RESEARCH STATEMENT

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My area of research is algebraic combinatorics. I apply representation theory of finite groups to compute some numerical invariants of combinatorial structures.

The problem of determining the isomorphism of two incidence structures (such as designs/graphs) is a difficult one. The algebraic invariants of the corresponding incidence matrices may help distinguish two non-isomorphic structures. One such invariant is the Smith normal form of the incidence matrix. In [6], the authors proved that two symmetric designs were nonisomorphic by computing the Smith normal forms of the corresponding incidence matrices. Therefore it is of some interest to compute the Smith normal forms of incidence matrices arising in combinatorics.

My current focus is on computing the Smith normal forms of the adjacency and Laplacian matrices of families of Strongly Regular Graphs. The Smith group of a graph is the abelian group with the same invariant factors as the smith normal form of its Adjacency matrix. The *critical* group of a graph is the finite part of the abelian group with the same invariant factors as the Smith normal form of its Laplacian matrix. The order of the *critical* group of a connected graph is equal to the number of spanning trees. The *critical* groups of various graphs arise in combinatorics in the context of chip firing games (cf. [2]), as the abelian sandpile group in statistical mechanics (cf. [7]), and also in arithmetic geometry. One may refer to [14] for a discussion on these connections.

It is therefore of some interest to compute the Smith groups and *critical* groups of graphs.

Background

Let Γ be a simple connected graph. Let A be the adjacency matrix with respect to some arbitrary but fixed order of V. Then the Laplacian matrix of Γ is the matrix $L := D - A$, where D is the diagonal matrix whose ith diagonal entry is the valency of the ith vertex. The Smith group S of Γ is the Cokernal of the integer matrix A (treated as a $\mathbb{Z}$-linear map), and the *critical* group $\mathcal{K}$ is the finite part of the Cokernel of the matrix L. Kirchoff's Matrix-Tree theorem implies that the order of the *critical* group of Γ is equal to the number of spanning trees of Γ (see for eg. [23]).

Some Families of graphs with known *critical* groups. An early author on the *critical* group was Vince, who in [25] computed them for Wheel graphs and complete bipartite graphs. In the same paper, it was shown that the group depends only on the cycle matroid of the graph. There are relatively few classes of graphs with known Smith and *critical* groups. Other papers that include computation of *critical* groups of families of graphs include [5], [1], [10], [8], [4], and [17].

Recent work

As a part of my Ph.D. thesis under the supervision of Prof.Peter Sin (Univ. of Florida), I determined the Smith and *critical* groups of two families of Strongly regular graphs.

1) In joint work (c.f [17]) with Peter Sin, I computed the Smith and *critical*groups of Polar graphs. Polar graphs are Strongly Regular collinearity graphs of finite classical polar spaces. These graphs admit certain finite classical groups as automorphisms. The action of finite classical groups on Polar spaces is a rank 3 permutation actions. General properties of Strongly Regular graphs and the modular representation theory of finite classical groups helped us determine the Smith and *critical* groups of these graphs.

2) In [16], I determined the *critical* groups of the family of van-Lint Schrijver cyclotomic strongly regular graphs. These are a family of Strongly regular graphs which are Cayley graphs on the additive group of a finite field with a multiplicative subgroup as the "connection" set. General properties of strongly regular graphs and the representation theory of abelian groups were used to determine the *critical* groups of these graphs.

Methodology. Let $\Gamma = (\tilde{V}, \tilde{E})$ be a simple undirected graph. Fix an ordering on the set of vertices $\tilde{V}$, and let A be the adjacency matrix with respect to this ordering. Let D be the diagonal matrix with D_{ii} being the degree of the ith vertex of Γ. Then $L := D - A$ is called the Laplacian matrix of Γ. With some abuse of notation we may assume that A and L

are elements of $End_{\mathbb{Z}}(\mathbb{Z}\tilde{V})$. The cokernal of A is the Smith group S of , and the finite part of the cokernal of L is the critical group $\mathcal{K}$ of Γ. The computation of S and K is equivalent to finding the elementary divisors of A and L, so can be carried out one prime at a time.

Let ℓ be a prime number, and $\mathbb{Z}_\ell$ be the ring of ℓ-adic integers. We may assume that A and L are elements of $End_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell\tilde{V})$. Given $i \in \mathbb{Z}_{\geq 0}$, define $M_i(A) := \{x \in \mathbb{Z}_\ell\tilde{V} \mid Ax \in \ell^i\mathbb{Z}_\ell\tilde{V}\}$, and define $M_i(L)$ in a similar fashion. Given any submodule M of $\mathbb{Z}_\ell\tilde{V}$, define $\overline{M} := (M + \ell\mathbb{Z}_\ell\tilde{V})/\mathbb{Z}_\ell\tilde{V}$. We observe that $\overline{M}$ is a subspace of the vector space $\mathbb{F}_\ell\tilde{V}$. The structure theorem of finitely generated modules over PIDs yields the following result.

Lemma 1. *Let e_0 be the ℓ rank of the adjacency matrix A (respectively Laplacian L). Also for $i \in \mathbb{Z}_{>0}$, let e_i be the multiplicity of ℓ^i as an elementary divisor of S (respectively $\mathcal{K}$) . Then* $\dim\left(\overline{M_i(A)}/\overline{M_{i+1}(A)}\right) = e_i$ *(respectively* $\dim\left(\overline{M_i(L)}/\overline{M_{i+1}(L)}\right) = e_i$*).*

The the above Lemma reduces computing these groups to finding the dimensions of certain finite vector spaces. More over if G is a group of automorphisms of Γ, then $\overline{M_i(A)}$'s and $\overline{M_i(L)}$'s are $\mathbb{F}_\ell$G-submodules of the permutation module $\mathbb{F}_\ell\tilde{V}$.

Results.

(1) Let V be a finite vector space endowed with either a non degenerate quadratic form or a non degenerate hermitian form, or a non-degenerate symplectic form. Let $\hat{V}$ be the set of isotropic one dimensional subspaces of V. A polar graph is a graph $\Gamma(V)$ with vertex $\hat{V}$, whose adjacency is defined by orthogonality with respect to the underlying form. Let $\mathrm{G}(V)$ be the group of automorphisms of V. This action of $\mathrm{G}(V)$ on $\hat{V}$ is a rank 3 permutation action (cf. [13] and [12]) and thus $\Gamma(V)$ is a strongly regular graph. Let $\mathcal{K}$ be the critical group of $\Gamma(V)$. In [17], I and Prof. Sin described the Smith and *critical* groups of $\Gamma(V)$. An outline of the our argument is in the paragraph below.

Let p be the characteristic of the underlying field (of V), then by some standard linear algebra, we deduced that the p-parts of S is cyclic and that of $\mathcal{K}$ is trivial. Let $\ell \neq p$ be a prime dividing $|\mathcal{K}|$ (resp. $|S|$). By the discussion above, the dimensions of $\mathbb{F}_\ell\mathrm{G}(V)$-modules $\overline{M_i(L)}$'s (resp. $\overline{M_i(A)}$'s) yield the ℓ-part of $\mathcal{K}$ (resp. S). These modules are submodules of the cross characteristic permutation module $\mathbb{F}_\ell\hat{V}$. The submodule structure of the cross characteristic permutation module corresponding to the action of $G(V)$ on $\hat{V}$ was determined in [13], [12], [11], and [21]. This submodule structure was used to determine the dimensions of $\overline{M_i(L)}$'s. Using those dimensions and Lemma 1 we obtained the ℓ-part of $\mathcal{K}$.

The following table encodes the multiplicities of elementary divisors of $\mathcal{K}$ when V is a symplectic space.

$(f,g) = \left(\frac{q(q^m-1)(q^{m-1}+1)}{2(q-1)},\ \frac{q(q^m+1)(q^{m-1}-1)}{2(q-1)}\right)$		
$(a,b,c,d) = \left(v_\ell(\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q),\ v_\ell(\begin{bmatrix} m \\ 1 \end{bmatrix}_q),\ v_\ell(q^m+1),\ v_\ell(q^{m-1}+1)\right)$		
Prime	Arithmetic conditions	Divisor multiplicities
$\ell = 2$	m is even	$e_0 = g+1$, $e_1 = f-g-1$, $e_{d+1} = 1$, $e_{d+b+1} = g-1$, and $e_i = 0$ for all other i.
	m is odd	$e_0 = g$, $e_a = 1$, $e_{a+c} = f-g-1$,$e_{a+c+1} = g$, and $e_i = 0$ for all other i.
$\ell \neq 2$	$b = d = 0$	$e_0 = g+\delta_{a,0}$, $e_a = \delta_{c,0}(f-1)+1+\delta_{a,0}(g)$, $e_{a+c} = f-1+\delta_{c,0}$, and $e_i = 0$ for all other i.
	$a = c = 0$	$e_0 = f+\delta_{d,0}$, $e_d = \delta_{b,0}(g)+1+\delta_{d,0}(f)$, $e_{b+d} = g-1+\delta_{b,0}$, and $e_i = 0$ for all other i

TABLE 1. *Critical* group of $\Gamma_s(q,m)$.

The descriptions for Smith and *critical* groups of $\Gamma(V)$ in other cases can be found in [17].

(2) Let $q = p^f$ be a prime power, let $N > 1$ be a divisor of $q-1$. Let D be a subgroup of multiplicative group of the finite field $\mathbb{F}_q$. Let $\mathrm{Cay}(\mathbb{F}_q, D)$ be the Cayley graph on the additive group of $\mathbb{F}_q$ with "connection" set D. If $\mathrm{Cay}(G,D)$ is a strongly regular graph, then we speak of a *cyclotomic strongly regular graph*. The Paley graph is a well known example of a *cyclotomic strongly regular graph*. Extensive scholarship on these graphs include [24], [3], [19], and [9]. In [24], van Lint and Schrijver define a family of *cyclotomic Strongly*

Regular Graphs whose construction is similar to that of the Paley Graph. Let p and ℓ be primes,with $\ell > 2$ and p primitive $\pmod{\ell}$. Let $t \in \mathbb{N}$ and $q = p^{(\ell-1)t}$. Consider the field $K = \mathbb{F}_q$, and let S be the unique subgroup of K^* of order $k = (q-1)/\ell$. Then by $G(p, \ell, t)$ we denote the graph with vertex set K and edge set $\{\{x, y\} \mid x, y \in K \;\&\; x - y \in S\}$. This is the undirected Cayley graph associated with (K, S). We call these families of graphs, *Van Lint-Schrijver cylclotomic Strongly Regular Graphs*. Extending the methods used in [5] and [20], I obtained a description of the *critical* groups of *Van Lint-Schrijver cylclotomic Strongly Regular Graphs* (cf. [16]). The outline of the method is in the paragraph below.

The additive group K, and multiplicative groups S and K^* are automorphism groups for $G(p, \ell, t)$. Let R be the ring of integers of the unique unramified extension of degree $(\ell-1)t$ over $\mathbb{Q}_P$. Let R^K be the permutation module over R, associated with the action of S on vertex set K. Let T be the Teichmller character of the multiplicative group K^*. Given $x \in K$, by $[x]$ we denote the basis element of R^K corresponding to x. Let $f_i = \sum_{x \in K^*} T^i(x^{-1})[x]$ We obtained the decomposition $R^K = \oplus_{i=0}^{k-1} N_i$, where each N_i is a submodule of R^K with basis $\{f_{i+mk} \mid 0 \le m \le \ell - 1\}$. Each N_i is an isotopic component of for the character $T^i|_S$. By L_i we denote the restriction $L|_{N_i}$ of the Laplacian L of $G(p, \ell, t)$. With respect to the basis of L_i describe above, the matrix of L_i has certain Jacobi sums as entries.

Computing the Smith Normal forms over R of the smaller matrices L_i, give us a description of the SNF over R of L. This information was used to compute the p-part of the *critical* group C of $G(p, \ell, t)$. The p'-part was obtained by conjugating L by $\frac{1}{q}X$, where X is the complex character table K.

The entries of matrices L_i are certain Jacobi sums. Classical results by Stickelberger and Gauss describe the p-adic valuations of Jacobi sums in combinatorial terms. The following describes the p part of the *critical* group of $G(p, \ell, t)$.

Theorem 2. *Consider the graph $G(p, \ell, t)$ with $p^{(\ell-1)t/2} \neq \ell - 1$ whenever t is odd. Given integers a, b not divisible by $p^{(\ell-1)t} - 1$, let $c(a, b)$ denote the number of carries when adding the p-adic expansions of a and b $\pmod{q-1}$. Let L be the Laplacian matrix and C be the* critical *group of $G(p, \ell, t)$. For $1 \le i < k - 1$, let*

$$c(i) = \min\left(\{c(i + mk, nk) | 0 \le m \le \ell - 1 \;\&\; 0 \le n \le \ell - 1\}\right).$$

Let e_j be the multiplicity of p^j as an elementary divisor of C. Then we have the following.

(a) $e_0 = |\{i \mid 1 \le i \le k-1 \;\& c(i) = 0\}| + 2$ *and* $e_{(\ell-1)t+d} = e_0 = |\{i \mid c(i) = 0\}|$.
(b) $e_j = |\{i \mid 1 \le i \le k-1 \;\& c(i) = j\}|$ *for* $0 < j < \frac{(\ell-1)t}{2}$.
(c) $e_j = e_{(\ell-1)t+d-j}$ *for* $0 < j < \frac{(\ell-1)t}{2}$.
(d) *If* $p \nmid \ell - 1$, *then* $e_{\frac{(\ell-1)t}{2}} = q + 1 - 2\sum_{j<t} e_j$.
(e) *If* $p \mid \ell - 1$, *then*
  (i) $e_{\frac{(\ell-1)t}{2}+d} = k + 2 - \sum_{j<t} e_j$ *and*
  (ii) $e_{\frac{(\ell-1)t}{2}} = (\ell-1)k - \sum_{j<t} e_j$.
(f) $e_j = 0$ *for all other* j.

In the case of $G(p, 3, t)$, using the transfer matrix method (cf. Section 4.7 of [22]) we were able to determine a closed form for the p-rank (i.e e_0 in the context of the Theorem above) of the Laplacian. The following theorem gives a quick recursive algorithm to compute other p-elementary divisors.

Let $P = \left(\left(\frac{p+1}{3}\right)^2 (x^2y^2 + x^2y + xy^2 + x + y + 1) + \left(\frac{p-2}{3}\right)^2 3xy\right)$, $R = p^2x^3y^3$ and

$Q = \left(\left(\frac{p+1}{3}\right)^2 (xy)(x^2y^2 + x^2y + xy^2 + x + y + 1) + \left(\frac{2p-1}{3}\right)^2 3x^2y^2\right)$. We define the polynomial $C(2t) \in \mathbb{C}[x, y]$ recursively as follows:

$$\begin{gathered} C(2) = 2P \\ C(4) = 2(P^2 - 2Q), \\ C(6) = 6R + 2(P^3 - 2QP) - 2PQ, \\ \text{and } C(2t) = PC(2t-2) - QC(2t-4) + RC(2t-6) \text{ for } t > 3. \end{gathered} \tag{1}$$

Theorem 3. *Let C_p be the p-part of the critical group of the graph $G(p,3,t)$ (with $(p,t) \neq (2,1)$). Let e_j denote the multiplicity of p^j in the elementary factor form of C_p. Let E_{ab} be the coefficient of $x^a y^b$ in $C(2t)$. Then we have the following.*

(a) $e_0 = e_{2t+\delta_{2,p}} + 2 = \left(\frac{(p+1)}{3}\right)^{2t}(2^{t+1} - 2)$.
(b) *For $a < t$, we have* $e_a = e_{2t+\delta_{2,p}-a} = \sum_{a<b\leq t} E_{ab}$
(c) $e_{t+\delta_{2,p}} = (k + 2 - \sum_{j<t} e_j) + (1 - \delta_{2,p})(2k - \sum_{j<t} e_j)$.
(d) $e_t = (1 - \delta_{2,p})(k + 2 - \sum_{j<t} e_j) + (2k - \sum_{j<t} e_j)$.
(e) $e_a = 0$ *for all other a.*

Future Work

I plan to find the Smith and *critical* groups of other families of Strongly Regular Graphs using the methodology described in the previous section.

My current focus is to try to obtain the *critical* groups of some other families of Cyclotomic Strongly Regular graphs. The methods I used to find the *critical* groups of the van Lint-Schrijver family were extensions of to those used in [5] to compute the same for Paley Graphs. I believe these methods can be generalized to other families of Cyclotomic Strongly Regular Graphs.

I am also invested in computing the *critical* groups of directed graphs. I am particularly interested in *critical* groups of Doubly Regular Tournaments(DRT). It was shown in [18] that the existence of a DRT on n vertices is equivalent to the existence of a skew Hadamard Matrix of size $n + 1$. It was shown in [15] that the Smith normal forms of all skew Hadamard matrices of a certain size have the same Smith normal forms. Let H be a skew Hadamard matrix, and let $T(H)$ be the corresponding DRT. The *critical* group of $T(H)$ is an invariant of H. I believe that this invariant could be useful in showing inequivalence of skew Hadamard matrices.

I am open to work on most problem in Algebraic combinatorics and Representation theory. In particular, I believe that I am well equipped to apply algebraic methods to solve problems in areas such as Designs and Finite geometry.

References

[1] Hua Bai. On the critical group of the n-cube. *Linear algebra and its applications*, 369:251–261, 2003.
[2] N.L. Biggs. Chip-firing and the critical group of a graph. *Journal of Algebraic Combinatorics*, 9(1):25–45, Jan 1999.
[3] AE Brouwer, RM Wilson, and Qing Xiang. Cyclotomy and strongly regular graphs. *Journal of Algebraic Combinatorics*, 10(1):25–28, 1999.
[4] Andries Brouwer, Joshua Ducey, and Peter Sin. The elementary divisors of the incidence matrix of skew lines in PG$(3, q)$. *Proceedings of the American Mathematical Society*, 140(8):2561–2573, 2012.
[5] David B. Chandler, Peter Sin, and Qing Xiang. The smith and critical groups of paley graphs. *Journal of Algebraic Combinatorics*, 41(4):1013–1022, Jun 2015.
[6] David B Chandler and Qing Xiang. The invariant factors of some cyclic difference sets. *Journal of Combinatorial Theory, Series A*, 101(1):131–146, 2003.
[7] Deepak Dhar. Self-organized critical state of sandpile automaton models. *Physical Review Letters*, 64(14):1613, 1990.
[8] Joshua E Ducey, Jonathan Gerhard, and Noah Watson. The smith and critical groups of the square rook's graph and its complement. *arXiv preprint arXiv:1507.06583*, 2015.
[9] Tao Feng, Koji Momihara, and Qing Xiang. Constructions of strongly regular cayley graphs and skew hadamard difference sets from cyclotomic classes. *Combinatorica*, 35(4):413–434, 2015.
[10] Brian Jacobson, Andrew Niedermaier, and Victor Reiner. Critical groups for complete multipartite graphs and cartesian products of complete graphs. *Journal of Graph Theory*, 44(3):231–250, 2003.
[11] JM Lataille, Peter Sin, and Pham Huu Tiep. The modulo 2 structure of rank 3 permutation modules for odd characteristic symplectic groups. *Journal of Algebra*, 268(2):463–483, 2003.
[12] Martin W Liebeck. Permutation modules for rank 3 unitary groups. *Journal of Algebra*, 88(2):317–329, 1984.
[13] Martin W Liebeck. Permutation modules for rank 3 symplectic and orthogonal groups. *Journal of Algebra*, 92(1):9–15, 1985.
[14] Dino Lorenzini. Smith normal form and laplacians. *Journal of Combinatorial Theory, Series B*, 98(6):1271 – 1300, 2008.
[15] TS Michael and WD Wallis. Skew-hadamard matrices and the smith normal form. *Designs, Codes and Cryptography*, 13(2):173–176, 1998.
[16] Venkata Raghu Tej Pantangi. Critical group of van lint-schrijver cyclotomic strongly regular graphs. 2018.
[17] Venkata Raghu Tej Pantangi and Peter Sin. Smith and critical groups of polar graphs. *arXiv preprint arXiv:1706.08175*, 2017.
[18] KB Reid and Ezra Brown. Doubly regular tournaments are equivalent to skew hadamard matrices. *Journal of Combinatorial Theory, Series A*, 12(3):332–338, 1972.
[19] Bernhard Schmidt and Clinton White. All two-weight irreducible cyclic codes? *Finite Fields and Their Applications*, 8(1):1–17, 2002.
[20] Peter Sin. The critical groups of the peisert graphs $\mathrm{P}^*(q)$. *arXiv preprint arXiv:1606.00870*, 2016.
[21] Peter Sin and Pham Huu Tiep. Rank 3 permutation modules of the finite classical groups. *Journal of Algebra*, 291(2):551–606, 2005.
[22] Richard P. Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge University Press, New York, NY, USA, 2nd edition, 2011.

[23] Richard P Stanley. Smith normal form in combinatorics. *Journal of Combinatorial Theory, Series A*, 144:476–495, 2016.
[24] Jacobus H van Lint and Alexander Schrijver. Construction of strongly regular graphs, two-weight codes and partial geometries by finite fields. *Combinatorica*, 1(1):63–73, 1981.
[25] A. Vince. Elementary divisors of graphs and matroids. *European Journal of Combinatorics*, 12(5):445 – 453, 1991.

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