Stability of Geodesics of Black holes

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by

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CERTIFICATE

*This is to certify that the work contained in this thesis entitled "**Stability of Geodesics of Black holes**" is a bonafide work of **S Venkat Bharadwaj (Roll No. 200121045)**, carried out in the Department of Physics, Indian Institute of Technology Guwahati under my supervision and that it has not been submitted elsewhere for a degree.*

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Abstract

Geodesics around the black hole can be used to describe important features of the space time. Since they also describe the path of photons, they are used to describe the appearance as seen by an observer. Conventionally, people study geodesics by obtaining an effective potential. In this paper, different approaches using the intrinsic geometry and the chaotic nature of the trajectory of the particle to analyse the stability of the geodesics of a black hole is being reviewed. Furthermore, the equivalence between these approaches and the conventional approach is also discussed.

Contents

Chapter 1

Introduction

The theory of relativity predicts very compact objects of large mass around which the gravity is so strong that even light is not able to escape it. These masses are termed as black holes. In general relativity, gravity is not regarded as a force, rather it is seen as a consequence of curved space-time. By solving the Einstein's Field Equations, the metric tensor that defines this curved space time can be calculated, and hence the space-time for a compact mass at the centre can be defined.

Geodesics are curves that define the shortest path between 2 points. For a flat space-time, the shortest path (or the geodesic) is given by a straight line between the two points. However, the space-time around the black hole is curved, and the geodesics corresponding to these are also curved. This curve can be estimated by solving the equation of motion of a particle in the space time, also known as geodesic equations.

Stability refers to the ability of particles to fall back to their initial path when they are perturbed. The path is said to be unstable if the particle continuously moves farther from the initial path. The geodesic around a black hole is said to be unstable if a perturbation makes the particle collapse into the black hole or allows the particle to escape its orbit around the black. Generally, this stability is studied by assuming an effective potential due to the curved space time, then analysing the concavity of the effective potential.

In this paper, we will analyse the stability of the geodesics using different approaches. We will start with the conventional approach, where we use the Lagrangian of a particle in the space-time to find the effective potential to analyse its stability. Then we analyse the stability of the null like geodesics using optical metric and looking at the geometry of the metric. Finally, we will compute the Lyapunov exponents to study the stability of the geodesics. In the last section, we will also show the equivalence between these approaches.

1.1 The Metric

The metric tensor refers to a fundamental object that defines the structure of the space-time. Generally, the metric tensor is expressed in the form of line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, where $g_{\mu\nu}$ are the elements of the metric tensor. For Minkowski (Flat) space time, this element in Cartesian coordinates is given as $-ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$.

For a space-time with a stationary object of mass, M at the center, the solution to Einstein's Field Equations gives the metric in polar coordinates as

$$d\tau^2 = f(r)c^2dt^2 - \frac{1}{f(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2; f(r) = (1 - \frac{2GM}{c^2r}) \tag{1.1}$$

For objects with a radius less than $2GM/c^2$, the line element blows up to infinity at r = $2GM/c^2$. This is referred to as the Schwarzschild radius, r_s. Stationary objects with $r < r_s$ can be considered as Schwarzschild black holes in the simplest case with the event horizon at r = r_s. In this paper, we will work with the Schwarzchild black hole with the metric of this form unless stated otherwise. We will also work in the natural units (G=1, c=1) for simplicity.

Chapter 2

Effective Potential Approach

2.1 Equation of Motion of Test Particle

Using the metric (1.1), the Lagrangian of a particle can be defined as

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda}$$

$$\implies 2\mathcal{L} = f(r)(\frac{dt}{d\lambda})^2 - \frac{1}{f(r)}(\frac{dr}{d\lambda})^2 - r^2(\frac{d\theta}{d\lambda})^2 - r^2 \sin^2\theta(\frac{d\phi}{d\lambda})^2 \tag{2.1}$$

where λ is an affine parameter. From equation (2.1) we can see that the Lagrangian is independent of t and ϕ. Therefore they are cyclic coordinates with conserved quantities p_t, p_ϕ. As the angular momentum is conserved, the motion of the particle will be confined in a plane. Hence, we can set $\theta = \pi/2$ without loss in generality. Thus, the Lagrangian reduces to

$$2\mathcal{L} = f(r)\dot{t}^2 - \frac{1}{f(r)}\dot{r}^2 - r^2\dot{\phi}^2 \tag{2.2}$$

and the momenta are given by

$$p_q = \frac{\delta\mathcal{L}}{\delta(\frac{dq}{d\lambda})}$$

$$p_r = \frac{1}{f(r)}\dot{r}; \quad E = -p_t = f(r)\dot{t}; \quad L = p_\phi = r^2\dot{\phi} \tag{2.3}$$

Here $\dot{x}$ refers to derivative of x with respect to affine parameter λ. Thus the Hamiltonian is written as

$$\begin{aligned} 2\mathcal{H} &= 2(p_t\dot{t} + p_\phi\dot{\phi} + p_r\dot{r} - \mathcal{L}) \\ &= f(r)\dot{t}^2 - \frac{1}{f(r)}\dot{r}^2 - r^2\dot{\phi}^2 \\ &= -\frac{1}{f(r)}\dot{r}^2 + \frac{E^2}{f(r)} - \frac{L^2}{r^2} = \epsilon = const \end{aligned} \tag{2.4}$$

Here ϵ = 0,1 for null-like and time-like geodesics respectively. Re-arranging equation (2.4) we get the radial equation of the particle as

$$\dot{r}^2 - E^2 + f(r)\frac{L^2}{r^2} + f(r)\epsilon = 0 \tag{2.5}$$

2.2 Effective Potential

The equation (2.5) is similar to an energy conservation equation with $\dot{r}^2$ term as Kinetic Energy. Thus, we can consider this as a motion in an effective potential V_r. Thus we get

$$\dot{r}^2 + V_r = 0; \quad V_r = -E^2 + f(r)\frac{L^2}{r^2} + f(r)\epsilon \tag{2.6}$$

Analogous to a particle in potential r, the condition for circular orbits is $V_r = 0$, and since the forces will be balnced, $V_r' = 0$. Here x' denotes differentiation of x with respect to r.

2.2.1 Null-like Geodesic

For $\epsilon = 0$, the above equations are reduced to give the solution

$$\begin{aligned} V_r = 0 &\implies \frac{f(r)}{r^2} = \frac{E^2}{L^2} \\ V_r' = 0 \implies \frac{f'(r)}{r^2} - \frac{2f(r)}{r^3} = 0 &\implies r = 3M \end{aligned} \tag{2.7}$$

Hence, for a photon in circular orbit, the radius of the orbit is given by $r_{ph} = 3M$.

2.2.2 Time-like Geodesic

For massive object, ϵ =1 , the equation are reduced to (with r_c = radius of circular orbit)

$$V_r' = 0 \implies \frac{f'(r)L^2}{r^2} - \frac{2f(r)L^2}{r^3} + f'(r) = 0 \implies L^2 = \frac{r^3 f'(r)}{2f(r) - rf'(r)} = \frac{2Mr_c^2}{2r_c - 6M}$$
$$V_r = 0 \implies E^2 = f(r)[\frac{L^2}{r^2} + 1] = \frac{2f(r)^2}{2f(r) - rf'(r)} = \frac{2(r_c - 2M)^2}{r_c(2r_c - 6M)} \tag{2.8}$$

2.3 Stability Analysis

Now that we have the radius of the circular orbit, we can analyse its stability by looking at the concavity of the effective potential V_r. i.e. The value of V_r'' at $r = r_c$ gives the stability of the curve. Similar to a classical system there are 3 possibilities:

- if $V_r'' > 0$, it corresponds to a local minimum of the effective potential. Hence, the particle is in stable equilibrium.
- if $V_r'' < 0$, it corresponds to a local maximum of the effective potential. Hence, the particle is in unstable equilibrium.
- if $V_r'' = 0$, it corresponds to an extremum of the effective potential. Hence, the particle is in neutral equilibrium.

The expression for V_r'' is given by

$$V_r'' = \frac{f''(r)L^2}{r^2} - \frac{4f'(r)L^2}{r^3} + \frac{6f(r)L^2}{r^4} + \epsilon f''(r) \tag{2.9}$$

2.3.1 Null-like

For a photon, the expression reduces to

$$V_r'' = (\frac{f''(r)}{r^2} - \frac{4f'(r)}{r^3} + \frac{6f(r)}{r^4})$$

Substituting from equation (2.7b)

$$V_r'' = (\frac{f(r)f''(r)}{2} - (\frac{f'(r)}{4})^2)L^2 \tag{2.10}$$

At the photon radius $r = r_{ph}$

$$V_r'' = -\frac{2L^2}{(3M)^4} < 0 \tag{2.11}$$

Therefore, the circular photon orbit is unstable

2.3.2 Time-like

The effective potential for a time-like particle becomes

$$\begin{aligned} V_r'' &= \frac{f''(r)L^2}{r^2} - \frac{4f'(r)L^2}{r^3} + \frac{6f(r)L^2}{r^4} + f''(r) \\ &= L^2(\frac{6r-24M}{r^5}) - \frac{4M}{r^3} \\ &= \frac{4Mr - 24M^2}{(2r-6M)r^3} \end{aligned} \tag{2.12}$$

Therefore, for $r_c > 6M$, the circular orbit is stable. Otherwise for $3M < r_c < 6M$, the orbit is unstable.

Chapter 3

Geometric Approach

3.1 Optical Metric

The Optical metric is a tool that is used to analyse the motion of photons and other mass-less particles. We use the null constraint $ds^2 = 0$ to get this new modified metric. We can reduce the Schwarchild metric (1.1) by removing θ dependence ($\theta = \pi/2$)due to the spherically symmetric nature of the space-time. Thus the metric becomes

$$d\tau^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\phi^2 \tag{3.1}$$

Also using the null constraint, we get the optical metric as

$$\begin{aligned} 0 &= f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\phi^2 \\ \implies dt^2 &= \frac{1}{f(r)^2}dr^2 + \frac{r^2}{f(r)}d\phi^2 \end{aligned} \tag{3.2}$$

For analysing the photon geodesic, we will define 2 important geometric quantities: Geodesic Curvature($\mathbf{k}_g$) and Gaussian Curvature ($\mathcal{K}_G$)

3.2 Geodesic Curvature

The geodesic curvature of a curve measures how far the curve is from being a geodesic. As we are dealing with the optical metric, we will define the geodesic in this 2D plane. Let us consider a 2D curved surface S, embedded in a 3 dimensional Euclidean space. The tangent vector is the defined by $dR/d\lambda$, where $R(x^1, x^2, \hat{n})$ is the position vector and λ is an affine parameter used to parameterise the curve. Then the curvature of this curve is defined as $dT/d\lambda$. According to the definition of geodesic curvature, it is defines as the tangential component of this curvature.

$$\mathbf{k}_g = (\frac{dT}{d\lambda})^{tan} = (\frac{d^2R}{d\lambda^2})^{tan} \tag{3.3}$$

Now, we can calculate the explicit expression from the above relation.

$$\begin{aligned} \frac{d^2R}{d\lambda^2} &= \frac{d}{d\lambda}(\frac{dR}{dx^i}\frac{dx^i}{d\lambda}) \\ &= \frac{\partial R}{\partial x^i}\frac{d^2x^i}{d\lambda^2} + \frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\frac{\partial^2 R}{\partial x^i \partial x^j} \end{aligned}$$

Now we can resolve the equation in tangential and normal component by taking

$$\frac{\partial^2 R}{\partial x^i \partial x^j} = \Gamma^k_{ij}\frac{\partial R}{\partial x^k} + L_{ij}\hat{n}$$

Where Γ^k_{ij} are the Christoffel symbols and give us the tangential component, while L_{ij} gives us the normal components. Substituting this, we get

$$\begin{aligned} \frac{d^2R}{d\lambda^2} &= \frac{\partial R}{\partial x^i}\frac{d^2x^i}{d\lambda^2} + \frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}(\Gamma^k_{ij}\frac{\partial R}{\partial x^k} + L_{ij}\hat{n}) \\ \implies \frac{d^2R}{d\lambda^2} &= \frac{\partial R}{\partial x^i}(\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}) + \frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}L_{ij}\hat{n} \end{aligned}$$

Thus, geodesic curvature is

$$\mathbf{k}_g = \frac{\partial R}{\partial x^i}\left(\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}\right) \tag{3.4}$$

By definition, a geodesic should have the geodesic curvature equal to zero, as there is no deviation. Hence, $\mathbf{k}_g = 0$ for geodesics. This gives us the condition that

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0 \tag{3.5}$$

which is the well-known geodesic equation.

3.2.1 Photon Orbit

From the previous definition, we can now expand the geodesic equation to calculate the radius of circular photon orbit. To do so, we'll calculate the Christoffel symbols, and impose the circular condition, i.e, $\dot{r} = 0$, $\ddot{r} = 0$.

$$\Gamma^l_{ij} = \frac{1}{2}g^{lk}(\partial_j g_{ki} + \partial_k g_{ji} - \partial_k g_{ij}) \tag{3.6}$$

$$\begin{gathered}\Gamma^r_{rr} = -\frac{f'(r)}{f(r)}; \quad \Gamma^r_{r\phi} = 0; \quad \Gamma^r_{\phi\phi} = \frac{r^2 f'(r)}{2} - rf(r) \\ \Gamma^\phi_{\phi\phi} = 0; \quad \Gamma^\phi_{r\phi} = -\frac{f'(r)}{2f(r)} + \frac{1}{r}; \quad \Gamma^\phi_{rr} = 0\end{gathered} \tag{3.7}$$

Now, using the geodesic equation with $\dot{r} = 0$, $\ddot{r} = 0$,

$$\begin{aligned} \frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda}\Gamma^r_{\phi\phi} &= 0 \\ \implies \Gamma^r_{\phi\phi} &= 0 \\ \implies \frac{r^2 f'(r)}{2} - rf(r) &= 0 \\ \implies \frac{r}{2}\left(\frac{2M}{r^2}\right) - \left(1 - \frac{2M}{r}\right) &= 0 \\ \implies r_{ph} &= 3M \end{aligned} \tag{3.8}$$

As expected, we get the photon orbit as $r_{ph} = 3M$

3.3 Gaussian Curvature

This geometric quantity is the one we use to define the stability of the geodesic. This is the most general form of curvature in 2D, which tells us how far a surface is from being intrinsically flat.

Taking any two vector fields, $\vec{v}$, $\vec{s}$, from definition, we can calculate the value of Gaussian curvature by taking the dot product of Riemannian Curvature Tensor with $\vec{s}$ and dividing the expression by square of area of parallelogram made by $\vec{v}, \vec{s}$. Hence we get the relation for Gaussian curvature as (using area of parallelogram $= |\vec{v} \times \vec{s}|$)

$$\mathcal{K}_G = \frac{R(\vec{s}, \vec{v})\vec{v}.\vec{s}}{|\vec{v}|^2|\vec{s}|^2 - |\vec{v}.\vec{s}|^2} \tag{3.9}$$

Now for deriving the relation of stability with this curvature, we need to look at geodesic deviation.

3.3.1 Geodesic Deviation

The geodesic deviation is defined by taking double covariant derivative of separation vector, $\vec{s}$ with respect to the vector field $\vec{v}$ in which the geodesics lie. This term defines the rate of divergence of the separation vectors.

$$Geodesic\, Deviation = \nabla_{\vec{v}}\nabla_{\vec{v}}\vec{s} \tag{3.10}$$

Since, the separation vector and the vector field form a closed parallelogram,

$$[\vec{v}, \vec{s}] = \nabla_{\vec{v}}\vec{s} - \nabla_{\vec{s}}\vec{v} = 0 \tag{3.11}$$

By definition of geodesics, we also know that $\nabla_{\vec{v}}\vec{v} = 0$. Then using equation (3.11), we write

$$\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{s} - \nabla_{\vec{v}}\nabla_{\vec{s}}\vec{v} + \nabla_{\vec{s}}\nabla_{\vec{v}}\vec{v} = 0 \tag{3.12}$$

We can now use the expression for Riemmanian Curvature tensor

$$R(\vec{s},\vec{v})\vec{v} = \nabla_{\vec{s}}\nabla_{\vec{v}}\vec{v} - \nabla_{\vec{v}}\nabla_{\vec{s}}\vec{v} - \nabla_{[\vec{v},\vec{s}]}\vec{v} = 0 \tag{3.13}$$

Using equation (3.11), (3.12), (3.13) we can write

$$R(\vec{s},\vec{v})\vec{v} = -\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{s} \tag{3.14}$$

Now that we have a relation between Riemann Tensor and geodesic deviation, we can define the stability of the geodesics.

3.4 Stability Analysis

Depending on the sign of $\mathcal{K}_G$, the stability can be determines as follows:

- if $\mathcal{K}_G = 0$, then, $\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{s} = 0$. This implies that the curves deviate at a constant rate. i.e intrinsically flat surface

- if $\mathcal{K}_G > 0$, then, $\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{s} < 0$. This implies that the separation between the curves reduces. Hence, it is stable.

- if $\mathcal{K}_G < 0$, then, $\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{s} > 0$. This implies that the separation between the curves diverges. Hence, it is unstable.

Since our concern is with analysing the stability for the optical black hole metric, we can choose $\vec{v},\vec{s}$ as the 2 basis vectors $\vec{e_1},\vec{e_2}$. So the Gaussian curvature in the 2D manifold is given by:

$$\mathcal{K}_G = \frac{R(\vec{e_1}, \vec{e_2})\vec{e_2}.\vec{e_1}}{|\vec{e_1}|^2|\vec{e_2}|^2 - |\vec{e_1}.\vec{e_2}|^2}$$
$$\mathcal{K}_G = \frac{R^l_{212}e_l.e_1}{|\vec{e_1}.\vec{e_1}||\vec{e_2}.\vec{e_2}| - |\vec{e_1}.\vec{e_2}|^2} \tag{3.15}$$
$$\mathcal{K}_G = \frac{R^1_{212}g_{11}}{|g_{11}||g_{22}| - |g_{12}|^2}$$

3.5 For the Optical Metric

Now that we have the explicit expression for $\mathcal{K}_G$, we can directly compute its functional form. For the optical metric, we have

$$g_{11} = g_{rr} = \frac{1}{f(r)^2}; \quad g_{22} = g_{\phi\phi} = \frac{r^2}{f(r)}; \quad g_{12} = g_{r\phi} = 0 \tag{3.16}$$

The expression of Riemann Tensor $R^r_{\phi r\phi}$ can be calculated using

$$R^l_{ijk} = \Gamma^l_{mj}\Gamma^m_{ik} - \Gamma^l_{mk}\Gamma^m_{ij} - \partial_k\Gamma^l_{ij} + \partial_j\Gamma^l_{ik} \tag{3.17}$$

Thus, we get,

$$\begin{aligned}
R^r_{\phi r\phi} &= \Gamma^r_{mr}\Gamma^m_{\phi\phi} - \Gamma^r_{m\phi}\Gamma^m_{\phi r} - \partial_\phi\Gamma^r_{\phi r} + \partial_r\Gamma^r_{\phi\phi} \\
&= \Gamma^r_{rr}\Gamma^r_{\phi\phi} - \Gamma^\phi_{r\phi}\Gamma^r_{\phi\phi} + \partial_r\Gamma^r_{\phi\phi} \\
&= (-\frac{f'(r)}{f(r)})(\frac{r^2f'(r)}{2} - rf(r)) - (\frac{r^2f'(r)}{2} - rf(r))(-\frac{f'(r)}{2f(r)} + \frac{1}{r}) + \frac{\partial}{\partial r}(\frac{r^2f'(r)}{2} - rf(r)) \\
&= (\frac{f(r)f''(r)}{2} - \frac{(f'(r))^2}{4})\frac{r^2}{f(r)}
\end{aligned} \tag{3.18}$$

Therefore, from equation (3.15) and (3.18), the Gaussian curvature is given by

$$\mathcal{K}_G = \frac{f(r)f''(r)}{2} - \frac{(f'(r))^2}{4} \tag{3.19}$$

Chapter 4

Lyapunov Exponents

Lyapunov exponents are quantities that determine the rate of separation of 2 infinitesimally close trajectories. In simpler terms, it determines the rate at which the separation vector between 2 trajectories evolves over time. Mathematically,

$$\delta X_i(t) = \delta X_i(0) e^{\lambda t} \tag{4.1}$$

where $\delta X_i(t)$ is the separation vector at time, t, λ is the Lyapunov exponent. From the equation (4.1), the value of λ can be estimated as

$$\implies e^{\lambda t} = \frac{\delta X_i(t)}{\delta X_i(0)} \implies \lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\delta X_i(t)}{\delta X_i(0)} \tag{4.2}$$

Now, by definition of (4.1), we can see that for positive real values, the separation will continuously increase. Hence, the orbit is deemed unstable. However, if the exponents is imaginary, the separation will be oscillatory, hence it neither increases nor decreases. Thus, the orbit is stable.

4.1 Lyapunov Exponents from Equation of Motion

Let $\frac{dX_i}{dt} = H_i(X_j)$ be any equation of motion where X_i's are the phase space variables. Taking a perturbation $X_i \to X_i + \delta X_i$, we get

$$\frac{d(X_i + \delta X_i)}{dt} = H_i(X_j) + \frac{\partial H_i}{\partial X_j}\Big|_{X_i} \delta X_i + H.O.$$

Taking up to linear order, we get

$$\implies \frac{d\delta X_i(t)}{dt} = \frac{\partial H_i}{\partial X_j}\Big|_{X_i} \delta X_i(t) = K_{ij}(t)\delta X_i(t) \tag{4.3}$$

Assuming a linear solution

$$\delta X_i(t) = L_{im}\delta X_m(0) \tag{4.4}$$

we get,

$$\frac{dL_{im}(t)}{dt}\delta X_m(0) = K_{ij}L_{jm}\delta X_m(0)$$
$$\implies (\frac{dL_{im}(t)}{dt} - K_{ij}L_{jm})\delta X_m(0) = 0$$

Since δX_m are linearly independent, each equation in bracket should be 0

$$\implies \frac{dL_{im}(t)}{dt} - K_{ij}L_{jm} = 0 \tag{4.5}$$

From equation (4.5) we get the simultaneous equations

$$\begin{aligned}
\frac{dL_{11}}{dt} &= K_{11}L_{11} + K_{12}L_{21} \\
\frac{dL_{21}}{dt} &= K_{21}L_{11} + K_{22}L_{21} \\
\frac{dL_{12}}{dt} &= K_{11}L_{12} + K_{12}L_{22} \\
\frac{dL_{22}}{dt} &= K_{21}L_{12} + K_{22}L_{22}
\end{aligned} \tag{4.6}$$

Differentiating (4.6a) with respect to t and substituting (4.6b), we get

$$\frac{d^2L_{11}}{dt^2} = (K_{11} + K_{22})\frac{dL_{11}}{dt} + (K_{12}K_{21} - K_{22}K_{11})L_{11} \tag{4.7}$$

Substituting $L_{11}(t) = e^{\alpha t}$, we get the equation $\alpha^2 - (K_{11} + K_{22})\alpha - (K_{12}K_{21} - K_{22}K_{11}) = 0$ with solutions

$$\alpha_{\pm} = \frac{1}{2}[(K_{11} + K_{22}) \pm \sqrt{(K_{11} + K_{22})^2 + 4K_{12}K_{21}}] \tag{4.8}$$

From equation (4.4) we get the initial condition that $L_{im}(0) = \delta_{im}$. Using this, we get solutions of L_{ij}

$$\begin{aligned} L_{11} = L_{22} &= \frac{1}{2}(e^{\alpha_+ t} + e^{\alpha_- t}) \\ L_{12} = L_{21} &= \frac{1}{2}(e^{\alpha_+ t} - e^{\alpha_- t}) \end{aligned} \tag{4.9}$$

and hence we get,

$$\begin{aligned} \delta X_1(t) &= L_{11}(t)\delta X_1(0) + L_{12}(t)\delta X_2(0) \\ \delta X_2(t) &= L_{21}(t)\delta X_1(0) + L_{22}(t)\delta X_2(0) \end{aligned} \tag{4.10}$$

Substituting equation (4.10), (4.9), (4.8) in equation (4.2)

$$\begin{aligned} \lambda &= \lim_{t\to\infty} \frac{1}{t} \ln \frac{L_{11}(t)\delta X_1(0) + L_{12}(t)\delta X_2(0)}{\delta X_1(0)} \\ &= \lim_{t\to\infty} \frac{1}{t} \ln \frac{\frac{1}{2}(e^{\alpha_+ t} + e^{\alpha_- t})\delta X_1(0) + \frac{1}{2}(e^{\alpha_+ t} - e^{\alpha_- t})\delta X_2(0)}{\delta X_1(0)} \\ &= \lim_{t\to\infty} \frac{1}{t}[\ln e^{\frac{K_{11}+K_{22}}{2}t} + \ln e^{\frac{\sqrt{(K_{11}+K_{22})^2+4K_{12}K_{21}}}{2}t} + c] \\ &= \frac{(K_{11} + K_{22}) + \sqrt{(K_{11} + K_{22})^2 + 4K_{12}K_{21}}}{2} \end{aligned} \tag{4.11}$$

4.2 Lyapunov Exponents in Schwarzschild Metric

Recalling equation (2.2) and rewriting it in terms of p_r from equation (2.3), the Lagrangian for the system is given by

$$2\mathcal{L} = -f(r)p_r^2 + \frac{E^2}{f(r)} - \frac{L^2}{r^2}$$

Taking the phase space variables as r,p_r, we can get the Equations of motion by using the Euler-Lagrange equations

$$2\dot{p_r} = 2\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 2\frac{\delta \mathcal{L}}{\delta r} = -f'(r)p_r^2 + f'(r)\frac{E^2}{f(r)^2} - 2\frac{L^2}{r^3}$$

$$\dot{r} = f(r)p_r$$

By multiplying with $1/\dot{t}$ and using equation (2.3), we get the equation of motion with respect to time as

$$\frac{r}{t} = \frac{\dot{r}}{\dot{t}} = H_1 = \frac{f(r)^2}{E}p_r \tag{4.12}$$

$$\frac{p_r}{t} = \frac{\dot{p_r}}{\dot{t}} = H_2 = -f'(r)f(r)\frac{p_r^2}{2E} + f'(r)\frac{E}{2f(r)} - f(r)\frac{L^2}{Er^3} \tag{4.13}$$

From equations (4.12), (4.13) and (4.3), we get

$$K_{11} = \frac{\partial H_1}{\partial r} = \frac{2f(r)f'(r)p_r}{E} = \frac{4M}{E}\frac{r-2M}{r^3}p_r \tag{4.14}$$

$$K_{12} = \frac{\partial H_1}{\partial p_r} = \frac{f(r)^2}{E} = \frac{(r-2M)^2}{Er^2} \tag{4.15}$$

$$\begin{aligned} K_{21} &= \frac{\partial H_2}{\partial r} \\ &= -(f''(r)f(r) + f'(r)^2)\frac{p_r^2}{2E} + (f''(r) - \frac{f'(r)^2}{f(r)})\frac{E}{2f(r)} - (f'(r) - 3\frac{f(r)}{r})\frac{L^2}{Er^3} \\ &= E[\frac{2Mr-2M^2}{r^2(r-2M)^2}] - \frac{L^2}{E}[\frac{3r-8M}{r^5}] + \frac{p_r^2}{E}[\frac{2Mr-6M^2}{r^4}] \end{aligned} \tag{4.16}$$

$$K_{22} = \frac{\partial H_2}{\partial p_r} = -f'(r)f(r)\frac{p_r^2}{E} = -\frac{-2M(r-2M)p_r}{Er^3} \tag{4.17}$$

Using equation (4.14) to equation (4.17) in equation (4.11)

$$\lambda = \frac{r-2M}{r^3}\frac{Mp_r}{E}+\sqrt{\frac{2Mr-2M^2}{r^4}+\frac{p_r^2}{E^2}\frac{(r-2M)^2}{r^6}(2Mr-5M^2)-\frac{L^2}{E^2}\frac{(r-2M)^2}{r^7}(3r-8M)} \tag{4.18}$$

4.3 Stability Analysis

From the arguments in Chapter 2, we know that for a Circular photon orbit, $V_r \propto \dot{r}^2 = 0$, and $V_r' \propto \ddot{r} = 0$. We also know from equation (2.3) that $p_r \propto \dot{r}$. Hence for an orbit of constant radius, r_c, the lyapunov exponent reduces to the form

$$\lambda = \sqrt{(\frac{2Mr-2M^2}{r^4}) - \frac{L^2}{E^2}\frac{(r-2M)^2}{r^7}(3r-8M)} \tag{4.19}$$

4.3.1 Null-like

From equation (2.7b), we can reduce the expression of lyapunov exponent to

$$\begin{aligned}\lambda &= \sqrt{(\frac{2Mr-2M^2}{r^4}) - \frac{r^3}{(r-2M)}\frac{(r-2M)^2}{r^7}(3r-8M)} \\ &= \sqrt{\frac{-3r^2+16Mr-18M^2}{r^4}}\end{aligned} \tag{4.20}$$

It can be seen from the above equation that at $r = r_{ph}$,

$$\lambda = \sqrt{\frac{1}{27M^2}} \tag{4.21}$$

As the lyapunov exponent is real, it describes an unstable photon orbit as expected.

4.3.2 Time-like

From equation (2.8), the lyapunov exponent reduces to the

$$
\begin{aligned}
\lambda &= \sqrt{(\frac{2Mr-2M^2}{r^4}) - \frac{Mr^3}{(r-2M)^2}\frac{(r-2M)^2}{r^7}(3r-8M)} \\
&= \sqrt{\frac{6M^2-Mr}{r^4}}
\end{aligned}
\tag{4.22}
$$

As expected, we get the condition that for $r_c > 6M$, the Lyapunov exponent is complex, hence the orbits will be stable.

Chapter 5

Equivalence Between the Approaches

5.1 Between Chapter 2 and Chapter 4

To show the equivalence, we start with a perturbation in the circular orbit. Then by Taylor Series Expansion, we can write effective potential V_r around r_c as

$$V_r(r = r_c + dr) = V_r|_{r_c} + V_r'|_{r_c}(r - r_c) + V_r''|_{r_c}\frac{(r - r_c)^2}{2} + H.O. \tag{5.1}$$

Since $V_r, V_r' = 0$ at $r = r_c$, we can approximate the potential as

$$V_r \approx V_r''|_{r_c}\frac{(r - r_c)^2}{2} \tag{5.2}$$

Then the equation of motion (2.6) becomes,

$$\begin{aligned}
\dot{r}^2 + V_r''|_{r_c}\frac{(r - r_c)}{2} &= 0 \\
\implies \frac{dr}{d\alpha} &= \sqrt{-V_r''|_{r_c}\frac{(r - r_c)^2}{2}} \\
\implies \frac{1}{\dot{t}}\frac{dr}{dt} &= \sqrt{\frac{-V_r''|_{r_c}}{2}}(r - r_c) \\
\implies r - r_c &\approx ce^{\sqrt{\frac{-V_r''|_{r_c}}{2\dot{t}^2}}}
\end{aligned} \tag{5.3}$$

Comparing Equation (5.3) and Equation (4.1), we can see that

$$\lambda = \sqrt{\frac{-V_r''}{2\dot{t}^2}} \tag{5.4}$$

Thus, it can be seen that when $V_r < 0$, λ is real. Hence, the orbit is unstable. Similarly, when $V_r > 0$, λ is imaginary and the orbit is stable.

In fact, we can also get the equation (5.4) from the derivation of Lyapunov exponents. To verify this, we can calculate the equations in a simpler manner. Equation (4.2) can be written as

$$\begin{aligned} H_1 &= \frac{\dot{r}}{\dot{t}} = \frac{-p_r}{g_{rr}\dot{t}} \\ \implies K_{11} &= 0 \quad (since\, p_r = 0) \\ \implies K_{12} &= \frac{-1}{g_{rr}\dot{t}} \end{aligned} \tag{5.5}$$

$$\begin{aligned} H_2 &= \frac{\partial\mathcal{L}/\partial r}{\dot{t}} \\ \implies K_{21} &= \frac{d}{dr}(\frac{\partial\mathcal{L}/\partial r}{\dot{t}}) \\ \implies K_{22} &= \frac{d}{dp_r}(\frac{\partial\mathcal{L}/\partial r}{\dot{t}}) = 0 \quad (since\, p_r = 0) \end{aligned} \tag{5.6}$$

Substituting these in equation (4.11), we get

$$\lambda = \sqrt{K_{21}K_{12}} \tag{5.7}$$

From Euler-Lagrange equation and the equation of motion(2.6),

$$\frac{d}{d\tau}\frac{\delta\mathcal{L}}{\delta\dot{r}} = \frac{\delta\mathcal{L}}{\delta r} \implies \frac{d}{d\tau}(-g_{rr}\dot{r}) = -\dot{r}\frac{d}{dr}g_{rr}\dot{r} = -\frac{1}{2g_{rr}}\frac{d}{dr}(g_{rr}^2\dot{r}^2) \implies \frac{\delta\mathcal{L}}{\delta r} = \frac{1}{2g_{rr}}\frac{d}{dr}(g_{rr}^2V_r) \tag{5.8}$$

Substituting in equation (5.7), we get back the relation (5.4)

$$\lambda = \sqrt{\frac{-1}{g_{rr}\dot{t}}\frac{d}{dr}[(\frac{1}{\dot{t}})(\frac{1}{2g_{rr}}\frac{d}{dr}(g_{rr}^2 V_r))]}$$

With the condition that $V_r = 0$ and $V_r' = 0$, it simplifies to

$$\lambda = \sqrt{\frac{-V_r''}{2\dot{t}^2}} \tag{5.9}$$

5.2 Between Chapter 2 and Chapter 3

In the conventional approach, $V_r = 0$ and $V_r' = 0$ gave us the condition for circular orbit with $r_{ph} = 3M$. In Chapter 3, the same result was obtained by setting $\mathbf{k}_g = 0$. Hence, from the result directly, we see that the 2 approaches are equivalent. However, it has to be noted that this is only true for the null-like geodesic since the optical metric is valid only for mass-less particles such as photons.

The equivalence between the effective potential and the Gaussian curvature is very straight forward. From equations (3.19) and (2.10), it can be seen that the expressions for Gaussian curvature and the V_r'' are the same.

$$\mathcal{K}_G = \frac{f(r)f''(r)}{2} - \frac{(f'(r))^2}{4}$$

$$V_r'' = (\frac{f(r)f''(r)}{2} - (\frac{f'(r)}{4})^2)L^2$$

$$\implies V_r'' = \mathcal{K}_G L^2 \tag{5.10}$$

Hence, when the orbit is unstable, $V_r'' < 0$ and consequently $\mathcal{K}_G < 0$, which implies the curves are moving farther away which is as expected of an unstable orbit. The vice-versa is also true.

5.3 Between Chapter 3 and Chapter 4

By definition, lyapunov exponent is a parameter that defines the separation of the trajectories or in this case the geodesic curves. Similarly, the Gaussian curvature also is parameter that defines the rate of separation of the curves. So, it is intuitive to understand that there is a proportional relation between these quantities. Mathematically, one can directly substitute equation (5.10) in equation (5.9) to get the relation between the Lyapunov exponent and the Gaussian curvature.

$$\lambda = \sqrt{-\frac{\mathcal{K}_G L^2}{\dot{t}^2}} \tag{5.11}$$

Hence, as expected the Lyapunov exponent being real implies $\mathcal{K}_G < 0$ which corresponds to the unstable orbit and vice versa. However, it has to be noted that the equation (5.9) is valid for both time-like and null-like geodesics, but, since the Gaussian curvature $\mathcal{K}_G$ is defined in the optical metric, the equation (5.11) is only valid for Null-like geodesics.

Chapter 6

Conclusion and Future Work

In this paper, we defined different parameters to calculate and analyse the stability of the geodesics around the Schwarszchild black hole. We calculated the effective potential and used the concavity of it, V_r'' to define the stability. We also used the optical metric to define geometric parameters like the Gaussian curvature to define stability. Finally, we calculated the stability parameter, Lyapunov exponents to define the stability. In conclusion, the equivalence and the important parameter are as listed below:

	Conventional Approach	Geometric Approach	Chaotic Exponent Approach
Parameter	Effective Potential, V_r	Gaussian Curvature, $\mathcal{K}_G$	Lyapunov Exponent, λ
Stable Orbit	$V_r'' > 0$	$\mathcal{K}_G > 0$	λ is imaginary
Unstable Orbit	$V_r'' < 0$	$\mathcal{K}_G < 0$	λ is positive and real

6.1 Future Work

While the relation between V_r and λ is easily defined, the description of it using just the geometric approach is more complex. Moving forward, we will try to derive the explicit relation between them. Lyapunov exponents have also been related with Quasi-normal modes of black holes. We will also be reviewing this relation.

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