

Remarks on the Spatial Smoothing Step in Coarray MUSIC

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Abstract—Sparse arrays such as nested and coprime arrays use a technique called spatial smoothing in order to successfully perform MUSIC in the difference-coarray domain. In this paper it is shown that the spatial smoothing step is not necessary in the sense that the effect achieved by that step can be obtained more directly. In particular, with $\tilde{\mathbf{R}}_{ss}$ denoting the spatial smoothed matrix with finite snapshots, it is shown here that the noise eigenspace of this matrix can be directly obtained from another matrix $\tilde{\mathbf{R}}$ which is much easier to compute from data.

Index Terms—Coprime arrays, nested arrays, sparse arrays, spatial smoothing.

I. INTRODUCTION

SPARSE arrays open a new approach to sensor array processing because of the high degrees of freedom offered in the difference-coarray domain. Nested arrays [1] and coprime arrays [2] are examples of sparse arrays obtained from a union of two uniform linear arrays (ULAs) with different interelement spacings. The increased freedom has been used to identify $O(N^2)$ sources (DOAs) from only N sensors [1], [2]. Sparse arrays can be used in various applications, including DOA estimation [1]–[4], line spectrum estimation using MUSIC algorithms [5], super resolution [6], [7], two dimensional array design [8]–[10], beamforming and coprime spatial filter bank design [11]–[13].

In DOA estimation using the MUSIC algorithm [5] or any gridless algorithm [14], a technique called spatial smoothing [15] is sometimes used to construct a positive definite matrix on which MUSIC operates. For sparse arrays which use the MUSIC algorithm in the difference-coarray domain, it was proved in [1], [5] that the spatially smoothed matrix \mathbf{R}_{ss} in the coarray domain is a perfect square of a positive definite matrix $\tilde{\mathbf{R}}$ which contains noise-subspace information. Using this fact it was possible to separate the signal subspace and the noise subspace based on the eigenvalues of \mathbf{R}_{ss} . This leads to a successful implementation of MUSIC in the coarray domain. It should be mentioned herein that when DOA estimation based

on coarray domain is performed by formulating a dictionary based approach [16], spatial smoothing is not necessary. It has been used in the past only when the MUSIC algorithm or other gridless algorithms [14] is to be employed in the coarray domain.

In this paper, we will show that spatial smoothing is not needed even to implement the MUSIC algorithm in the coarray domain. The performance of [1], [5] can be achieved without it. This is done as follows: based on the snapshot-based covariance estimate of the data, a new matrix $\tilde{\mathbf{R}}$ is introduced which can be directly used to find the noise eigenspace associated with $\tilde{\mathbf{R}}_{ss}$ (the finite-snapshot version of \mathbf{R}_{ss} of [1], [5]). Even with finite number of snapshots, these matrices are related as $\tilde{\mathbf{R}}_{ss} = \tilde{\mathbf{R}}^2/L$, where L is a constant factor, unlike [1], [5] where such a relation is proved for the ideal infinite snapshot scenario. The MUSIC spectrum which is usually computed based on $\tilde{\mathbf{R}}_{ss}$ can therefore be directly computed based on $\tilde{\mathbf{R}}$. The construction of $\tilde{\mathbf{R}}$ is much simpler than that of $\tilde{\mathbf{R}}_{ss}$ while the performance is guaranteed to be exactly the same for a fixed number of snapshots. So the complexity of the algorithm is less than that of [1], [5]. It turns out that the intermediate matrix $\tilde{\mathbf{R}}$ is *indefinite* (although Hermitian), but we show that this is of no consequence.

While computational reduction is an advantage, the insight provided by the simplification is perhaps more important, as it might lead to considerable theoretical simplification in the case of multidimensional arrays [9], [10], multiple level nested arrays [17], and other future developments of coarray applications.

The paper outline is as follows: Basic ideas from sparse arrays are reviewed in Section II. The new matrix $\tilde{\mathbf{R}}$ is introduced in Section III, and it is shown how coarray MUSIC can be successfully performed from certain eigenspaces computed from this matrix, as well as the computational complexity analysis. In Section IV, the results are further discussed, before Section V concludes the paper.

II. PRELIMINARIES

Consider a sparse array whose sensors are located at nd , $n \in \mathbb{S}$. Here d stands for the minimum spacing between sensors and \mathbb{S} is an integer set. As an example, a two-level nested array [1] is specified by the following

$$\mathbb{S}_{\text{nested}} = \{1, 2, \dots, N_1, (N_1 + 1), 2(N_1 + 1), \dots, N_2(N_1 + 1)\}, \quad (1)$$

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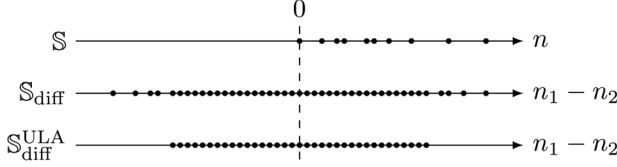


Fig. 1. An illustration of the sets \mathcal{S} , $\mathcal{S}_{\text{diff}}$, and $\mathcal{S}_{\text{diff}}^{\text{ULA}}$. Here we consider a coprime array with $M = 3$ and $N = 5$.

where N_1 and N_2 are the number of sensors at the first level and the second level, respectively. On the other hand, coprime arrays are represented by [2]

$$\mathcal{S}_{\text{coprime}} = \{0, M, 2M, \dots, (N-1)M, N, 2N, \dots, (2M-1)N\}. \quad (2)$$

Here M and N are coprime integers. To add clarity, the geometry of nested arrays and coprime arrays can be found in Fig. 1 in [1] as well as Fig. 1 in [5], respectively. Assume D monochromatic sources with direction-of-arrival (DOA) information $-\pi/2 \leq \theta_i < \pi/2$ for $i = 1, \dots, D$ impinge on a sparse array \mathcal{S} . Then the received signal vector $\mathbf{x}_{\mathcal{S}}$ is modeled as

$$\mathbf{x}_{\mathcal{S}} = \sum_{i=1}^D A_i \mathbf{v}_{\mathcal{S}}(\bar{\theta}_i) + \mathbf{n}_{\mathcal{S}}, \quad (3)$$

where $\mathbf{x}_{\mathcal{S}}, \mathbf{v}_{\mathcal{S}}(\bar{\theta}), \mathbf{n}_{\mathcal{S}} \in \mathbb{C}^{|\mathcal{S}|}$, where $|\mathcal{S}|$ denotes the cardinality of \mathcal{S} . The spatial steering vectors $\mathbf{v}_{\mathcal{S}}(\bar{\theta})$ are column vectors with entries $e^{j2\pi\bar{\theta}n}$ for $n \in \mathcal{S}$. The subscript \mathcal{S} is appended to indicate that these entries are defined over the grid \mathcal{S} . Here $\{A_i\}_{i=1}^D$ and $\mathbf{n}_{\mathcal{S}}$ are zero-mean uncorrelated random vectors satisfying $\mathbb{E}[A_k^* A_\ell] = \sigma_k^2 \delta_{k,\ell}$ and $\mathbb{E}[\mathbf{n}_{\mathcal{S}} \mathbf{n}_{\mathcal{S}}^H] = \sigma^2 \mathbf{I}$. The parameter $\bar{\theta}$ is the normalized DOA, defined as $\bar{\theta} = (d/\lambda) \sin \theta \in [-1/2, 1/2]$.

The essence of sparse array processing is to convert the data to their second-order (or higher-order) statistics. The covariance matrix of $\mathbf{x}_{\mathcal{S}}$ is defined as

$$\mathbf{R}_{\mathbf{x}_{\mathcal{S}}} = \mathbb{E}[\mathbf{x}_{\mathcal{S}} \mathbf{x}_{\mathcal{S}}^H] = \sum_{i=1}^D \sigma_i^2 \mathbf{v}_{\mathcal{S}}(\bar{\theta}_i) \mathbf{v}_{\mathcal{S}}^H(\bar{\theta}_i) + \sigma^2 \mathbf{I}. \quad (4)$$

Note that in (4), the matrix $\mathbf{v}_{\mathcal{S}}(\bar{\theta}) \mathbf{v}_{\mathcal{S}}^H(\bar{\theta})$ has entries $e^{j2\pi\bar{\theta}(n_1 - n_2)}$ for $n_1, n_2 \in \mathcal{S}$, where n_1 and n_2 are sensor locations in our sparse array \mathcal{S} . It can be seen that such quantity depends only on the differences of sensor locations ($n_1 - n_2$).

Definition 1: (Difference-Coarray). Let \mathcal{S} be an integer set specifying the sensor locations. The set $\mathcal{S}_{\text{diff}} = \{n_1 - n_2 | n_1, n_2 \in \mathcal{S}\}$ is called the difference-coarray of the sparse array \mathcal{S} .

Definition 2: (Bracket notations). Given a support set \mathcal{S} , the signal defined on it is denoted as a column vector $\mathbf{x}_{\mathcal{S}}$. We use $[\mathbf{x}_{\mathcal{S}}]_i$ to denote the i th component of this vector. For $n \in \mathcal{S}$ we use the triangular bracket notation $\langle \mathbf{x}_{\mathcal{S}} \rangle_n$ to denote the value of the signal at the support location n . For example, if $\mathcal{S} = \{1, 3, 7\}$ and $\mathbf{x}_{\mathcal{S}} = [0.1, 0.2, 0.3]^T$, then $[\mathbf{x}_{\mathcal{S}}]_1 = 0.1, [\mathbf{x}_{\mathcal{S}}]_2 = 0.2, [\mathbf{x}_{\mathcal{S}}]_3 = 0.3$, whereas $\langle \mathbf{x}_{\mathcal{S}} \rangle_1 = 0.1, \langle \mathbf{x}_{\mathcal{S}} \rangle_3 = 0.2, \langle \mathbf{x}_{\mathcal{S}} \rangle_7 = 0.3$. The notations extend to covariance matrices as follows: $[\mathbf{R}_{\mathbf{x}_{\mathcal{S}}}]_{i_1, i_2} = \mathbb{E}[[\mathbf{x}_{\mathcal{S}}]_{i_1} [\mathbf{x}_{\mathcal{S}}]_{i_2}^*]$ and $\langle \mathbf{R}_{\mathbf{x}_{\mathcal{S}}} \rangle_{n_1, n_2} = \mathbb{E}[\langle \mathbf{x}_{\mathcal{S}} \rangle_{n_1} \langle \mathbf{x}_{\mathcal{S}} \rangle_{n_2}^*]$.

The vectorized version of $\mathbf{v}_{\mathcal{S}}(\bar{\theta}) \mathbf{v}_{\mathcal{S}}^H(\bar{\theta})$ can be viewed as a spatial steering vector defined over $\mathcal{S}_{\text{diff}}$. Then, it is plausible to rewrite (4) into the vector expression as

$$\mathbf{x}_{\mathcal{S}_{\text{diff}}} = \sum_{i=1}^D \sigma_i^2 \mathbf{v}_{\mathcal{S}_{\text{diff}}}(\bar{\theta}_i) + \sigma^2 \mathbf{e}_0,$$

where $\langle \mathbf{e}_0 \rangle_m = \delta_{m,0}$ for $m \in \mathcal{S}_{\text{diff}}$. In convention, $\mathcal{S}_{\text{diff}}$ is arranged in ascending order. That is, the first entry in $\mathbf{x}_{\mathcal{S}_{\text{diff}}}$ corresponds to $\min(\mathcal{S}_{\text{diff}})$ while the last entry corresponds to $\max(\mathcal{S}_{\text{diff}})$. Note that the degrees of freedoms can be increased due to the increase in the number of sample points, from $|\mathcal{S}|$ to $|\mathcal{S}_{\text{diff}}|$. Sparse arrays are designed properly so that the difference-coarray $\mathcal{S}_{\text{diff}}$ contains mainly a ULA part around the origin, which is denoted by $\mathcal{S}_{\text{diff}}^{\text{ULA}}$. Fig. 1 illustrates an example of $M = 3$ and $N = 5$ coprime array. Based on the relations between $\mathcal{S}_{\text{diff}}$ and $\mathcal{S}_{\text{diff}}^{\text{ULA}}$, the signal over $\mathcal{S}_{\text{diff}}^{\text{ULA}}$ is then constructed to be

$$\mathbf{x}_{\mathcal{S}_{\text{diff}}^{\text{ULA}}} = \sum_{i=1}^D \sigma_i^2 \mathbf{v}_{\mathcal{S}_{\text{diff}}^{\text{ULA}}}(\bar{\theta}_i) + \sigma^2 \mathbf{e}'_0, \quad (5)$$

where $\langle \mathbf{e}'_0 \rangle_m = \delta_{m,0}$ for $m \in \mathcal{S}_{\text{diff}}^{\text{ULA}}$.

To estimate the normalized DOAs $\{\bar{\theta}_i\}_{i=1}^D$, previous works perform (forward) spatial smoothing on $\mathbf{x}_{\mathcal{S}_{\text{diff}}^{\text{ULA}}}$ to obtain a spatial smoothed matrix \mathbf{R}_{ss} and then apply the MUSIC algorithm [1], [5]. \mathbf{R}_{ss} is defined as

$$\mathbf{R}_{ss} = \frac{1}{L} \sum_{p=0}^{L-1} \mathbf{J}_p \mathbf{x}_{\mathcal{S}_{\text{diff}}^{\text{ULA}}} \mathbf{x}_{\mathcal{S}_{\text{diff}}^{\text{ULA}}}^H \mathbf{J}_p^H, \quad (6)$$

where \mathbf{J}_p is a selection matrix defined as

$$\mathbf{J}_p = [\mathbf{0}_{L \times (L-1-p)} \quad \mathbf{I}_{L \times L} \quad \mathbf{0}_{L \times p}] \in \{0, 1\}^{L \times (2L-1)}, \quad (7)$$

and

$$L = (|\mathcal{S}_{\text{diff}}^{\text{ULA}}| + 1) / 2. \quad (8)$$

\mathbf{R}_{ss} is a positive semidefinite matrix [1], [5]. Hence, \mathbf{R}_{ss} is suitable for the MUSIC algorithm, where we can separate signal/noise subspaces and then define a valid MUSIC spectrum.

In practice, the signal vector measured at sparse arrays is denoted by $\tilde{\mathbf{x}}_{\mathcal{S}}$, where the tilde notation, throughout this paper, stands for observed or measured quantities. Then the second-order statistics is estimated using K snapshots,

$$\tilde{\mathbf{R}}_{\mathbf{x}_{\mathcal{S}}} = \frac{1}{K} \sum_{k=1}^K \tilde{\mathbf{x}}_{\mathcal{S}}(k) \tilde{\mathbf{x}}_{\mathcal{S}}^H(k), \quad (9)$$

where k is an index for snapshots. If we start from $\tilde{\mathbf{R}}_{\mathbf{x}_{\mathcal{S}}}$ instead of $\mathbf{R}_{\mathbf{x}_{\mathcal{S}}}$ then the quantities $\mathbf{x}_{\mathcal{S}_{\text{diff}}}$, $\mathbf{x}_{\mathcal{S}_{\text{diff}}^{\text{ULA}}}$, and \mathbf{R}_{ss} are replaced with the finite snapshot versions $\tilde{\mathbf{x}}_{\mathcal{S}_{\text{diff}}}$, $\tilde{\mathbf{x}}_{\mathcal{S}_{\text{diff}}^{\text{ULA}}}$, and $\tilde{\mathbf{R}}_{ss}$, which are used in the following developments. Then, $\tilde{\mathbf{x}}_{\mathcal{S}_{\text{diff}}}$ can be determined from $\tilde{\mathbf{R}}_{\mathbf{x}_{\mathcal{S}}}$ as follows:

Definition 3: The measurement vector $\tilde{\mathbf{x}}_{\mathcal{S}_{\text{diff}}}$ is defined as

$$\langle \tilde{\mathbf{x}}_{\mathcal{S}_{\text{diff}}} \rangle_m = \frac{1}{|\mathcal{M}(m)|} \sum_{(n_1, n_2) \in \mathcal{M}(m)} \langle \tilde{\mathbf{R}}_{\mathbf{x}_{\mathcal{S}}} \rangle_{n_1, n_2},$$

for all $m \in \mathbb{S}_{\text{diff}}$. The set $\mathbb{M}(m)$ collects every pair (n_1, n_2) that has contribution to the coarray index m , which is

$$\mathbb{M}(m) = \{(n_1, n_2) \in \mathbb{S}^2 | n_1 - n_2 = m \in \mathbb{S}_{\text{diff}}\}.$$

Similarly, the measurement vector over $\mathbb{S}_{\text{diff}}^{\text{ULA}}$ is

$$\langle \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}} \rangle_m = \frac{1}{|\mathbb{M}(m)|} \sum_{(n_1, n_2) \in \mathbb{M}(m)} \langle \tilde{\mathbf{R}}_{\mathbf{x}_s} \rangle_{n_1, n_2},$$

for all $m \in \mathbb{S}_{\text{diff}}^{\text{ULA}}$.

According to (6), the measured spatial smoothed matrix $\tilde{\mathbf{R}}_{ss}$ is computed from $\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}$ and then the MUSIC algorithm can be applied to obtain line spectrum or DOA estimates.

III. SPARSE ARRAYS WITHOUT SPATIAL SMOOTHING

In this section, we will propose a new matrix $\tilde{\mathbf{R}}$ such that it produces exactly the same MUSIC spectra as [1], [5] without implementing spatial smoothing.

Lemma 1: $\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}}$ is Hermitian symmetric. That is, $\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}} = \mathbf{J} \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}}^*$, where \mathbf{J} is the anti-identity matrix that has ones along its anti-diagonal entries and zeros elsewhere.

Proof: Hermitian symmetry is equivalent to $\langle \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}} \rangle_m = \langle \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}} \rangle_{-m}^*$ for $m \in \mathbb{S}_{\text{diff}}$. Starting with Definition 3, we obtain

$$\langle \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}} \rangle_m = \frac{1}{|\mathbb{M}(-m)|} \sum_{(n_2, n_1) \in \mathbb{M}(-m)} \langle \tilde{\mathbf{R}}_{\mathbf{x}_s}^H \rangle_{n_1, n_2},$$

which is based on these properties: $|\mathbb{M}(m)| = |\mathbb{M}(-m)|$, $(n_1, n_2) \in \mathbb{M}(m)$ if and only if $(n_2, n_1) \in \mathbb{M}(-m)$, and $\tilde{\mathbf{R}}_{\mathbf{x}_s}$ is a Hermitian matrix. Pulling out the complex conjugate yields

$$\langle \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}} \rangle_m = \left(\frac{1}{|\mathbb{M}(-m)|} \sum_{(n_2, n_1) \in \mathbb{M}(-m)} \langle \tilde{\mathbf{R}}_{\mathbf{x}_s} \rangle_{n_2, n_1} \right)^*.$$

Therefore, we obtain $\langle \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}} \rangle_m = \langle \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}} \rangle_{-m}^*$, which completes the proof. ■

Theorem 1: Let $\tilde{\mathbf{R}}$ be the following Toeplitz matrix,

$$\tilde{\mathbf{R}} = \begin{bmatrix} [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_L & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_{L-1} & \cdots & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_1 \\ [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_{L+1} & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_L & \cdots & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_2 \\ \vdots & \vdots & \ddots & \vdots \\ [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_{2L-1} & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_{2L-2} & \cdots & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_L \end{bmatrix},$$

where L is defined in (8). Then $\tilde{\mathbf{R}}$ is Hermitian and $\tilde{\mathbf{R}}_{ss} = \tilde{\mathbf{R}}^2/L$.

Proof: It was proved in Lemma 1 that $\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}}$ follows the Hermitian symmetric property. The same property also holds true for $\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}$ by replacing \mathbb{S}_{diff} with $\mathbb{S}_{\text{diff}}^{\text{ULA}}$ in the proof. The Hermitian of $\tilde{\mathbf{R}}$ is, by definition,

$$\tilde{\mathbf{R}}^H = \begin{bmatrix} [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_L^* & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_{L+1}^* & \cdots & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_{2L-1}^* \\ [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_{L-1}^* & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_L^* & \cdots & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_{2L-2}^* \\ \vdots & \vdots & \ddots & \vdots \\ [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_1^* & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_2^* & \cdots & [\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}]_L^* \end{bmatrix},$$

Since $\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}} = \mathbf{J} \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}^*$, each entry in $\tilde{\mathbf{R}}^H$ is replaced with one term in $\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}$. We obtain $\tilde{\mathbf{R}}^H = \tilde{\mathbf{R}}$, implying $\tilde{\mathbf{R}}$ is Hermitian.

The next part involves expression of $\tilde{\mathbf{R}}$ in terms of \mathbf{J}_p , which is defined in (7). Since $\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}$ is the ULA part in the coarray domain, $\mathbf{J}_0 \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}$ extracts the responses on $0, 1, \dots, \max(\mathbb{S}_{\text{diff}}^{\text{ULA}})$. According to the definition of $\tilde{\mathbf{R}}$, we obtain

$$\tilde{\mathbf{R}} = [\mathbf{J}_0 \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}} \quad \mathbf{J}_1 \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}} \cdots \quad \mathbf{J}_{L-1} \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}] .$$

The square of $\tilde{\mathbf{R}}$ is then evaluated as

$$\begin{aligned} \tilde{\mathbf{R}}^2 &= \tilde{\mathbf{R}} \tilde{\mathbf{R}}^H \\ &= \mathbf{J}_0 \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}} (\mathbf{J}_0 \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}})^H + \mathbf{J}_1 \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}} (\mathbf{J}_1 \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}})^H \\ &\quad + \cdots + \mathbf{J}_{L-1} \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}} (\mathbf{J}_{L-1} \tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}})^H \\ &= L \tilde{\mathbf{R}}_{ss}, \end{aligned}$$

which is equivalent to $\tilde{\mathbf{R}}_{ss} = \tilde{\mathbf{R}}^2/L$. ■

The importance of Theorem 1 is that the MUSIC spectrum can be computed directly from $\tilde{\mathbf{R}}$, rather than the spatially smoothed matrix $\tilde{\mathbf{R}}_{ss}$. It is a direct consequence of $\tilde{\mathbf{R}}_{ss} = \tilde{\mathbf{R}}^2/L$ that (i) eigenvalues of $\tilde{\mathbf{R}}_{ss}$ are proportional to the square of eigenvalues of $\tilde{\mathbf{R}}$, and (ii) $\tilde{\mathbf{R}}_{ss}$ and $\tilde{\mathbf{R}}$ share the same eigenspace. These claims lead to the following corollary:

Corollary 1: MUSIC spectra based on either $\tilde{\mathbf{R}}_{ss}$ or $\tilde{\mathbf{R}}$ are identical if the signal and noise subspaces of $\tilde{\mathbf{R}}$ are determined by magnitudes of its eigenvalues.

A. Computational Complexity Analysis

Our proposed method reduces the complexity of the existing approaches [1], [5]. Here, a more detailed comparison on the number of multipliers will be made to demonstrate the computational savings. The DOA estimation over sparse arrays can be divided into the following three stages:

- 1) *Construct $\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}$ from $\tilde{\mathbf{x}}_s$.* Once the sensor array collects K snapshots, $\tilde{\mathbf{R}}_{\mathbf{x}_s}$ is estimated from (9), taking $O(K|\mathbb{S}|^2)$ operations. Based on Definition 3, $|\mathbb{S}_{\text{diff}}^{\text{ULA}}|$ multipliers are involved. Using (8) and the fact that $O(|\mathbb{S}|^2) = O(|\mathbb{S}_{\text{diff}}^{\text{ULA}}|)$, as proved in [1], [5], we see that the total complexity is $O(KL)$.
- 2) *Establish $\tilde{\mathbf{R}}_{ss}$ or $\tilde{\mathbf{R}}$.* In [1], [5], $\tilde{\mathbf{R}}_{ss}$ is implemented according to (6), where $\mathbf{J}_p \mathbf{x}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}$ is of size L . Since each term takes $O(L^2)$ multiplications, the cost for $\tilde{\mathbf{R}}_{ss}$ is $O(L^3)^1$. On the contrary, to evaluate $\tilde{\mathbf{R}}$, *no multiplication* is needed, since from Theorem 1, $\tilde{\mathbf{x}}_{\mathbb{S}_{\text{diff}}^{\text{ULA}}}$ is reshaped into a Toeplitz matrix $\tilde{\mathbf{R}}$ without further arithmetic operations.
- 3) *MUSIC spectra.* This step is dominated by the eigen-decomposition of an $L \times L$ Hermitian matrix, which can be either $\tilde{\mathbf{R}}_{ss}$ or $\tilde{\mathbf{R}}$. It is known that eigen-decomposition requires around $O(L^3)$ operations [20].

¹Note that there are some matrix multiplication algorithms with complexity $O(L^\alpha)$, $\alpha < 3$ [18], [19] but it still takes some resources to do so.

Thus, the computational complexity for the two approaches are

$$\begin{aligned} \text{Conventional}(\tilde{\mathbf{R}}_{ss}) &: O(KL + L^3 + L^3), \\ \text{Proposed}(\tilde{\mathbf{R}}) &: O(KL + L^3). \end{aligned}$$

As we can see, our proposed method saves $O(L^3)$ operations. However, the overall complexity is still dominated by the eigen-decomposition in both methods, which requires $O(L^3)$ computations.

Nevertheless, the computational reduction can become more prominent when we go for multidimensional arrays [8], [10] or higher order statistics [17]. For instance, in a $2q$ -level nested array [8], the ULA segment is of length $O(|\mathcal{S}|^{2q}) = O(L^q)$ so that the corresponding spatially smoothed matrix has dimension $O(L^q) \times O(L^q)$. Following the same analysis as in this subsection, the complexity for $2q$ -level nested array becomes $O(KL^q + L^{3q} + L^{3q})$ while the proposed method has complexity $O(KL^q + L^{3q})$. It can be seen that $O(L^{3q})$ operations will be deducted using the proposed method, which is not negligible especially for large q . But still, eigen-decomposition ($O(L^{3q})$) governs the overall complexity.

IV. DISCUSSIONS

- 1) Note that $\tilde{\mathbf{R}}$ is an indefinite square root of $\tilde{\mathbf{R}}_{ss}$. The finite snapshot square root matrix implicitly appears in [21]. While it is not obvious, it can be shown that this matrix in [21] is the same as the matrix $\tilde{\mathbf{R}}$ in Theorem 1. This matrix has been used in [21] for the convenience of analysis of root-MUSIC in coarray domain. However, the fact that coarray MUSIC can be obtained directly from an eigen computation of the data-based matrix $\tilde{\mathbf{R}}$ has not been observed in the past. Instead of this matrix, the spatially smoothed matrix $\tilde{\mathbf{R}}_{ss}$ has been used in all recent works [1] and [5] for obtaining MUSIC spectrum.
- 2) Our approach is applicable to any sparse array that has a ULA section in the coarray domain. For instance, the minimum-redundancy arrays [22] and Golomb arrays [23], [24] both satisfy such conditions. However, these arrays do not have a simple formulation on the sensor locations, which can only be obtained from table look-up [22], [23].
- 3) In [25], the authors proposed to construct a positive-definite Toeplitz matrix $\hat{\mathbf{T}}$, based on the estimated covariance matrices. The way that [25] establishes $\hat{\mathbf{T}}$ is identical to Definition 3 in this paper. However, in [25], $\hat{\mathbf{T}}$ is restricted to positive-definite Toeplitz matrices but here, $\tilde{\mathbf{R}}$ is an *indefinite* Toeplitz matrix. In addition, the goal of [25] is to fill missing lags in $\hat{\mathbf{T}}$. Nevertheless, in our work, all entries of $\tilde{\mathbf{R}}$ are known.
- 4) Our results are derived for finite snapshots. However, in the proofs of [1], [5], infinite snapshots are assumed in making the argument $\mathbf{R}_{ss} = \tilde{\mathbf{R}}^2$. Hence the statements here are stronger than [1], [5]. For instance, Theorem 1 serves as a finite-snapshot generalization of Theorem 2 in [1] as well as Theorem 1 in [5]. To show that our proposed method is consistent with [1], [5] under the infinite snapshot assumption, taking $K \rightarrow \infty$ yields $\lim_{K \rightarrow \infty} \tilde{\mathbf{R}}_{\mathbf{x}_s} = \mathbf{R}_{\mathbf{x}_s}$ and

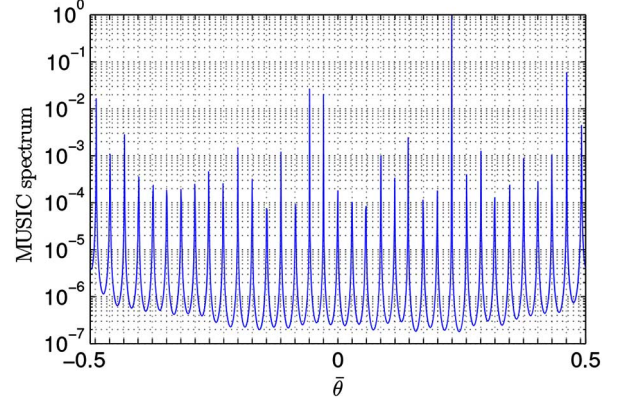


Fig. 2. The MUSIC spectrum based on an $M = 5$, $N = 7$ coprime array, 0 dB SNR, $K = 500$ snapshots, and the Hermitian Toeplitz matrix $\tilde{\mathbf{R}}$. $D = 35$ sources are placed uniformly over $\bar{\theta} \in [-0.49, 0.49]$. The number of sensors is $N + 2M - 1 = 16$.

$\lim_{K \rightarrow \infty} \tilde{\mathbf{x}}_{\text{diff}}^{\text{ULA}} = \mathbf{x}_{\text{diff}}^{\text{ULA}}$. According to Theorem 1, the limit of $\tilde{\mathbf{R}}$ becomes

$$\lim_{K \rightarrow \infty} \tilde{\mathbf{R}} = \mathbf{A}_{11} \mathbf{\Lambda} \mathbf{A}_{11}^H + \sigma^2 \mathbf{I}_{L \times L},$$

where \mathbf{A}_{11} and $\mathbf{\Lambda}$ follow similar definitions in [1], [5]. Note that $\lim_{K \rightarrow \infty} \tilde{\mathbf{R}}$ is proportional to $\hat{\mathbf{R}}$ in [1], [5].

As an example, consider a nested array with $N_1 = N_2 = 8$ and a coprime array with $M = 5$ and $N = 7$. The total number of sensors is then 16, as defined in (1) and (2). SNR is chosen to be 0 dB and there are $K = 500$ snapshots. $D = 35$ sources are selected uniformly over $\bar{\theta} = [-0.49, 0.49]$ in our first experiment. As seen in Fig. 2, there are 35 distinguishable peaks even though D is greater than the number of sensors, $|\mathcal{S}| = 16$. These 35 peaks match with the ideal peaks, which are marked by ticks on the $\bar{\theta}$ axis.

The same configuration with 1000 runs is conducted for both $\tilde{\mathbf{R}}_{ss}$ (spatial smoothing) and $\tilde{\mathbf{R}}$ (no spatial smoothing). On a Ubuntu 12.04 workstation with a Intel Core i7-2600 3.40 GHz processor and 8 GB RAM, conventional approaches take 417.59 seconds for nested arrays and 85.73 seconds for coprime arrays. Nevertheless, our new method spends 399.07 seconds (4.4% reduction) for nested arrays and 75.02 seconds (12.5% less) for coprime arrays. We see that the computational reduction is rather minor. This is because, the computation time is dominated by the eigenspace computation in the MUSIC stage. So, the computational advantage obtained by replacing the spatial smoothing step with the matrix $\tilde{\mathbf{R}}$ is minor. However, the insight provided by the fact that spatial smoothing can be avoided is more fundamental, and might have theoretical impact in future developments. The matrix $\tilde{\mathbf{R}}$ also makes the performance analysis more tractable (as compared to the use of the spatial smoothing matrix), as seen from [21]. This is because, $\tilde{\mathbf{R}}_{ss}$ requires 4th order statistics, whereas $\tilde{\mathbf{R}}$ is in terms of 2nd order statistics (correlations).

V. CONCLUSION

We have shown that coarray MUSIC, which is used for sparse arrays such as nested and coprime arrays can be performed without the use of spatial smoothing. While there are some computational advantages, the insight provided by the simplification is also important, as it might lead to theoretical simplification in future work.

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