Remainder Concept

REMAINDER THEROEM
Any number can be written in the form given below:
Dividend=Divisor x Quotient + Remainder
So When 86 is divided by 10 it can be written in the form
86 = 10 x 8 + 6
Consider the following question:
17×23
Suppose you have to find the remainder of this expression when divided by 12.
We can write this as:
17×23 = (12+5) × (12+11)
You will realise that, when this expression is divided by 12, the remainder will only depend on the last term above:
Thus, $\frac{12\times12+12\times11+5\times12+5\times11}{12}$ gives the same remainder as $\frac{5\times11}{12} \rightarrow Remainder = +7$
This is the remainder when 17 × 23 is divided by 12.
Learning Point: In order to find the remainder of 17 × 23 when divide by 12, you need to look at the individual remainders of 17 and 23 when divided by 12 and then successively divide by 12 to find the remainder of the original expression
Mathematically, this can be written as:
The remainder of the expression [A × B × C + D × E]/M, will be the same as the remainder of the expression [AR × BR × CR × ER]/M.
When AR is the remainder when A is divided by M,
BR is the remainder when B is divided by M,
CR is the remainder when C is divided by M
DR is the remainder when D is divided by M and
ER is the remainder when E is divided by M,
We call this transformation as the remainder theorem transformation and denote it by the sign
R→
Thus, the remainder of
1421 × 1423 × 1425 when divided by 12 can be given as:
$\frac{1421\times1423\times1425}{12}$ R→ $\frac{5\times7\times9}{12}$ R→ $\frac{35\times9}{12}$ R→ $\frac{11\times9}{12}$ R→ +3
In the above question, we have a series of remainder theorem transformations (denoted by R→)
and equality transformations to transform a difficult looking expression into a simple expression.
USING NEGATIVE REMAINDERS
Consider the following question.
Find the remainder when: 14 × 15 is divided by 8.
The obvious approach in this case would be
$\frac{14\times15}{8}$ R→ $\frac{6\times7}{8}$ R→ $\frac{42}{8}$ R→ +2
However, there is another option by which you can solve the same question:
When 14 is divide by 8, the remainder is normally seen as + 6. however, there might be times when using the negative value of the remainder might give us more convenience. Which is why you should know the following process.
Concept Note: Remainders by definition are always non-negative. Hence, even when we divide a number like – 27 by 5 we say that the remainder is 3 (and not – 2). However, looking at the negative value of the remainder–it has its own advantage in Mathematics at it results in reducing calculations.
Thus, when a number like 13 is divided by 8, the remainder being 5, the negative remainder is –3.
Note: It is in this context that we mention numbers like 13, 21, 29, etc. as 8n + 5 or 8n – 3 numbers.
Thus, $\frac{14\times15}{8}$ will give us $\frac{-2\times-1}{8}$ R → 2
Consider the advantage this process will give you in the following question:

$\frac{51\times52}{53} \to \frac{-2\times-1}{53}$ R $\to$ 2

(The alternative will involve long calculation. Hence, the principle is that you should use is negative remainders wherever you can. They can make life much simpler.)

What if the answer comes out to be negative?

For instance, $\frac{62\times63\times64}{66} \to \frac{-4\times-3\times-2}{66}$ R $\to -\frac{24}{66}$

But we know that a remainder of –24, equals a remainder of 42 when divided by 66. Hence, the answer is 42.

Of course, nothing stops you from using positive and negative remainders at the same time in order to solve the same question:

$\frac{17\times19}{9} \to \frac{-1\times+1}{9}$ R $\to -\frac{1}{9}$ R $\to$ +8

Try to solve the following question on Remainder theorem:

Find the remainder in each of the following cases:

1. 243 × 245 × 247 × 249 × 251 divided by 12.

Solution:

$\frac{243\times245\times247\times249\times251}{12} \to \frac{+3\times+5\times+7\times+9\times+11}{12} \to \frac{15\times63\times11}{12} \to \frac{+3\times+3\times-1}{12} \to -\frac{9}{12} R \to +3$

2. $\frac{173\times261}{13} + \frac{248\times249\times250}{15}$

Solution:

$\frac{173\times261}{13} + \frac{248\times249\times250}{15} \to \frac{+4\times+1}{13} + \frac{+8\times+9\times+10}{15} \to +\frac{4}{13} + \frac{0}{15}$ R $\to$ +4

3. $\frac{37\times43\times51}{7} + \frac{137\times143\times151}{9}$

Solution:

$\frac{37\times43\times51}{7} + \frac{137\times143\times151}{9} \to \frac{+2\times+1\times+2}{7} + \frac{+2\times-1\times-2}{9} \to (+4) + (+4)\ R \to\ +8$

Dealing with large power: There are two tools which are effective in order to deal with large powers:

(A) If you can express the expression in the form $\frac{((ax+1))^n}{a}$ then the remainder will become 1 directly. In such a case, no matter how large the value of the power n is, the remainder is 1. In such a case the value of the power does not matter.

Example: $\frac{37^{12635}}{9}$ R $\to \frac{1^{12635}}{9}$ R $\to$ +1

(B) $\frac{(ax-1)^n}{a}$ In such a case using –1 as the remainder it will be evident that the remainder will be +1 if n is even and it will be –1 (Hence a – 1) when n is odd.

Example: $\frac{31^{127}}{8}$ R $\to \frac{(-1)^{127}}{8}$ R $\to$ +7

ANOTHER IMPORTANT POINT:

Suppose you were asked to find the remainder of 14 divided by 4. It is clearly visible that the answer should be 2.

But consider the following process:

$\frac{14}{4} \to \frac{7}{2}$ R $\to$ +1 (The answer has changed)

What has happened?

We have transformed 14/4 into 7/2 by dividing the numerator and the denominator by 2. The result is that the original remainder 2 is also divided by 2 giving us 1 as the remainder. In order to take care of this problem we need to reverse the effect of the division of the remainder by 2. This is done by multiplying the final remainder by 2 to get the correct answer.

Note: In any question on remainder theorem, you should try to cancel out parts of the numerator and denominator as much as you can, since it directly reduces the calculation required.

THE APPLICATION OF REMAINDER THEOREM:

Finding the last two digits of an expression:

Suppose you had to find the last 2 digits of the expression:

22×31×44×27×37×43

The remainder the above expression will give when it is divided by 100 is the answer to the above question.

Hence, to answer the question above find the remainder of the expression when it is divided by 100.

Solution: $\frac{22\times31\times44\times27\times37\times43}{100}$

$= \frac{22\times31\times11\times27\times37\times43}{25}$ (on dividing by 4)

R $\rightarrow \frac{22\times6\times11\times2\times12\times18}{25}$ R $\rightarrow \frac{132\times22\times216}{25}$ R $\rightarrow \frac{7\times22\times16}{25}$ R $\rightarrow \frac{154\times16}{25}$ R $\rightarrow \frac{4\times16}{25}$ R $\rightarrow$ +14

Thus, the remainder being 14, (after division by 4). The actual remainder should be 56. (Don't forget to multiply by 4)

Hence, the last 2 digits of the answer will be 56.

Using negative remainders here would have helped further.

Note: Similarly finding the last three digits of an expression means finding the remainder when the expression is divided by 1000.

THE PRIME NUMBER DIVISOR RULE:

This rule states that: If 'P' is a prime number then:

The remainder of the expression $\frac{A^{P-1}}{P}$ is 1. (Provided A is not a multiple of P)

Example: The remainder of $\frac{24^{82}}{83}$ = 1

SPLITTING THE DENOMINATOR INTO CO-PRIME NUMBERS:

This is also sometime referred to as the 'Chinese Remainder Theorem'. It is useful when you have to find the remainder when there is a large denominator, and no other short cuts are working. It is best explained through an example.

Suppose you were trying to find the remainder of $\frac{107^{1444}}{136}$. You can split the denominator into two co-prime numbers as 17 and 8.

First find the remainder of $\frac{107^{1444}}{17} \rightarrow \frac{5^{1444}}{17} \rightarrow \frac{5^{16n}\times5^4}{17} \rightarrow \frac{1\times5^4}{17}$ R$\rightarrow$ +13. This means that 107^{1444} is a 17n + 13 number.

Next, find the remainder of $\frac{107^{1444}}{8} \rightarrow \frac{3^{1444}}{8} \rightarrow \frac{3^{2n}}{8} \rightarrow$ R$\rightarrow$ +1. This means that 107^{1444} is an 8n + 1 number.

The next step is to find a number below 136 that is both a 17n + 13 as well as an 8n + 1 number. That number would be the answer.

The list of 17n + 13 numbers below 136 is: 13, 30, 47, 64, 81, 98, 115 and 132. 81 can be seen to be an 8n + 1 number too.

Thus, the correct answer is 81.

Euler's Totient Function:

The totient φ(*n*) of a positive integer *n* greater than 1 is defined to be the number of positive integers less than *n* that are coprime to *n*. φ(*1*) is defined to be 1.

The general formula to compute φ(n) is the following:

If the prime factorisation of n is given by n =$p_1^{e_1}$*...*$p_n^{e_n}$, then φ(n) = n *(1 - 1/p_1)* ... (1 - 1/p_n).

For example:

- 9 = 3^2, φ(*9*) = 9* (1-1/3) = 6
- 4 =2^2, φ(*4*) = 4* (1-1/2) = 2
- 15 = 3*5, φ(*15*) = 15* (1 - 1/3)*(1 - 1/5) = 15*(2/3)*(4/5) =8
- 48 = 2^4×3, φ(*48*) = 48*(1 – ½)(1 – 1/3) = 16

Note: All prime numbers (p) have Euler's Totient = (p – 1)

- 73 has Euler's Totient = 73(1 – 1/73) = 72 which is 1 less than 73.

When x and y are two co-prime numbers, φ(*x***y*) = φ(*x*)* φ(*y*)

- 77 has Euler's Totient = φ(77) = φ(11*7) = φ(11)*φ(7) = 10x6 = 60.

Application of Euler's Totient:

Example: Find the remainder of $\frac{2^{79}}{37}$.

Solution:

Here, Euler's Totient φ(37) = 36 which means 37 has total 36 numbers that are co-prime to 37.

Thus, $\frac{2^{36}}{37}$ *will leave remainder* 1.

Also, $2^{72}, 2^{108}, 2^{144} \ldots \ldots \ldots$ will leave remainder as 1 when divided by 37.
Thus, 2^{79} will be converted to 2^{7}.
So, $\frac{2^7}{37} = \frac{128}{37}\, R \rightarrow +17$