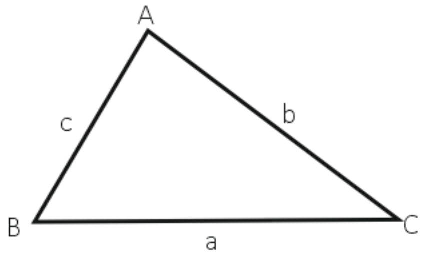
Triangle

Triangle

A triangle is a simplest polygon enclosed by three sides. As in the figure below three sides AB, BC, and CA represent triangle ABC or denoted by ΔABC. A, B and C represent the vertices of the ΔABC. Also, a, b and c represent the lengths of the sides BC, AC and AB respectively and ∠A, ∠B and ∠C represent the values of angles of the ΔABC.



Properties of a Triangle:

1. The sum of all the angles of a triangle = 180°

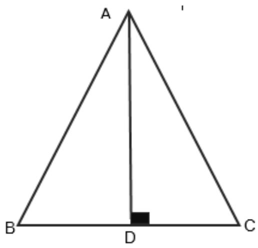
2. The sum of lengths of two sides > length of the third side of the triangle i.e.

$$a + b > c$$
$$b + c > a$$
$$c + a > b$$

3. The difference of lengths of two sides < length of the third side of the triangle i.e.

$$a - b < c$$
$$b - c < a$$
$$c - a < b$$

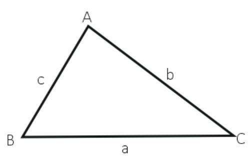
4. **Area of the triangle:**
   1. If in a triangle ΔABC, a perpendicular is drawn from vertex A to side BC which meets the side BC at point D. If the length of the perpendicular AD is 'h', then the Area of the ΔABC –



$$= \frac{1}{2} \times Base\ of\ the\ Triangle \times Length\ of\ the\ Perpendicular\ on\ the\ base$$

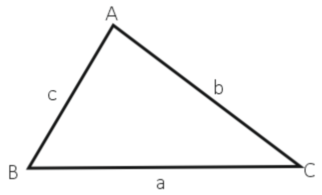
$$= \frac{1}{2} \times BC \times AD$$

   2. If the length of Two sides of Triangle and the angle between them is given, then the Area of the Triangle:



$$= \frac{1}{2} \times a \times b \times Sin\ C = \frac{1}{2} \times b \times c \times Sin\ A = \frac{1}{2} \times a \times c \times Sin\ B$$

3. Hero's Formula:



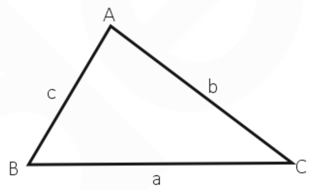
Area of ΔABC = $\sqrt{S(S-a)(S-b)(S-c)}$

Here, a, b, and c are the lengths of sides AB, BC, and CA respectively and "S" is the semi-perimeter of the ΔABC. Thus, S = $\frac{a+b+c}{2}$

Classification of Triangles:

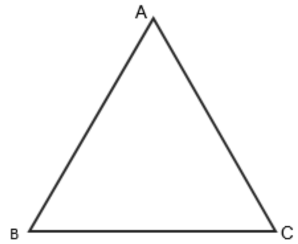
1. On the basis of Sides of the Triangle

(i) **Scalene Triangle:** If all the sides of a triangles are of unequal lengths then the triangle is termed as Scalene Triangle. In the figure, Δ ABC is a scalene triangle.



Area of Δ ABC = $\sqrt{S(S-a)(S-b)(S-c)}$

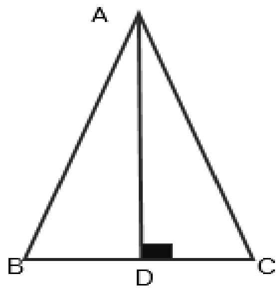
(ii) **Equilateral Triangle:** If all the three sides of a Triangle are equal in length then the Triangle is called an Equilateral Triangle. If the length of the side is 'a'.



ABC is an Equilateral Triangle. Where, AB = BC = CA = a

Then the area of Triangle ABC = $\frac{\sqrt{3}}{4} \times a^2$

(iii) **Isosceles Triangle:** If any two sides of a triangle are of equal length "a" unit and the third side is of "b" unit length then the triangle is said to be an Isosceles Triangle.



AB = AC = a; BC = b

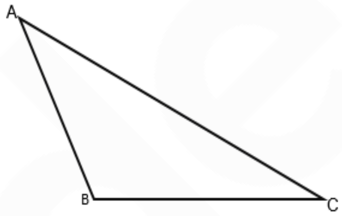
Height of the perpendicular AD = $\sqrt{a^2 - \frac{b^2}{4}} = \frac{\sqrt{4a^2-b^2}}{2}$

Area of Δ XYZ = $\frac{1}{2} \times b \times \frac{\sqrt{4a^2-b^2}}{2} = \frac{b\sqrt{4a^2-b^2}}{4}$

Note: All equilateral triangles are also isosceles triangles.

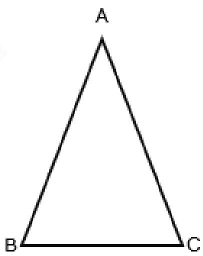
2. On the basis of Angle of a Triangle:

(i). Obtuse-angled Triangle: In a triangle, if one angle is more than 90° or other two angles are less than 90° then the triangle is said to be an obtuse angled triangle.



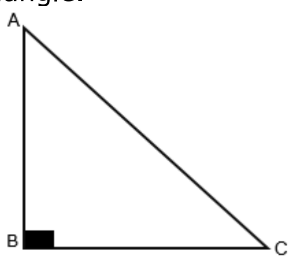
In ΔABC; ∠ABC >90° and ∠BAC, ∠BCA < 90°

(ii). Acute-angled Triangle: In a triangle, if all the angles are less than 90° then the triangle is said to be an acute angled triangle.



In ΔABC; ∠ABC, ∠BAC and ∠BCA < 90°

(iii). Right Angled Triangle: If in a triangle, two sides are perpendicular to each other or make 90° to each other than the triangle is said to be a Right-Angled Triangle. Here in the figure Δ ABC, AB is perpendicular, BC is base and AC is the hypotenuse of Right-Angled triangle.



Area of Δ ABC = $\frac{1}{2} \times \mathbf{Base} \times \mathbf{Perpendicular} = \frac{1}{2} \times \mathbf{BC} \times \mathbf{AB}$

Pythagoras Theorem:
Theorem: In a Right-Angled triangle, the sum of the squares of base and perpendicular is equal to the square of the hypotenuse. i.e.

$$(Hypotenuse)^2 = (Base)^2 + (Perpendicular)^2$$
$$AC^2 = BC^2 + AB^2$$

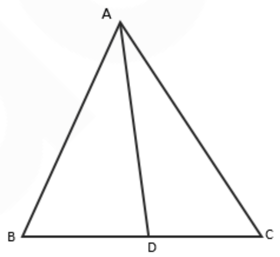
Ex.: A triangle with sides 3 cm, 4 cm and 5 cm form a right-angle triangle as $3^2 + 4^2 = 5^2$.

Pythagoras Triplets:
It is a set of three positive whole numbers that represent the lengths of the three sides of the right-angle triangle**.**
List of some Known and commonly used Pythagorean Triplets:
(3,4,5), (5,12,13), (6,8,10), (8,15,17), (7,24,25), (9,40,41), (11,60,61), (20,21,29) etc.

Application of Pythagoras Theorem:
- If $a^2 + b^2 = c^2$ then the triangle is right-angled triangle.
- If $a^2 + b^2 < c^2$ then the triangle is obtuse-angled triangle.
- If $a^2 + b^2 > c^2$ then the triangle is acute-angled triangle.

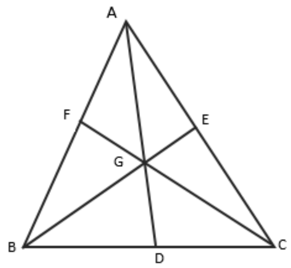
Median & Centroid

Median: It is the line segment that is drawn from a vertex of the triangle and joins the midpoint of the opposite side. There can be three medians that can be drawn from three vertices of the triangle.



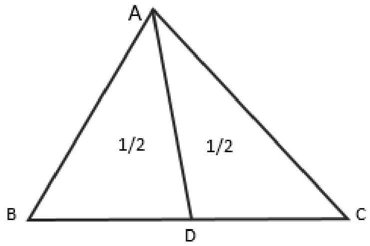
In the above diagram, AD is median of ΔABC which divides BC in two equal parts i.e. BD = DC

Centroid or Gravity Centre: The intersection points of all the three medians of a Triangle is called Centroid or Gravity Centre. It is denoted by 'G'. It is also called as "Gravity Centre". It divides each of the median in the ratio 2:1.

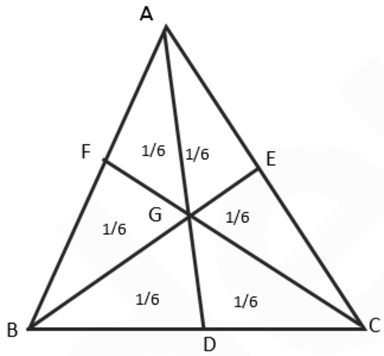


$$\frac{AG}{GF} = \frac{BG}{GD} = \frac{CD}{GE} = \frac{2}{1}$$

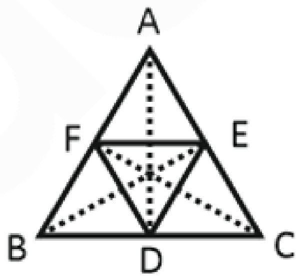
Property 1: A median bisects the area of a Triangle in two equal parts.



Property 2: Centroid of a Triangle divides the area of Triangle in six equal parts.



Property 3: If DEF are the mid-points of the sides of a Triangle ABC (AB, BC and CA).



Then the area of Triangle DEF = $\frac{Area\ of\ \Delta\ ABC}{4}$

Property 4: Sum of sides of a Triangle is always greater than the sum of medians of the same triangle.

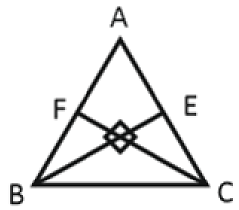
AB+BC+CA > AD+BE+CF

Property 5: If m_1, m_2 and m_3 are the length of medians of a Triangle.

Then, The area of Triangle = $\frac{4}{3}\sqrt{S(S-m_1)(S-m_2)(S-m_3)}$

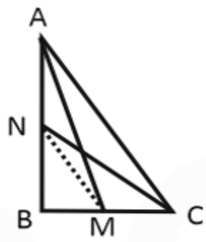
Where, S = $\frac{(m_1+m_2+m_3)}{2}$

Property 6: If medians of Triangle intersect at 90° then,



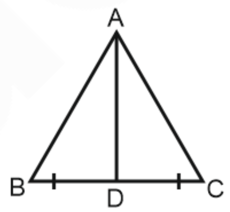
$$AB^2+AC^2 = 5\ BC^2$$

Property 7: In a Right- Angled Triangle ABC, If AM and CM are medians of the triangle. Then,



$$AM^2+CN^2 = \frac{5}{4}AC^2$$

Apollonius' Theorem: According to the Apollonius' theorem, in a triangle, the sum of squares of two sides is equal to the twice the sum of squares of half of the third side and median to that side.

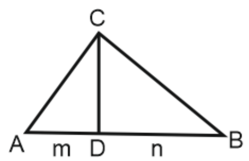


$$AB^2 + AC^2 = 2(BD^2 + AD^2)$$

Alternate Apollonius' Theorem:

$$3(AB^2 + BC^2 + AC^2) = 4(AD^2 + BE^2 + CF^2)$$

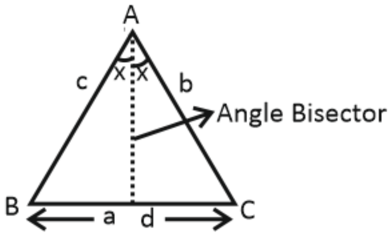
Stewarts' Theorem:



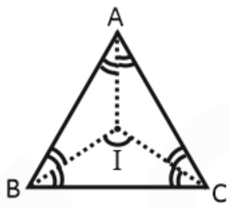
$$m \times BC^2 + n \times AC^2 = (m + n)AD^2 + mn(m+n)$$

Angle Bisector and Incentre

Angle Bisector: It is a line segment that originates from a vertex and also bisect the same angle into equal parts.



Incentre (I): It is the point of intersection of all the three angle bisectors of the triangle.

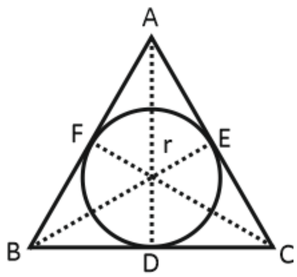


Angle made at Incentre ⇒

$$\angle BIC = 90° + \frac{\angle A}{2}$$
$$\angle AIC = 90° + \frac{\angle B}{2}$$
$$\angle AIB = 90° + \frac{\angle C}{2}$$

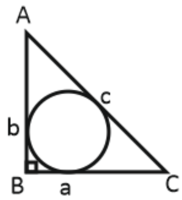
In-radius(r): The distance between the Incentre and the three sides of the triangle is equal to the perpendicular drawn from the Incentre to the three sides and this length is said as the Inradius (r).

In-Circle: The circle drawn with Inradius as the radius of the circle, is called the Incircle of the triangle. And it touches all the three sides of the triangle from the inside.



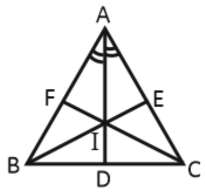
(I) For any triangle, Inradius ⇒ $r = \frac{A}{s}$; where "A" is the Area of the Triangle and "S" is the Semi-Perimeter of the triangle.

(II) In a Right-angled Triangle



$r = (a+b-c)/2$; Where a, b is the perpendicular and base and c is the hypotenuse of the Right-angled triangle.

Property 1: If in a triangle ABC with sides AB(=c), BC(=a) and AC(=b) also AD, BE and CF are the angle bisectors and I is Incentre. Then,

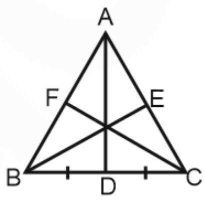


$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}; \frac{CE}{EA} = \frac{BC}{BA} = \frac{a}{c}; \frac{BF}{FA} = \frac{CB}{CA} = \frac{a}{b}$$

Property 2: If in a triangle ABC with sides AB(=c), BC(=a) and AC(=b) also AD, BE and CF are the angle bisectors and I is Incentre. Then,

$$\frac{AI}{ID} = \frac{b+c}{a}; \frac{CI}{IF} = \frac{a+b}{c}; \frac{BI}{IE} = \frac{a+c}{b}$$

Interior Bisector Theorem: In a triangle, an angle bisector of an angle divides the opposite side in the ratio of two other sides of the triangle.

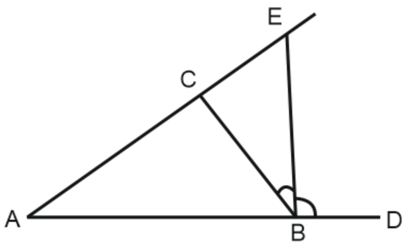


$$\frac{\mathbf{AB}}{\mathbf{AC}} = \frac{\mathbf{BD}}{\mathbf{CD}}$$

It can also be written as:

$$\mathbf{AB \times CD = AC \times BD = AD^2}$$

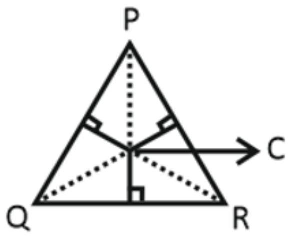
Exterior Bisector Theorem: In a triangle, an external angle bisector divides the opposite side of the external angle in the ratio of two other sides of the triangle.



$$\frac{\mathbf{AE}}{\mathbf{CE}} = \frac{\mathbf{AB}}{\mathbf{BC}}$$

Side Perpendicular Bisector and Circumcentre

Perpendicular Bisector: It is a line segment that bisects the side of a triangle perpendicularly.

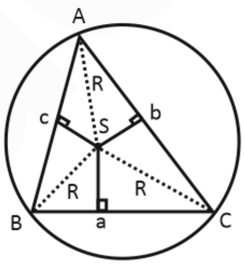


Circumcentre: It is the point of intersection of all three perpendicular bisectors of the triangle.

(i) For Acute-angled triangle, the Circumcentre lies inside the triangle.
(ii) For Obtuse-angled triangle, the Circumcentre lies outside the triangle.
(iii) For Right-angled triangle, the Circumcentre lies on the midpoint of the hypotenuse of the given triangle.

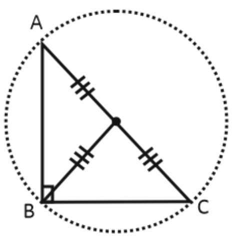
Circum-Radius: The distance between the circumcentre and the three vertices of the triangle is always equal in length and this length is said as the Circumradius (R) of the triangle.

Circum-Circle: The circle drawn with circumradius as the radius of the circle, is called the circumcircle of the triangle. And it passes through all the three vertices of the triangle.



For any triangle: $R = abc/4A$;
where a, b, c are the lengths of the sides and "A" is the Area of the Triangle.

In case of **Right-Angle Triangle**, Circumradius will be half the length of the hypotenuse of the right-angled triangle.



$$R = Hypotenous/2$$

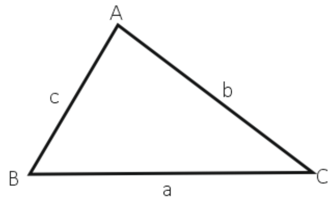
Angles made at Circumcentre:

∠BSC = 2∠A

∠ASC = 2∠B

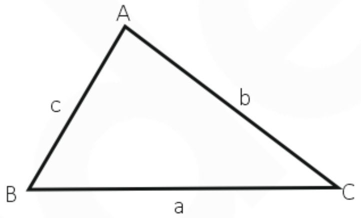
∠ASB = 2∠C

Sine Rule: In a Δ ABC, the lengths of the sides AB, BC and CA are a, b, and c respectively and the angles on the vertices A, B, and C are ∠A, ∠B, and ∠C. Then according to the Sine Rule,



$$\frac{a}{SinA} = \frac{b}{SinB} = \frac{c}{SinC} = 2R$$

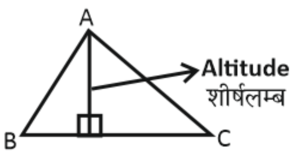
Cosine Rule: In a Δ ABC, the lengths of the sides AB, BC and CA are a, b, and c respectively and the angles on the vertices A, B, and C are ∠A, ∠B, and ∠C. Then according to the Cosine Rule,



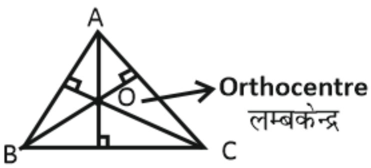
$$Cos\ A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$Cos\ B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$Cos\ C = \frac{b^2 + a^2 - c^2}{2ab}$$

Altitude and Orthocentre

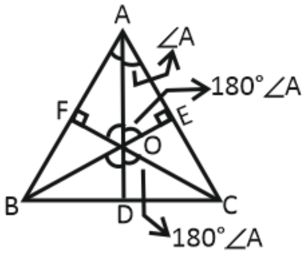
Altitude or Height: It is the line segment that is drawn from a vertex perpendicularly on the opposite side of the triangle. Thus, there are three altitudes in the triangle.



Orthocentre: The intersection point of all the three altitudes of a Triangle is called Orthocentre of the Triangle.



Angles made at Circumcentre:



∠BOC = 180°-∠A

∠AOC = 180°-∠B

∠AOB = 180°-∠C

Note:
1. For right angle triangle, orthocentre lies at the vertex containing right angle.
2. In obtuse angle triangle it lies opposite to largest side and outside the triangle.
3. In acute angle triangle it lies inside the Triangle.

Distance Between Inradius and Circumradius

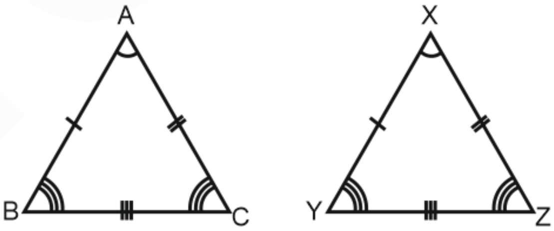
If "O" and "I" are Circumcentre and Incentre of the circle respectively and "R" and "r" are Circumradius and Inradius of the circle respectively. Then,

Distance between O and I:

$$d^2 = R(R - 2r)$$
$$\Rightarrow \mathbf{d} = \sqrt{\mathbf{R(R - 2r)}}$$

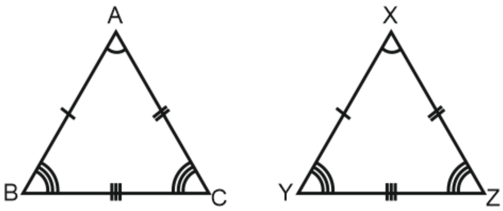
Concept of Congruency of Triangles

Any two triangles ABC and XYZ are said to be **congruent** when every corresponding side has the same length, and every corresponding angle has the same measure. It is denoted by ΔABC ≅ ΔXYZ.

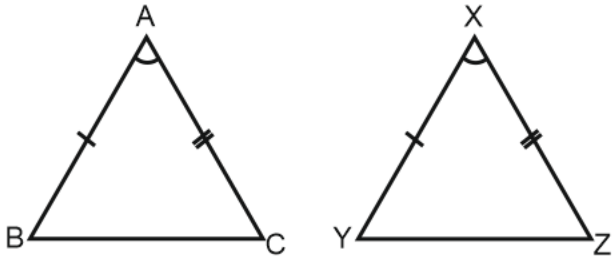


Congruency Rules:

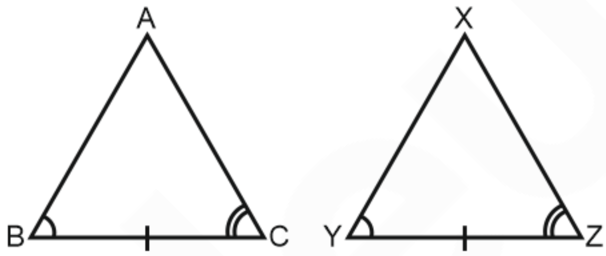
1. **S-S-S Rule:** If in two triangles, all three sides are equal to their corresponding sides of the other triangle. Then the triangles are congruent to each other by S-S-S Rule.



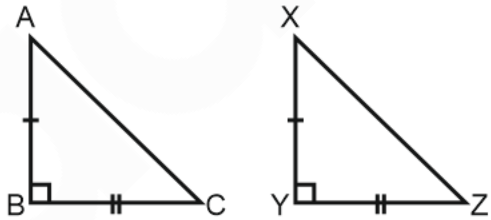
2. **S-A-S Rule:** If two sides and the angle included in between of one triangle are equal to two sides and the angle included in between of another triangle, then the triangles are congruent by S-A-S Rule.



3. **A-S-A Rule:** If two angles and the side included in between of one triangle are equal to two angles and the side included in between of another triangle, then the triangles are congruent by A-S-A Rule.



4. **R-H-S Rule:** This rule is applicable to only Right-angled triangle. If the hypotenuse and a side is equal to the hypotenuse and a side of the other right-angled triangle, then the two right triangles will be congruent by R-H-S rule.



Concept of Similarity of Triangles

Two triangles are said to be similar if -

(i) Their corresponding angles are equal

(ii) Their corresponding sides are in proportion.

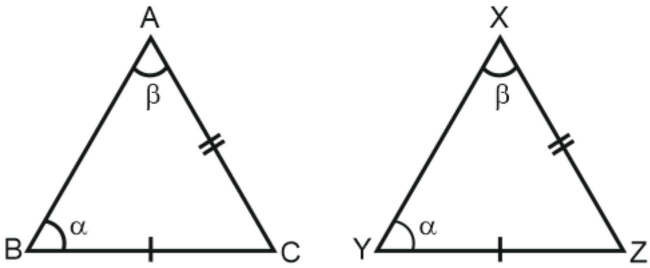
In two triangles, ΔABC and ΔXYZ are similar triangles when

∠A = ∠X, ∠B = ∠Y, ∠C = ∠Z and $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$

It is denoted by ΔABC ~ ΔXYZ.

Similarity Rules:

1. **AAA Rule or AA Rule:** If in two triangles, all three (two) corresponding angles are equal then the two triangles are termed as similar triangles by AAA (or AA) Rule.



In two triangles, ΔABC and ΔXYZ are similar triangles when
∠A = ∠X, ∠B = ∠Y.

2. **SAS Rule:** If in two triangles, two corresponding sides are in proportion and the corresponding angles contained between the two sides are equal then the two triangles are termed as similar triangles by SAS Rule.

3. **SSS Rule:** If in two triangles, all three sides and the corresponding sides are in proportion then the two triangles are termed as similar triangles by SSS Rule.

Properties of Similar Triangles –
If Triangle ABC and XYZ are similar, then

1. **The Ratio of their perimeters:**

$$\frac{\textbf{Perimeter of }\Delta\textbf{ABC}}{\textbf{Perimeter of }\Delta\textbf{XYZ}} = \frac{\textbf{AB}}{\textbf{XY}} = \frac{\textbf{BC}}{YZ} = \frac{\textbf{AC}}{XZ}$$

Also, If m_1, m_2 are the length of medians of ΔABC and ΔXYZ respectively and similarly h_1 and h_2 are Altitudes, I_1 and I_2 are angle bisectors, r_1 and r_2 are inradii and R_1 and R_2 are Cricumradii of respective triangles then –

$$\frac{\textbf{Perimeter of }\Delta\textbf{ABC}}{\textbf{Perimeter of }\Delta\textbf{DEF}} = \frac{\textbf{AB}}{\textbf{XY}} = \frac{\textbf{BC}}{YZ} = \frac{\textbf{AC}}{XZ} = \frac{m_1}{m_2} = \frac{h_1}{h_2} = \frac{I_1}{I_2} = \frac{r_1}{r_2} = \frac{R_1}{R_2}$$

2. **The Ration of their Areas:**

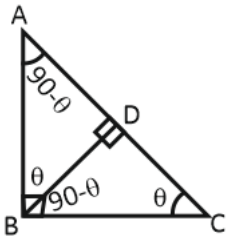
$$\frac{\textbf{Area of }\Delta\textbf{ABC}}{\textbf{Area of }\Delta\textbf{DEF}} = \frac{AB^2}{XY^2} = \frac{BC^2}{YZ^2} = \frac{AC^2}{XZ^2}$$

Also, If m_1, m_2 are the length of medians of ΔABC and ΔXYZ respectively and similarly h_1 and h_2 are Altitudes, I_1 and I_2 are angle bisectors, r_1 and r_2 are inradii and R_1 and R_2 are Cricumradii of respective triangles then –

$$\frac{\textbf{Area of }\Delta\textbf{ABC}}{\textbf{Area of }\Delta\textbf{DEF}} = \frac{AB^2}{XY^2} = \frac{BC^2}{YZ^2} = \frac{AC^2}{XZ^2} = \frac{{m_1}^2}{{m_2}^2} = \frac{{h_1}^2}{{h_2}^2} = \frac{{I_1}^2}{{I_2}^2} = \frac{{r_1}^2}{{r_2}^2} = \frac{{R_1}^2}{{R_2}^2}$$

Similarity in Right-Angle Triangle:

In a Right Angle Triangle ABC, ∠B = 90° and a perpendicular is drawn on Hypotenuse AC from vertex BC.



Then,

(i) BD $= \frac{AB \times BC}{AC}$

(ii) CD $= \frac{BC^2}{AC}$

(iii) $BD^2 = AD \times CD$

(iv) AD $= \frac{AB^2}{AC}$