Optimal Water Distribution Network Design Using the Newton-Raphson Method

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1 Problem statement

Urban infrastructure is incomplete, without the inclusion of water distribution networks which are essential for supplying water to homes, industries and businesses. To ensure a water supply while minimizing costs and resource wastage these networks need to be designed and operated efficiently. Given their intricacies involving pipe layouts, varying demand patterns and hydraulic constraints advanced optimization methods are required.

The Newton Raphson method is a technique widely employed across various engineering domains to solve nonlinear equations. This research aims to utilize the Newton Raphson method to address optimization challenges encountered during the construction of water distribution networks to acquire optimal cost.

Existing approaches for planning such networks often rely on approximations and simplifications that may not adequately consider the interactions between pipe diameters, nodal pressures, flow rates and limitations. By employing the Newton Raphson method to solve the equations governing network hydraulics a more accurate approach can be achieved. This technology allows for consideration of variables, like demand fluctuations, pressure losses and pipe friction.

We will be using a univariate method, called cost-head loss ratio criterion method, for noncomputer optimization of water distribution networks. The univariate cost-head loss ratio criterion approach is updated here for rapid convergence using the Newton-Raphson method. Any existing software for analysing water distribution networks using the Newton-Raphson approach can be easily modified for optimal water distribution network design.

The main objectives of this project are as follows:

- Develop a comprehensive model to apply the Newton-Raphson method in flow analysis of water distribution networks.
- Analyze the shortcomings of current design approaches for water distribution networks and identify the challenges in achieving optimal design.
- Refine and extend the cost-head loss ratio criterion method to incorporate the Newton-Raphson technique for rapid convergence and improved design accuracy.
- Formulate a mathematical framework that integrates the modified optimization method with the network's hydraulic behavior equations.
- Implement the proposed method into existing software tools for water distribution network analysis, enhancing their capabilities for optimal design.
- Validate the effectiveness and efficiency of the upgraded software through a series of case studies involving multi-source looped water distribution networks.

Through this project, we aim to revolutionize the field of water distribution network design by harnessing the power of the Newton-Raphson method to achieve efficient and optimal solutions, ultimately contributing to the sustainability and effectiveness of water supply systems for communities.

2 Mathematical Model and Equations

Consider a single source, branching gravity network with a source node labelled 0, demand nodes labelled j, j = 1,..., N, and links labelled x, x = 1,..., X, with link labels identical to downstream node labels. Let H0 be the available **HGL** (Hydraulic grade line) at the source node and Hjmin, j = 1,..., N be the minimum HGL values at the demand nodes. Taking link diameter as a continuous variable and nodal heads as decision variables, the optimization equation is

$$MinC_T = \sum B_x A_x^{m/r} L_x^{1+m/r} Q_x^{pm/r} (H_i - H_j)^{-m/r} \tag{1}$$

where x denotes any pipe connected between nodes i and j
B and A are constants
p and r are exponents of discharge (Q) and pipe diameter (D), respectively, in head loss formula
m is exponent of diameter in cost-diameter relationship
L is pipe length
Hi and Hj are HGL values at nodes i and j, respectively.

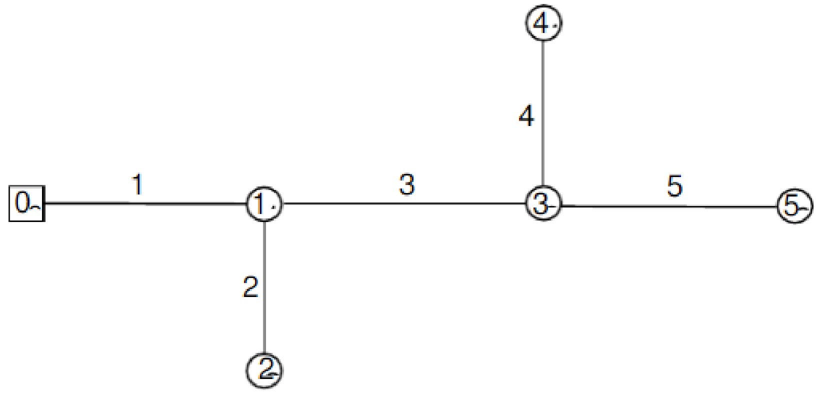

Figure 1: A Branched Water Distribution Network

3 Parameters

The parameters involved in the optimization problem for designing a water distribution network

- Nodes: These represent points within the network. Node 0 is the source node, where water is supplied, and nodes 1 to N are demand nodes, where water is consumed or distributed. Links (Pipes): These are the physical pipes that connect nodes in the network. Each link is labeled from 1 to X.
- Available HGL at Source Node (H0): The hydraulic grade line (HGL) at the source node represents the available energy or pressure head at the point where water enters the distribution network.
- Minimum HGL Values at Demand Nodes (Hj_min): These are the minimum hydraulic grade line values required at the demand nodes (consumption points) to ensure that water reaches those points with adequate pressure.
- Pipe Diameter: This parameter is treated as a continuous variable, and it represents the diameter of the pipes in the network. It is optimized as part of the design process.
- Constants (B, A, etc.): These are numerical values used in the equations and formulas that govern the behavior of the water distribution system. They are constants that influence factors such as head loss and cost.
- Exponents (p, r, m): p and r: These are exponents related to the discharge (Q) and pipe diameter (D) in the head loss formula. They determine how these factors affect head loss in the network. m: This exponent is related

to the diameter in the cost-diameter relationship, influencing how the diameter of pipes affects their cost.

- Pipe Length (L): This parameter represents the length of each pipe in the distribution network. It plays a role in determining head loss and costs.
- HGL Values at Nodes (Hi and Hj): These values represent the hydraulic grade line (HGL) at various nodes in the network. Hi represents the HGL at node i, and Hj represents the HGL at node j. These values are essential for ensuring that water flows with adequate pressure throughout the network.

These parameters collectively define the characteristics and constraints of the water distribution network optimization problem, where the goal is to minimize the total cost while meeting hydraulic constraints and ensuring adequate pressure at demand nodes

4 Optimality Criteria

for minimum cost design of distribution network a criterion, termed as cost-head loss ration criterion, must be satisfied at all nodes other than source nodes and critical nodes (available HGL = minimum required HGL). The criterion is

$$\sum mC/(H_i - H_j)_{ij} = \sum mC/(H_j - H_k)_{jk} \tag{2}$$

where, C is the cost of link and subscripts ij and jk respectively denote supply and distribution links incident at node j.

5 Optimization procedure

Current available iterative methods can give good approximation but take too long and many iterations to solve the problem. One such famous iterative method is univariate method. In this method HGL values are assumed initially at all nodes. With the assumed HGL values, the cost-head loss ratio requirement would be rarely, if ever, satisfied at all nodes at the same time. As a result, changes to these anticipated HGL values are required. Correction to assumed HGL value was calculated individually at each node by treating HGL at other nodes as fixed. The term "correction" was formed as

$$\Delta H_j = \frac{-\sum mC/h_{ij} + \sum mC/h_{jk}}{\sum mC/h^2{}_{ij} + \sum mC/h^2{}_{jk}} \tag{3}$$

where h is head loss at link.
When

(i) all HGL-corrections become insignificantly tiny, or
(ii) successive decreases in network cost become negligible, and the iterative method is halted.

6 Application of Newton Raphson method

The NM technique expands the non-linear terms in Taylor's series, ignores the residues after two terms, and thus takes into account only the linear terms. Thus, the NR approach linearizes nonlinear equations using partial differentiation and solves them. The iterative process is repeated until satisfactory precision is achieved. As a result, while using the NR approach to achieve adjustment in the cost-head loss ratio criterion method, all correction equations are examined and solved concurrently. We use multi-functional NM method to solve this problem.

The continuous correction in $\Delta x_1, \Delta x_2, ..., \Delta x_n$ to assumed values of n variables $x_1, x_2, \ldots, x_n$, in solution of n no of equations, $F_1(x_1, x_2, \ldots x_n) = 0; F_2(x_1, x_2, \ldots, x_n) = 0; ....; F_n(x_1, x_2, \ldots x_n) = 0$ can be acquired using NR method as The first ma-

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix} = - \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}$$

Figure 2: Newton Raphson using Jacobian matrix

trix ie. the partial differentials matrix is called as called the Jacobian of the n functions $F_1, F_2, \ldots, F_n$ with respect to the n variables $\Delta x_1, \Delta x_2, ..., \Delta x_n$. The second matrix is a column matrix representing the corrections, and the matrix on the right-hand side is a column matrix giving the values or the residue of the functions $F_1, F_2, \ldots, F_n$.

In our problem we iterate over H instead of x. Let $H_j, j = 1 \ldots Y$ be the assumed HGL values. Let $\Delta H_j, j = 1 \ldots Y$ be the additive correction so as to satisfy the optimality criteria given by Equation 2. Thus,

$$\sum_i mC/(H_i - H_j)_{ij} = \sum_j mC/(H_j - H_k)_{jk}, j = 1, 2.....Y \tag{4}$$

This can be written as:

$$F_j = \sum_i mC/(H_i - H_j)_{ij} - \sum_j mC/(H_j - H_k)_{jk} = 0, j = 1, 2.....Y \tag{5}$$

where F_j denotes the function equation at a node j. To get the Jacobian matrix as above we partially differentiate Equation 5 to get the diagonal terms ($\frac{\partial F_i}{\partial H_j}$, $i = j$) will be positive summation of $m\frac{C}{h^2}$ values of all the links (incoming and outgoing) incident at that node, and non-diagonal elements ($\frac{\partial F_i}{\partial H_j}$, $i \neq j$) will be the negative summation of the links connecting node i and j. Thus...

$$\frac{\partial F_i}{\partial H_j} = k. \sum mC_x/h_x^2$$

where x denotes the pipe connecting nodes i and j, and k is defined as:

$$k = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases}$$

After forming the Jacobian matrix, all the ΔH values are found in each iteration, and F values are updated according to equation 5 and iterated till the stopping condition is satisfied.

7 Procedure

As shown in figure 2 newton raphson method can be used for multiple functions all at a time.

- Step 1: Find the values of F's according to equation 5.
- Step 2: Form the Jacobian matrix by partial differentiation of respective terms.
- Step 3: Find ΔH values from the matrix using any system of linear equation methods

- Step 4: Update the values of F's again according to equation 5
- Step 5: Repeat the same process with the new updated F values until the terminating criterion is achieved.

8 Illustration

Consider a water distribution network shown in Fig. 3. Node 0 is the source node. The links labeled 1 to 5 have lengths 300, 400, 450, 300, and 350 m, respectively. The cost-diameter relationship for a link is taken as $c = 1.952D^{1.222}$, where c is the unit cost of pipe in Rupees, and D is the pipe diameter in millimetres. Required values of m C/h and m C/h2 for different pipes are obtained as given in Fig. 4 .

Since the number of iterations to final convergence depends on the initial

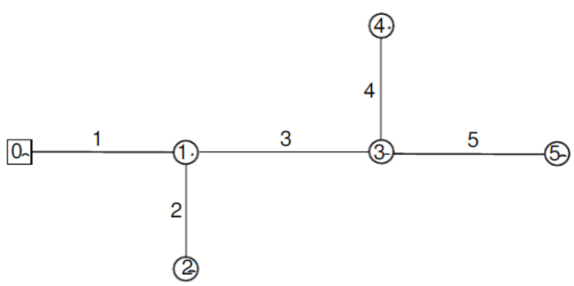

Figure 3: A Branched Water Distribution Network

Pipe no.	Length L (m)	Discharge Q (m^3/min)	Headloss h (m)	Diameter D (mm)	Cost C (Rs)	mC/h	mC/h^2
1	300	9.5	3.54	315.9	663865	229160	64740
2	400	2.0	8.46	155.0	370646	53538	6328
3	450	6.0	5.31	265.3	804317	185100	34860
4	300	1.5	6.15	139.8	245110	48703	7919
5	350	2.5	4.15	190.0	415976	122490	29520

Figure 4: Various parameters for sample example

assumption of HGL values and the critical path method provides a better estimate of initial trial HGL values, herein a critical path method is used to obtain initial trial HGL values. With the initial trial HGL values of H1 =96.46 m, H3 = 91.95 m. Substituting the required values in the Newton-Raphson equation we get

$$\begin{pmatrix} 105.928 & -34.86 \\ -34.86 & 72.229 \end{pmatrix} \begin{pmatrix} \Delta H1 \\ \Delta H2 \end{pmatrix} = - \begin{pmatrix} -9.478 \\ 13.907 \end{pmatrix}$$

On the first iteration, we get $\Delta H1$=0.0310 m, $\Delta H2$=10.177 m. We have to find the cost of the system for every iteration using Equation 1. But for this particular set of data, the A and B values are unavailable. We cannot take random values because it might affect the cost value. Instead, we use the proportionality between min C_T and $H_i - H_j$.

We measure the $H_i - H_j$ value for every iteration, indicating the effect on cost as they are directly related. Plotting the variation of $H_i - H_j$ vs. the number of iterations we get

after 0 iterations H_iff: 4.51

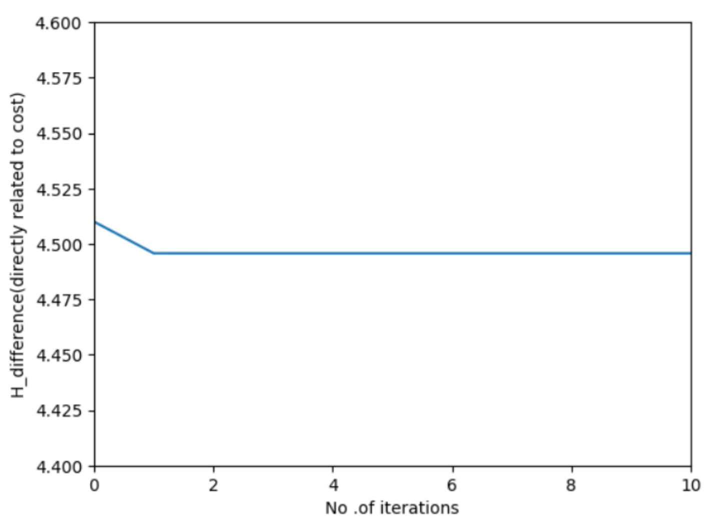

Figure 5: Plot of $H_i - H_j$ vs. no of iterations

after 1 iterations H_iff: 4.495751408460843
after 2 iterations H_iff: 4.49573678411825
after 3 iterations H_iff: 4.495736769493908
after 4 iterations H_iff: 4.495736768031473
after 5 iterations H_iff: 4.49573676803001

As we can observe from the above data, the change is almost negligible after the first few iterations. This shows the efficiency of the NM method and its ability to measure in a very small number of iterations in comparison with any other iterative methods.

9 Results and Conclusions

Newton raphson method is a very efficient method for cost optimization in water distribution networks in very small number of iterations in comparision with any other iterative methods. This can be applied to both branched and looped networks and can be used to design multi-source networks including pumped source nodes. These programs can be customized to design WDNs. Because the number of iterations is determined by the initial trial HGL values, the fewer the iterations the closer these values are to optimal values. The critical path method produces better initial trial values and should be utilized for HGL values in beginning trials.

Group Members

- Reddybathuni Venkat - 22110220
- Oruganti Dheekshitha - 22110176
- Rathod Vatsalraj Vikramsinh - 22110218
- Raiyani Vedanshi Sanjay - 22110213
- Tarun Meena - 22110271

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