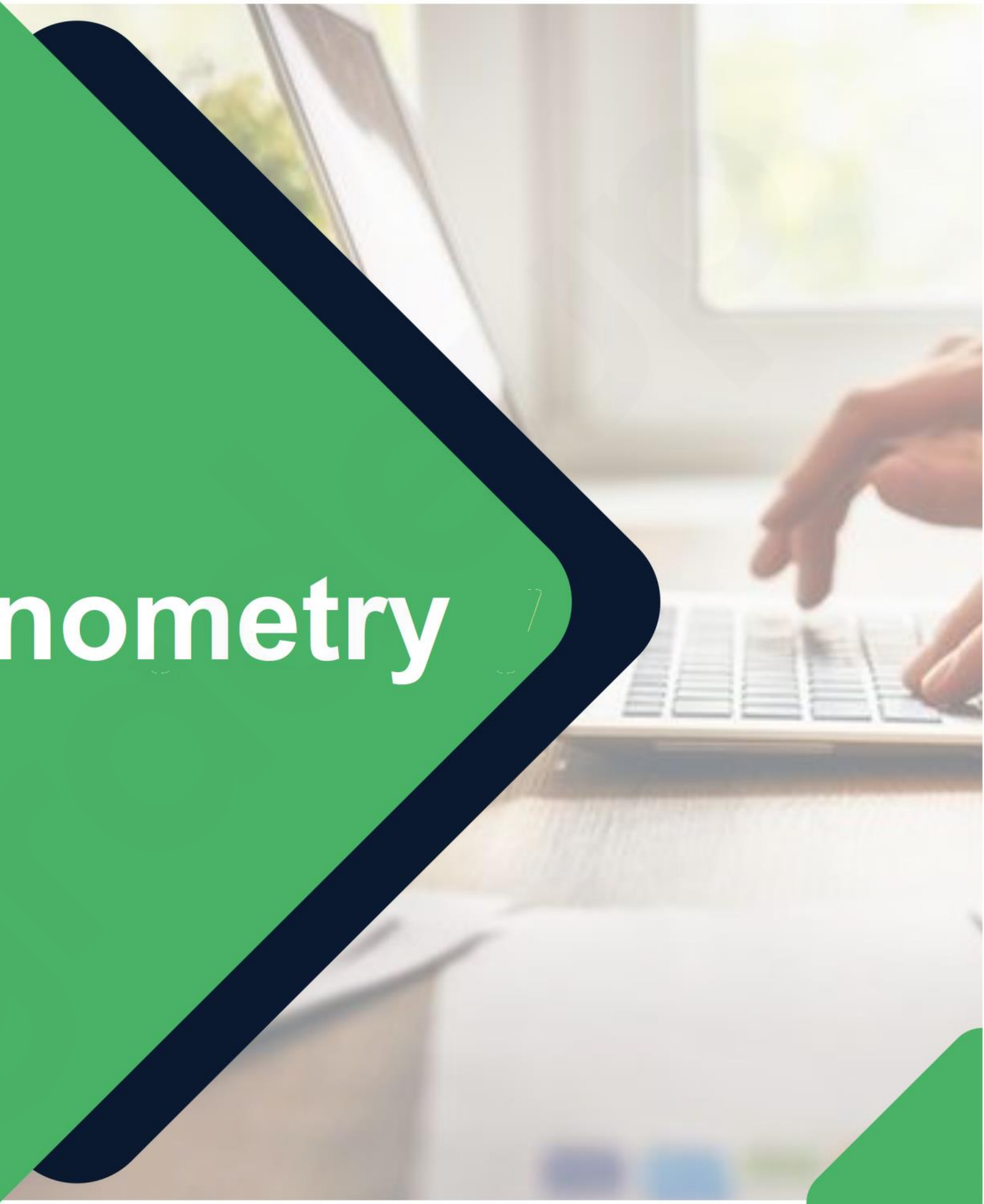


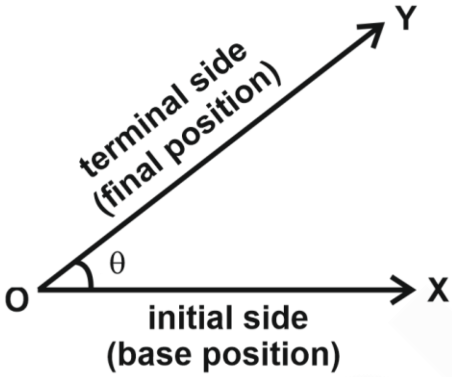
Trigonometry

Trigonometry

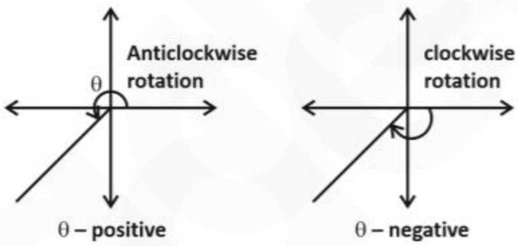
Definition: Trigonometry means Trigonon: Triangle and Metron: Measure. It is a branch of mathematics that deals with the angles and the sides of the triangle.

Angle: Angle is defined by the particular rotation of line from initial position to a final position. The end point O about which the line rotates is called the vertex of the angle.

Let, OX – Base position OY – Final position θ – Angle

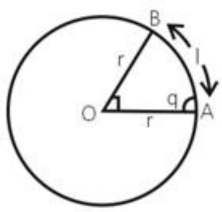


Sign Convention for measurement of angle: Generally, we consider anticlockwise rotation as positive and clockwise rotation as negative.



Circular system:

The measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.



Consider a circle of radius *r* having centre at *O*. Let *A* be a point on the circle. Now cut off an arc *A*B whose length is equal to the radius *r* of the circle. Then by the definition the measure of ∠AOB is 1 radian $(=1^c)$.

$$\textbf{1 radian} = \frac{180^\circ}{\pi} \Rightarrow \pi \textbf{ radians} = 180^\circ$$

Note:

- **1 Radian =** $\frac{180^\circ}{\pi}$ **degree ≈ 57°17′15″ (approx.)**

- **1 degree = $\frac{\pi}{180°}$ radian ≈ 0.0175 radian**

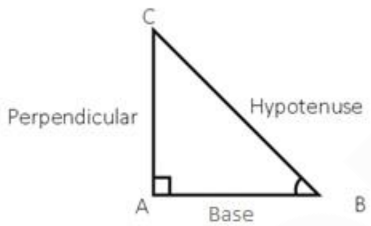
Relationship between an arc and an angle:

If "*l*" is the length of an arc of a circle of radius "r", then the angle $\boldsymbol{\theta}$ (in radians) subtended by this arc at the centre of the circle is given by

$$\boldsymbol{\theta(in\ radians)} = \frac{\boldsymbol{l}}{\boldsymbol{r}}$$

Trigonometric Ratios and Functions:

Let **∠ABC = θ** (Acute angle of right-angle triangle) Here, AB = Base (B) AC = Perpendicular (P) BC = Hypotenuse (H) There are six trigonometric ratios,



Pythagoras theorem for the given right-angled theorem, we have:

(Perpendicular)^2 + (Base)^2 = (Hypotenuse)^2
⇒ (P)^2 + (B)^2 = (H)^2

$$\textbf{Sin}\ \boldsymbol{\theta} = \frac{\textbf{Perpendicular}}{\textbf{Hypotenuse}} = \frac{\textbf{AC}}{\textbf{BC}}$$

$$\textbf{Cos}\ \boldsymbol{\theta} = \frac{\textbf{Base}}{\textbf{Hypotenuse}} = \frac{\textbf{AB}}{\textbf{BC}}$$

$$\textbf{Tan}\ \boldsymbol{\theta} = \frac{\textbf{Perpendicular}}{\textbf{Base}} = \frac{\textbf{AC}}{\textbf{AB}}$$

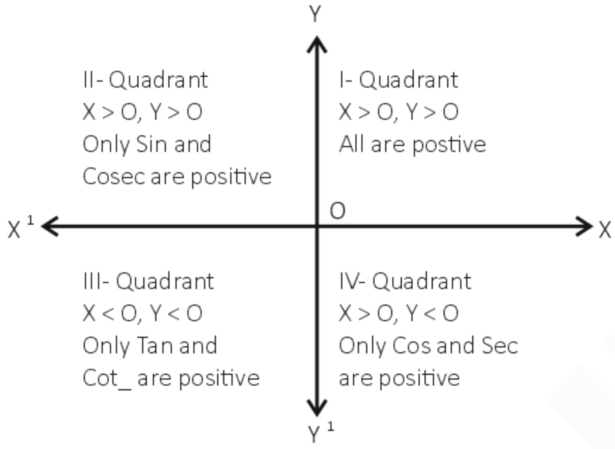
$$\textbf{Cot}\ \boldsymbol{\theta} = \frac{\textbf{Base}}{\textbf{Perpendicular}} = \frac{\textbf{AB}}{\textbf{AC}}$$

$$\textbf{Sec}\ \boldsymbol{\theta} = \frac{\textbf{Hypotenuse}}{\textbf{Base}} = \frac{\textbf{BC}}{\textbf{AB}}$$

$$\textbf{Cosec}\ \boldsymbol{\theta} = \frac{\textbf{Hypotenuse}}{\textbf{Perpendicular}} = \frac{\textbf{BC}}{\textbf{AC}}$$

Values/Signs of Trigonometric ratio (functions) in different Quadrants:

The signs depend on the quadrant in which the terminal side of the angle lies.



1. $\sin(90° - \theta) = \cos\theta$
2. $\cos(90° - \theta) = \sin\theta$
3. $\sin(90° + \theta) = \cos\theta$
4. $\cos(90° + \theta) = -\sin\theta$
5. $\sin(180° - \theta) = \sin\theta$
6. $\cos(180° - \theta) = -\cos\theta$
7. $\sin(180° + \theta) = -\sin\theta$
8. $\cos(180° + \theta) = -\cos\theta$
9. $\sin(270° - \theta) = -\cos\theta$
10. $\cos(270° - \theta) = -\sin\theta$
11. $\sin(270° + \theta) = -\cos\theta$
12. $\cos(270° + \theta) = \sin\theta$

Note:

1. $\sin(2n\pi + \theta) = sin\theta$ and $\cos(2n\pi + \theta) = cos\theta$ where n is an integer.
2. $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$ and $\tan(n\pi) = 0$ where n is an integer.
3. $\sin(2n + 1)\left(\frac{\pi}{2}\right) = (-1)^n$ and $\cos(2n + 1)\left(\frac{\pi}{2}\right) = 0$ where n is an integer.

θ	0°	30°	45°	60°	90°	180°	270°	360°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	0	1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0
cot θ	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	−1	∞	1

cosec θ	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	−1	∞

Basic trigonometric identities:

1. $\text{Sin }\theta.\text{Cosec }\theta = 1$
2. $\text{Cos }\theta.\text{Sec }\theta = 1$
3. $\text{Tan }\theta.\text{Cot }\theta = 1$
4. $\text{Tan }\theta = \frac{\text{Sin }\theta}{\text{Cos }\theta}$
5. $\text{Cot}\theta = \frac{\text{Cos}\theta}{\text{Sin}\theta}$
6. $\text{Sin}^2\theta + \text{Cos}^2\theta = 1$
7. $\text{Sec}^2\theta - \text{Tan}^2\theta = 1$

   Thus, $(\text{Sec}\theta - \text{Tan}\theta)(\text{Sec}\theta + \text{Tan}\theta) = 1$

   If $(\text{Sec}\theta + \text{Tan}\theta) = x$ then $(\text{Sec}\theta - \text{Tan}\theta) = \frac{1}{x}$
8. $\text{Cosec}^2\theta - \text{Cot}^2\theta = 1$

   Thus, $(\text{Cosec}\theta - \text{Cot}\theta)(\text{Cosec}\theta + \text{Cot}\theta) = 1$

   If $(\text{Cosec}\theta - \text{Cot}\theta) = x$ then $(\text{Cosec}\theta + \text{Cot}\theta) = \frac{1}{x}$

Important trigonometric formulas:

1. $(\text{Sin}\theta + \text{Cos}\theta)^2 + (\text{Sin}\theta - \text{Cos}\theta)^2 = 2$
2. $\text{Sin}^4\theta + \text{Cos}^4\theta = 1 - 2\text{Sin}^2\theta.\text{Cos}^2\theta$
3. $\text{Sin}^6\theta + \text{Cos}^6\theta = 1 - 3\text{Sin}^2\theta.\text{Cos}^2\theta$
4. $\text{Sec}^2\theta + \text{Cosec}^2\theta = \text{Sec}^2\theta.\text{Cosec}^2\theta$
5. $\text{Tan}\theta + \text{Cot}\theta = \text{Sec}\theta.\text{Cosec}\theta$

Trigonometric Formulas involving sum or difference of angles:

1. $\text{Sin}(A + B) = \text{Sin}(A)\text{Cos}(B) + \text{Cos}(A)\text{Sin}(B)$
2. $\text{Sin}(A - B) = \text{Sin}(A)\text{Cos}(B) - \text{Cos}(A)\text{Sin}(B)$
3. $\text{Cos}(A + B) = \text{Cos}(A)\text{Cos}(B) - \text{Sin}(A)\text{Sin}(B)$
4. $\text{Cos}(A - B) = \text{Cos}(A)\text{Cos}(B) + \text{Sin}(A)\text{Sin}(B)$
5. $\text{Tan}(A + B) = \frac{\text{Tan A} + \text{Tan B}}{1 - \text{Tan A} \times \text{Tan B}}$
6. $\text{Tan}(A - B) = \frac{\text{Tan A} - \text{Tan B}}{1 + \text{Tan A} \times \text{Tan B}}$
7. $\text{Tan}(45° + A) = \frac{1 + \text{Tan A}}{1 - \text{Tan A}} = \frac{(\text{CosA} + \text{SinA})}{(\text{CosA} - \text{SinA})}$

8. $\text{Tan}(45° - A) = \frac{1 - \text{Tan } A}{1 + \text{Tan } A} = \frac{(\text{Cos}A - \text{Sin}A)}{(\text{Cos}A + \text{Sin}A)}$
9. $\cot(A + B) = \frac{\cot A.\cot B - 1}{\cot B + \cot A}$
10. $\cot(A - B) = \frac{\cot A.\cot B + 1}{\cot B - \cot A}$
11. $\text{Cos}(A + B).\text{Cos}(A - B) = \text{Cos}^2 A - \text{Sin}^2 B$
12. $\text{Sin}(A + B).\text{Sin}(A - B) = \text{Sin}^2 A - \text{Sin}^2 B = \text{Cos}^2 B - \text{Cos}^2 A$
13. $\text{Sin}(2A) = 2\text{Sin}(A)\text{Cos}(A) = \left[\frac{2\text{Tan}A}{1 + \text{Tan}^2 A}\right]$
14. $\text{Cos}(2A) = \text{Cos}^4(A) - \text{Sin}^4(A) = \text{Cos}^2(A) - \text{Sin}^2(A) = \left[\frac{1 - \text{Tan}^2 A}{1 + \text{Tan}^2 A}\right]$
15. $\text{Cos}(2A) = 2\text{Cos}^2(A) - 1 = 1 - 2\text{Sin}^2(A)$
16. $\text{Tan}(2A) = \frac{[2\text{Tan}(A)]}{[1 - \text{Tan}^2(A)]}$
17. $\text{Sec}(2A) = \frac{\text{Sec}^2 A}{2 - \text{Sec}^2 A}$
18. $\text{Cosec } (2A) = \frac{\text{Sec A.Cosec A}}{2}$
19. $\text{Sin}A \cdot \text{Cos}B = \frac{\sin(A+B) + \sin(A-B)}{2}$
20. $\text{Cos}A \cdot \text{Cos}B = \frac{\cos(A+B) + \cos(A-B)}{2}$
21. $\text{Sin}A \cdot \text{Sin}B = \frac{\cos(A+B) - \cos(A-B)}{2}$
22. $\text{Cos}A.\text{Cos}B = \frac{\text{Cos}(A+B) + \text{Cos}(A-B)}{2}$
23. $\text{Sin}A + \text{Sin}B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
24. $\text{Sin}A - \text{Sin}B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
25. $\text{Cos}A + \text{Cos}B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
26. $\text{Cos}A - \text{Cos}B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

Important Results:

1. $\text{Sin}3A = 3\text{Sin}A - 4\text{Sin}^3 A = 4\text{Sin}(60° - A).\text{Sin}A.\text{Sin}(60° + A)$
2. $\text{Cos}3A = 4\text{Cos}^3 A - 3\text{Cos}A = 4\text{Cos}(60° - A).\text{Cos}A.\text{Cos}(60° + A)$
3. $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} = \tan(60° - A).\tan A.\tan(60° + A)$
4. $\text{Cot}3A = \text{Cot}(60° - A).\text{Cot}A.\text{Cot}(60° + A)$

Some important values to remember:

1. $\textbf{Sin}(75°) = \frac{\sqrt{3}+1}{2\sqrt{2}} = \textbf{Cos }(15°)$

2. $\mathbf{Sin(18°) = \frac{\sqrt{5}-1}{4} = Cos(72°) = Sin(\frac{\pi}{10})}$
3. $\mathbf{Cos(75°) = \frac{\sqrt{3}-1}{2\sqrt{2}} = Sin(15°)}$
4. $\mathbf{Cos(36°) = \frac{\sqrt{5}+1}{4} = Sin(54°) = Cos(\frac{\pi}{5})}$
5. $\mathbf{Tan(75°) = \frac{\sqrt{3}+1}{\sqrt{3}-1} = (2+\sqrt{3}) = Cot(15°)}$
6. $\mathbf{Tan(22.5°) = \sqrt{2}-1 = Cot(67.5°) = Cot\left(\frac{3\pi}{8}\right) = tan(\frac{\pi}{8})}$
7. $\mathbf{Tan(67.5°) = \sqrt{2}+1 = Cot(22.5°)}$

Conditional Trigonometric Identities:

1. $\mathbf{If\ A + B = 90°\ then}$

(i) $\mathrm{SinA = CosB\ or\ CosA = SinB,}$

(ii) $\mathrm{TanA = CotB\ or\ TanB = CotA,}$

(iii) $\mathrm{SecA = CosecB\ or\ SecB = CosecA}$

(iv) $\mathrm{Sin^2A + Sin^2B = 1\ and\ Cos^2A + Cos^2B = 1}$

(v) $\mathrm{SinA.SecB = 1\ and\ CosA.CosecB = 1}$

(vi) $\mathrm{TanA.TanB = 1\ and\ CotA.CotB = 1}$

(vii) $\mathrm{SinA = CosB,\ TanA = CotB\ and\ SecA = CosecB}$

2. $\mathbf{If\ A + B = 45°\ then}$

(i) $\mathrm{(CotA - 1)(CotB - 1) = 2}$

3. $\mathbf{If\ A + B + C = 180°\ then}$

(i) $\mathrm{TanA + TanB + TanC = TanA.TanB.TanC}$

(ii) $\frac{1}{\mathrm{TanA.TanB}} + \frac{1}{\mathrm{TanB.TanC}} + \frac{1}{\mathrm{TanA.TanC}} = 1$

(iii) $\mathrm{CotA.CotB + CotB.CotC + CotA.CotC = 1}$

(iv) $\mathrm{Tan(\frac{A}{2}).Tan(\frac{B}{2}) + Tan(\frac{B}{2})Tan(\frac{C}{2}) + Tan(\frac{C}{2})Tan(\frac{A}{2}) = 1}$

(v) $\mathrm{Cot(\frac{A}{2}) + Cot(\frac{B}{2}) + Cot(\frac{C}{2}) = Cot(\frac{A}{2})Cot(\frac{B}{2})Cot(\frac{C}{2})}$

4. If $\mathrm{aSin\theta + bCos\theta = p\ then\ bSin\theta - aCos\theta = \sqrt{(a^2 + b^2 - p^2)}}$