MATHS ASSIGNMENT

MA200 - Assignment 2018

1. How many automorphisms do the following (labeled) graphs have?

(a) The complete graph of n vertices.

(b) The cycle of n vertices, with $n \geq 3$.

(c) The path of n vertices.

Sol:- An automorphism of a labelled graph is a form of symmetry in which the graph is mapped onto itself while preserving the edge-vertex connectivity.

Formal definition → Automorphism of labelled graph G is a permutation ϕ on the set of vertices V_G that satisfies the property that $\{u_i, u_j\} \in E_G$ if & only if $\{\phi(u_i), \phi(u_j)\} \in E_G$

Now, →

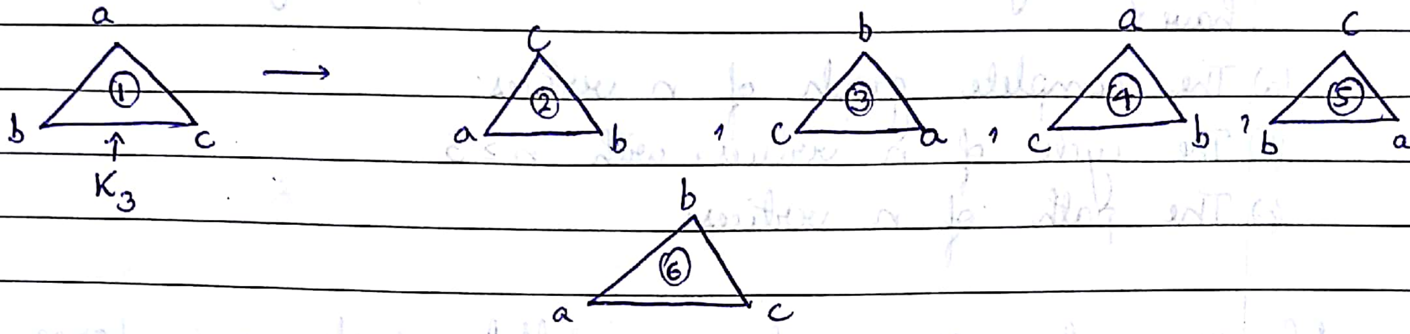
a) In a complete graph of n vertices that is K_n, all the vertices are connected to all other vertices.
So, even on permuting vertices, they will still be connected to all other vertices, thus retaining the edge-vertex connectivity. So, we just need to find the total no. of permutation of vertices of a Complete graph K_n.

Consider n empty places in space. Put 1^{st} vertex taken at random on any one place i.e. we can put it on n places, 2^{nd} vertex can be put on remaining n-1 ~~plas~~ places, 3^{rd} vertex on n-2 places
So,
finally, total permutation of vertices of a Complete graph $K_n = n \times (n-1) \times (n-2) \cdots 2 \times 1$

$= n!$

So, automorphism of complete labelled graph K_n is $n!$

Example → Take $n = 3$, So, No. of automorphisms $= 3! = 6$



b) In A cycle on n vertices, $n \geq 3$, as any vertex has degree 2, it is connected to two vertices & its edge-vertex connectivity with those two vertices should be maintained.
Lets assume that, the automorphisms of a cycle with n vertices are only due to either rotations or flipping. So, while doing this, we get n automorphisms due to rotation. In case of flipping, if n is odd, then flip across vertex - opposite edge midpoint line makes n automorphisms & if n is even, then flip across two opposite vertex, line makes n/2 automorphisms & flip across the line joining the midpoint of opposite edges makes n/2 automorphisms. In either case of n being odd or even we get n automorphisms due to flipping. So, in total we get 2n automorphisms.

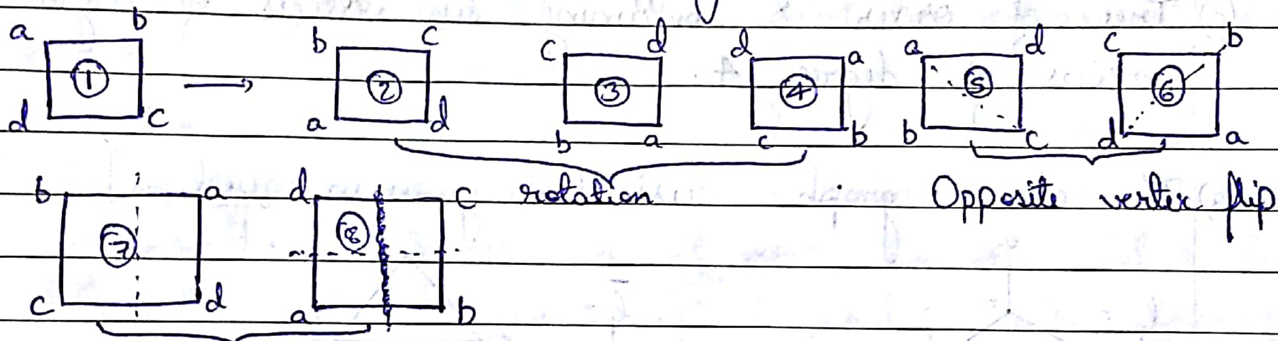
Now, to prove that there doesn't exist any other automorphism, we can prove that any automorphism f is already a member of the above found 2n automorphisms.

Firstly, label an arbitrary node (vertex) x in the cycle, f must map x

to some node (vertex) y in the cycle, & must map neighbours to neighbours, so either it must go one way around the cycle or the other way, since each node(vertex) has only two neighbours. ~~The direction would correspond to the flipping while the offset of y from x would correspond to the rotation.~~
So, going one way will correspond to rotation & the other way will correspond to flipping. Hence, there are no other automorphisms of the cycle of n vertices, $n \geq 3$

No. of automorphisms of the cycle with n vertices = $\underline{2n}$

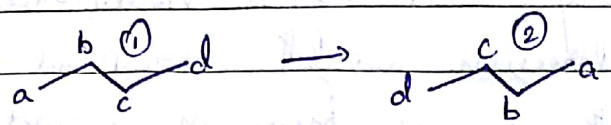
Example → $n = 4$



Opposite edges midpoint flip

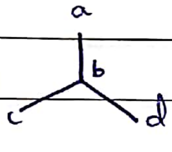
(c) In a path of n vertices, there are two vertices which have degree only 1. And $n-2$ vertices have degree 2, the automorphism formed should also be a path with degree 1 vertices at the end & all the $n-2$ degree vertices inside. If we assume the auto-morphism is not a path & not a cycle, there will exist a vertex with degree 3 which is contradicting that the vertex [not the corner vertex] is connected to only 2 other vertices & if we assume the automorphism is a cycle then there will be a contradiction to the fact that there are two vertices with degree 1.
So, as the automorphisms of a path can only be a path, this gives us with only one other graph which is the reverse path i.e. all the vertices are written in reverse.

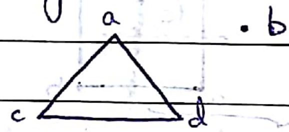
order. Hence we get,
No. of automorphisms of a path with n vertices is 2

Example →


2. Give an example of each of the following or explain why no such sample exists.
(a) A tree of order 4 whose complement is not a tree.
(b) A tree of order 6 containing four vertices of degree 1 & two vertices of degree 3.
(c) Tree of order 8 containing six vertices of degree 1 & two vertices of degree 4.

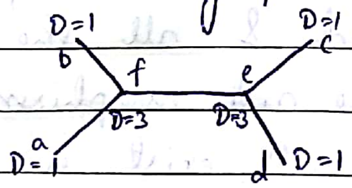
(a) The above graph exists [as given in question]

G =


$\bar{G}$ =


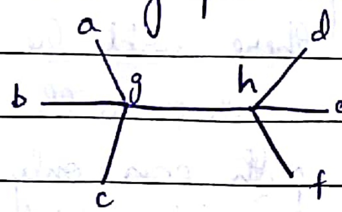
G is a tree of order 4, whereas $\bar{G}$ is a cycle and a no-edge vertex. not a tree even though it has 3 (n-1) edges.

(b) The above graph exists [as given in question]

G =


Vertices with degree 1 are a, b, d, c
Vertices with degree 3 are e, f

(c) The above graph exists [as given in question]

G =


Vertices with degree 1 are a, b, c, d, e & f
Vertices with degree 4 are g & h.

URBAN EDGE ___/___/___

3. The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in atleast one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

Sol 1→

Sol:- Consider a graph with 20 vertices ~~And as~~ which represents 20 members. As each member plays atleast one game and total number of edges = 14 [edges represent match between two people]

Sum of degrees of the vertices is 28. So, at least 12 vertices will have degrees of 1, and at most 8 vertices will have ~~8 verti~~ degree greater than 1. [∴ As 28 degrees needed to be distributed over 20 vertices where every vertex has atleast 1 degree, so 20 degrees are distributed to 20 vertices, rest 8 degrees can at max be distributed to 8 different vertices, giving us with atleast 12 vertices with degree 1.]

Now, If we keep deleting edges of vertices with degree greater than 1 (a maximum of 8 such edges as a vertex may have degree more than 2), then we are left with atleast 6 edges, and all of the vertices have degree either 0 or 1. These 6 edges represent the 6 games with 12 distinct players.

Hence Proved.

Sol 2→

Sol:- Let a slot be a place we can put a member in a game, so there are two slots per game, and 28 slots total. We begin by filling exactly 20 slots each with a distinct member since each member must play at least one game. Let there be m game with both slots filled and n games with only one slot filled, so $2m+n=20$. Since there are only 14 games.

$m+n \leq 14 \Rightarrow 2m+n \leq 14+m \Rightarrow 20 \leq 14+m \Rightarrow m \geq 6$.

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So, there must be atleast 6 games with two distinct members each, & we must have our desired set of 6 games.
Hence Proved.

4. There are k people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29), & that people's birthdays are independent (we assume there are no twins in the room). What is the probability that two or more people in the group have the same birthday? Also, state the minimum value of k for which this probability exceeds 0.5.

Sol:- $P(A)$ = Probability that atleast two people in the room have the same birthday. [There are k people]
$P(\bar{A})$ = Probability that no two people in the room have the same birthday. [There are k people]

For computing $P(\bar{A})$ for k people in the room,
Event 1 → Take a person at random and as his birthday can be on any day of the year, $P(\text{Event } 1) = \frac{365}{365}$

Event 2 → Take another person at random, and as the first person already has his birthday on some date, the second person has 365-1 days to have his birthday. So, $P(\text{event } 2) = \frac{364}{365}$

Similarly, $P(\text{Event } 3) = \frac{363}{365}$

⋮

$$P(\text{Event } k) = \frac{365 - k + 1}{365}$$

If $k > 365$, $P(\text{Event } k) = 0$

And by principle of conditional probability,

$$P(\bar{A}) = P(\text{event } 1) \times P(\text{event } 2) \cdots P(\text{event } k)$$

$$P(\bar{A}) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \cdots \frac{365-k+1}{365}$$

$$P(\bar{A}) = \frac{(365)!}{365^k \times (365-k)!}$$

So, $P(A) = 1 - P(\bar{A}) = 1 - \dfrac{365!}{365^k \times (365-k)!}$

So, probability that atleast two people in the room have the same birthday given there are k people in the room is

$$\boxed{P(A) = 1 - \frac{365!}{365^k \times (365-k)!}}$$

For the probability P(A) to exceed 0.5, minimum value of k can be found by hit & trial or by approximation $e^x \approx 1 + x$ if $x \ll 1$.

So, $P(A) = 1 - \left(1 - \dfrac{1}{365}\right)\left(1 - \dfrac{2}{365}\right) \cdots \left(1 - \dfrac{(k-1)}{365}\right)$

$$\approx 1 - e^{-\frac{1}{365}} \times e^{-\frac{2}{365}} \times e^{-\frac{3}{365}} \cdots e^{-\frac{(k-1)}{365}}$$

$$= 1 - e^{-\frac{k(k-1)}{365 \times 2}}$$

$$P(A) = 1 - e^{-\frac{k(k-1)}{720}}$$

Now, $P(A) > 0.5$

$$1 - e^{-\frac{k(k-1)}{720}} > 0.5$$

$$\Rightarrow 0.5 > e^{-\frac{k(k-1)}{720}}$$

Taking ln on both sides

$$\Rightarrow -0.69315 > -\frac{k(k-1)}{720} \Rightarrow k(k-1) > 0.69315 \times 720$$

$k^2 - k - 499 > 0$ [Approx]

Solving quadratic inequality we get

$$k \in (-\infty, -21.844) \cup (22.845, \infty)$$

But as k is minimum positive integer value. So, $\boxed{k = 23}$

By hit & trial also,

$$P(A) = 1 - \frac{365!}{365^{23}(342)!}$$

$$\approx 1 - 0.492703$$

$$\boxed{P(A) = 0.507297}$$

Hence, k = 23 for the probability to be just greater than 0.5.

5. Consider two probability spaces, (S_1, F_1, P_1) and (S_2, F_2, P_2). Here S_1 & S_2 are finite sets, & F_1 & F_2 are their power sets. Let $S_1 = \{x_1, \dots, x_m\}$ & $S_2 = \{y_1, \dots, y_n\}$. Let $S := S_1 \times S_2$ & let F be the power set of S. Define an additive function $P: F \longrightarrow R$ such that

$$P((x_i, y_j)) = P_1(x_i) \cdot P_2(y_j)$$

for each $(x_i, y_j) \in S$. Use additivity to define P for each element of F. Show that P is a probability function.

Sol:- $S_1 = \{x_1, x_2 \dots x_m\}$

$F_1 = \{\{\phi\}, \{x_1\}, \{x_2\} \dots \{x_m\}, \{x_1, x_2\} \dots, \{x_1, x_2 \dots x_m\}\}$

$S_2 = \{y_1, y_2 \dots y_n\}$

$F_2 = \{\{\phi\}, \{y_1\}, \{y_2\} \dots \{y_n\}, \{y_1, y_2\} \dots \{y_1, y_2, y_3 \dots y_n\}\}$

$S = S_1 \times S_2 = \{(x_1, y_1), (x_1, y_2), \dots (x_m, y_n)\}$

$F = \{\{\phi\}, \{(x_1, y_1)\}, \{(x_1, y_2)\}, \dots \{(x_1, y_1)(x_1, y_2) \dots (x_m, y_n)\}\}$

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Additivity $\rightarrow$ If $A_1, A_2 \cdots \in \mathcal{F}_k$ are pairwise disjoint (i.e., $A_i \cap A_j = \phi$, for all $i \neq j$), then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Using additivity, $P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k)$ where $A_1, A_2 \cdots \in \mathcal{F}$ are pairwise adjoint

And acc ~~to~~ to question, $A_k = ((x_i, y_j))$

So, $$P\Big(\bigcup_{\substack{i=1\\j=1}}^{\infty} (x_i, y_j)\Big) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} P(x_i, y_j)$$

$$\Rightarrow P\Big(\bigcup_{\substack{i=1\\j=1}}^{\infty} (x_i, y_j)\Big) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}[P_1(x_i) \cdot P_2(y_j)]$$ [According to question]

$$\Rightarrow P\Big(\bigcup_{\substack{i=1\\j=1}}^{\infty} (x_i, y_j)\Big) = \Big[\sum_{j=1}^{\infty} P_2(y_j)\Big] \times \Big[\sum_{i=1}^{\infty} P_1(x_i)\Big]$$ [$\because$ as $P_2(y_j)$ & $P_1(x_i)$ dont depend on each other & can the whole expression can be seen as a factor with some manipulation]

So, $$\boxed{P\Big(\bigcup_{\substack{i=1\\j=1}}^{\infty} (x_i, y_j)\Big) = \sum_{i=1}^{\infty} P_1(x_i) \cdot \sum_{j=1}^{\infty} P_2(y_j)}$$

Probability of each element of F is given by

Now,

To show that P is a probability function, we need to show that the sum of all the probabilities of the events in the set S is equal to 1.

i.e.

$$\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j) = 1 \longrightarrow \text{To prove}$$

Taking L.H.S.

$$\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j) = \sum_{j=1}^{n}\sum_{i=1}^{m} P_1(x_i) P_2(y_j)$$ [given in question]

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} P_1(x_i) \cdot P_2(y_j)$$

- As $P_1(x_i)$ & $P_2(y_j)$ are not dependent on each other. And on a closer analysis, we can observe that $P_1(x_1)$ is getting multiplied with all the $P_2(y_j)$ $j \in \{1,2,3 \cdots n\}$. Similarly $P_1(x_2)$ also. So, we observe that there are two factors of the above expression.

i.e. $(P_1(x_1) + P_1(x_2) + P_1(x_3) \cdots P_1(x_m))$ &

$(P_2(y_1) + P_2(y_2) + \cdots P_2(y_m))$

So, $\sum_{j=1}^{n} \sum_{i=1}^{m} P_1(x_i) \cdot P_2(y_j) = \left(\sum_{j=1}^{n} P_2(y_j)\right)\left(\sum_{i=1}^{m} P_1(x_i)\right)$

And as we know, $\sum_{j=1}^{n} P_2(y_j) = 1$ & $\sum_{i=1}^{m} P_1(x_i) = 1$ as

Sum of probabilities of all events in the sample space set is equal to 1.

So, $\sum_{j=1}^{n} \sum_{i=1}^{m} P_1(x_i) \cdot P_2(y_j) = 1 \times 1 = \underline{\underline{1}} = R.H.S.$

So, $L.H.S. = R.H.S.$

also, $P\left(\bigcup_{\substack{i=1 \\ j=1}}^{\infty} (x_i, y_j)\right) = \underbrace{\sum_{i=1}^{\infty} P_1(x_i)}_{\geq 0} \cdot \underbrace{\sum_{j=1}^{\infty} P_2(y_j)}_{\geq 0}$ is greater than equal to zero

So, P is a probability function.

Hence proved.