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Sahi Prep Hai Toh Life Set Hai

Special Series

Agenda

* Leftover Part of Geometric Progression → 8 – 10 min

* Special Series

(i) $\sum n, \sum n^2, \sum n^3$

(ii) T_n $\quad S_n = \sum T_n$

(iii) Telescopic → 85 min – 88 min

GEOMETRIC MEAN

If a and b are two numbers and G is their GM (Geometric Mean)
a, G & b are in GP.

$$\frac{G}{a} = \frac{b}{G}$$

$$G^2 = ab$$

$$G = \sqrt{ab}$$

a, G, b in G.P

Both a > 0 &
b > 0

Eg1. Find GM of 4 & 9.

$$GM = \sqrt{4 \cdot 9}$$

$$= 6$$

GM of

(i) $4 \,\&\, 9 = \sqrt{4 \cdot 9}$

$= 6$

(ii) $4 \,\&\, -9$

$\rightarrow$ Not possible

(iii) $-4 \,\&\, -9$

$= -6$

v. imp

Note : If both the numbers are positive, GM = $\sqrt{ab}$

If one of them is negative and other is positive, GM doesn't exist.

If both are negative, GM is negative.

If there are n positive numbers : a, b, c, d, ...

$$GM = \sqrt[n]{a \cdot b \cdot c \cdot d \ldots\ldots\ldots\ldots\ldots}$$

eg find G.M 12, 6 & 3

$\sqrt[3]{12 \cdot 6 \cdot 3} = 6$

Eg2. Find the GM of 2, 6, 16 & 108.

$$G \cdot M = \sqrt[4]{2 \cdot 6 \cdot 16 \cdot 108}$$

$$= \sqrt[4]{2^1 \cdot 2^1 \cdot 3^1 \cdot 2^4 \cdot 2^2 \cdot 3^3}$$

$$= \sqrt[4]{2^8 \, 3^4}$$

$$= 2^2 \cdot 3^1 \Rightarrow 12$$

RELATIONSHIP BETWEEN AM AND GM

If a and b are two positive numbers:

$$AM = \frac{a+b}{2}$$

$$GM = \sqrt{ab}$$

$$AM \geq GM$$

Reason

$$\frac{a+b}{2} - \sqrt{ab}$$

$$\frac{a+b-2\sqrt{ab}}{2}$$

$$\frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0$$

$$A.M - GM \geq 0$$

$$A.M \geq GM$$

SPECIAL SERIES

Some Basic formulas which will be used in Special Series:

$$(1+2+3+4+\ldots\ldots n)=\frac{n(n+1)}{2}$$

$$(1^2+2^2+3^2+\ldots\ldots n^2)=\frac{n(n+1)(2n+1)}{6}$$

$$(1^3+2^3+3^3+\ldots\ldots n^3)=\left[\frac{n(n+1)}{2}\right]^2$$

Eg1. Find the value of

$1 + 2 + 3 + 4 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 50$

$$1 + 2 + 3 + \text{------} n = \frac{n(n+1)}{2}$$

$$= \frac{\overset{25}{\cancel{50}} \cdot 51}{\cancel{2}}$$

$$= \underline{\underline{1275}}$$

Eg2. Find the value of

$1^2 + 2^2 + 3^2 + \dots\dots\dots\dots\dots\dots + 60^2$

$$1^2 + 2^2 + 3^2 + \text{-----} \; n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\frac{\overset{10}{\cancel{60}} \cdot 61 \cdot 121}{\cancel{6}}$$

$$= \underline{\underline{73810}}$$

Eg3. Find the value of

$1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 20^3$

$$1^3 + 2^3 + 3^3 + \text{-----} \; n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\left[\frac{\cancel{20}^{10} \cdot 21}{\cancel{2}}\right]^2$$

$$= \underline{\underline{44100}}$$

Eg4. Find the value of

21 + 22 + 23 + 24 + ... + 60

Ist

$(1+2+3+ \cdots\cdots\ 60) - (1+2+3+4+ \cdots 20)$

$= \dfrac{\cancel{60}^{30} \cdot 61}{\cancel{2}} - \dfrac{\cancel{20}^{10} \cdot 21}{\cancel{2}}$

$1830 - 210 = \boxed{1620}$

IInd

$(20+1)+(20+2)+(20+3) + \cdots (20+40)$

$20 \cdot 40 + 1+2+3+ \cdots 40$

$800 + \dfrac{\cancel{40}^{20} \cdot 41}{\cancel{2}} = \boxed{1620}$

IIIrd

$$\underline{21} + 22 + 23 + \cdots \underline{60}$$

$$\text{Sum} = \frac{n}{2}\left[2a + (n-1)d\right]$$

OR

$$\frac{n}{2}\left[f + L\right]$$

$$\frac{40}{2}\left[21 + 60\right] = 20 \cdot 81$$

$$= \underline{\underline{1620}}$$

Eg5. Find the value of

$2^2 + 4^2 + 6^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 40^2$

Sol^n

$$(2\cdot1)^2 + (2\cdot2)^2 + \ldots\ldots (2\cdot20)^2$$

$$2^2\left[1^2 + 2^2 + 3^2 + \ldots 20^2\right]$$

$$4\left[\frac{\cancel{20}^{10}\cdot\cancel{21}^{7}\cdot 41}{\cancel{6}\,\cancel{2}}\right]$$

$$= \underline{\underline{11480}}$$

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Eg6. Find the value of

$1^3 + 3^3 + 5^3 + 7^3 + \ldots\ldots\ldots\ldots + 29^3$

2min

$$\left(1^3+2^3+3^3+4^3+\ldots\ldots 29^3\right)-\left(2^3+4^3+\ldots 28^3\right)$$

$$\left[\frac{29\cdot \cancel{30}^{15}}{\cancel{2}}\right]^2-2^3\left(1^3+2^3+\ldots 14^3\right)$$

$$29^2 15^2-8\cdot\left(\frac{\cancel{14}^{7}\cdot 15}{\cancel{2}}\right)^2$$

$$29^2 15^2-8\cdot 7^2\cdot 15^2$$

$$15^2\left(841-392\right)=225\times 449$$

$$=101025$$

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Eg7. Find the value of

$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + 49^2 - 50^2 + 51^2$

$(1-2)(1+2) + (3-4)(3+4) + \cdots (49-50)(49+50) + 51^2$

$-1[1+2+3+4+\cdots 49+50] + 51^2$

$-\left(\frac{\cancel{50}^{25} \times 51}{\cancel{2}}\right) + 51^2$

$51(-25+51) = 51 \times 26$

$= 1326$

If you know the n^{th} term of a sequence then you can calculate its sum:

Let $\quad T_n = an^3 + bn^2 + cn + d$

So, $\quad S_n \rightarrow$ sum of n terms

$$S_n = \sum T_n$$

$$= a\sum n^3 + b\sum n^2 + c\sum n + d\sum 1$$

$$= a\left[\frac{n(n+1)}{2}\right]^2 + b\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{c(n \cdot n+1)}{2} + d \cdot n$$

Eg1. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots\ldots\ldots\ldots$

Find the sum of first 10 terms.

Sol^n $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \text{--------}$

Recognize the pattern of series & find n^{th} term

$$n^{th} \text{ term} = n(n+1)$$

$$= n^2 + n$$

$$1^2 + 1 + 2^2 + 2 + 3^2 + 3 + \text{---} \; 10^2 + 10$$

$$\frac{\cancel{10}^{5} \cdot 11 \cdot \cancel{21}^{7}}{\cancel{6}\cancel{2}} + \frac{\cancel{10}^{5} \cdot 11}{\cancel{2}}$$

$$385 + 55 = 440$$

Eg2. $1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7$ …………………… n terms.

Find the sum of first n terms.

Sol^n Analyze the pattern $\rightarrow n(n+3)$

$$\therefore T_n \rightarrow n^2 + 3n$$

$$S_n = \Sigma n^2 + 3\Sigma n$$

$$= \frac{n(n+1)(2n+1)}{6} + 3\frac{n(n+1)}{2}$$

$$\frac{n(n+1)}{2}\left[\frac{2n+1}{3} + 3\right]$$

$$\frac{n(n+1)}{2}\left[\frac{2n+10}{3}\right]$$

$$\frac{n(n+1)(n+5)}{3}$$

Eg3. Find the sum of first 10 terms.

$$2\cdot 5 + 5\cdot 7 + 8\cdot 9 + 11\cdot 11 + 14\cdot 13 + \ldots\ldots\ldots\ldots\ldots\ldots$$

Time 2min

Sol^n →

$2, 5, 8, 11, 14, \ldots \rightarrow 2+(n-1)3$

$\Rightarrow (3n-1)$

$5, 7, 9, 11, 13, \ldots \rightarrow 5+(n-1)2$

$\Rightarrow 2n+3$

$$T_n' = (3n-1)(2n+3) = 6n^2+7n-3$$

$$S_n = \sum T_n = \cancel{6}\left[\frac{10\cdot 11\cdot 21}{\cancel{6}}\right] + 7\cdot\frac{\overset{5}{\cancel{10}}\cdot 11}{\cancel{2}} - 30$$

$$= 2310 + 385 - 30 = \underline{\underline{2665}}$$

Remember,

In these kind of questions,

First calculate $T_n \rightarrow n^{th}$ terms

and then to calculate

$$\boxed{S_n = \Sigma T_n}$$

SPECIAL SERIES

I. TELESCOPIC SERIES:

Telescopic series is a series whose partial sums eventually only have a fixed number of terms after cancellation.

This will be illustrated with some examples.

V. Imp

Eg1. $$S=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots+\frac{1}{19\cdot 20}$$

Detailed App

$$S = \frac{2-1}{1\cdot 2}+\frac{3-2}{2\cdot 3}+\ldots\ldots \frac{20-19}{19\cdot 20}$$

$$= \left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\ldots\left(\cancel{\frac{1}{19}}-\frac{1}{20}\right)$$

$$1-\frac{1}{20} = \frac{19}{20}$$

$$S = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots \frac{1}{19\cdot 20}$$

Shortcut $\longrightarrow$ $\frac{1}{1} - \frac{1}{20} = \frac{19}{20}$

eg $$S = \frac{1}{100\cdot 101} + \frac{1}{101\cdot 102} + \cdots \frac{1}{149\cdot 150}$$

Shortcut $\frac{1}{100} - \frac{1}{150} = \frac{1}{300}$

Eg2. $S=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots+n$ terms.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\text{-----}\frac{1}{n(n+1)}$$

Shortcut

$$\frac{1}{1}-\frac{1}{n+1}=\frac{n}{n+1}$$

Eg3. $A = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \infty$ terms.

Shortcut $\longrightarrow$ $\frac{1}{1} -$

$= 1$

Eg4. $P = \frac{1}{14 \cdot 15} + \frac{1}{15 \cdot 16} + \frac{1}{16 \cdot 17} + \ldots\ldots + \frac{1}{48 \cdot 49}$

Shortcut $\rightarrow$ $\frac{1}{14} - \frac{1}{49} = \frac{5}{98}$

Eg5. $S = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots\dots\dots + \frac{1}{240}$

$$S = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots \frac{1}{15\cdot 16}$$

$$= \frac{1}{1} - \frac{1}{16} = \frac{15}{16}$$

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Eg6. $A = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots\ldots + \frac{1}{19\cdot 21}$

Time 90sec

Detailed Approach

Sol^n

$$\frac{1}{2}\left[\frac{2}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} + \cdots \frac{2}{19\cdot 21}\right]$$

$$= \frac{1}{2}\left[\frac{3-1}{1\cdot 3} + \frac{5-3}{3\cdot 5} + \frac{7-5}{5\cdot 7} + \cdots \frac{21-19}{19\cdot 21}\right]$$

$$= \frac{1}{2}\left[\left(1-\cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right) + \cdots \left(\cancel{\frac{1}{19}}-\frac{1}{21}\right)\right]$$

$$= \frac{1}{2}\left[1-\frac{1}{21}\right] = \frac{1}{2}\cdot\frac{20}{21} = \frac{10}{21}$$

$$S = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots \frac{1}{19\cdot 21}$$

Shortcut

$$\frac{1}{2}\left(\frac{1}{1} - \frac{1}{21}\right)$$

$$\frac{1}{\cancel{2}} \cdot \frac{\cancel{20}}{21} = \frac{10}{21}$$

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Now, the same examples can be tested in exams like:

$$A = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots\ldots + \frac{1}{19\cdot 21}$$

Or

$$B = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\ldots\ldots\ldots + \frac{1}{399}$$

Or

$$C = \frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \ldots\ldots\ldots\ldots + \frac{1}{20^2-1}$$

Eg7. $$B = \frac{1}{2\cdot 4} + \frac{1}{4\cdot 6} + \frac{1}{6\cdot 8} + \ldots\ldots + \frac{1}{18\cdot 20}$$

Shortcut

$$\frac{1}{2}\left[\frac{1}{2} - \frac{1}{20}\right]$$

$$\frac{1}{2}\left[\frac{10-1}{20}\right] = \frac{9}{40}$$

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Eg8. $$C = \frac{1}{11\cdot 13} + \frac{1}{13\cdot 15} + \frac{1}{15\cdot 17} + \ldots\ldots + \frac{1}{97\cdot 99}$$

$$\frac{1}{2}\left[\frac{1}{11} - \frac{1}{99}\right]$$

$$\frac{1}{\cancel{2}}\left[\frac{\cancel{8}}{99}\right] = \frac{4}{99}$$

Eg9. $D = \dfrac{1}{5\cdot 8} + \dfrac{1}{8\cdot 11} + \dfrac{1}{11\cdot 14} + \ldots\ldots + \dfrac{1}{47\cdot 50}$

$$\frac{1}{3}\left[\frac{1}{5} - \frac{1}{50}\right]$$

$$\frac{1}{\cancel{3}}\left[\frac{\cancel{9}}{50}\right] = \frac{3}{50}$$

Eg10. $Q=\frac{1}{1\cdot3\cdot5}+\frac{1}{3\cdot5\cdot7}+\frac{1}{5\cdot7\cdot9}+\dots\dots\dots+\frac{1}{17\cdot19\cdot21}+\frac{1}{19\cdot21\cdot23}$

Detailed App $\rightarrow$ $\frac{1}{4}\left[\frac{4}{1\cdot3\cdot5}+\frac{4}{3\cdot5\cdot7}+\text{------}\ \frac{4}{19\cdot21\cdot23}\right]$

$=\frac{1}{4}\left[\frac{5-1}{1\cdot3\cdot5}+\frac{7-3}{3\cdot5\cdot7}+\text{-----}\ \frac{23-19}{19\cdot21\cdot23}\right]$

$=\frac{1}{4}\left[\frac{1}{1\cdot3}-\cancel{\frac{1}{3\cdot5}}+\cancel{\frac{1}{3\cdot5}}-\cancel{\frac{1}{5\cdot7}}+\text{---}\ \cancel{\frac{1}{19\cdot21}}-\frac{1}{21\cdot23}\right]$

$=\frac{1}{4}\left[\frac{1}{1\cdot3}-\frac{1}{21\cdot23}\right]\Rightarrow\frac{1}{\cancel{4}}\left[\frac{\cancel{161}-1}{483}\right]$

$=40/483$

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$$Q = \frac{1}{1\cdot 3\cdot 5} + \frac{1}{3\cdot 5\cdot 7} + \cdots \frac{1}{19\cdot 21\cdot 23}$$

Sol^n

$$\frac{1}{4}\left[\frac{1}{1\cdot 3} - \frac{1}{21\cdot 23}\right]$$

$$= \frac{1}{\cancel{4}}\left[\frac{\cancel{161-1}}{483}\right] = \frac{40}{483}$$

Eg11. $R = \frac{1}{1\cdot 4\cdot 7} + \frac{1}{4\cdot 7\cdot 10} + \ldots\ldots\ldots + \frac{1}{10\cdot 13\cdot 16}$

Solⁿ

$$\frac{1}{6}\left[\frac{1}{1\cdot 4} - \frac{1}{13\cdot 16}\right]$$

$$\frac{1}{6}\left[\frac{52-1}{208}\right]$$

$$\frac{1}{\cancel{6}\,2} \cdot \frac{\cancel{51}\,17}{208} = \frac{17}{416}$$

Eg12. $$S=\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\dots\dots\dots+\frac{1}{18\cdot19\cdot20}$$

$$\frac{1}{2}\left[\frac{1}{1\cdot2}-\frac{1}{19\cdot20}\right]$$

$$\frac{1}{2}\left[\frac{190-1}{380}\right]=\frac{189}{760}$$

Practice Questions

Q1. $A = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \frac{9}{400} + \dots\dots\dots\dots\dots \frac{19}{8100}$

Find the value of A.

Sol^N

$$A = \frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \text{---} \frac{19}{9^2 \cdot 10^2}$$

$$\frac{2^2 - 1^2}{1^2 \cdot 2^2} + \frac{3^2 - 2^2}{2^2 \cdot 3^2} + \frac{4^2 - 3^2}{3^2 \cdot 4^2} + \text{---} \frac{10^2 - 9^2}{9^2 \cdot 10^2}$$

$$A = \left(1 - \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}} + \text{---} \cancel{\frac{1}{9^2}} - \frac{1}{10^2}\right) = \frac{99}{100}$$

Ans. $\frac{99}{100}$

Q2. $B=\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\ldots\ldots\ldots\ldots\ldots 10\ terms$

Find the value of B.

$$B = \frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+ - - - \frac{1}{1+2+3+\cdots n}$$

$$T_n = \frac{2}{n(n+1)} = 2\left[\frac{1}{n(n+1)}\right]$$

$$2\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+ - - - \frac{1}{10\cdot 11}\right]$$

$$2\left[1-\frac{1}{11}\right] = \frac{20}{11}$$

Ans. $1\frac{9}{11}$

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Q3. $C = \frac{1}{3\cdot 7} + \frac{1}{7\cdot 11} + \frac{1}{11\cdot 15} + \cdots\cdots\cdots\cdots \frac{1}{899\cdot 903}$

Find the value of C.

$$\frac{1}{4}\left[\frac{1}{3} - \frac{1}{903}\right]$$

$$\frac{1}{4}\left[\frac{300}{903}\right] = \frac{\cancel{75}^{\,25}}{\cancel{903}_{\,301}}$$

Ans. $\frac{25}{301}$

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Q4. $S=\frac{1}{1\cdot 2}+\frac{1}{1\cdot 4}+\frac{1}{2\cdot 3}+\frac{1}{4\cdot 7}+\frac{1}{3\cdot 4}+\frac{1}{7\cdot 10}+\ldots\ldots\ldots\ldots\ldots\ldots 20\ terms$

Find the value of S.

2min

$$\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\frac{1}{10\cdot 11}\ \text{10 terms}\right)+\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\ldots\frac{1}{28\cdot 31}\ \text{(10 terms)}$$

$$\left(1-\frac{1}{11}\right)+\frac{1}{3}\left[1-\frac{1}{31}\right]$$

$$\frac{10}{11}+\frac{1}{\cancel{3}}\cdot\frac{\cancel{30}^{10}}{31}=\frac{420}{341}$$

Ans. $\dfrac{420}{341}$

Q5. Find the value of M.

$$M = \frac{1}{1\cdot3\cdot5} + \frac{1}{1\cdot4} + \frac{1}{3\cdot5\cdot7} + \frac{1}{4\cdot7} + \frac{1}{5\cdot7\cdot9} + \frac{1}{7\cdot10} + \ldots\ldots\ldots\ldots\ldots\ldots 20 \textit{ terms}$$

Homework

Ans. $\frac{6070}{14973}$