Shadows of Black Holes

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Abstract

In this paper, the analytical calculation for photon shadows around a black hole are reviewed. In particular, the propagation of photons in Schwarzschild metric and Kerr metric are studied to calculate the shadow. These calculations are used to derive the angular size of the shadow in these space-times. It also compares these shadows for various parameters such as the observer distance and spin parameter. A general method to outline a photon orbit in a generic space-time is stated. This paper also describes current status and the further scope of research in this topic.

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Motivation

Black holes, and GTR in general, have been a significant research field for theoretical physicists. Since the recent discoveries, simulating images of black holes has been an important task. We can perform this task by studying photon orbits around a black. Any progress in this will help us understand black holes better and build upon existing theories. This paper reviews shadows of black holes using fundamentals of GTR and classical mechanics to simulate photon orbits.

1 Introduction

Black holes are compact objects of very large mass, around which there exists a region of space time, where the gravity so strong that even light is not able to escape the event horizon. Such objects have been theorized as early as 1784. A concrete theory came into existence after Einstein's Theory of Relativity. By solving the Einstein's Field Equations, Karl Schwarzschild was able to find such a solution which consisted of a point-like object with mass M.

In 2016, LIGO and Virgo collaboration announced the first detection of gravitational waves, representing a black hole merger. In 2019, the first image of Black hole was published form the observational data of the Event horizon telescope. These were the first observational proofs for the existence of Black Holes In this image, one could observe a dark region in the center surrounded by bright disk-like regions. This bright region is termed as the photon radius surrounding a black hole and the central dark region is the shadow cast by it.

In this article, the calculations behind simulating a theoretical model for such Black hole orbits has been outlined. Such calculations have been done in the past and are currently being done for various models of space-times. Simulations of such shadows help in identifying black holes and distinguishing them from other compact objects. This also helps in identifying some basic properties surrounding the propagation of light in such space-times.

As a black hole pulls everything close to it to a point of no escape, an observer expects to see a black spot in place of the black hole. Therefore, there exist photons that are unstably orbiting a black hole just beyond the point of no escape. This orbit as seen by a observer is defined as the photon radius. However, due to strong gravity of the Black hole, the size and shape of the photon orbit as seen by the observer are dissimilar to what one might expect the shadow of a trivial object will look like. The difference in these images is due to the bending of light and is termed as 'Gravitational Lensing'. The expected shadow can be computed from calculating the motion of photons in the black hole metric by solving the Lagrangian. However, to simulate the image as seen by the observer, one has to define a set of locally inertial coordinates around the observer and compute the apparent shadow as seen by the observer.

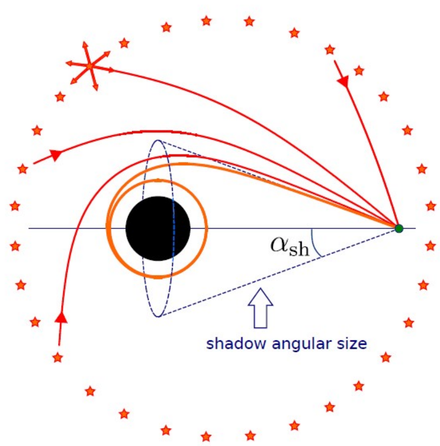

Figure 1: This image shows the orbit of photons and the apparent angular radius seen by the observer. The orange paths are rays that are captured by the BH to an unstable orbit around it

In particular, the analytical calculation for the photon radius of Schwarzschild metric and the Kerr metric has been reviewed in this paper. The scope of this paper is limited to propagation of light in highly idealised conditions and in vacuum. For more complex systems, one needs to employ numerical methods for asymptotically getting the photon orbits.

2 Static and Spherically Symmetric Metric

For the simplest black hole metric, we consider a system with a spherical symmetry at the centre of space time. Since this system is spherically symmetric, the metric to this space time will also be spherically symmetric, i.e. the metric does not depend on the polar and azimuthal angle. It also means that this space-time is built upon a 2-sphere. As we are also considering a static case, the metric should be independent of time. So, the metric is reduced to the form

$$ds^2 \;=\; -g_{tt}(r)dt^2 + g_{tr}(r)dtdr + g_{rr}(r)dr^2 + r^2 d\Omega^2\;;\;\; d\Omega^2 \;=\; d\theta^2 + \sin^2\theta d\phi^2$$

By choosing an appropriate co-ordinate transformation, we can always diagonalize the t,r components of the metric. Thus, we end up with a metric ansatz for a static, spherically symmetric system of the form,

$$ds^2 \;=\; -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

2.1 Schwarzschild Metric

For the simplest solution, we consider a charge-less spherical mass in vacuum. Since, we are considering a vacuum solution, the energy-momentum tensor ($T_{\mu\nu}$) is equal to zero. As a consequence of this, the Ricci Tensor in a vacuum tensor also vanishes. Thus by computing the Ricci tensor and equating it to zero, we get a solution unique to this case. Let $g_{tt} = f(r)$ and $g_(rr) = g(r)$. By solving $R_{\mu\nu} = 0$, we get the solution as

$$f(r) = -(1-\frac{A}{r})f(r) = -\frac{1}{g(r)}$$

Considering the metric in weak-field approximation we get $A = 2M$. Thus the schwarzschild metric is

$$ds^2 \;=\; -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

From the following metric, we can get the Lagrangian of a photon moving in this space-time from the equation

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

$$\mathcal{L} = \frac{1}{2}[-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\dot{\phi}^2]$$

From the Lagrangian, we can understand that it is independent of t,ϕ. Therefore, they are killing vectors in this space-time with conserved quantities E, p_ϕ. For any given null geodesic, we can choose the coordinates such that θ is constant along it. Due to this symmetry of the space-time, we can solve the equations of motion in $\theta = \pi/2$ plane without loss is generality. By solving the Euler-Lagrange equation for t,ϕ components, we get the expression for these constants as

$$\frac{d}{d\tau}(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}) - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0$$

$$E = f(r)\dot{t}\;;\;\; p_\phi = r^2\dot{\phi}$$

Instead of solving the other Euler-Lagrange equations, we can solve the first integral of the geodesic equation (here m=0)

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -m^2 = 0$$

$$-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = 0$$

By substituting above constants, we get the equation for $\frac{dr}{d\phi}$ as

$$-\frac{E^2}{f(r)} + f(r)^{-1}\frac{\dot{r}^2}{\dot{\phi}^2}(\frac{p_\phi}{r^2})^2 + p_\phi^2/r^2 = 0$$

$$(\frac{\dot{r}}{\dot{\phi}})^2 = (\frac{dr}{d\phi})^2 = r^4[\frac{E^2}{p_\phi^2} - \frac{f(r)}{r^2}]$$

This equation is similar to a 1D energy conservation equation with ϕ playing the role of time. Thus, we can consider this as the motion an effective potential, V_{eff}

$$(\frac{\dot{r}}{\dot{\phi}})^2 + V_{eff} = 0$$

Thus by solving the equations $V_{eff} = 0$, $(\frac{dV_{eff}}{dr})|_{r_{ph}} = 0$ we get the closest point of approach of the photon (r_{ph}) and the constant of motion (here the impact parameter) $\eta = (p_\phi/E)$. Here we get the solution as

$$r_{ph} = 3M; \quad \eta = 3\sqrt{3}M$$

Plotting the function $\frac{dr}{d\phi} = 0$ with y-axis as EM/p_ϕ and x-axis as r/M, we see that there is a turning point at $r = r_{ph} = 3M$, $\frac{p_\psi}{E} = \eta = 3\sqrt{3}M$. We learn that for $r > 3M$, all photons can escape the barrier and reach the observer at infinity and for $r < 3M$ only photons with $p_\phi/E > \eta$ reach the observer. Thus, we see that $r = r_{ph}$ is the closest the photons can reach and escape the gravity of the black hole. This matches with the result that we calculated analytically.

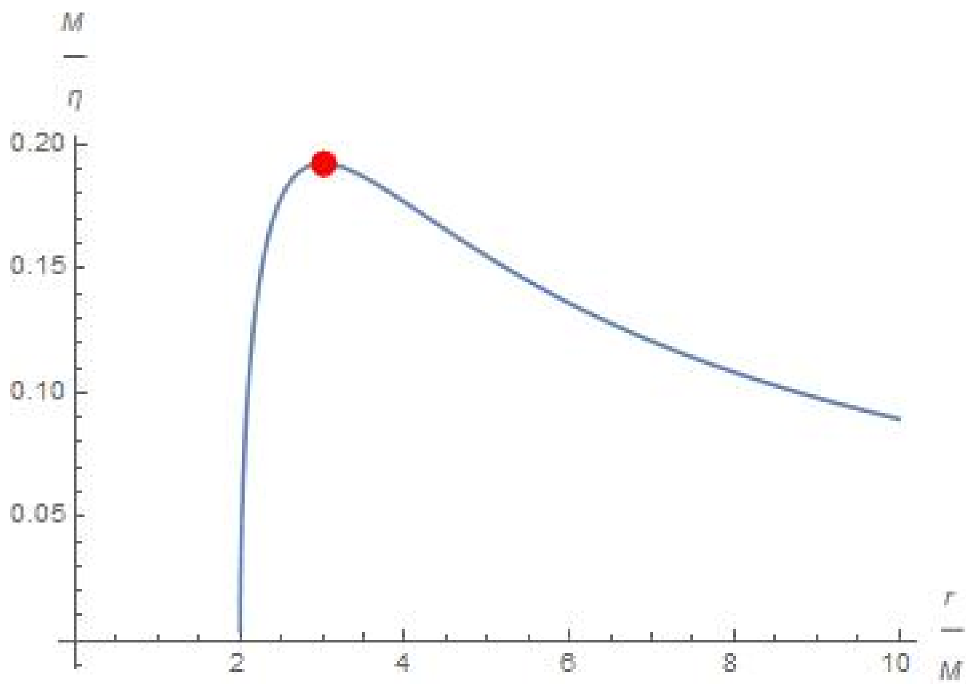

Figure 2: Graph for Effective Potential = 0. The x-axis is r/M and the y-axis is M/ImpactParameter. The maxima for the function is at (3,1/3$\sqrt{3}$)

For calculating the angular size of the shadow we look at the metric in t = constant 3-Space. Here, considering an infinitesimal triangle (r, θ, ϕ), $(r + dr, \theta, \phi)$, $(r, \theta, \phi + d\phi)$, we get

$$\cot \Psi = \frac{1}{rf(r)^{1/2}} \frac{dr}{d\phi}$$

$$=> \cot^2 \Psi = \frac{1}{r^2 f(r)} (\frac{dr}{d\phi})^2 = \frac{r^2}{f(r)} \frac{E^2}{p_\phi^2} - 1$$

Using the trignometric identities $\cot^2 \Psi = \csc^2 \Psi - 1$ and $\csc \Psi = 1/\sin \Psi$

$$\sin^2 \Psi = \frac{p_\phi^2 f(r)}{E^2 r^2}$$

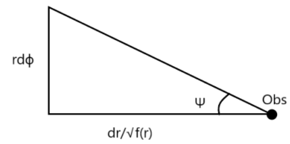

Figure 3: Defining Coordinates as seen by observer in Schwarzschild Metric

Thus for a photon with fixed impact parameter ($p_\phi/E = \eta$), the angular radius as observed by the observer is given as

$$\sin\Psi = \frac{3\sqrt{3}M(1-\frac{2M}{r})}{r}$$

For a observer that is very far (r>>2M), the observed photon radius ($\Psi * r$) is constant ($= 3\sqrt{3}M$)

$$\Psi = \frac{3\sqrt{3}M}{r}$$

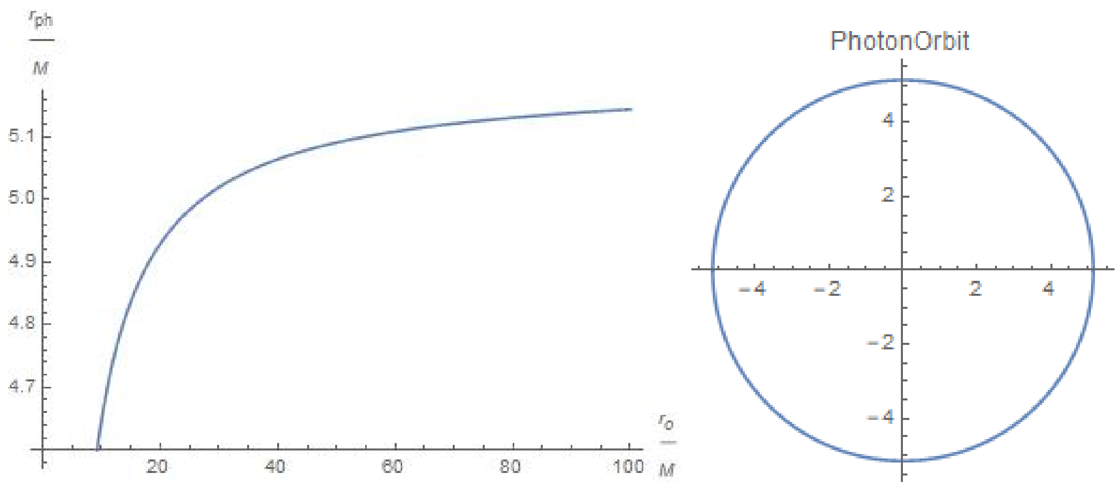

Figure 4: (Left) Angular radius of Photon orbit for various observer distance. (Right) Photon Orbit as seen by observer at $r_o = 10M$

3 Rotating and Axially Symmetric Metric

In the precious section, we constructed the shadow in a spherically symmetric and static space-time. Because of this symmetry, the photons formed a spherical boundary with constant r. In this section we look at the shadow of a black hole that is rotating about a fixed axis.

3.1 Kerr Metric

The Kerr metric is one such such Space-time that is axially symmetric. It describes the geometry around an uncharged rotating black hole. The difference from the Schwarszchild Metric is due to the rotating nature of this black hole, because of which the body has as intrinsic angular momentum J. This gives rise to a cross-term in the t-ϕ coordinate, where we assume the black hole to rotate about the ϕ axis. The metric can be derived form the basic definition using Newman-Penrose formalism, Ernst equation.The Kerr Metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1-\frac{2M}{\Sigma})dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2+a^2+\frac{2Ma^2\sin^2\theta}{\Sigma})\sin^2\theta d\phi^2 - \frac{4Mra\sin^2\theta}{\Sigma}dtd\phi$$

$$\Sigma = r^2 + a^2\cos^2\theta;\ \Delta = r^2 + a^2 - 2Mr;\ a = \frac{J}{M}$$

By putting a = 0, which corresponds to no angular momentum (J=0), we get back the spherically symmetric Schwarzschild Metric. For the following metric, the corresponding Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}[-(1-\frac{2M}{\Sigma})\dot{t}^2 + \frac{\Sigma}{\Delta}\dot{r}^2 + \Sigma\dot{\theta}^2 + (r^2 + a^2 + \frac{2Ma^2\sin^2\theta}{\Sigma})\sin^2\theta\dot{\phi}^2 - \frac{4Mra\sin^2\theta}{\Sigma}\dot{\phi}\dot{t}]$$

From the Lagrangian, we can see two obvious conserved quantities p_t, p_ϕ, they can be obtained by solving the Euler-Lagrange for t,ϕ

$$=> E = (1-\frac{2M}{\Sigma})\dot{t} + \frac{2Mra\sin^2\theta}{\Sigma}\dot{\phi}$$

$$=> L = (r^2 + a^2 + \frac{2Ma^2\sin^2\theta}{\Sigma})\sin^2\theta\dot{\phi} - \frac{2Mra\sin^2\theta}{\Sigma}\dot{t}$$

Another not so obvious conserved quantity in this space-time is the Carter constant K that arises from the separability of the Hamiltonian

$$K = \Sigma^2\dot{\theta}^2 + [(\frac{L}{\sin\theta}) - aE]^2$$

Using these quantities, we can define the Equation of motion of a photon from its light-like geodesics as as

$$\dot{t} = \frac{a(L - Ea\sin^2\theta)}{\Sigma} + \frac{(r^2+a^2)[(r^2+a^2)E - aL]}{\Sigma\Delta}$$

$$\dot{\phi} = \frac{L - aE\sin^2\theta}{\Sigma\sin^2\theta} + \frac{a[(r^2+a^2)E - aL]}{\Sigma\Delta}$$

$$\dot{r}^2 = -\frac{\Delta}{\Sigma^2}[K] + \frac{1}{\Sigma^2}[aL - (r^2+a^2)E]^2$$

$$\dot{\theta}^2 = K - \frac{aE\sin^2\theta - L}{\sin^2\theta}$$

To find the photon boundary determining the black hole shadow, we solve the equations corresponding to $\dot{r} = 0$, $\ddot{r} = 0$, which yields

$$\frac{\Delta}{\Sigma^2}[K] = \frac{1}{\Sigma^2}[aL - (r^2+a^2)E]^2$$

$$4Er[aL - (r^2+a^2)E] = 2(M-r)[K]$$

From these, we find the constants η (impact parameter) and σ as

$$\eta = \frac{L}{E} = \frac{r^2(r-3M) + a^2(r+M))}{a(M-r)}$$

$$\sigma = \frac{K}{E^2} = \frac{4r^2(r^2+a^2-2Mr)}{(r-M)^2}$$

So far we considered the motion of the photon. For constructing the shadow, we will consider light rays which are sent from the observers position into the past. By doing this we can distinguish between 2 photon orbits. In the case of Schwarzschild metric, it was spherically symmetric and all light rays had similar orbits. However, in this case, Light rays that orbit the black hole parallel to its rotation and anti-parallel to its rotation have 2 different orbits. Therefore, we see a non-spherical photon orbit. For calculating the shadow analytically, we will consider a Local inertial frame near the observer defined by the orthonormal tetrad

$$e_o = \frac{(r^2+a^2)\partial_t + a\partial_\phi}{\sqrt{\Sigma\Delta}};$$

$$e_1 = \frac{\partial_\theta}{\sqrt{\Sigma}}$$

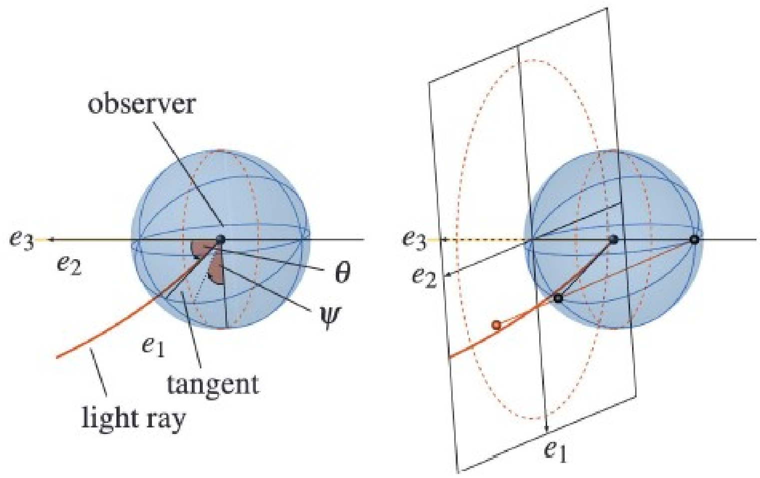

Figure 5: Defining Celestial Coordinates at the observer in Kerr Metric

$$e_2 = \frac{-(a\sin^2\theta)\partial_t - \partial_\phi}{\sqrt{\Sigma\Delta}};$$

$$e_3 = -\frac{\Delta}{\Sigma}$$

This tetrad has been chosen such that $e_0 \pm e_3$ are tangential to in-going principal null ray and out-going principal null ray. With this tetrad we can define the tangent vector at the observer(r_o, θ_o)as

$$\dot{\lambda} = A[-e_0 + \sin\alpha\cos\Psi e_1 + \sin\alpha\sin\Psi e_2 + \cos\alpha e_3]$$

$$A = (\frac{aL - (r^2 + a^2 E)}{\sqrt{\Sigma\Delta}}) \ at \ (r_o, \theta_o)$$

For a light ray, we can write the tangent vector as

$$\dot{\lambda} = \dot{t}\partial_t + \dot{r}\partial_r + \dot{\theta}\partial_\theta + \dot{\phi}\partial_\phi$$

Comparing Coefficients, we get

$$\cos\alpha = \frac{\Sigma\dot{r}}{(r^2 + a^2)E - aL}|_{r_o,\theta_o}$$

$$\sin\Psi = \frac{\sin\theta}{\sqrt{\Delta}\sin\alpha}(\frac{\Sigma\Delta\dot{\phi}}{(r^2 + a^2)E - aL} - a)|_{r_o,\theta_o}$$

Substituting $\dot{\phi}$ and $\dot{t}$ from the light-like geodesic equation,

$$\sin\alpha = \frac{\sqrt{\Delta\sigma}}{r^2 + a^2 - a\eta}|_{r=r_o}$$

$$\sin\Psi = \frac{L - a\sin^2\theta}{\sqrt{\sigma}\sin\theta}|_{\theta=\theta_o}$$

Since, η and σ are constants of the motion, they will be same as that for a limiting case spherical light-like geodesic. i.e.

$$\eta = \frac{r^2(r - 3M) + a^2(r + M))}{a(M - r)}|_{r=r_p}$$

$$\sigma = \frac{4r^2(r^2 + a^2 - 2Mr)}{(r - M)^2}|_{r=r_p}$$

where r_p is the radius of the limiting spherical photon orbit. Substituting these expressions for the angles, we get the boundary curve $(\Psi(r_p),\, \alpha(r_p))$

$$\sin\alpha(r_p) = \frac{2r_p\sqrt{r_p^2+a^2-2Mr_p}\sqrt{r_o^2+a^2-2Mr_o}}{r_o^2 r_p - r_o^2 M + r_p^3 - 3r_p^2 M + 2r_p a^2}$$

$$\sin\Psi(r_p) = -\frac{r_p^3 - 3r_p^2 M + r_p a^2 + a^2 M + a^2 sin^2\theta_o(r_p - M)}{2ar_p\sin\theta_o\sqrt{r_p^2+a^2-2Mr_p}}$$

The minimum and maximum value of r_p can be computed by varying $\sin\Psi(r_p)$ from its maximum to minimum. i.e for $r_{p,min}$, $\sin\Psi(r_p) = 1$ and for $r_{p,max}$, $\sin\Psi(r_p) = -1$

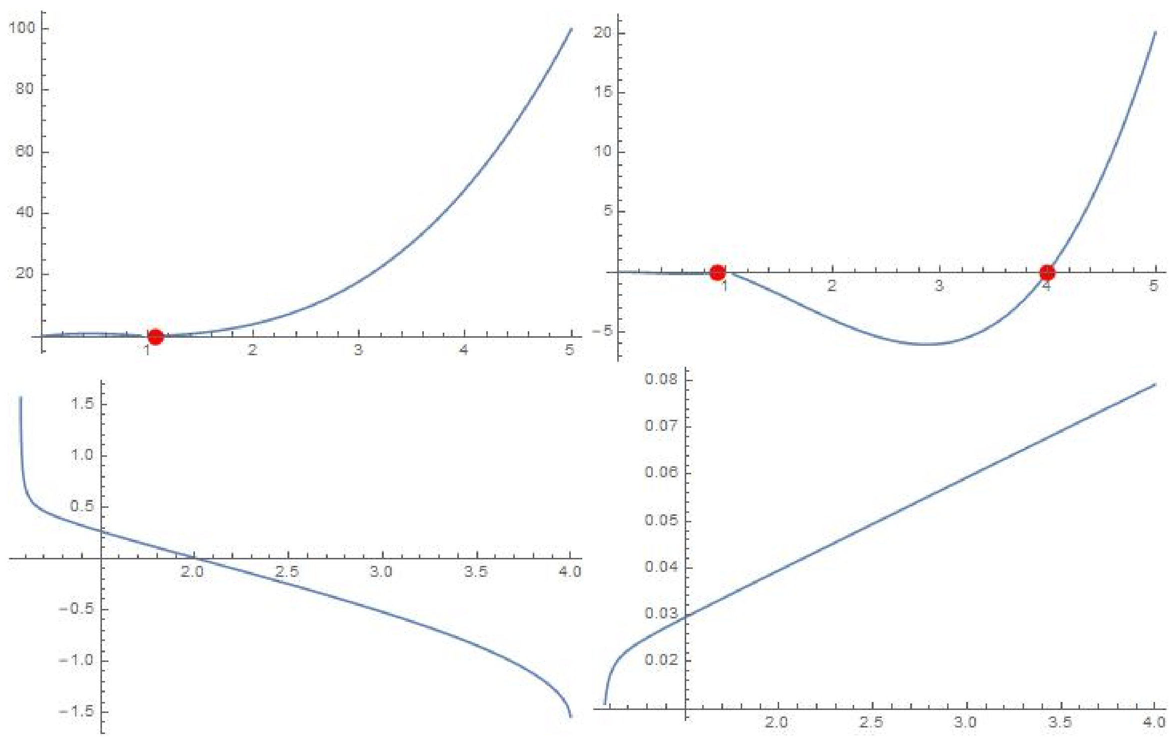

Figure 6: (Top)[a = 0.999, $\theta = \pi/2$] Graphs for $\sin\Psi = 1, -1$ for finding $r_{p,min}$ and $r_{p,max}$ respectively. (Bottom) Values of Ψ, α as r_p goes from is minimum to maximum.

For plotting the shadow, we use a stereographic projection which maps the celestial sphere of the observer onto a plane that is tangent to this sphere. In this plane, the cartesian co-ordinates are given as

$$x(r_p) = -2\tan\left(\frac{\alpha(r_p)}{2}\right)\sin\Psi(r_p); \quad y(r_p) = -2\tan\left(\frac{\alpha(r_p)}{2}\right)\cos\Psi(r_p)$$

For distant observer, the celestial angles gets a simple form, and the cartesian co-ordinates simplifies as

$$\alpha(r_p) = \frac{2r_p\sqrt{r_p^2+a^2-2Mr_p}}{r_o(r_p - M)}$$

$$x(r_p) = \frac{a^2\sin^2\theta_o(r_p - M) + r_p^3 + 3Mr_p^2 + r_o a^2 - a^2 M}{a(r_p - M)r_o\sin\theta_o}$$

$$y(r_p) = \pm\frac{1}{r_o}\sqrt{\frac{4r_p^2 r_p^2 + a^2 - 2Mr_p}{(r_p - M)^2} - \frac{(a^2\sin^2\theta_o(r_p - M) + r_p^3 + 3Mr_p^2 + r_o a^2 - a^2 M)^2}{a^2(r_p - M)^2\sin^2\theta_o}}$$

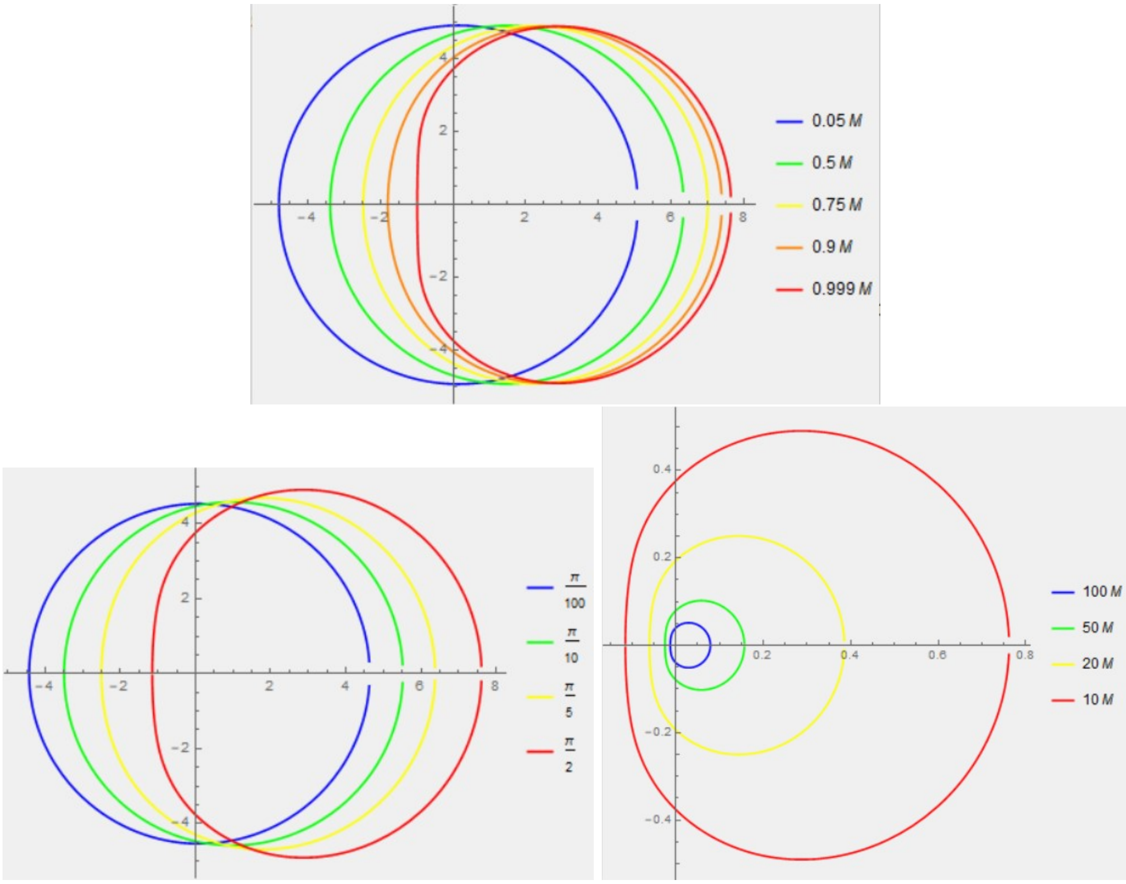

Figure 7: Photon Orbit around Kerr Black hole for various parameters. a) $[\theta_o = \pi/2, r_o = 10M]$(x-axis scaled by r_o) For different spin parameter(a). b) $[a = 0.99M, r_o = 10M]$(x-axis scaled by r_o) For different θ. c) $[\theta = \pi/2, a = 0.99M]$ For different Observer distances.

4 Further Studies

The generalization of this calculation has been done for various other space-times in the Plebanski-Demianski class in a paper by Grenzebach et al. This class consists of Black holes with various parameters such as Spin, Charge, a NUT parameter, Cosmological constant. For all such black holes, one has to find the constants of motion and a set of proper tetrad for the observer. By doing these, one can parameterize the motion of photons in an orbit with respect to the constants and select a proper chart to represent it in the cartesian plane.

Apart from Black holes, lots of other compact objects also cast such shadows. One can repeat a similar process for such Space-times. There have been calculations done for various compact objects such as worm-holes and collapsing stars. In case of Ellis wormhole, the metric is very much similar to the Schwarzschild metric and the shadow cast by it is similar as such. Schneider and Perlick have also discussed the shadow cast by a transparent star. It was found that the shadow for the simplest spherically symmetric case imitates a Schwarzschild Blackhole asymptotically.

For further studies, one can solve the Equation of motion for a photon in any space time numerically. One can trace the path of light in any medium using such numerical methods. Analytical calculations can be done for non-magnetised plasma medium. However, for more complex media, one has to solve the Equations of motion of the photon numerically in an iterative manner to asymptotically reach the exact solution. Such methods can also be used fot understanding the effects in magnetised plasma media.

5 Conclusion

In this paper, the analytical calculations of Black Hole shadow in Schwarzschild and Kerr Metric is seen. The photon radius is first found out and the apparent shadow seen by the observer is calculated. A general process outline to calculating the shadow cast by a compact object was specified. The constants of motions are calculated from the Lagrangian, and these constants are used to parameterize the shadow boundary. One also has to find a suitable tetrad for the actual measurement of the photon orbit. Further studies to these shadows con be done by numerically solving the Equation of motion for more complex Space-Times.

6 Appendix

6.1 A) Mathematica code for Plotting shadow

```
a = 0.999;
tho =  Pi/2;
ro = 100;

plt = Plot[f1 = rp^3 - 3*rp^2 + rp*a^2 + a^2 + a^2*Sin[tho]*Sin[tho]*(rp - 1)
                   + 2*a*rp*Sin[tho]*Sqrt[rp^2 + a^2 - 2*rp], {rp, 0, 5},
                   MeshFunctions -> {#2 &}, Mesh -> {{0}},
                   MeshStyle -> Directive[Red, PointSize[.03]]];
Normal[plt] /. Point[rp_] :> Tooltip[Point[rp], rp[[1]]]
plt2 = Plot[f2 = rp^3 - 3*rp^2 + rp*a^2 + a^2 + a^2*Sin[tho]*Sin[tho]*(rp - 1)
                   -  2*a*rp*Sin[tho]*Sqrt[rp^2 + a^2 - 2*rp], {rp, 0, 5},
                   MeshFunctions -> {#2 &}, Mesh -> {{0}},
                   MeshStyle -> Directive[Red, PointSize[.03]]];
Normal[plt2] /. Point[rp_] :> Tooltip [Point[rp], rp[[1]]]

th = ArcSin[(2*rp*Sqrt[rp^2 + a^2 - 2*rp]*Sqrt[ro^2 + a^2 - 2*ro])/
              (ro^2*rp - ro^2 + rp^3 - 3*rp^2 + 2*rp*a^2)];
phi = ArcSin[-(rp^3 - 3*rp^2 + rp*a^2 + a^2 + a^2*Sin[tho]*Sin[tho]*(rp - 1))/
               (2*a*rp*Sin[tho]*Sqrt[rp^2 + a^2 - 2*rp])];
rpmin = 2.34705
rpmax = 3.5307

Plot[phi, {rp, rpmin, rpmax}];
Plot[th, {rp, rpmin, rpmax}];

gr = ParametricPlot[{-2*Tan[th/2]*Sin[phi]*ro, -2*Tan[th/2]*Cos[phi]*ro},
                    {rp, rpmin, rpmax}, PlotStyle -> Blue, PlotLegends -> { 0.5 M}];
grb = ParametricPlot[{-2*Tan[th/2]*Sin[phi]*ro, 2*Tan[th/2]*Cos[phi]*ro},
                    {rp, rpmin, rpmax}, PlotStyle -> Blue];
Show[gr, grb, PlotRange -> All]
```

For Plotting the shadow:

- Fix the spin parameter for the Black hole and the observer coordinates (r_o, θ_o)
- Compute $r_{p,min}$ and $r_{p,max}$ from the maximum and minimum values for Ψ.
- Using this as the limits and the parametric coordinates, plot the graph. The upper half of the photon orbit will be obtained.
- For the other half, mirror the above plot about the y-axis.

7 References

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