Lyapunov Exponents, Jacobi Metric Stability and Quasi-normal modes of Black hole

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<u>CERTIFICATE</u>

*This is to certify that the work contained in this thesis entitled "**Lyapunov Exponents, Jacobi Metric Stability and Quasi-normal modes of Black hole**" is a bonafide work of **S Venkat Bharadwaj** (**Roll No. 200121045**), carried out in the Department of Physics, Indian Institute of Technology Guwahati under my supervision and that it has not been submitted elsewhere for a degree.*

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Abstract

Geodesics around the black hole can be used to describe important features of the space time. This paper discusses studying the geodesic of Schwarzchild black hole using its Jacobi metric and determines the stability of the same using lyapunov exponents in the same metric. The later part of this paper shows the relation between the quasi-normal modes of the black hole and lyapunov exponents, and thus gives a relation between the unstable null geodesic and the quasi-normal mode of the black hole.

Contents

Chapter 1

Introduction

The theory of relativity predicts very compact objects of large mass around which the gravity is so strong that even light is not able to escape it. These masses are termed as black holes. In general relativity, gravity is not regarded as a force, rather it is seen as a consequence of curved space-time. By solving the Einstein's Field Equations, the metric tensor that defines this curved space time can be calculated, and hence the space-time for a compact mass at the centre can be defined.

Geodesics are curves that define the shortest path between 2 points. For a flat space-time, the shortest path (or the geodesic) is given by a straight line between the two points. However, the space-time around the black hole is curved, and the geodesics corresponding to these are also curved. This curve can be estimated by solving the equation of motion of a particle in the space time, also known as geodesic equations.

Stability refers to the ability of particles to fall back to their initial path when they are perturbed. The path is said to be unstable if the particle continuously moves farther from the initial path. The geodesic around a black hole is said to be unstable if a perturbation makes the particle collapse into the black hole or allows the particle to escape its orbit around the black. Generally, this stability is studied by assuming an effective potential due to the curved space time, then analysing the concavity of the effective potential.

In the phase-1 of the project, we analysed the stability of the particle using different approaches and introduced lyapunov exponents to determine the same. We had uses the optical metric of the Scwarszchild metric to derive its null geodesic and the derived radius was substituted in the expression of lyapunov exponent derived from the complete metric. In this paper, we discuss a more robust way in which the geodesic is completely discussed in the Jacobi metric of the Schwarzchild black hole.

While we discuss black holes using its basic parameters, real black holes are always in a perturbed state. A perturbed black hole responds to its perturbation by emitting gravitational waves. Quasinormal modes are solutions of wave equations with specific boundary conditions derived from the perturbed metric. In the later part of this paper, the computed lyapunov exponents are also shown to be related to the quasinormal mode in the eikonal limit, thus its relation to the unstable null geodesic is discussed.

Chapter 2

Jacobi Metric and Geodesic

The Jacobi metric is a Riemannian metric that encodes information about the geodesics of the original metric restricted to a surface of constant energy. For a metric

$$ds^2 = -V^2dt^2 + g_{ij}dx^i dx^j, \tag{2.1}$$

the Jacobi metric is given by

$$j_{ij}dx^i dx^j = (E^2 - m^2V^2)V^{-2}g_{ij}dx^i dx^j \tag{2.2}$$

At the zero mass limit, the Jacobi metric coincides with the Fermat's metric with an additional factor of E^2. As it is an overall factor, the null geodesic is not parameterized by the constant energy. However, for a massive object, the geodesic depends on the energy E. The Jacobi metric for the Schwarzchild black holes is thus given by,

$$ds^2 = (E^2 - m^2f)(\frac{dr^2}{f^2} + \frac{r^2}{f}(d\theta^2 + sin^2\theta d\phi^2)) \tag{2.3}$$

where $f = (1 - \frac{2M}{r})$. Considering the plane $\theta = \pi/2$, the metric reduces to

$$ds^2 = (E^2 - m^2 f)(\frac{dr^2}{f^2} + \frac{r^2}{f} d\phi^2) \tag{2.4}$$

2.1 Geodesic using the Metric

To derive the radius of the circular geodesic from the Jacobi metric, we will calculate the geodesic curvature from the metric and equate it to zero. From the phase-1 of the projects, we know that the geodesic curvature $\mathbf{k}_g$ is given by

$$\mathbf{k}_g = \frac{\partial R}{\partial x^i}(\frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}) \tag{2.5}$$

For the circular geodesic, we use the conditions $V_r = 0$, which gives the relation $\dot{r} = 0$, i.e. radius is constant. Hence the expression for the geodesic curvature reduces to

$$\mathbf{k}_g \propto \Gamma^r_{\phi\phi}\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda} \tag{2.6}$$

Calculating $\Gamma^r_{\phi\phi}$ for the Jacobi metric, we get,

$$\begin{aligned} \Gamma^r_{\phi\phi} &= \frac{1}{2}g^{rk}(-\partial_k g_{\phi\phi} + \partial_\phi g_{k\phi} + \partial_\phi g_{k\phi}) \\ &= \frac{1}{2}g^{rr}(\partial_r g_{\phi\phi}) \\ &= \frac{1}{2}\frac{f^2}{E^2 - m^2 f}\partial_r \frac{(E^2 - m^2 f)r^2}{f} \end{aligned} \tag{2.7}$$

Using the condition that $\mathbf{k}_g = 0$ for a geodesic, we get,

$$\begin{aligned} \Gamma^r_{\phi\phi} = \frac{1}{2}\frac{f^2}{E^2 - m^2 f}\partial_r \frac{(E^2 - m^2 f)r^2}{f} &= 0 \\ \implies \partial_r \frac{(E^2 - m^2 f)r^2}{f} &= 0 \end{aligned} \tag{2.8}$$

2.1.1 Null Geodesic

We already discussed the fact that the Jacobi metric reduces to the optical metric with an E^2 factor at the zero mass limit. Hence it is expected that we get the same radius for the null geodesic. Using the above equation and that $m = 0$, we get,

$$\begin{aligned} \frac{2r}{f} - \frac{r^2 f'}{f^2} &= 0 \\ \implies 2r &= \frac{r^2 2M}{(1-\frac{2M}{r})r^2} \\ \implies 2r - 4M = 2M \implies r &= 3M \end{aligned} \tag{2.9}$$

Therefore we get the same result that the radius of the null geodesic r_c = 3M.

2.1.2 Time like Geodesic

For a massive object, we can compute the geodesic radius using the above expression and we get

$$\begin{aligned} \partial_r \frac{E^2 r^2}{f} &= \partial_r m^2 r^2 \\ \implies E^2\left(\frac{2r}{f} - \frac{r^2 f'}{f^2}\right) &= 2m^2 r \\ \implies E^2(2r(1-\frac{2M}{r}) - 2M) &= 2m^2 r(1-\frac{2M}{r})^2 \\ \implies E^2(r-3M) &= m^2(r + \frac{4M^2}{r} - 4M) \\ \implies E^2(r^2 - 3Mr) &= m^2(r^2 + 4M^2 - 4Mr) \\ \implies r^2(E^2 - m^2) + (4m^2 - 3E^2)Mr - 4M^2m^2 &= 0 \end{aligned} \tag{2.10}$$

Thus, the solution to the above quadratic equation gives the radius of the time-like geodesic of a body of mass m. As expected, we can also see that the radius is parameterized by the constant energy E of the massive particle unlike the null-like radius.

2.2 Stability using Lyapunov Exponents

To determine the stability using lyapunov, we begin our calculations by deriving the Equations of motion from the Lagrangian of the system.

$$2\mathcal{L} = (E^2 - m^2 f)(\frac{\dot{r}^2}{f^2} + \frac{r^2}{f}(\dot{\phi}^2)) = \frac{\dot{r}^2}{g(r)} + \frac{r^2 f(r)\dot{\phi}^2}{g(r)}; \quad g(r) = \frac{f(r)^2}{E^2 - m^2 f(r)} \tag{2.11}$$

Generalised momenta from the equation can be written as

$$p_r = \frac{\dot{r}}{g(r)} \tag{2.12}$$

$$p_\phi = L = \frac{r^2 f(r)\dot{\phi}]}{g(r)} \tag{2.13}$$

Thus the hamiltonian can be written as

$$\begin{aligned} 2\mathcal{H} &= 2(p_\phi\dot{\phi} + p_r\dot{r}) - 2\mathcal{L} \\ &= \frac{L^2 g(r)}{r^2 f(r)} + p_r^2 g(r) \end{aligned} \tag{2.14}$$

For the general Schwarzchild metric, we calculated the equations of motion using time as the affine parameter. In the Jacobi metric, we have reduced to spatial co-ordinates by taking a constant energy surface. Hence, the equation of motion is computed using an affine parameter α. Hence we get,

$$H_1 = \frac{\dot{r}}{\dot{\alpha}} = \frac{g p_r}{\dot{\alpha}} \tag{2.15}$$

$$H_2 = \frac{\dot{p_r}}{\dot{\alpha}} = \frac{1}{\dot{\alpha}}\frac{\partial\mathcal{L}}{\partial r} = \frac{1}{\dot{\alpha}}[-\frac{1}{2}g' p_r^2 - \frac{1}{2}\frac{L^2 g'}{r^2 f} + \frac{1}{2}(2rf + r^2 f')\frac{g(r)}{r^4 f^2}] \tag{2.16}$$

From the equation of motion, we get the components of the stability matrix as

$$K_{11} = \frac{\partial H_1}{\partial r} = \frac{g'(r)p_r}{\dot{\alpha}} \tag{2.17}$$

$$K_{12} = \frac{\partial H_1}{\partial p_r} = \frac{g(r)}{\dot{\alpha}} \tag{2.18}$$

$$K_{21} = \frac{\partial H_2}{\partial r} = \frac{1}{2\dot{\alpha}}[-g''p_r^2 - \frac{L^2 g''}{r^2 f} + 2(2rf + r^2 f')\frac{L^2 g'}{f^2 r^4} - 2(2rf + r^2 f')^2 \frac{g(r)}{r^6 f^3} + (2f + 4rf' + r^2 f'')\frac{g(r)}{r^4 f^2}] \tag{2.19}$$

$$K_{22} = \frac{\partial H_2}{\partial p_r} = -\frac{g'(r)p_r}{\dot{\alpha}} \tag{2.20}$$

Now we can substitute the above values in the expression

$$\lambda = \frac{(K_{11} + K_{22}) + \sqrt{(K_{11} + K_{22})^2 + 4K_{12}K_{21}}}{2} \tag{2.21}$$

Since we are concerned with the circular geodesic, we can substitute $p_r = 0$ as an universal assumption to simplify the expression. Hence we get,

$$\lambda = \sqrt{-\frac{L^2 r^3(-2M + r)^2(5m^4(2M - r)^3 + E^2(12M - 5r)r^2 + 2Em^2 r(24M^2 - 21Mr + 5r^2))}{\dot{\alpha}^2(m^2(2M - r) + Er)^4}} \tag{2.22}$$

2.2.1 Null Geodesic

For the null-like geodesic, we know that the mass m =0 and that the circular radius is given by $r_c = 3M$. To determine the stability of the null circular geodesic, we now substitute the values. Thus we get,

$$\lambda = 3\sqrt{\frac{L^2 M^4}{\dot{\alpha}^2 E^2}} \tag{2.23}$$

We can see that the lyapunov exponent is real and positive. Thus, the geodesic is unstable.

2.2.2 Time-like Geodesic

Similarly, for determining the stability for the time-like geodesic, we can substitute the radius from the solution of the quadratic equation discussed in the previous section. Since the radius is parameterized by the mass and energy of the particle, the lyapunov exponent also implicitly depends on those parameters. Hence, the stability can be determined.

$$\lambda = \sqrt{-\frac{L^2 r_{ct}^3(-2M + r_{ct})^2(5m^4(2M - r_{ct})^3 + E^2(12M - 5r_{ct})r_{ct}^2 + 2Em^2 r_{ct}(24M^2 - 21Mr_{ct} + 5r_{ct}^2))}{\dot{\alpha}^2(m^2(2M - r_{ct}) + E)^4}} \tag{2.24}$$

$$r_{ct} = \frac{3E^2M - 4m^2M + E\sqrt{9E^2 - 8m^2M}}{2(E^2 - m^2)} \tag{2.25}$$

Chapter 3

Quasinormal modes & Lyapunov Exponents

Quasinormal modes are solutions to wave equations of a perturbed black hole metric satisfying the boundary conditions

$$
\begin{aligned}
r_* \to \infty \, (purely\ reflected) \implies \Psi \to e^{\iota\omega t} \\
r_* \to -\infty \, (purely\ transmitted) \implies \Psi \to e^{-\iota\omega t}
\end{aligned}
\tag{3.1}
$$

The perturbation of a black hole space-time can be done in the form of adding a scalar field to it. For a specific mass μ, the equation can be written in the form of the general Klein-Gordon equation

$$(\nabla^\nu \nabla_\nu - \mu^2)\Psi = 0 \tag{3.2}$$

The above equation can be written as,

$$\frac{1}{\sqrt{-g}}\partial_\nu(g^{\mu\nu}\sqrt{-g}\partial_\mu \Psi) - \mu^2\Psi = 0 \tag{3.3}$$

3.1 Schwarzschild Black Hole

For the Schwarzschild metric,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2; \;\; f(r) = 1 - \frac{2M}{r} \;\; g = -r^4 sin^2\theta \tag{3.4}$$

The Klein-Gordon equation can be written as

$$\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\alpha}(g^{\alpha\alpha} r^2 sin\theta \frac{\partial\Psi}{\partial\alpha}) - \mu^2\Psi = 0 \tag{3.5}$$

Assuming the ansatz $\Psi(t, r, \theta, \phi) = e^{-\iota\omega t} Y_l(\theta, \phi)\frac{R(r)}{r}$,where $Y_l(\theta, \phi)$ are the spherical harmonics given by,

$$\Delta_{\theta,\phi} Y_l(\theta, \phi) = -l(l + 1)Y_l(\theta, \phi) \tag{3.6}$$

we can separate the variables and get equations as:

t co-ordinate

$$\begin{aligned} &\frac{1}{r^2 sin\theta}\frac{\partial}{\partial t}(g^{tt} r^2 sin\theta \frac{\partial\Psi}{\partial t}) \\ \implies &-\frac{1}{f(r)}\frac{\partial^2}{\partial t^2}(e^{-\iota\omega t} Y_l(\theta, \phi)R(r)/r) \\ \implies -\frac{1}{f(r)}\omega^2(e^{-\iota\omega t} Y_l(\theta, \phi)R(r)/r) = &\frac{\omega^2}{f(r)}\Psi \end{aligned} \tag{3.7}$$

θ, ϕ co-ordinate

$$\begin{aligned} &\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}(g^{\theta\theta} r^2 sin\theta \frac{\partial\Psi}{\partial\theta}) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\phi}(g^{\phi\phi} r^2 sin\theta \frac{\partial\Psi}{\partial\phi}) \\ \implies [&\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial Y_l(\theta, \phi)}{\partial\theta}) + \frac{1}{r^2 sin\theta}\frac{\partial^2}{\partial\phi^2}(Y_l(\theta, \phi))]\frac{R(r)}{r}e^{-\iota\omega t} \\ \implies &\frac{1}{r^2}[\Delta_{\theta,\phi} Y_l(\theta, \phi)]\frac{R(r)}{r}e^{-\iota\omega t} = -\frac{l(l + 1)}{r^2}\frac{R(r)}{r}e^{-\iota\omega t} \\ &\implies -\frac{l(l + 1)}{r^2}\Psi \end{aligned} \tag{3.8}$$

r co-ordinate

$$\begin{aligned}
&\frac{1}{r^2 sin\theta}\frac{\partial}{\partial r}(g^{rr}r^2 sin\theta\frac{\partial\Psi}{\partial r}) \\
\implies &\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f(r)\frac{\partial}{\partial r}(\frac{R(r)}{r}))Y_l(\theta,\phi)e^{-\iota\omega t} \\
\implies &\frac{Y_l(\theta,\phi)e^{-\iota\omega t}}{r^2}[\frac{\partial}{\partial r}(rf(r)\frac{\partial R(r)}{\partial r}) - \frac{\partial f(r)R(r)}{\partial r}] \\
\implies &\frac{Y_l(\theta,\phi)e^{-\iota\omega t}}{r^2}[r\frac{\partial f(r)\frac{\partial R(r)}{\partial r}}{\partial r} + f(r)\frac{\partial R(r)}{\partial r} - \frac{\partial f(r)R(r)}{\partial r}]
\end{aligned} \tag{3.9}$$

$$\implies \frac{Y_l(\theta,\phi)e^{-\iota\omega t}}{r^2}[r\frac{\partial f(r)\frac{\partial R(r)}{\partial r}}{\partial r} - R(r)\frac{\partial f(r)}{\partial r}] \tag{3.10}$$

Using a co-ordinate transformation $dr_* = \frac{dr}{f(r)}$,

$$\implies \frac{Y_l(\theta,\phi)e^{-\iota\omega t}}{r^2}[\frac{r}{f(r)}\frac{\partial^2 R(r)}{\partial r_*^2} - \frac{2M}{r^2}R(r)] \tag{3.11}$$

Here r_* is called the tortoise co-ordinate and maps the semi infinite region from horizon to infinity into $(-\infty,\infty)$ region.

Now re-writing the equations, we get

$$\begin{aligned}
&\frac{\omega^2}{f(r)}\Psi - \frac{l(l+1)}{r^2}\Psi + \frac{Y_l(\theta,\phi)e^{-\iota\omega t}}{r^2}[\frac{r}{f(r)}\frac{d^2R(r)}{dr_*^2} - \frac{2M}{r^2}R(r)] - \mu^2\Psi = 0 \\
\implies &\frac{1}{rf(r)}\frac{d^2R(r)}{dr_*^2} + \frac{\omega^2}{rf(r)}R(r) - \frac{2M}{r^4}R(r) - \frac{l(l+1)}{r^3}R(r) - \mu^2\frac{R(r)}{r} = 0 \\
\implies &\frac{d^2R(r)}{dr_*^2} + \omega^2R(r) - f(r)[\frac{2M}{r^3} + \frac{l(l+1)}{r^2} + \mu^2]R(r) = 0 \\
\implies &-\frac{d^2R(r)}{dr_*^2} + V(r)R(r) = \omega^2R(r); \quad V(r) = f(r)[\frac{2M}{r^3} + \frac{l(l+1)}{r^2} + \mu^2] \\
\implies &\frac{d^2R(r)}{dr_*^2} + Q(r)R(r) = 0; \quad Q(r) = \omega^2 - V(r)
\end{aligned} \tag{3.12}$$

3.2 Solution to the Radial Equation

Close to the turning point, Q(r) can be expanded as,

$$Q(r) = Q_o + \frac{1}{2}Q''_o(r - r_o)^2 + .. \tag{3.13}$$

The radial equation can be written as

$$\frac{1}{R(r)}\frac{d^2R(r)}{dr_*^2} + \frac{1}{2}Q''_o(r - r_o)^2 = -Q_o \tag{3.14}$$

Taking the radial equation, and comparing it to the harmonic oscillator equation of frequency ω

$$\omega^2 = -\frac{\hbar^2 Q''_o}{2m} \quad E = \frac{\hbar^2 Q_o}{2m} \tag{3.15}$$

Since for a harmonic oscillator,

$$E = \hbar\omega(n + \frac{1}{2}) \tag{3.16}$$

We get the relation,

$$\frac{\hbar^2 Q_o}{2m} = \iota(n + \frac{1}{2})\frac{\hbar^2}{2m}\sqrt{(}2Q''_o) \tag{3.17}$$

$$\implies \frac{Q_o}{\sqrt{2Q''_o}} = \iota(n + \frac{1}{2}) \tag{3.18}$$

3.3 Lyapunov Exponent and Quasi normal mode

In the eikonal limit (l $\rightarrow \infty$), the potential reduces to the form

$$V(r) = \frac{f(r)}{r^2}l^2 \tag{3.19}$$

which gives the equation

$$Q(r) = \omega^2 - f(r)\frac{l^2}{r^2} \tag{3.20}$$

For the following Q, the turning point (Q' = 0) gives the condition $2f(r_o) = r_o f'(r_o)$. This condition coincides with the condition for the null circular geodesic. Thus, at $r = r_c$, the Quasi normal frequency is written as

$$\omega_{QNM}^2 = f(r_c)\frac{l^2}{r_c^2} + \iota(n + \frac{1}{2})\sqrt{2Q_o''} \tag{3.21}$$

$$Q_o'' = \frac{d^2 Q_o}{dr_*^2} = -l^2(\frac{d^2}{dr_*^2}\frac{f}{r^2}) \tag{3.22}$$

Therefore at the turning point, we get

$$\sqrt{2Q_o''} = \pm\sqrt{-2l^2(\frac{d^2}{dr_*^2}\frac{f}{r^2})_{r_c}} \tag{3.23}$$

Hence,

$$\omega_{QNM}^2 = f(r_c)\frac{l^2}{r_c^2} \pm \iota(n + \frac{1}{2})\sqrt{-2l^2(\frac{d^2}{dr_*^2}\frac{f}{r^2})_{r_c}} \tag{3.24}$$

Since l is very large, we can complete the square, thus we get the quasi normal frequency as

$$\omega_{QNM} = l\sqrt{\frac{f(r_c)}{r_c^2}} \pm \iota(n + \frac{1}{2})\frac{1}{\sqrt{2}}\sqrt{-\frac{r_c^2}{f(r_c)}(\frac{d^2}{dr_*^2}\frac{f}{r^2})_{r_c}} \tag{3.25}$$

The real part of the frequency can be recognised as the co-ordinate angular frequency, $(\frac{\dot{\phi}}{\dot{t}})$, at the unstable null radius, which is denoted as Ω_c. From the phase-1 of this project, the lyapunov exponent of the null circular geodesic was found to be

$$\lambda = \sqrt{-\frac{V_r''}{2\dot{t}^2}} \tag{3.26}$$

Expanding the above expression with $V_r = -E^2 + f(r)\frac{L^2}{r^2}$ and the conditions $V_r = 0$, $V_r' = 0$, we get the lyapunov exponent to be

$$\lambda = \sqrt{-r_c^2\frac{d}{dr}(f(r)\frac{d}{dr}(\frac{f}{r^2}))} \tag{3.27}$$

Using the tortoise co-ordinate, $dr_* = \frac{dr}{f(r)}$, we see that the expression becomes

$$\lambda = \sqrt{-\frac{r_c^2}{f(r_c)}(\frac{d^2}{dr_*^2}\frac{f}{r^2})_{r_c}} \tag{3.28}$$

We see that it is now related to the ω_{QNM} as,

$$\omega_{QNM} = l\Omega_c - \iota(n + \frac{1}{2})|\lambda| \tag{3.29}$$

While we can see the relation directly, we can also implicitly understand from the expression of Q(r) how this relation comes forth. At the eikonal limit, the potential defined from the Klein-Gordon equation simplifies to the Effective potential defined in the conventional approach. Thus the Q_o'' in turn is equivalent to the double derivative of the effective potential (V_r'') and hence, the Lyapunov exponent can be related from it.

Chapter 4

Conclusion

In chapter 2, we used the Jacobi metric of the Schwarzchild black hole to compute the circular geodesic radius and determine the stability using the Lyapunov exponent. While the time-like geodesic gave a complex expression depending on the particle mass energy, the null geodesic was found to be unstable. The comparison between the phase-1 and phase-2 methods for the null geodesic is tabulated below.

	Schwarzchild Metric	Jacobi Metric
Geodesic Radius	Effective Potential, V_r $(r = 3M)$	Geodesic Curvature, $\mathbf{k}_g$ $(r = 3M)$
Lyapunov Exponent	$\lambda = \sqrt{\frac{1}{27M^2}}$	$\lambda = 3\sqrt{\frac{L^2M^4}{\dot{\alpha}^2E^2}}$
Stability	Unstable	Unstable

In chapter 3, the quasinormal mode frequency in the eikonal limit was seen to be completely determined by the parameters of the null geodesic. The real part was determined by the co-ordinate angular velocity, whereas the complex part of the frequency was dependent on the lyapunov exponent of the null geodesic. In general, the calculation can be repeated for any asymptotically flat spherically symmetric space time, and the results will be similar.

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