Reducing Noise in Gravitational Wave Detectors by Utilizing Quantum Optics

S Venkat Bharadwaj, Sudhir S, Soham Avinash Atkar, Pratham Kulkarni, Prateek Ranjan Gupta

November 2022

Abstract

Einsten's General Theory of Relativity predicts the existence of gravitational waves, which are the ripples of spacetime. However, detecting these waves require very sensitive instruments which can detect a change in mechanical energy of 10^{-32}J. These detectors are based on the principles of Michelson interferometer. The change in the arm length of the detectors during a gravitational wave event would produce changes in the interference pattern. The reason for low frequency of detecting such events is because the amplitude of signature of gravitational waves are far less than the zero point fluctuations. We discuss here about the sources of various noise in gravitational wave detectors. To improve the accuracy in detecting such events, we can couple the mechanical motion of test mass to optical field through an optomechanical coupling. We discuss about squeezed states of light and its application in improving the accuracy of detectors. At last, we have considered the use of optomechanical interaction to dilute the effect of thermal noise through optical dilution scheme and to narrow the bandwidth of a filter cavity for frequency dependent squeezing.

Contents

1 Quantization of Electromagnetic Field

1.1 Introduction

At the beginning of the 20th century, Max Planck derived a formula that described the distribution of wavelengths emitted from a blackbody, depending on the temperature T. According to his formula light could only be absorbed or emitted in discrete chunks or quanta, whose energy depended on the frequency or wavelength. The idea that electromagnetic waves are quantized into discrete packets called photons was revolutionary as it could explain many phenomena such as the Photoelectric effect, Compton effect, etc. In this section, we will discuss about the Second quantization of electromagnetic field and explore cavity field quantization.

1.2 Second Quantization of Electromagnetic Field

According to the Classical Theory of Electrodynamics,source-free electromagnetic fields can be decomposed as a collection of infinite harmonic oscillators. But in order to define a quantized electromagnetic field we need to apply the theory of the quantum harmonic oscillator. The ground-state energy of a quantum harmonic oscillator with eigenfrequency ω is given by $\hbar\omega/2$. This means that a quantized electromagnetic field is not static but fluctuates with a frequency ω. The electromagnetic vector potential A(t, x) and energy E can be expressed in terms of the annihilation operator $\hat{\mathbf{a}}^\dagger$ and creation operator $\hat{\mathbf{a}}$ as,

$$A(t,x) = \sum_k (2\pi c^2/\omega_k)^{1/2}[\hat{\mathbf{a}} \exp(ikx) + \hat{\mathbf{a}^*} \exp(-ikx)]$$

$$\varepsilon = \sum_k \hbar\omega_k (\hat{\mathbf{a}}^\dagger \, \hat{\mathbf{a}} + 1/2)$$

The energy ε can be defined as the energy of an ensemble of harmonic oscillators with different frequencies ω_k. Now let us find the electric field and magnetic field whose components are aligned in the x and y direction respectively for an EM wave propagating in the z-direction. the representations of these fields in terms of creation and annihilation operators are:

$$E_x(z,t) = i\sum_k (2\pi\hbar\omega_k)^{1/2}[\hat{\mathbf{a(t)}} \exp(ikz) - \hat{\mathbf{a(t)}}^\dagger \exp(-ikz)]$$

$$H_y(z,t) = -i\sum_k (2\pi\hbar c^2/\omega_k)^{1/2} k[\hat{\mathbf{a(t)}} \exp(ikz) - \hat{\mathbf{a(t)}}^\dagger \exp(-ikz)]$$

The following commutation relations can be derived from these expressions:

$$[E_x(z,t), E_x(z',t)] = 0$$

$$[E_x(z,t), H_y(z',t)] = -ihc^2 \frac{\partial\delta(z-z')}{\partial z}$$

$$[H_y(z,t), H_y(z',t)] = 0$$

$$[E_x(z,t), H_y(z,t')] = ih\frac{\partial\delta(t-t')}{\partial t}$$

From these relations, we can see that there are two problems that arise during the quantum measurement of these quantities. 1. Simultaneously measuring the electric and magnetic fields at two space-likely separated points are impossible 2. Measurement of an electric field at one spacetime point (x, t) will introduce disturbance to the measurement result of the electric field at a time-likely separated spacetime point. To solve these issues a resolution limit is set for the measuring devices.

1.3 Cavity field quantization and Quadrature operators

1.3.1 Cavity modes in Quantum Mechanics

As discussed earlier, the quantum treatment of light as a particle describes the energy of the light source which is proportional to the frequency ω. The photon of this frequency is associated with a cavity mode with wavevector k = ω/ c that describes the number of oscillations that the wave can make in a cube with length L. For a very large cavity there can be a continuous range of allowed k values. The cavity is important for considering the energy density of a light field, since the electromagnetic field energy per unit volume will clearly depend on the wavelength $\lambda = 2\pi$/k of the light. Higher the frequency of light, the more modes you can fit into the cavity.

1.3.2 Amplitude and phase quadratures

The optical modes supported by the arm-cavity of the interferometer are described by the Hermite-Gauss mode. The expression for intracavity electric field is given by:

$$E(x,y,z,t) = ix\sum_j (2\pi\hbar\omega_j)^{1/2}[\mathbf{a}\hat{(\mathbf{t})}\exp(-i(w_jt-k_jz)) - \mathbf{a}\hat{(\mathbf{t})}^\dagger \exp(i(w_jt-k_jz))]u(x,y,z)$$

, where u(x, y, z) is the slowly changing (compare to light wavelength) spatial profile of the intra-cavity field. Let us take the case of a single spatial mode with j=0 and $\hat{\mathbf{a_o}}^\dagger$= -iA/$2^{1/2}$.Then,

$$E(x,y,z,t) = E_o cos(\omega_o t - k_o z)$$

While making a measurement using high precision laser interferometers,the carrier light gets perturbed by the signal and noise sources. Hence, the expression of the new electric field is now,

$$E(z,t) = (E_o + \delta A)cos(\omega_o t - k_o z + \delta\phi)$$

. The δA and $\delta\phi$ here represent the amplitude and phase perturbation of the carrier light field. By applying the first and second order pertubation correction methods we get,

$$\delta A = (2\hbar\omega)^{1/2}\int_0^\infty d\Omega[\hat{\mathbf{a_1}}\exp(-i(\Omega t - kz)) + \hat{\mathbf{a_1}}^\dagger \exp(i(\Omega t - kz))]$$

$$\delta\phi = (2\hbar\omega)^{1/2}\int_0^\infty d\Omega[\hat{\mathbf{a_2}}\exp(-i(\Omega t - kz)) + \hat{\mathbf{a_2}}^\dagger \exp(i(\Omega t - kz))]$$

, where Ω=ω-ω_o and $\hat{\mathbf{a_1}}$=($\hat{\mathbf{a_\Omega}}$+$\hat{\mathbf{a}}_{-\mathbf{\Omega}}^\dagger$)/$2^{1/2}$, $\hat{\mathbf{a_2}}$=($\hat{\mathbf{a_\Omega}}$-$\hat{\mathbf{a}}_{-\mathbf{\Omega}}^\dagger$)/$i2^{1/2}$. These creation and annihilation operator combinations $\hat{\mathbf{a_1}}$ and $\hat{\mathbf{a_2}}$ are named "amplitude" and "phase" quadrature operators respectively.

2 Quantum States of Light

We discuss here about different light states based on the theory formalised in previous section which are used in an interferometric gravitational wave detector. These are vacuum state, coherent state and squeezed state.

2.1 Vacuum State

The vacuum state of an electromagnetic field correspond to the ground state of harmonic oscillator for all modes present i.e $|0\rangle = \prod_{\otimes k}|0\rangle_k$. They have the lowest energy and also minimum uncertainity.

2.2 Coherent State

These are the eigenstates of the annihilation operator. For a coherent state $|\alpha\rangle$, $a|\alpha\rangle = \alpha|\alpha\rangle$. These can also be thought as displaced vacuum state in the phase phase representation i.e $|\alpha\rangle = D(\alpha)|0\rangle$ where $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$ is the displacement operator. The correspond to minimum uncertainity states and in a sense most closely resemble the classical behaviour among all quantum states.

2.3 Squeezed State

We know that coherent state has minimum uncertainity of $\frac{\hbar}{2}$ with each quadrature contributing equally towards the uncertainity. What squeezing does is that it reduces the fluctuation (or) uncertainity in one of the quadratures and increases the uncertainity in the other quadrature to respect the Heisenberg's uncertainity principle.

This means that the error in one of the quadrature is reduced and hence precise measurements can be done. But the problem is that the squeezed state evolves with time which in turn rotates the phase space and hence measurements must be performed at time when the fluctuation is minimum. The laser interferometer itself is a squeezed light generator. The ponderomotive interaction between the light field and test mass

An intuition for squeezing can be given by taking an analogy in classical mechanics. Consider a harmonic oscillator oscillating at frequency w_0. Now, let's drive it parametrically at $2w_0$ with low amplitude. The potential will be

$$V(x) = \frac{1}{2}mw_0x^2(1 + \epsilon \sin 2w_0t), \epsilon << 1$$

Now, the solution of the harmonic oscillator becomes,

$$x(t) = c(t)\cos w_0t + s(t)\sin w_0t$$

Since the parametric driving is small, the coeffecients c & s will vary at a constant rate i.e $c\ddot{(t)}, s\ddot{(t)}$ is nearly zero. The equation of motion then satisfies,

$$-w_0\dot{c}\sin w_0t + w_0\dot{s}\cos w_0t = -\frac{\epsilon w_0^2}{2}[c\sin w_0t + s\cos w_0t]$$

By solving this the coeffecients become, c(t) = $c_0 \exp\left(\frac{\epsilon w_0 t}{2}\right)$ and s(t) = $s_0 \exp -\left(\frac{\epsilon w_0 t}{2}\right)$. We can see that the cos quadrature is amplified and sin quadrature is damped. So we can measure the sin quadrature with high accuracy.

2.4 Generating Squeezed States

As seen in the previous section, we need parametric driving at $2w_0$. What we do is that we couple photon at w_0 to $2w_0$ using non-linear process. There are certain optical parametric oscillator such as non-linear crystals such as KDP which perform these parametric down conversion i.e excite the crystal to $2w_0$ energy level and during de-excitation they emit two entangled squeezed photons at a frequency of w_0.

We realise parametric down conversion in a crystal by exploiting the second order non-linearity $\chi^{(2)}$. This is by far the preferred means to produce squeezed vacuum in GW band. Driving non-linear materials induces the following polarisation.

$$\vec{P} = \epsilon_0(\chi^{(1)}\vec{E} + \chi^{(2)}\vec{E}^2 + ...)$$

In classical picture, this oscillating dipole will re-radiate. Due to quadratic dependence on $\vec{E}$, the resulting field can depend on incident field with components at different frequency based on the value of $\chi^{(2)}$. Here we will have field at the fundamental laser frequency will be w_0 which is that of interferometer input frequency and at second harmonic frequency. We use the second harmonic generation to convert two photons at fundamental frequency to one photon at second harmonic. The field is then used to pump an Optical Parametric Oscillator, where photons at second harmonic are converted to two correlated/entangled squeezed photons at fundamental frequency.

Since squeezing is a two photon process, the hamiltonian for such process would have 2 creation at w_0 mode and one annihilation at mode $2w_0$. The hamiltonian looks as,

$$H = {a^\dagger}^2 b + a^2 b^\dagger$$

The mode b is a strong powerful coherent state i.e $b\,|\beta\rangle = \beta\,|\beta\rangle$ and $b^\dagger\,|\beta\rangle = \beta^*\,|\beta\rangle$. Let, $\beta = \frac{r}{2}e^{i\phi}$. The the hamiltonian can be roughly be written as,

$$H = \tfrac{r}{2}[a^2 e^{-i\phi} + {a^\dagger}^2 e^{i\phi}]$$

The squeezing operator then for $\phi = \frac{\pi}{2}$ and t $= \frac{\pi}{2}$ can be represented as,

$$S(r) = \exp -\tfrac{r}{2}(a^2 - {a^\dagger}^2)$$

In the Heisenberg picture the position and momentum operators become,

$$\hat{X_H} = S(r)\hat{X}S^\dagger(r) = \hat{X}e^r$$

$$\hat{P_H} = S(r)\hat{P}S^\dagger(r) = \hat{P}e^{-r}$$

The noise in the position quadrature is reduced while the same amount is being increased in the position so as to satisfy the hyperbolic Heisenberg uncertainity relation. By changing the value of ϕ other quadratures can be squeezed. While evolving in time, the squeezing operator S(r) does rotation in phase space and the squeezing is varying periodically.

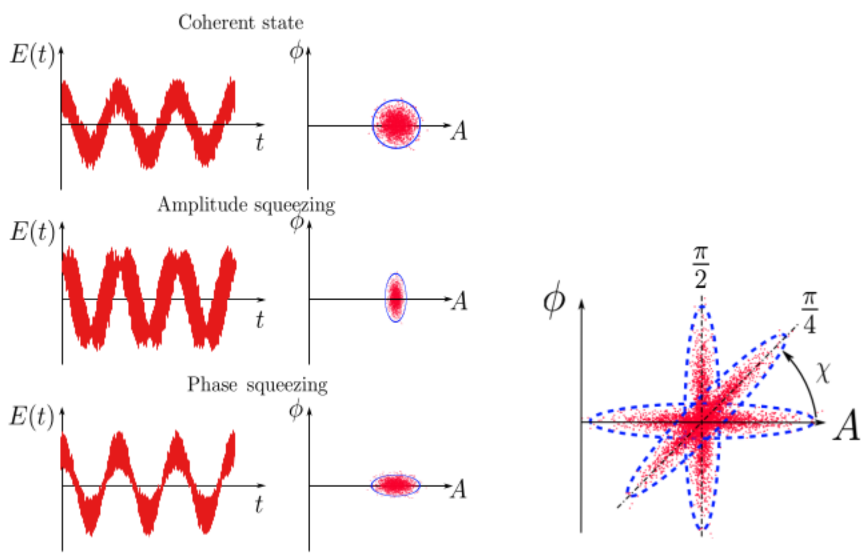

Figure 1: Squeezing in different quadrature and it's time dependence in phase space

2.5 Homodyne Detection

Since the squeezing evolves with time, the error in particular quadrature is minimum only at certain instants of time. The photodetector measures the intensity fluctuations rather than the quadrature fluctuations. By interfering quadrature signal with strong laser field, we can read the quadrature fluctuations. This is called Homodyne detection. As the name suggests we couple the light mode with a parametric oscillator of same frequency.

We use beam splitter to couple two modes of light. One input to the beam splitter will be the squeezed light(a) and the other will be a strong coherent light(b). The output from the beam splitter will be $\frac{a+b}{\sqrt{2}}$ and $\frac{a-b}{\sqrt{2}}$. Taking the difference of signal $I_{-} = ab^{\dagger} + a^{\dagger}b$. Then,

$$X_{\theta} = \frac{I_{-}}{2|\beta|} = \frac{ae^{i\theta} + a^{\dagger}e^{-i\theta}}{2}$$

where, $\beta = |\beta|e^{i\theta}$. We can measure now the position and momentum for $\theta = 0$ and $\frac{\pi}{2}$. We can have squeezed measurements at all instants of time. In an interferometer, the phase quadrature usually carries the displacement information of test masses. Using the homodyne detection we can perform readout to extract this information.

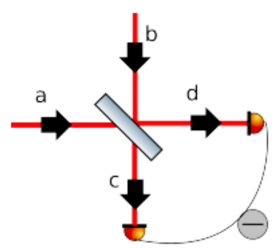

Figure 2: Schematic of Homodyne Detection

3 Sources of Noise in a System

Our aim in this section is to introduce reader common sources of noise in classical as well as quantum systems. When we talk about noise we usually refer to random noise. types of noise can be divided into two regimes one where $\hbar\omega/kT >> 1$, which is the quantum regime where noise due to zero-point fluctuation (vacuum fluctuation), etc. are primarily important. Another is $\hbar\omega/kT < 1$, this is the region where thermal noise has precedence over quantum noise. The nature of both these noise is very different but same mathematical functions can be used to describe and analyse them. In the next section we introduce basic tools to analyse and understand noise in a system.

3.1 Classical treatment of Noise

Before delving into details about noise sources and possible remedies we must carefully understand how to describe noise. As noise is a generally defined as deviation from an ideal signal, one might consider defining noise in a system as variance of the signal. We define signal as X, whose variance is given by,

$$\sigma = < X^2 > - < X >^2$$

where ¡.¿ represents mean. In most real systems average of noise signal tends to be zero, so that mean of the real signal is nearly equal to mean of the ideal signal. Variance is a good indicator when we are only concerned about time averaged quantities, a more firm understanding of noise in a signal can be understood by calculating auto-correlation function of the signal in some time range. Let X be a random variable, then auto-correlation can be defined as,

$$C_{XX}(t-t') =< X(t)X(t') >$$

The above definition may seem absurd at first sight but provides more intuition about the noise in the system. It basically correlates the function at two different time instants. Now, this helps us to check the periodicity of the signal in presence of noise. Consider a variable at two time instants $t = t'$ and $t' = t-1$, the function C_{XX} will always be one at $t = t'$ as essentially its the same instance of the function but for $t' = t-1$, if the output function are nearly same the function is said to be auto-correlated. And generally the graph of auto-correlation function and lag time (i.e the difference in the time intervals) has a decreasing trend which signifies that signal gradually goes out of phase with respect to its initial phase over time. If the consecutive terms do not follow a trend then signal is said to consist of white noise which means signals at different time instants are not related and it is a purely random process. Another interesting quantity that can be derived from auto-correlation function is noise power spectral density which holds great importance. Now, with the help of Wiener-Khintchine theorem, we can define power spectral density S_{XX} as,

$$S_{XX} = \int_{-\infty}^{\infty} < X(t)X(t') > e^{i\omega(t-t')}dt$$

which is nothing but Fourier transform of auto-correlation function. From S_{XX} we can understand the distribution of noise in frequency (or ω) domain which is nothing but, distribution of noise with respect to energy. In the next subsection we will try to apply these concepts learnt here to quantum domain.

3.2 Quantum noise

We know that variables in classical physics become operators in quantum physics, same applies here. Instead of a random variable X we will introduce a corresponding operator $\hat{X}$ and power spectral density function will now be given by,

$$S_{XX} = \int_{-\infty}^{\infty} < \hat{X}(t)\hat{X}(t') > e^{i\omega(t-t')}dt$$

Now there are some points about quantum mechanical S_{XX} that we should take note of. First of all in classical treatment auto-correlation function was real, hence we are assured that its Fourier transform i.e Power spectral density would be even function in ω. In quantum mechanical treatment, the auto-correlation function might not be a strictly real function hence, there will be anti-symmetric part too in S_{XX}. Also, a classical system at zero temperature will have no noise, but quantum mechanically there would be zero-point fluctuations noise. In order to make these propositions more concrete, we would take up an example of simple harmonic oscillator. This example is very suitable as we saw in previous section that electromagnetic fields can be quantized into a system of N-harmonic oscillator.
First lets consider the classical harmonic oscillator problem.
Consider a harmonic oscillator with mass m and angular frequency Ω in thermal equilibrium with a large heat bath at temperature T. Now $x(t)$ for such a system would be given by,

$$x(t) = x(0)cos\Omega t + \frac{p(0)}{m\Omega}sin\Omega t$$

We take reference time $t' = 0$ for simplicity of calculation.Hence, the auto-correlation for such a system would be given by,

$$C_{XX}(t) =< x(t)x(0) >=< x(0)x(0) > cos\Omega t+ < p(0)x(0) > \frac{sin\Omega t}{m\Omega}$$

There would be no correlation between $x(0)$ and $p(0)$ classically, hence we can safely neglect that term. Also note that, we can get $< x^2 >$ from equi-partition theorem (i.e $\frac{1}{2}k_Bt = \frac{1}{2}m\Omega^2 < x^2 >$) .From the above expression we can calculate the spectral density of the function as defined earlier.

$$S_{XX}(\omega) = \frac{\pi k_B T}{m}(\delta(\omega - \Omega) + \delta(\omega + \Omega))$$

This is an even function in frequency as explained earlier.
For the quantum case, the variables will become operators and $\hat{x}(t)$ will be defined similarly. So auto-correlation function in this case would be given by,

$$C_{XX}(t) =< \hat{x}(t)\hat{x}(0) >=< \hat{x}(0)\hat{x}(0) > cos\Omega t+ < \hat{p}(0)\hat{x}(0) > \frac{sin\Omega t}{m\Omega}$$

A very important thing to note here would be that second term in above expression won't be zero as $[\hat{x}(0), \hat{p}(0)] = i\hbar$. If we write $\hat{x}$ and $\hat{p}$ in terms of ladder operators and some clever manipulation would give,

$$C_{XX}(t) = \frac{\hbar}{2m\Omega} < e^{i\Omega t}a^\dagger(0)a(0) + e^{-i\Omega t}a(0)a^\dagger(0) >$$

Here as there are number operators, their average would correspond to average number of particles in a particular state. So in case of photons this average number of bosons in a state is given by Bose-Einstein distribution,

$$n_{BE}(\hbar\Omega) = \frac{1}{e^{\frac{\hbar\Omega}{kT}} - 1}$$

Now, the final auto-correlation function would look like,

$$C_{XX}(t) = \frac{\hbar}{2m\Omega} < e^{i\Omega t}n_{BE}(\hbar\Omega) + e^{-i\Omega t}(n_{BE}(\hbar\Omega) - 1) >$$

Hence taking a Fourier transform of this function would give us spectral power density,

$$S_{XX}(\omega) = \frac{\pi\hbar}{m\Omega} < n_{BE}(\hbar\Omega)\delta(\omega + \Omega) + n_{BE}(\hbar\Omega) - 1) >$$

This function is anti-symmetric as expected because we had imaginary component in auto-correlation function.

3.3 Noise in GW detector

In the preceding section we saw basic function definition and examples to define and understand noise in the system. In this subsection we will try to explore noise in GW detector system.

3.3.1 Phase Noise

We know that ground state energy of an harmonic oscillator is non-zero. Hence by Heisenberg uncertainty principle, this state would contribute to zero-point random fluctuation energies. These fluctuation will cause phase fluctuation in light field travelling out of dark port. In quantum mechanics, time fluctuation is associated with energy fluctuation as $\Delta E \Delta t >= \hbar$. For a monochromatic light field with frequency ω, we have the number-phase uncertainty relation is given by $\Delta N \Delta\phi >= \hbar$ and since $\Delta E \approx \Delta N \hbar\omega$, $\phi \approx \omega\Delta t$. For a coherent laser where number of photons follows a Poisson distribution, we have $\Delta N \approx \sqrt{N}$ and $\Delta\phi \approx 1/\sqrt{N}$. Thus, higher laser power ensures lower phase noise. Using $\Delta\phi \approx k_p\Delta x$, we can estimate the power spectral density as, $S^{ph}_{XX} \approx \frac{\hbar c^2}{I_0\omega}$, where I_0 is the intra-cavity laser power, which is frequency independent white noise spectrum.

3.3.2 Displacement noise

As we already stated, the fluctuation in photon number is given by $\approx \sqrt{N}$. This fluctuation will induce a fluctuation in momentum and thereby a random fluctuation in radiation pressure force over a period of time τ, given by, $\Delta F \approx \sqrt{N}\hbar k_p/\tau$

The mechanical response function of test masses is $-1/m\Omega^2$. Therefore, the displacement noise spectrum by fluctuation radiation pressure noise is

$$S_{XX}^{rp} \approx \frac{\Delta F_{rp}^2 \tau}{m^2\Omega^4} \approx \frac{I_0\hbar\omega}{m^2c^2\Omega 4}$$

Once we have understood the noise spectrum due to phase noise and displacement noise we can now calculate total spectrum noise, which will be give by $S_{XX}^{total} = S_{XX}^{ph} + S_{XX}^{rp}$

A peculiar thing to note here is that there is lower limit to this total noise spectrum of quantum shot noise and quantum radiation pressure noise given by,

$$S_{XX}^{ph} + S_{XX}^{rp} \approx \frac{\hbar c^2}{I_0\omega} + \frac{I_0\hbar\omega}{m^2c^2\Omega 4} \approx \frac{2\hbar}{m\Omega^2}$$

This lower limit is referred to as standard quantum limit. It is a consequence of Heisenberg uncertainty principle.

4 Gravitational Wave Detectors

In this section, we will provide the reader a brief introduction to the nature of gravitational waves and a general overview of the working of gravitational wave detectors using the Michelson-Morley Interferometer set-up, same as the one used in LIGO GW detectors.

4.1 The Nature of Gravitational Waves

Einstein, in his General Theory of Relativity (GTR), predicted the existence of Gravitational Waves (GWs), which were predicted to be "ripples" in the space-time curvature. They are basically waves of intensity of the gravitational force. GTR predicts gravity to be a consequence of space-time curvature, thereby implying that we can view space-time as an "elastic-medium" of sorts. It's "stiffness" can be deduced from the *Einstein-Field Equations*:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

The proportionality factor κ between the curvature and energy tensor essentially gives us the stiffness of space-time, $1/\kappa \sim 10^{43}$. Due to this extremely high degree of stiffness, we can say that the scale of events required to generate these GWs must be extremely large and of the cosmic-order. Some examples of these cosmic events include:

- Binary star-system/black hole in-spirals or collisions
- Fast spinning black holes
- Cosmological background from the universe's inflationary stage
- Supernovae/Hypernovae explosions

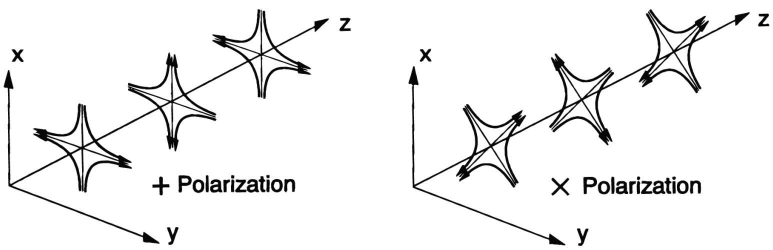

Figure 3: The plus (+) and cross (×) polarizations of GWs

Now, in order to understand how GW detectors works, we need to understand the basics of polarization of GWs. As these waves do not oscillate in just one dimension (Like E or B of an EM wave) but in two. Writing the Einstein Field Equations as a wave equation for perturbations, we end up with two matrices for the basis of the polarization tensor:

$$\epsilon^{+} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\epsilon^{\times} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pictorially, we can visualize these two kinds of polarizations as shown in figure 1.

GWs provide us with a completely new angle to study major astrophysical processes and could possibly give astro-physicists new insight into the origin of the universe. Which is why it becomes important realize a physical system that can detect these waves.

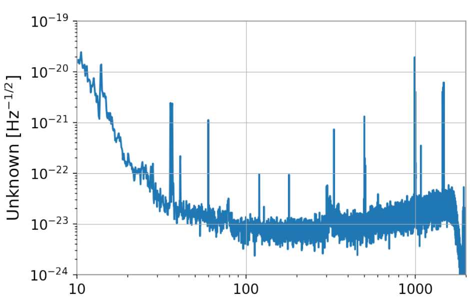

Figure 4: The Amplitude Spectral Density (ASD) of the GW strain for the LIGO detector in Hanford for the event GW150914, the first recorded GW

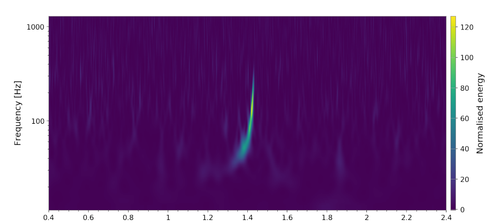

Figure 5: The q-transform of the ASD white bandpassed signal for the same event

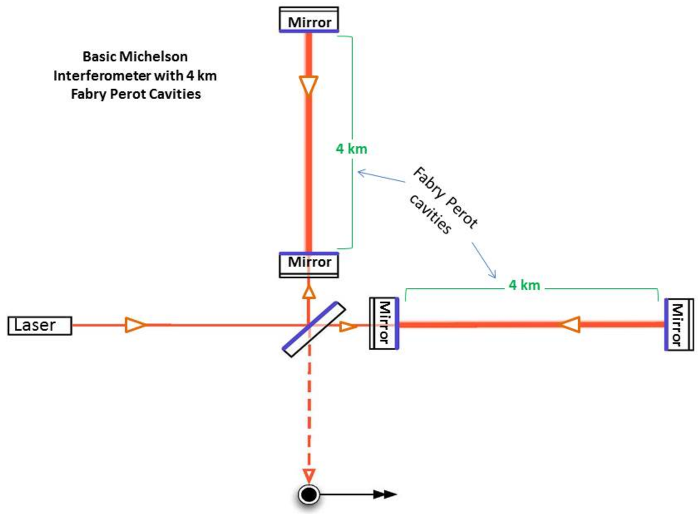

Figure 6: A schematic diagram of the LIGO Interferometer set-up

4.2 Gravitational Wave Detectors

The experimental detection of these waves is extremely difficult. Due to the high stiffness of space-time and the sheer distance from which these GW-generating events occur away from us results in a strain amplitude of the order of 10^{-21}, which would result in a length-contraction of the order of 10^{-18}m, which 1/1000 times that of a radius of a nucleus, thereby demanding an extremely sensitive experimental technique to detect GWs.

There have been several proposed experimental set-ups designed to detect these elusive GWs, however in our study we will be confining ourselves to *Terrestrial Laser Interferometer Detectors*. In this set-up, a Michelson interferometer is used, now a normal Michelson Interferometer design would require an arbitrarily large distance to be sensitive enough to detect GWs (For reference, LIGO's arms are 4km long), this was resolved with the help of *Fabry-Perot Cavities*, which allows the light to undergo multiple reflections while travelling each arm, thereby amplifying the the path-difference (Δx) and making it more sensitive.

In this design, the GWs induce a motion or a path-difference (Δx) between the mirrors as a result of their polarization (as discussed previously) and the Fabry-Perot cavity (FPC) amplifies this signal. In a static configuration, there will be no output since the light wave will be precisely cancelled out at the photodetector, known as the "dark port," and all of the light field will return to the laser source, known as the "bright port," following a round-trip propagation inside the interferometer. Suppose a GW with "+" polarization passes throught the detector, then the interferomter plane gets squeezed and creates a length difference between the two arms, therefore there won't be a complete destructive interference at the photodetector resulting in a light field with information about the motion.

Suppose our GW has an amplitude strength s and the distance of the end mirror to the beam splitter is L, then our resultant path difference will be $\Delta x = 2s(t)L$, generating a phase shift of $\Delta\phi = 2k_o s(t)L$ where $k_o = \omega_o/c$ is the wave-vector of the laser. Now the resultant phase difference, classically, is too small unless the length L is arbitrarily large. This is resolved with the help of a cavity in-between both arms but drastically reduces signal bandwidth.

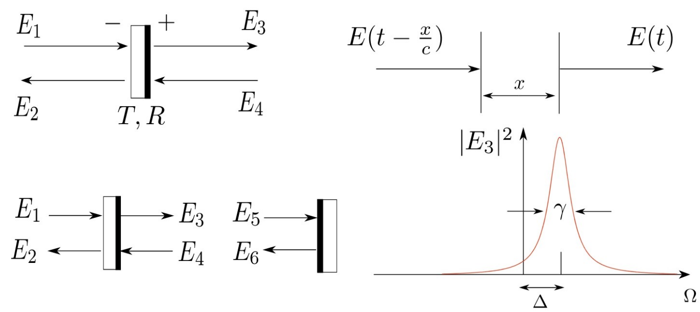

Figure 7: Light propagation inside a Fabry-Perot cavity.(a)the transmission and reflection of the light field by a mirror;(b)the propagation of light by distance x in the free-space;(c)light propagation inside a lossless Fabry-Perot cavity;(d) the single-mode resonant structure of a Fabry-Perot cavity.

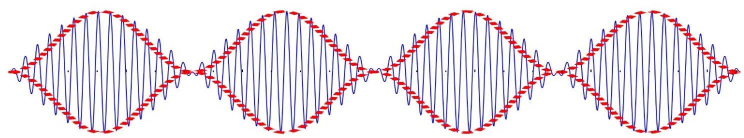

Figure 8: Slowly varying amplitude

4.2.1 The Fabry-Perot Cavity

The different quantum states of light, when injected into the detector, will circulate within the FPC. To understand the optomechanics of the set-up, it is crucial to understand the nature of light-propagation in these cavities. We start with some simple equations describing the propagation of the light field and its interaction with the mirrors.

$$E(t) = E(t - \frac{x}{x})$$

$$E_2(t) = -\sqrt{R}E_1(t) + \sqrt{T}E_4(t)$$

$$E_3(t) = \sqrt{T}E_1(t) + \sqrt{R}E_4(t)$$

Where R and T are the reflectivity and transmitivity of the mirror. Now, in optomechanics, the frequency scale of the various perturbations is considerably smaller than the light-field's frequency ω_o, allowing for the decompostion $E_i = \tilde{E_i}(t)e^{-i\omega_o t} + \tilde{E_i^*}(t)e^{i\omega_o t}$. Where ω_o is the central frequency which is detuned from the cavity resonance $\Delta = \omega_o - \omega_c$. Additionally, $\tilde{E_i}$ is slowly varying with the time-scale $1/\omega_o$.

We can proceed to write this expression in the frequency domain and employ certain approximations to the first-order $\Omega L/c, \Delta L/c << 1$ and R ~ 1,(Ω is the frequency of the GW and $\gamma = cT/L$) which ultimately leads to the following expressions:

$$\tilde{E_3}(\Omega) = \sqrt{\frac{c}{2L}} \frac{\sqrt{2\gamma}}{\gamma - i(\Delta + \Omega)} \tilde{E_1}(\Omega)$$

$$\tilde{E_2}(\Omega) = \frac{\gamma + i(\Delta + \Omega)}{\gamma - i(\Delta + \Omega)} \tilde{E_1}(\Omega)$$

We can see that the interaction between the cavity-field and the external field behaves like that of a harmonic oscillator with a driving force, thereby we can try and investigate these dynamics using a hamiltonian.
Now, we can write the Hamiltonian for FPC by drawing it's analogy to a driven harmonic oscillator.

$$\hat{H}_o = \hbar\omega_c \hat{a}^\dagger(t) a^\dagger(t) + i\hbar\sqrt{2\gamma}[\hat{a}_{in}(t)\hat{a}^\dagger(t) - \hat{a}^\dagger_{in}(t)\hat{a}(t)]$$

Our first term is the cavity-field (the quantum HO) and the second term represents the interaction between the cavity and the external field. $\hat{a}/\hat{a}^\dagger$ represent the creation/annihilation operators of the cavity with resonant frequency ω_c while $\hat{a}_{in}/\hat{a}^\dagger_{in}$ are the operators for the external field.
For a pumped cavity with frequency ω_o, $\hat{a}_{in}(t) =][\bar{a}_{in} + \delta\hat{a}_{in}(t)]e^{-i\omega_o t}$, the term inside the square-brackets tells us about the amplitude of the steady pumping field ($\bar{a}_{in}$) and the perturbation due to vacuum fluctuations ($\delta\hat{a}_{in}(t)$). We can proceed with the dynamics of the cavity-field by using Heisenberg's equation of motion $i\hbar\dot{\hat{a}} = [\hat{a}, \hat{H}]$ this gives

$$\dot{\hat{a}} = -i\omega_c\hat{a} - \gamma\hat{a} + \sqrt{2\gamma}[\bar{a}_{in} + \delta\hat{a}_{in}(t)]e^{-i\omega_o t}$$

We can work out our solutions in the rotating frame which would simplify our calculations by using the transformation $\hat{a} \longrightarrow \hat{a}^{-i\omega_o t}$ we finally solve the resulting EOM pertubatively giving us:

$$\bar{a} = \frac{\sqrt{2\gamma}}{\gamma - i\Delta}\bar{a}_{in}$$

$$\hat{a}(\Omega) = \frac{\sqrt{2\gamma}}{\gamma - i(\Delta + \Omega)}\hat{a}_{in}(\Omega)$$

5 Reducing the Quantum Noise in GWD

We have already seen how the advanced interferometric gravitational wave detectors (GWD) are affected by quantum noises over the complete detection range. Further improvement of the device's sensitivity is achieved by manipulating the optical field and the readout scheme at the quantum level. One proposed way (by Kimble) is to inject frequency-dependent squeezed light into the main interferometer. In this setup a series of optical cavities are used to filter the squeezed light and to create proper rotation of the squeezed angle at different frequencies. In order to maximize the reduction of quantum noise over the complete broadband, the frequency scale of the optical cavity system needs to match that of the quantum noise of the interferometer. For the advanced LIGO, the quantum noise is dominated by quantum radiation pressure noise at low frequencies (around 100Hz), which determines the required filter cavity bandwidth.

The original proposal uses filter cavities of kilometer length. Later a more compact (10 meters) filter cavity with high finesse (10^5) was proposed (by Evans) to achieve the required bandwidth. But using a system with such high finesse, meant that a small optical error can degrade the squeezing and becomes the limiting factor in the filter cavity performance. However, it was later experimentally proved that the optical losses in current mirror technology is small enough for it to be used in the current LIGO system. To further make the system compact, the optical loss should be much lower.

One other solution to make the system more compact is to actively narrow the cavity bandwidth by using Electromagnetically induced transparency (EIT) effect in a pumped atomic system. In principle, this can be used to produced to a Optical cavity in centimeter scale with the same required bandwidth. Additionally, since the optical properties can be varied dynamically by changing the power of the controlling field, it allows optimization of the filter cavity for different operational modes where the quantum noise has different frequncy dependencies.

This active atomic system is generally lossy, which will degrade the squeezing level. So in this paper, we propose to narrow the filter cavity bandwidth using Optomechanically induced transparency (OMIT). This cavity filter consists of a mirror-endowned mechanical oscillator with eigenfrequency ω_m that is much larger than the cavity bandwidth γ. A control pump laser drives the filter cavity at frequency ω_p, detuned from the cavity resonant frequency, ω_0 by $\omega_m - \delta$ with δ comparable to the gravitational-wave signal frequency Ω.

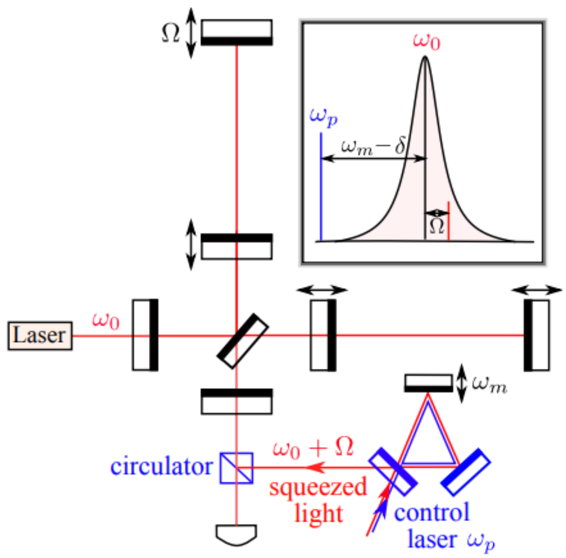

Figure 9: Schematic showing the configuration for achieving frequency-dependent squeezing

5.1 Optomechanical Induced Transparency

The optomechanical induced transparency effect comes from the mutual interference between different cavity field components. The signal light and the control light will beat up inside the cavity to create an AC radiation pressure on the mechanical oscillator at driving frequency $\omega_s - \omega_p$. The motion of the oscillator under this pondermotive force will modulate the control light and create upper and lower sidebands with frequency ω_c and $2\omega_p - \omega_c$. The upper sideband has the same frequency as the signal field and will destructively interfere with the signal field and decrease the intra cavity signal field. Therefore, more signal field are reflected because of which reflection of the cavity has a peak(Figure below). The width of the peak is determined by the amplitude of the mechanical motion which is equal to the total mechanical bandwidth. When the optomechanical dampening rate ($\gamma_o pt$) is much larger than the mechanical dampening rate (γ_m), the width of the central reflection peak is just equal to $\gamma_o pt$.

Compared to the situation with no optomechanical interaction, the signal field with frequency around the cavity resonance almost do not dissipate. This effect is called optomechanical induced transparency (OMIT). We shall see further how the OMIT effect combines the optomechanical effect with the optical property (reflectivty) and create an effective cavity of required bandwidth.

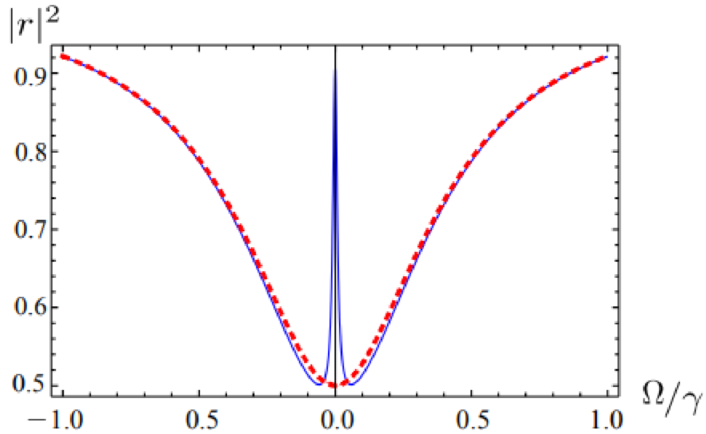

Figure 10: Optomechanical Transperancy

5.2 Optical Dynamics

We start our analysis with the linearised Hamiltonian

$$\mathcal{H} = \hbar\omega_0 a^\dagger a + \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \hbar\overline{G}_0 x(a^\dagger + a) + i\hbar\sqrt{2\gamma}(a^\dagger a_{in} e^{-i\omega_p t} - a a^\dagger e^{i\omega_p t})$$

here, a is the annihilation operator of the cavity mode and $a_i n$ is the annihilation operator for the squeezed light, x is the oscillator position operator, p is the oscillator momentum operator, $\overline{G}_0 = sqrt 2P_c\omega_0/\hbar cL$ with L being the cavity length. In the rotating frame at frequency ω_p of the control laser, the Heisenberg equation of motion reads,

$$m(\ddot{x} + \gamma_m\dot{x} + \omega^2 x) = -\hbar\overline{G}_0(a^\dagger + a) + F_{th},$$

$$\dot{a} + (\gamma + i\Delta)a = -i\overline{G}_0 x + \sqrt{2\gamma}a_{in}$$

where $\Delta = \omega_0 - \omega_p$ is the detuning frequency and F_{th} is the associated Langevin force. Solving these equations in the frequency domain, we get

$$x(\omega) = \chi_m(\omega)[\hbar\overline{G}_0(a^\dagger(-\omega) + a(\omega)) + F_{th}(\omega)]$$

$$a(\omega) = \chi_c(\omega)[-i\overline{G}_0 x(\omega) + \sqrt{2\gamma}a_{in}(\omega)]$$

where the susceptibilities are defined as $\chi_m = -[m(\omega^2 - \omega_m^2 + i\gamma_m\omega)]^{-1}$ and $\chi_c = [\gamma - i(\omega - \Delta)]^{-1}$.

We consider thee regime where, $\Delta = \omega_m - \delta$ with $\omega_m >> \delta$, and the so-called resolved side-band regime $\omega_m >> \gamma$. Correspondingly, the lower side-band mode $(a^\dagger(-\omega))$ is negligibly small. Then we obtain,

$$a(\omega) \approx \frac{\sqrt{2\gamma}a_{in}\omega - i\overline{G}_0\chi_m(\omega)F_{th}(\omega)}{\chi_c^{-1}(\omega) + i\hbar\overline{G}_0^2\chi_m(\omega)}$$

Since we are interested in the signal sidebands around ω_0, we substitute $\omega = \Delta + \Omega$. As $\Omega \approx \delta << \omega_m$, we have $\chi_m \approx -[2m\omega_m(\Omega - \delta + i\gamma_m)]^{-1}$ and $\chi_c = \gamma^{-1}$. Together with $a_{out} = -a_{in} + sqrt2\gamma a$ and using $\gamma_m << \gamma_{opt}$ (since high quality factor oscillator), we get final input-output relation as

$$a_{out}(\Omega) \approx \frac{\Omega - \delta - i\gamma_{opt}}{\Omega - \delta + i\gamma_{opt}} a_{in}(\Omega) + n_{th}(\Omega)$$

where $\gamma_{opt} = 4P_c\omega_0/m\omega_m c^2 T_f$ with P_c equal to the intracavity power of the control field, m is the mass of the mechanical oscillator and T_f the transmitivity of the front mirror.

The second term n_{th} arises from the thermal fluctuation of the mechanical oscillator. It is of the form

$$n_{th}(\Omega) = \frac{i\sqrt{2\gamma}\gamma_{opt}F_{th}(\Omega)}{\hbar \overline{G}_0(\Omega - \delta + i\gamma_{opt})}$$

It is uncorelated with the input optical field. So, in order for it to be negligible, we require $8k_BT/Q_m < \hbar\gamma_{opt}$, with Q_m the mechanical quality factor and T the temperature of the environment. As the required effective bandwidth($\gamma_{opt}/2\pi$) of the system is $\approx$ 100Hz, we need

$$\frac{T}{Q_m} < 6 * 10^{-10} K$$

This is hard to achieve even with low loss materials at cryogenic temperatures. A possible solution is to use optical dilution. It allows for a significant boost of the mechanical quality factor by using the optical field, to provide most of the restoring force.

5.3 Optical Dilution

As we have seen, the most significant issue in this system is the thermal noise n_{th}, which puts a stringent constraint on the mechanical quality factor and the environmental temperature. In this section, we explore optical dilution as a solution to meet this condition.

In order to reduce this thermal noise, the mirror must be weakly coupled with the environment as possible, which in turn means that the mirror should be suspended with the stiffness of suspension as small as possible. A laser beam is used to create a potential well by generating an optical radiation pressure to be used as restoring force (optical spring). This potential well can create a mode of oscillation with frequency up to a few kilohertz. Here, the mode is dynamically unstable and the system can be stabilised by application of electronic or optical feedback forces. The optical spring shifts the oscillator's resonant frequency while leaving its mechanical loss unchanged. The mechanical quality factor Q_m, as limited by these losses is increased. This is referred to as 'Optical Dilution'. The general idea of optical dilution is to use the optomechanical interaction to boost the rigidity of the mechanical mode. The quality factor after optical dilution is given by,

$$Q = \frac{\sqrt{\omega_m^2 + \omega_{opt}^2}}{\gamma_m}$$

where $\gamma_{opt}^2 = K_{opt}/m$ and K_{opt} is the pondermotive rigidity.

The mean thermal occupation number for the mechanical oscillator now becomes,

$$n_{th} = \frac{k_B}{\hbar\omega_m Q} = n_{0th}\frac{\omega_m}{\omega_{opt}}$$

where n_{0th} is the previous mean thermal occupation number before optomechanical interaction. From the above formula, it is clear that after optical dilution, ideally the mean thermal occupation number decreased by a factor of ω_{opt}/ω_m

6 Conclusion

In conclusion, we introduced the reader to the optomechanical functioning of the GW detector and characterized the different sources of noise signals from these GWDs could undergo, such as phase noise and displacement noise, which results from zero-point fluctuation (in the vacuum) and thermal noise. We begin by introducing the reader to squeezed states of light by which we can tune the uncertainty between the phase and amplitude quadratures, allowing for more precise measurements. Further, we delve into how these squeezed states are generated using non-linear dielectric crystals such as KDP (here the non-linearity up to the second order was considered) to induce parametric driving at $2\omega_o$ which emits two entangled squeezed photons at a frequency of ω_o. As a solution to the previously mentioned sources of noise in the GWD, we propose injecting this squeezed light into the main interferometer and setting up a series of optical cavities designed to filter this squeezed light in order to decrease the quantum noise over the overall broadband. Suggestions were made on how to make the system more compact by using EIT effect in a pumped atomic system. Lastly in order to tackle the issue of thermal noise we propose certain optomechanical restraints on the interferometer mirror set-up through optical dilution. All these noise reduction mechanisms were rigorously studied and presented in our report.

7 References

- A squeezed-state primer
- Noise in homodyne detection
- Generation of Squeezed States by Parametric Down Conversion
- Fabry Perot Cavity optomechanics
- A guide to LIGO detector noise and extraction of gravitational wave signals
- Precision measurement beyond the shot-noise limit
- Quantum Noise and measurement
- Optomechanical Physics in the Design of Gravitational Wave Detectors, Yiqiu Ma
- M. Evans, L. Barsotti, P. Kwee, J. Harms, and H. Miao, Phys. Rev. D 88, 022002 (2013).
- H. J. Kimble, Y. Levin, A. B. Matsko, K. S. Thorne, and S. P. Vyatchanin, Phys. Rev. D 65, 022002 (2001).
- Optical Dilution