Algebra

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Factors:

1. $(a+b)^2 = a^2 + 2ab + b^2$

2. $(a-b)^2 = a^2 - 2ab + b^2$

Note: $\sqrt{a^2-2ab+b^2}$ = (a - b) when a > b and (b - a) when b > a.

3. $(a+b)^2 = (a-b)^2 + 4ab$

4. $(a-b)^2 = (a+b)^2 - 4ab$

5. $(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$

6. $(a+b)^2 - (a-b)^2 = 4ab$

7. $\frac{(a+b)^2+(a-b)^2}{a^2+b^2} = 2$

8. $\frac{(a+b)^2-(a-b)^2}{ab} = 4$

9. $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

10. $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

11. $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

12. $a^3 - b^3 = (a-b)^3 + 3ab(a-b)$

13. $(a+b)^3 + (a-b)^3 = 2a(a^2+3b^2)$

14. $(a+b)^3 - (a-b)^3 = 2b(b^2+3a^2)$

15. $(a^2-b^2) = (a+b)(a-b)$

Similarly, $(a^4-b^4) = (a^2+b^2)(a^2-b^2) = (a^2+b^2)(a+b)(a-b)$
And $(a^8-b^8) = (a^4+b^4)(a^4-b^4) = (a^4+b^4)(a^2+b^2)(a+b)(a-b)$

16. $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

17. $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

If (a + b + c) = 0 then $a^3+b^3+c^3 = 3abc$
And a = b = c then also $a^3+b^3+c^3 = 3abc$

18. $a^3+b^3+c^3-3abc = \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

19. $(a^2+b^2+c^2-ab-bc-ca) = \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$

20. If $\left(\mathbf{x}+\frac{\mathbf{1}}{\mathbf{x}}\right) = \mathbf{m}$, then $\left(\mathbf{x}^2+\frac{\mathbf{1}}{\mathbf{x}^2}\right) = (\mathbf{m}^2-\mathbf{2})$

Proof: Given that, $\left(x+\frac{1}{x}\right) = m$
Squaring on the both the sides:
$\Rightarrow \left(x+\frac{1}{x}\right)^2 = m^2$
$\Rightarrow \left(x^2+\frac{1}{x^2}+2\times x\times\frac{1}{x}\right) = m$

$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) = m^2 - 2$

21. $\boldsymbol{If\ \left(x - \frac{1}{x}\right) = m}$**, then** $\boldsymbol{\left(x^2 + \frac{1}{x^2}\right) = (m^2 + 2)}$

Proof: Given that, $\left(x - \frac{1}{x}\right) = m$

Squaring on the both the sides:

$\Rightarrow \left(x - \frac{1}{x}\right)^2 = m^2$

$\Rightarrow \left(x^2 + \frac{1}{x^2} - 2 \times x \times \frac{1}{x}\right) = m$

$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) = m^2 + 2$

22. Similarly,

$\boldsymbol{If\ \left(x^2 + \frac{1}{x^2}\right) = n}$ **then** $\boldsymbol{\left(x + \frac{1}{x}\right) = \sqrt{n+2}}$

Proof: Given that, $\left(x^2 + \frac{1}{x^2}\right) = n$

Adding 2 on both the sides:

$\Rightarrow \left(x^2 + \frac{1}{x^2} + 2\right) = n + 2$

Or it can be written as:

$\Rightarrow \left(x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x}\right) = n + 2$

$\Rightarrow \left(x + \frac{1}{x}\right)^2 = n + 2$

Taking square root on both the sides:

$\Rightarrow \left(x + \frac{1}{x}\right) = \sqrt{n+2}$

23. If $\boldsymbol{\left(x^2 + \frac{1}{x^2}\right) = n}$ **then** $\boldsymbol{\left(x - \frac{1}{x}\right) = \sqrt{n-2}}$

Proof: Given that, $\left(x^2 + \frac{1}{x^2}\right) = n$

Subtracting 2 on both the sides:

$\Rightarrow \left(x^2 + \frac{1}{x^2} - 2\right) = n - 2$

Or it can be written as:

$\Rightarrow \left(x^2 + \frac{1}{x^2} - 2 \times x \times \frac{1}{x}\right) = n - 2$

$\Rightarrow \left(x - \frac{1}{x}\right)^2 = n - 2$

Taking square root on both the sides:

$\Rightarrow \left(x - \frac{1}{x}\right) = \sqrt{n-2}$

24. If $\boldsymbol{\left(x + \frac{1}{x}\right) = p}$ **and** $\boldsymbol{\left(x - \frac{1}{x}\right) = q}$ **then** $\boldsymbol{p = \sqrt{(q^2 + 4)}}$

Proof: We know that,

$\left(x + \frac{1}{x}\right)^2 - \left(x - \frac{1}{x}\right)^2 = \left(x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x}\right) - \left(x^2 + \frac{1}{x^2} - 2 \times x \times \frac{1}{x}\right)$

$\Rightarrow \left(x+\frac{1}{x}\right)^2 - \left(x-\frac{1}{x}\right)^2 = 4$

Putting values:

$\Rightarrow p^2 - q^2 = 4$

So, $p = \sqrt{(q^2+4)}$

25. If $x+\frac{1}{x} = m$ then $x^3+\frac{1}{x^3} = m^3 - 3m$

Proof: Given that, $x+\frac{1}{x} = m$

Cubing on both the sides:

$\left(x+\frac{1}{x}\right)^3 = m^3$

$\Rightarrow x^3+\frac{1}{x^3} + 3 \times x \times \frac{1}{x}\left(x+\frac{1}{x}\right) = m^3$

$\Rightarrow x^3+\frac{1}{x^3} + 3m = m^3$

$\Rightarrow x^3+\frac{1}{x^3} = m^3 - 3m$

26. If $x-\frac{1}{x} = m$ then $x^3-\frac{1}{x^3} = m^3 + 3m$

Proof: Given that, $x-\frac{1}{x} = m$

Cubing on both the sides:

$\left(x-\frac{1}{x}\right)^3 = m^3$

$\Rightarrow x^3-\frac{1}{x^3} - 3 \times x \times \frac{1}{x}\left(x-\frac{1}{x}\right) = m^3$

$\Rightarrow x^3-\frac{1}{x^3} - 3m = m^3$

$\Rightarrow x^3-\frac{1}{x^3} = m^3 + 3m$

27. If $x+\frac{1}{x} = 2$ then $x = 1$.

Proof: $x+\frac{1}{x} = 2$

$\Rightarrow x^2+1 = 2x$

$\Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x-1)^2 = 0$

Thus, $x = 1$

28. If $x+\frac{1}{x} = -2$ then $x = -1$.

Proof: $x+\frac{1}{x} = -2$

$\Rightarrow x^2+1 = -2x$

$\Rightarrow x^2 + 2x + 1 = 0$

$\Rightarrow (x+1)^2 = 0$

Thus, $x = -1$

Powers and Power roots:

1. $y = x^n$ (Here "x" is the base and "n" is the power)

It can also be written as: $x = (y)^{\frac{1}{n}} or \sqrt[n]{y}$

2. $y^m = x^n$ or $y = (x)^{\frac{n}{m}} = \sqrt[\left(\frac{n}{m}\right)]{x}$

3. $x^a . x^b . x^c . x^d \ldots \ldots \ldots .. = x^{(a+b+c+d+\cdots)}$

It means that when expressions where base is same but powers are different are multiplied together, their respective powers are joined i.e. summed up.

4. $a^n . b^n . c^n . d^n \ldots \ldots \ldots \ldots = (a.b.c.d \ldots \ldots \ldots)^n$

It means that when expressions where bases are different but powers are same then all the bases are multiplied to each other and the same power is used for whole expression.

5. $\frac{1}{x^n} = x^{-n}$

(As 1 can be written as x^0)
So, $\frac{1}{x^n} = \frac{x^0}{x^n} = x^{0-n} = x^{-n}$

6. $x^n \div x^m = \frac{x^n}{x^m} = x^n . x^{-m} = x^{(n-m)}$

7. $\sqrt[n]{x} \times \sqrt[m]{x} = (x)^{\frac{1}{n}} . (x)^{\frac{1}{m}} = (x)^{\frac{1}{n}+\frac{1}{m}}$

8. $\sqrt[n]{x} \div \sqrt[m]{x} = (x)^{\frac{1}{n}} \div (x)^{\frac{1}{m}} = (x)^{\frac{1}{n}} . (x)^{-\frac{1}{m}} = (x)^{\frac{1}{n}-\frac{1}{m}}$

9. $\sqrt[n]{x} \div \sqrt[n]{y} = \sqrt[n]{\left(\frac{x}{y}\right)}$

10. $\sqrt[n]{x^m} = (x)^{\frac{m}{n}} = \sqrt[\frac{n}{m}]{x}$

Note:

☐ $x^1 = x$ (Any Base has power of 1 equal to same base expression.)

☐ $x^0 = 1$ (Any Base has power of 0 is always equal to 1.)

☐ $(x^y)^z = x^{yz}$ whereas $x^{y^z} \neq x^{yz}$

Example: $(2^3)^4 = 2^{3\times4} \Rightarrow 8^4 = 2^{12} = 4096$

But $2^{3^4} = 2^{81}$ and $2^{3\times4} = 2^{12} = 4096$

Clearly, $2^{81} \neq 4096$ means that $x^{y^z} \neq x^{yz}$

Divisibility:

1. (a^n – b^n) will always be divisible by (a - b) & (a + b) when n = even no.

[Ex.: (a^2 – b^2) = (a – b)(a + b)]

2. (a^n – b^n) is divisible by (a – b) when n = odd no.

[Ex.: (a^3 – b^3) = (a – b)(a^2 + ab + b^2)]

3. (a^n + b^n) is divisible by (a + b) when n = odd no.

[Ex.: $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$]

Rational and Irrational Numbers:

A rational number is a number that can be expressed as a fraction (ratio) in the form $\frac{p}{q}$ where *p* and *q* are integers and *q* is not zero. Ex.: 7, $\frac{1}{2}$, $5\frac{1}{4}$, 12.25, etc.

When a rational number fraction is divided to form a decimal value, it becomes a **terminating** or **repeating decimal**.

Ex.: $\frac{1}{2}$ *can be written as* 0.5 which is a terminating decimal. And

An irrational number is a number that cannot be expressed as a fraction (ratio) in the form $\frac{p}{q}$ where *p* and *q* are integers and *q* is not zero. An irrational number can be written as a decimal, but not as a fraction. Ex.: √5, π, e, $\frac{13}{7}$ etc.

$\frac{8}{3}$ can be written as 2.66666…. which is a non-terminating, repeating decimal. Here 2.66666…. can be written as $2.\bar{6}$.

Polynomials:

Assume that a_1, a_2, a_3, a_4 ………. are real numbers and x is a real variable.

Then, $f(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + a_4x^{n-3}$ …………… $+ a_{n-1}x^2 + a_nx$ is called a polynomial.

Degree of Polynomial:

The maximum power of real variable 'x' is called the degree of the polynomial.

Note:

(i) Degree of polynomial is defined for both real and complex polynomials.

(ii) Degree of polynomial cannot be a fraction.

Quadratic Equation:

An equation in which the highest power of the unknown quantity is two is called quadratic equation.

Types of quadratic equation

Quadratic equations are of two types:

(i) **Purely Quadratic:**

$ax^2 + c = 0$; where a,c ∈ C and b = 0, a≠0

(ii) **Adfected quadratic**

$ax^2 + bx + c = 0$; where a,b,c ∈ C and a≠0, b≠0

Roots of a quadratic equation:

The values of variable *x* which satisfy the quadratic equation is called roots of quadratic equation.

Solution of quadratic equation

(1) **Factorization method**

Let $ax^2 + bx + c = a(x-\alpha)(x-\beta) = 0$.

Then $x = \alpha$ and $x = \beta$ will satisfy the given equation.

Hence, factorize the equation and equating each factor to zero gives roots of the equation.

Example **:** $x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0$; $x = 2, 3$

(2) **Sri Dharacharya method:**

By this method the solutions of quadratic equation $ax^2 + bx + c = 0$ are given as

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Hence the quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) has two roots,
given by $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Note: Every quadratic equation has two and only two roots.

Nature of roots

In a quadratic equation $ax^2 + bx + c = 0$, let us suppose that a, b, c are real and $a \neq 0$. The following is true about the nature of its roots.

(i) The equation has real and distinct roots if and only if D = $b^2 - 4ac > 0$.

(ii) The equation has real and coincident (equal) roots if and only if D = $b^2 - 4ac = 0$ and the equation will be a perfect square also.

(iii) The equation will have no real roots if and only if D = $b^2 - 4ac < 0$.

Relations between roots and coefficients

(1) **Relation between roots and coefficients of quadratic equation :** If α and β are the roots of quadratic equation $ax^2 + bx + c = 0$, ($a \neq 0$) then

Sum of roots $= S = \alpha + \beta = \frac{-b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of roots $= P = \alpha . \beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$

(2) **Formation of an equation with given roots :** A quadratic equation whose roots are α and β is given by $(x - \alpha)(x - \beta) = 0$.

$\therefore \; x^2 - (\alpha + \beta)x + \alpha\beta = 0$

i.e. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$\therefore \; x^2 - Sx + P = 0$

Condition for common roots

(1) **Only one root is common :** Let α be the common root of quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$.

$\therefore \; a_1\alpha^2 + b_1\alpha + c_1 = 0$, $a_2\alpha^2 + b_2\alpha + c_2 = 0$

By Crammer's rule : $\frac{\alpha^2}{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}} = \frac{\alpha}{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

or $\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$

$\therefore \; \alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2}$, $\alpha \neq 0$

$\therefore$ The condition for only one root common is

$(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$

(2) **Both roots are common:** Then required condition is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Remainder and Factor theorem:

Remainder Theorem: This theorem provides us the ways of finding remainders without the actual division of the Quadratic Equation or Polynomial.

(i) If a polynomial p(x) is divided by (x + a), then the remainder is the value of p(x) at x = -a, i.e. p(-a).

(ii) If a polynomial p(x) is divided by (ax – b), then the remainder is the value of p(x) at x =b/a, i.e. p(b/a)

(iii) If a polynomial p(x) is divided by (ax + b), then the remainder is the value of p(x) at x =-b/a, i.e. p(-b/a)

(iv) If a polynomial p(x) is divided by (b – ax), then the remainder is the value of p(x) at x =b/a, i.e. p(b/a)

Factor Theorem: According to this theorem;

"If g(x) is divided by f(x), then we say that f(x) is divisible by g(x) or g(x) is a factor of f(x).

(i) (x + a) is a factor of a polynomial f(x) if f(-a) = 0.

(ii) (ax - b) is a factor of a polynomial f(x) if f(b/a) = 0.

(iii) (ax + b) is a factor of a polynomial f(x) if f(-b/a) = 0.

(iv) (x -a).(x - b) is a factor of a polynomial f(x) if f(a) = 0 and f(b) = 0.

Maxima and Minima of Quadratic Equation:
The equation f(x)= ax^2+bx+c will have maxima and minima at following points-

(i) If a>0, then the maximum value of the function will be infinity (∞) and the minimum value of the function will be –D/4a.

(ii) If a<0, then the maximum value of the function will be (-D/4a) and the minimum value of the function will be (-∞).

Two variable Linear Equations:
A system of two linear equations in two unknowns is a system of two equations of the form –
$a_1x + b_1y = c_1$
and $a_2x + b_2y = c_2$
where, x and y are variables and a_1, b_1, c_1, a_2, b_2, c_2 are arbitrary real numbers.

Now, there are three possible cases in case of above two equations.

(i) **Both the Equations intersect each other:** If both the equations intersect each other than the system of linear equations has a unique solution, i.e.

$$a1/a2 \neq b1/b2$$

(ii) **The lines are parallel:** If both the equations are parallel to each other than the system of equations will be inconsistent or in other words the equations will have no solutions.

$$a1/a2 = b1/b2 \neq c1/c2$$

(iii) **The lines are coincident:** If both the lines are coincident to each other than the System of equations has infinitely many solutions.

$$a1/a2 = b1/b2 = c1/c2$$

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