical energy at the end of the motion would be the same that the beginning one, namely, the kinetic energy (15) equals to the initial potential energy (10).

2.2.1 Gravitational Potential Energy

This kind of energy can be interpreted as the energy that a body has according to its position on a reference system. In this way, if the double cone is studied as a particular dot (in other words, its center of mass), it may be noted that it descends throughout the z axis, while the motion through the rails is being carried out. In this way, the initial potential energy (10) has the maximum value for the reason that the Δz corresponds to the highest distance between the CM and the surface, and the final potential energy, has the minimum value because CM has the shortest distance with respect to the surface.

That behaviour is completely consistent with the physical principles, because, although the height of the rails is increasing, the center of mass is moving from the point of maximum potential energy (A) to a point of minimum potential energy (B), as it is shown in the image 4.

The last explanation could be confusing, due to apparent upward movement of the double cone, however, it is just the result of a geometrical setup that has been described before in sections 1.1 and 2.1.

2.2.2 Rotational and Translational Kinetic Energy

In the last section, the double cone has been analyzed as a particular dot, however, the study of the kinetic energy implies the consideration of the object as a body with geometry and volume. In this manner, the particles that compose the double cone would present two types of movement: A rotational and a translational one, where both conform the total kinetic energy of the system.

The rotational kinetic energy refers to the angular velocity associated to the rotation of the body (15), where the inertia momentum corresponds to the opposition of the rotational movement, being analogous to the mass for a translational movement.

3 Methodology

Once the mathematical description and the physical theory is explained, it is viable to perform it in order to get a realistic perspective of the model, taking into account the results obtained aiming to do an objective comparison between the theoretical data and the experimental one.

3.1 Experimental setup and data collection

The experiment will be done using a compact wooden double cone, as the rails employed, and mass equal to $0.255\ [kg]$ as seen in the figure (5). It is crucial to point the fact that the double cone worked in the article [1] had an aluminum hollow structure, so the inertia momentum would differ, being the correspondent to the present study that one mentioned in the mathematical description (16).

Figure 5: Picture of the experimental setup

The angles Ψ taken by the rails will be 15° and 20°, being the first one used in the article [1], and second one, an example to identify the change of the angular and translational velocity according to the angle.

The movement of the body will be recorded at a lateral view, with the purpose of follow a particular dot throughout the displacement across the Z and X axis using the software *tracker* (with a display as seen in figure 6), which provide data related to the distance traveled over the time, being essential to estimate the translational velocity of the double cone.

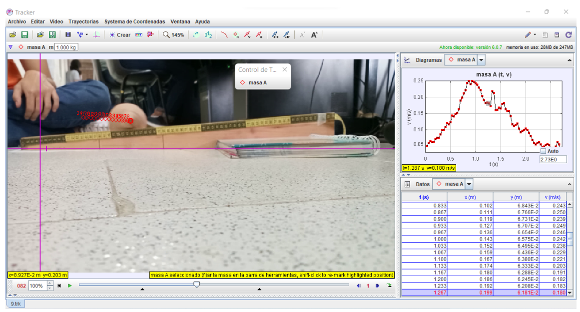


Figure 6: Picture of *tracker* software

3.2 Code Development

Data obtained by *tracker* about time, linear velocity, and X and Z distance traveled, will be loaded in order to standardize them on a notebook code, using *Python* as the programming language. In this way, it will be designed a function that calculates the mechanical energy behaviour throughout the movement of the body.

The angular speed will be calculated through its relation with the tangential velocity and the radius, which corresponds to the $\Delta\zeta$ distance, taking into account its expression depending on x displacement, that can be deduced from relation (5) by integrating:

$$\zeta(x) = R - \tan(\theta)\tan(\psi)x. \tag{18}$$

Ultimately, it is important to clarify the fact that the equilibrium angle, as the final angular velocity, is going to be verified experimentally on the process that has been already exposed. Also, all the data, python code and results will be held on a GitHub repository that can be consulted in Proyecto Double Cone.

4 Analysis and Results

As seen before in geometrical description, the condition 8 must be achieved. So on, it was proofed for this case, obtaining:

$$4.093^\circ < 6.909^\circ, when\ \Psi = 15^\circ \tag{19}$$

$$4.093^\circ < 9.384^\circ, when\ \Psi = 20^\circ, \tag{20}$$

which is true, so the phenomenon must occur.

On the other hand, the equilibrium angle given by 9 was calculated obtaining:

$$\Psi_{eq} = 9.02^\circ, \tag{21}$$

which was proofed experimentally, as seen in the video that can be downloaded here.

Now, more than seven videos were recorded for each angle studied, in order to have a better sample to get more viable results about how changes the behaviour of the phenomenon. It is important to highlight an aspect that was taken into account to make easier the process: painting one of the corners on the cone, although it does not correspond to its CM, it is a point whose rotation is practically null, being crucial to measure the translational velocity, that is shown in figures 7 and 8.

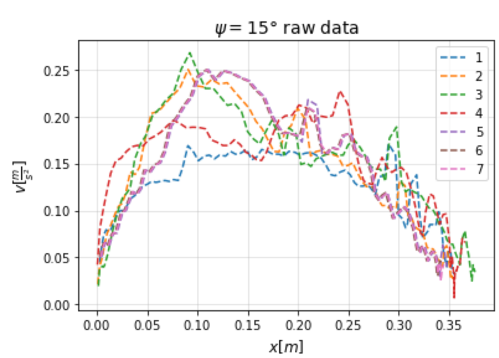


Figure 7: Translational velocity for a Ψ angle of 15°

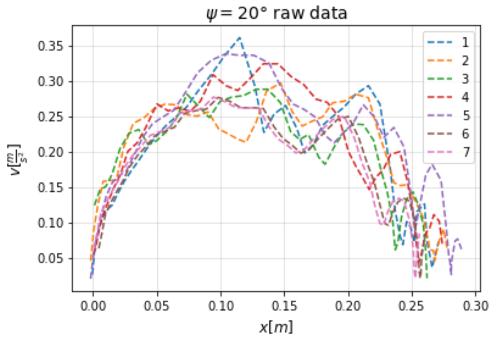


Figure 8: Translational velocity for a Ψ angle of 20°

It can be proven an optimal correlation between the values obtained, taking into account the standard deviation calculated for each measure (data_std_Ψ.csv in Results). On this wise, it results appropriated to standardize the data, getting a global graphic that permits to visualize the change of the velocity across the X displacement as seen in figure 9.

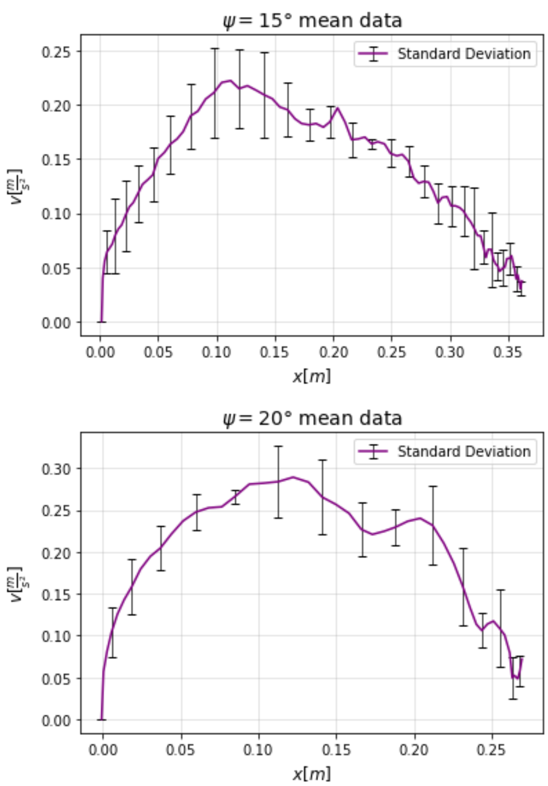


Figure 9: Translational standardized velocity for $\Psi = 15^{\circ}$ and 20°

Now, both behaviours should be contrasted, in order to recognize similarities and differences between them (Figure 10).

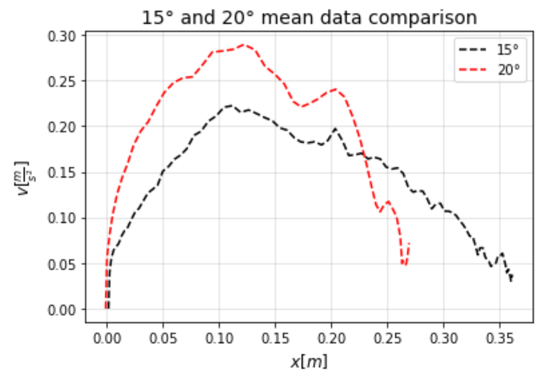


Figure 10: 15° and 20° velocities comparison

It is important to notice that the translational velocity starts from a minimum value 0 m/s, increasing progressively up to a maximum one, and then it decreases until tend to acquire the initial speed. In this case, the highest maximum value obtained by the double cone corresponds to the measurement done with a Ψ angle of 20°.

Likewise, the X displacement required at the Ψ angle of 20° to decrease, results shorter than the employed by the Ψ angle of 15°.

Using the movement across the X axis, it was carried out the calculation of the angular velocity (Figure 11).

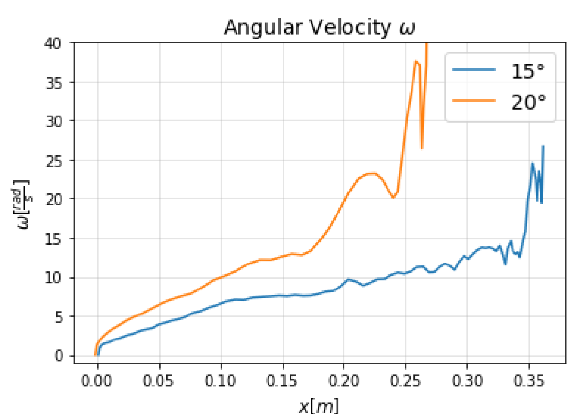


Figure 11: Angular velocity

It is interesting the fact that, the angular velocity is higher when the angle is lower, and it tends to increase while the X distance traveled augments.

All the behaviour of the translational velocity, angular velocity and the variation of the height of the CM is already known. Then it is possible to describe the mechanical energy through the movement, that can be visualized in figures 12 and 13.

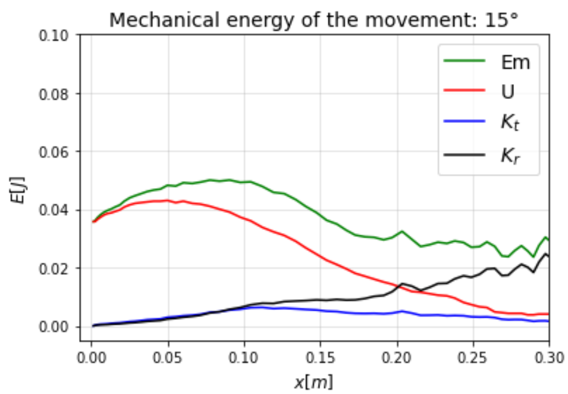


Figure 12: Mechanical Energy variation by $\Psi = 15°$

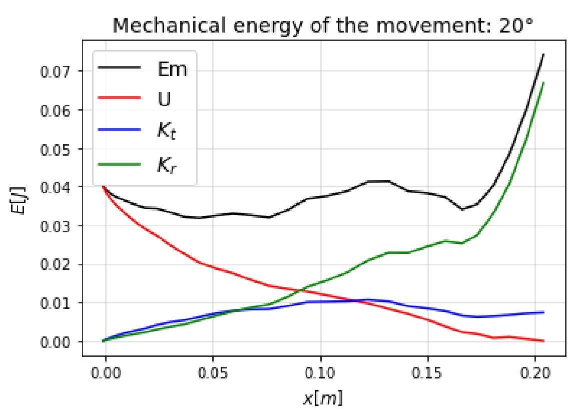


Figure 13: Mechanical Energy variation by $\Psi = 20°$

As it is shown, the mechanical energy from both of the angles studied tends to keep on a value, although the displacement is being carried out, it decreases slightly. In addition, it can be appreciated that the rotational kinetic energy increases at the end of the movement, while the translational velocity decreases. Similarly, the relation of the total mechanical energy as the sum of the potential and kinetic energy is notorious.

5 Conclusions

The mathematical description is an important step in the comprehension of the phenomenon, establishing the demonstration in the behaviour of the double cone, through the dynamical and geometrical understanding, which evidence its consistency with the maths and the physical principles.

Previously, as seen in the physical description the apparent paradox can be described as the results of a certain geometrical conditions, which imply the veracity of the mechanical physics theories, through the variation of the height of the CM reaching the point of least potential energy, which is at the point where the rails are highest, being it transformed into rotational and translational kinetic energy along its path.

Boarding the experimental results, there are many aspects that should be highlighted. Firstly, when the Ψ angle is higher, the maximum velocity will be too. However, the X displacement would be lower for that minor angle, so, in the same way, the duration of the movement decreases.

Besides, the final velocity of the double cone will be completely rotational, due to the fact that the transnational velocity tends to zero, being in agreement with the certainty that the distance between the CM and the point of contact with the rails are practically null at that moment. To this extent, it has a consistent relation with the behaviour of the mechanical energy conservation, although it is not absolutely conservative, is similarly enough to a pure translational and rotational movement, that is the considered one by the article ([1]). In parallel, it could be proven the equilibrium angle, showing one more time its coherence with the theoretical description.

Finally, it would be suggested to use in further investigations more precise tools for collecting the data in order to be more accurate with the real movement. So on, it would be an enhancement to measure more than two Ψ angles so more statements could be realized.

References

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