BS 617 MAT DSE (D) - E

B.Sc.

THIRD YEAR SEMESTER - VI

MATHEMATICS

DISCIPLINE SPECIFIC ELECTIVE COURSE - D

VECTOR CALCULUS

"We may forgo material benefits of civilization, but we cannot forgo our right and opportunity to reap the benefits of the highest education to the fullest extent…"

Dr.B.R.Ambedkar

Dr. B.R. AMBEDKAR OPEN UNIVERSITY

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PREFACE

The course material of “Vector Calculus” is prepared in accordance with the guidelines issued by the University Grants Commission and the Telangana State Commission of Higher Education to suit the Choice Based Credit Format in the Semester System. Keeping in view the diversity of learners in the Open University with regard to their age, learning abilities and the minimum academic pre-requirements, the course material is prepared in Self Instruction Mode with all the inputs to help the learner to learn by himself/herself. The text material is supplemented with limited face to face instruction and audio/video support. From the academic year 2017-18, an additional learning input in the form of PRACTICALS was introduced. The separate practical manual cum record book is prepared and supplied to the learner along with the course material. Learners are expected to attend the practical classes and practice the problem solving skills by solving the problems given in the material under the guidance of a counsellor. The record of the work carried out by the learner is to be submitted at the time of the practical examination for assessment. The text material or theory part carries weightage of 4 credits and the practicals carry 1 credit.

In the Open Distance Learning System, the onus of learning rests with the learner. The University provides an opportunity to realize the academic aspirations of the learner and facilitate the learner to ‘LEARN TO LEARN’.

Vector Calculus plays an important role in the study of physical sciences in describing electromagnetic and gravitational fields and fluid flows. The equations describing the behaviour of physical quantities such as electric fields and velocity of fluids are expressed in terms of the gradient, divergence and curl operators. Gauss’s divergence theorem, Stoke’s theorem and Green’s theorem have several applications in Engineering in calculating the flux through a closed surface enclosing a volume etc. The topics discussed in this book cover the core area of the subject spread over four blocks, each covering some specific topics intended for study in the degree curriculum.

Every effort is made to make learning a pleasure. Any suggestions to improve the content of the course material are welcome and the University would consider them seriously.

CONTENTS

BLOCK-I: VECTOR DIFFERENTIAL CALCULUS

Vector analysis is an essential branch of mathematics which provides useful tools to study mathematically model situations that arise in physics and geometry. Which studying physical phenomena like elasticity, electricity, magnetism, one has to deal with quantities which have both direction and magnitude. This leads to the notion of vectors. Vector Algebra extends the concepts of addition, subtraction, multiplication to vectors. The velocity of a moving object tangent line to a curve at points on the curve and normal to a surface are vector quantities which are not static but change with change in position. This leads to the idea of vector functions and vector fields. Vector calculus introduces the analogues of differentiation, integration from ordinary calculus to vectors.

In this block, the basic concepts of vector algebra which are required for further development of subject in subsequent units are discussed. The ideas of differentiation, partial differentiation of vector function and their applications to geometry and space curves explained. The vector differential operators gradient, divergence, curl are defined and relations among these operations are established. Application of these notions to physical and geometrical problems is illustrated with examples.

This block includes the following units:

Unit - 1 : Basic Vector Algebra

Unit - 2 : Differentiation of Vector Point Functions, Curves, Tangents, Arc Length

Unit - 3 : Gradient, Divergence, Curl Operators and their Algebra

UNIT-1: BASIC VECTOR ALGEBRA

Contents

1.0 OBJECTIVES

After studying this unit, you will be able to:

- Distinguish between scalars and vectors.
- Apply basic algebraic operations on to vectors.
- Learn different ways of multiplying vectors.
- Understand physical and geometrical interpretation of results using vector methods and efficiently use them to represent the real world problems.

1.1 INTRODUCTION

This unit is concerned with the mathematical description of physical quantities. We introduce the concepts of vectors and scalars and using them classify all physical quantities into vector quantities and scalar quantities. The vector addition, scalar multiplication and multiplication of vectors are explained.

1.2 VECTORS AND SCALARS

A physical quantity which has both direction and magnitude is called a **vector**.

Displacement, velocity, acceleration, force are some examples of vectors.

A physical quantity which has only magnitude is called a **scalar**.

Mass, temperature, pressure are some examples of scalars.

Vectors are usually denoted by $\vec{a}$ or ***a*.** The magnitude of a vector $\vec{a}$ is denoted by $|\vec{a}|$. The magnitude of a vector is also known as its **length or modulus**. Vectors are pictorially represented by directed line segments.

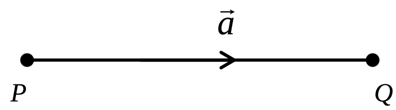

Fig: 1.1

In Fig. 1.1, the vector $\vec{a}$ is represented by the line segment PQ with direction from P to Q. P is called the initial point and Q is called the terminal point. The length of the line segment PQ gives the magnitude of the vector $\vec{a}$. With this understanding, we write $\vec{a} = \overrightarrow{PQ}$ or ***PQ*** and magnitude of $\vec{a} = |\vec{a}| = |\overrightarrow{PQ}|$ or |***PQ***|. A line of infinite length of which the directed line segment representing the vector $\vec{a}$ is a part is called the **support** of $\vec{a}$. Observe that the line segments $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ represent two vectors with same magnitude but opposite directions.

Scalars are usually denoted by *a, b, ...* etc, where *a, b, ...* are real numbers that represent the magnitude of the scalars. Thus we may identify a scalar with a real number.

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. Identify whether the following quantities are vectors or scalars.

   a) Energy

   b) Electric charge

   c) Electric current

   d) Momentum

   e) Weight

   f) Density

We now study the algebraic operations on the vectors.

1.3 TYPES OF VECTORS

We are familiar with the operations of addition, subtraction, multiplication of real numbers or scalars. With suitable modifications, we can extend these ideas to vectors. First, we give some fundamental definitions.

1.3.1 Equal vectors

Two vectors are **equal** if they have same direction and magnitude.

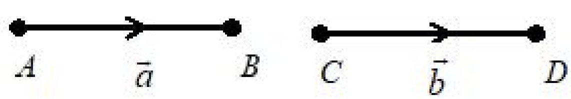

Fig: 1.2

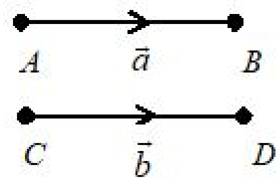

Fig: 1.3

In the above figures the vectors $\vec{a}$ and $\vec{b}$ are equal. So we write $\vec{a} = \vec{b}$.

1.3.2 Zero Vector

A vector which has zero magnitude is called a **zero vector or null vector**. It is denoted by $\vec{0}$ or **0**.

Note that the initial and terminal point of a zero vector coincide.

1.3.3 Unit Vector

A vector having magnitude as 1 unit is called a **unit vector**.

If $\vec{a}$ is a vector, then $\dfrac{\vec{a}}{|\vec{a}|}$ is vector with unit magnitude. This is called unit vector in the direction of $\vec{a}$ and it is denoted by $\hat{a}$.

1.3.4 Negative of a Vector

If $\vec{a}$ is a vector, a vector having same magnitude as $\vec{a}$ but having its direction opposite to that of $\vec{a}$ is called the negative of $\vec{a}$. It is denoted by $-\vec{a}$.

Fig. 1.4

In Fig. 1.4 if $\vec{a} = \overrightarrow{AB}$ then $-\vec{a} = \overrightarrow{BA}$.

1.3.5 Coinitial Vector

Vectors having same initial point are called **coinitial vector**.

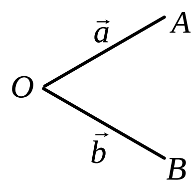

Fig: 1.5

In fig: 1.5 the vectors $\vec{a} = \overrightarrow{OA},\ \vec{b} = \overrightarrow{OB}$ are coinitial vectors.

1.3.6 Non-zero Vector

A vector whose magnitude is not zero is called a p**roper vector or non-zero vector**. A non-zero vector has a definite direction and magnitude.

Thus if $\vec{a}$ is a non-zero vector, then $|\vec{a}| = a > 0$.

1.3.7 Spatial Vector

A vector $\vec{a}$ is said to be **spatial** if it lies in the space.

1.3.8 Rectangular Unit Vectors

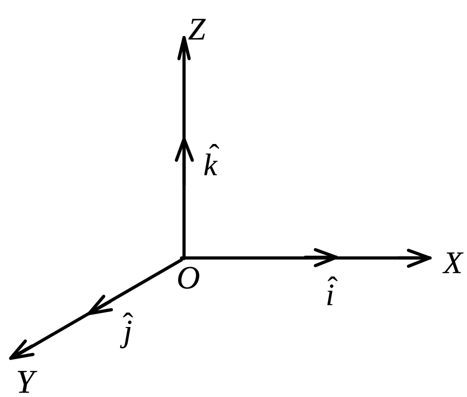

Fig: 1.6

In the rectangular Cartesian co-ordinate system (Fig. 1.6) the unit vectors along the positive directions of *X, Y, Z* axes are called **rectangular unit vectors**. They are denoted by $\hat{i}, \hat{j}, \hat{k}$ respectively. The vectors $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular and they form a right handed system, because a screw rotated through 90^0 from *OX* to *OY* will advance in the positive *Z* direction.

1.4 ALGEBRA OF VECTORS

1.4.1 Addition of Vectors

A single vector whose effect in equal to the combined effect of two or more vectors is called the resultant or sum of these vectors. The process of finding the sum is called **vector addition**.

If $\vec{a}, \vec{b}$ are two vectors, then their sum is denoted by $\vec{a}+\vec{b}$.

The sum of $\vec{a}$ and $\vec{b}$ can be found by using the triangle law or parallelogram law.

1.4.2 Triangle Law of Addition

If $\vec{a}$ and $\vec{b}$ are two given vectors (not necessarily coinitial), we first draw the vector $\vec{a}$ and place the initial point of $\vec{b}$ on the endpoint of $\vec{a}$. Joining the initial point of $\vec{a}$ to the end point of $\vec{b}$, we obtain the sum $\vec{a}+\vec{b}$.

Graphically, if $\vec{a}=\overrightarrow{AB}, \vec{b}=\overrightarrow{BC}$ represent two consecutive sides of a triangle *ABC*, then the sum of $\vec{a}$ and $\vec{b}$ is given by the third side $\overrightarrow{AC}$ of the triangle.

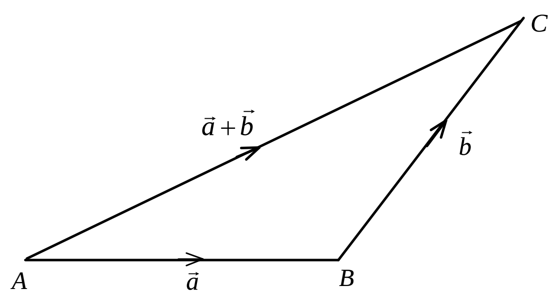

Fig: 1.7

Thus from Fig 1.7, $\vec{a}+\vec{b}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$.

1.4.3 Parallelogram Law of Addition

If $\vec{a} = \overrightarrow{AB}, \vec{b} = \overrightarrow{AD}$ represent two adjacent sides of a parallelogram *ABCD*, then the sum of $\vec{a}$ and $\vec{b}$ is given by the diagonal $\overrightarrow{AC}$ (Fig. 1.8).

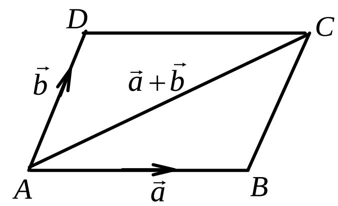

Fig: 1.8

Thus the sum of two coinitial vectors is found by using the parallelogram law.

From Fig 1.8, completing the parallelogram *ABCD*, we note that

$\overrightarrow{AB} = \vec{a}, \overrightarrow{BC} = \overrightarrow{AD} = \vec{b}$ and $\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Properties of Vector Addition

1.4.4 Commutative Law

For any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

Let $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{AB}$.

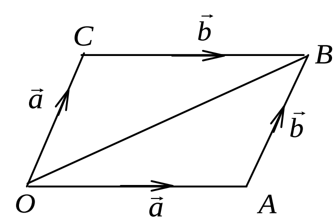

Fig: 1.9

From ΔOAB,

$\vec{a} + \vec{b} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ ..... (1)

completing the parallelogram, *OABC*,

we get, $\overrightarrow{OC} = \overrightarrow{AB} = \vec{b}$ and $\overrightarrow{CB} = \overrightarrow{OA} = \vec{a}$

$\vec{b} + \vec{a} = \overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$ ..... (2)

From (1) and (2), $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

1.4.5 Associative Law of Addition

For any three vectors $\vec{a}, \vec{b}, \vec{c}$, $\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$

Let $\vec{a}=\overrightarrow{OA}, \vec{b}=\overrightarrow{AB}$ and $\vec{c}=\overrightarrow{BC}$.

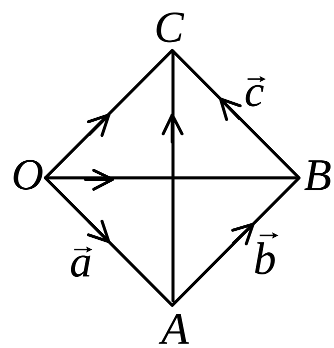

Fig: 1.10

From Fig 1.10, we have

$$\vec{a}+\left(\vec{b}+\vec{c}\right)=\overrightarrow{OA}+\left(\overrightarrow{AB}+\overrightarrow{BC}\right)=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC} \quad \text{...... (1)}$$

$$\left(\vec{a}+\vec{b}\right)+\vec{c}=\left(\overrightarrow{OA}+\overrightarrow{AB}\right)+\overrightarrow{BC}=\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OC} \quad \text{...... (2)}$$

From (1) and (2),

$$\vec{a}+\left(\vec{b}+\vec{c}\right)=\left(\vec{a}+\vec{b}\right)+\vec{c}.$$

1.4.6 Subtraction of Vectors

The difference of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}-\vec{b}$ and it is defined as the sum of $\vec{a}$ and $-\vec{b}$ where $-\vec{b}$ is the negative of $\vec{b}$.

Thus $\vec{a}-\vec{b}=\vec{a}+\left(-\vec{b}\right)$.

1.4.7 Multiplication of a Vector by a Scalar

Let $\vec{a}$ be a non-zero vector and m be a scalar. The product of $\vec{a}$ by m is denoted by $m\vec{a}$ and $m\vec{a}$ is a vector whose magnitude is $|m|$ times the magnitude of $\vec{a}$ and whose direction is same or opposite that of $\vec{a}$, according as m is positive or negative. If $m = 0$, $m\vec{a}$ is the null vector.

Properties of Scalar Multiplication

The following properties are true for any two vectors $\vec{a}, \vec{b}$ and scalars *m, n*.

$(-1)\vec{a} = -\vec{a}$

$m\vec{0} = \vec{0}$

$m\vec{a} = \vec{a}m$

$(m+n)\vec{a} = m\vec{a} + n\vec{a}$

$m\left(\vec{a}+\vec{b}\right) = m\vec{a} + m\vec{b}$

$(mn)\vec{a} = m(n\vec{a})$

1.4.8 Note

The properties of vector addition and scalar multiplication allow us to deal with vector equations in the same way as algebraic equations.

(i) If $\vec{a}$ is a non-zero vector, $\dfrac{\vec{a}}{|\vec{a}|}$ is a unit vector in the direction of $\vec{a}$. This is denoted by $\hat{a}$.

Thus, $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

i.e. $\vec{a} = |\vec{a}|\hat{a}$.

Thus any vector $\vec{a}$ can be represented as a scalar multiple of a unit vector in its direction.

(ii) Two vectors $\vec{a}, \vec{b}$ are collinear vectors $\Leftrightarrow$ we can write $\vec{a} = k\vec{b}$ for some scalar *k*.

(iii) If $\vec{a}, \vec{b}$ are non-collinear vectors and *x, y* are scalars such that

$x\vec{a} + y\vec{b} = \vec{0}$ then $x = 0, y = 0$.

If possible, we assume that $x \neq 0$, then

$x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x\vec{a} = -y\vec{b} \Rightarrow \vec{a} = \dfrac{-y}{x}\vec{b}$.

$\Rightarrow$ $\vec{a}$ and $\vec{b}$ are collinear, which is not true.

$\therefore$ $x = 0$.

Similarly, we can prove that $y = 0$.

1.4.9 Components of a Vector

Consider the rectangular cartesian coordinate system *OX, OY, OZ* with *O* as origin (Fig. 1.11). We know that $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors along *OX, OY, OZ* respectively.

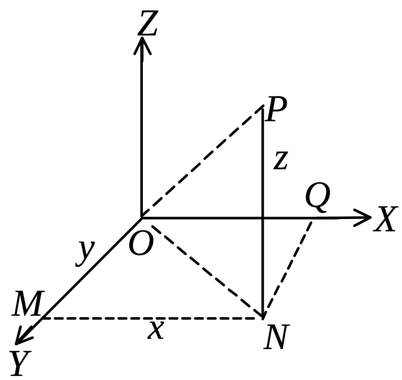

Fig: 1.11

Let $\vec{a}$ be a given vector in three dimensions. Then $\vec{a}$ can be represented by a vector $\overrightarrow{OP}$ with initial point at '*O*' and end point at *P*. Let *x, y, z* be the coordinates of *P* w.r.t. the rectangular system *OXYZ*. Draw *PN* perpendicular to the *XY*-plane. Let *NM, NQ* be perpendiculars drawn to *Y, X* axes respectively from *N*. Hence, we have

$$OM = QN = y,\ OQ = MN = x,\ NP = z\ .$$

$$\therefore \overrightarrow{OQ} = x\hat{i},\ \overrightarrow{QN} = y\hat{j},\ \overrightarrow{NP} = z\hat{k}\ .$$

From ΔOMN, by triangle law of addition, we get

$$\overrightarrow{ON} = \overrightarrow{OQ} = \overrightarrow{QN}$$

$$\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP}$$

$$= \overrightarrow{OQ} + \overrightarrow{QN} + \overrightarrow{NP}$$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

Thus, the position vector of the point $P(x, y, z)$ can be expressed as

$$\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}\ .$$

the vectors $x\hat{i}$, $y\hat{j}$, $z\hat{k}$ are called the rectangular component vectors or simply component vectors of $\vec{a}$ in the *X*, *Y*, *Z* directions respectively. The scalars *x*, *y*, *z* are called rectangular components simply components of $\vec{a}$ along *OX*, *OY*, *OZ* respectively.

Thus for every vector $\vec{a}$ in space, there exists a point $P(x,y,z)$ in space such that $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$. Also, given a point $P(x,y,z)$ in space, there is a vector $\overrightarrow{QP} = x\hat{i} + y\hat{j} + z\hat{k}$ associated with that. Thus there is a 1-1 correspondence between vectors and points in space. With this understanding, we write $\vec{a} = (x,y,z)$ in terms of components.

Also, from Fig 1.11, we have

$$OP^2 = ON^2 + NP^2 = OQ^2 + QN^2 + NP^2 = x^2 + y^2 + z^2$$

$$\Rightarrow OP = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow |\vec{a}| = \sqrt{x^2 + y^2 + z^2} \qquad (\because OP = |\overrightarrow{OP}| = \text{ magnitude of } \vec{a})$$

Check Your Progress:

2. Find the components of the unit vector in the direction of $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$.

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1.4.10 Expressing a given Vector in terms of Position Vectors

Let $\overline{AB}$ be a given vector. Suppose $\vec{a}$ and $\vec{b}$ are the position vectors of *A* and *B* with respect to the origin *O*. Then we have from Fig 1.12.

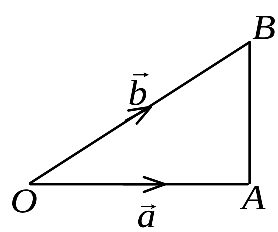

Fig: 1.12

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$$

= position vector of *B* w.r.t. *O* - position vector of *A* w.r.t *O*.

Also, if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\overline{AB} = \vec{b} - \vec{a} = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) - \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$

$$= (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}.$$

Check Your Progress:

3. If the position vectors of A and B w.r.t O are $\hat{i} + 3\hat{j} - 7\hat{k}$ and $5\hat{i} - 2\hat{j} + 4\hat{k}$ respectively, then find $\overline{AB}$ and its modulus. Also find the unit vector in the direction of $\overline{AB}$.

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1.5 COLLINEAR AND COPLANAR VECTORS

In this section we define collinear and coplanar vectors which are of fundamental importance.

1.5.1 Collinear vectors

Two vectors are collinear if their lines of support are either the same or parallel. If $\vec{a}$ and $\vec{b}$ are collinear vectors, then

(a) their directions are same or opposite and

(b) their magnitude may differ

(c) they may or may not be co-initial vectors.

If two collinear free vectors have the same length as well as direction, then they are equal.

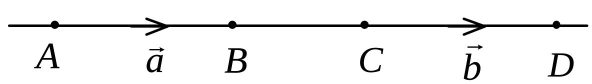

Fig: 1.13

Vectors which are not collinear are known as non-collinear vectors.

1.5.2 Theorem

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then either of them can be expressed as the product of the other by a suitable scalar, and conversely.

Proof: We know that $\vec{a} = |\vec{a}|\hat{a}$, $\vec{b} = |\vec{b}|\hat{b}$

where $\hat{a}, \hat{b}$ are unit vectors along $\vec{a}$ and $\vec{b}$ respectively.

If $\vec{a}$ and $\vec{b}$ are collinear, then $\hat{a} = \pm\hat{b}$ since $|\hat{a}| = |\hat{b}| = 1$.

Thus $\vec{b} = |\vec{b}|\hat{b} = \dfrac{|\vec{b}|}{|\vec{a}|}(|\vec{a}|)\hat{b} = \dfrac{|\vec{b}|}{|\vec{a}|}(\pm|\vec{a}|\hat{a})$

$= \pm\dfrac{|\vec{b}|}{|\vec{a}|}\vec{a} = k\vec{a}$, where $k = \pm\dfrac{|\vec{b}|}{|\vec{a}|}$.

conversely, if $\vec{a} \neq \vec{0}$, then by definition of multiplication of a vector by a scalar, $m\vec{a}$ represents a vector collinear with $\vec{a}$. Hence, if $\vec{b} = m\vec{a}$ for some scalar *m*, then $\vec{a}, \vec{b}$ are collinear vectors.

1.5.3 Theorem

If $\vec{a}$ and $\vec{b}$ are two non-zero, non-collinear vectors and *m* and *n* are two scalars such that $m\vec{a} + n\vec{b} = \vec{0}$, then $m = n = 0$.

Proof: If possible, let $m \neq 0$.

Then, $m\vec{a} + n\vec{b} = 0 \Rightarrow \vec{a} = -\dfrac{n}{m}\vec{b}$.

$\therefore \vec{a}$ is collinear to $\vec{b}$, a contradiction.

Thus $m = 0$.

So, $m\vec{a} + n\vec{b} = \vec{0} \Rightarrow n\vec{b} = 0 \Rightarrow n = 0$ as $\vec{b} \neq \vec{0}$.

Thus, $m = n = 0$ when $m\vec{a} + n\vec{b} = \vec{0}$

1.5.4 Note

From the above theorem, if $\vec{a}$ and $\vec{b}$ are non-collinear, and

$m\vec{a} + n\vec{b} = x\vec{a} + y\vec{b}$

$\Rightarrow (m - x)\vec{a} + (n - y)\vec{b} = \vec{0}$

$\Rightarrow m - x = 0,\ n - y = 0$

$\Rightarrow m = x,\ n = y$.

1.5.5 Theorem

If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, then every vector $\vec{r}$ lying in the plane determined by $\vec{a}$ and $\vec{b}$ can be uniquely expressed in the form $m\vec{a}+n\vec{b}$ for some scalars m and n.

Proof: Let 'O' be the origin of reference and $\vec{a}, \vec{b}, \vec{r}$ be represented by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OP}$ respectively, when *OA, OB, OP* lie on the same plane.

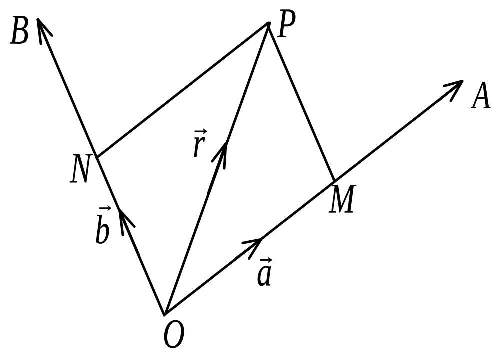

Fig: 1.14

From P, draw PM and PN parallel to OB and OA and complete the parallelogram $OMPN$ (Fig. 1.14).

Since $\overrightarrow{OM}$ and $\overrightarrow{ON}$ are collinear to $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively, there exist suitable scalars m and n, such that

$$\overrightarrow{OM} = m\overrightarrow{OA} = m\vec{a} \text{ and } \overrightarrow{ON} = n\overrightarrow{OB} = n\vec{b}$$

Thus, from ΔOMP, by triangle law of addition,

$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = \overrightarrow{OM} + \overrightarrow{ON}$$

$$\Rightarrow \vec{r} = m\vec{a} + n\vec{b} \cdot$$

Uniqueness: If possible, let $\vec{r} = m_1\vec{a} + n_1\vec{b}$ for some scalars m_1, n_1 respectively, then

$$m\vec{a} + n\vec{b} = m_1\vec{a} + n_1\vec{b}$$

$$\Rightarrow (m - m_1)\vec{a} + (n - n_1)\vec{b} = \vec{0} .$$

Since $\vec{a}, \vec{b}$ are non-collinear, we have

$$m - m_1 = 0,\ n - n_1 = 0 \Rightarrow m = m_1,\ n = n_1 .$$

Hence the representation of $\vec{r}$ is unique.

1.5.6 Note

The vectors $m\vec{a}, n\vec{b}$ are called components of $\vec{r}$ in the directions of $\vec{a}$ and $\vec{b}$ respectively. $\vec{a}$ and $\vec{b}$ are called base vectors in that plane.

1.5.7 Theorem: Condition for the Collinearity of three Points

The necessary and sufficient condition for 3 distinct points *A, B, C* with position vectors $\vec{a},\vec{b},\vec{c}$ to be collinear is that there exist 3 scalars *x, y* and *z* (not all zero) such that $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0},\ x+y+z=0$.

Proof:

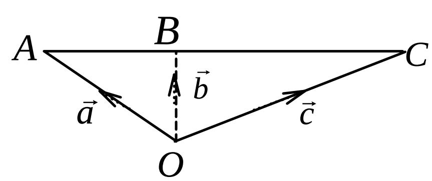

Fig: 1.15

Let *O* be the origin of reference, Fig. 1.15.

Using triangle law,

$$\overrightarrow{AB}=\vec{b}-\vec{a}$$

$$\overrightarrow{BC}=\vec{c}-\vec{b}$$

The points *A, B, C* are collinear iff the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear.

i.e. $\overrightarrow{AB}=m\overrightarrow{BC}$ for some scalar *m*.

$$\Rightarrow\left(\vec{b}-\vec{a}\right)=m\left(\vec{c}-\vec{b}\right)$$

$$\Rightarrow-\vec{a}+\left(1+m\right)\vec{b}-m\vec{c}=\vec{0}$$

taking $x=-1,\ y=1+m,\ z=-m$, we see that

$x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$, where $x+y+z=-1+1+m-m=0$.

conversely suppose that *x, y, z* are scalars, not all zero such that

$x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$, $x+y+z=0$.

Now, $x+y+z=0\Rightarrow y=-x-z$

$$\therefore x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$$

$$\Rightarrow x\vec{a} + (-x - z)\vec{b} + z\vec{c} = \vec{0}$$

$$\Rightarrow -x\left(\vec{b} - \vec{a}\right) - z\left(-\vec{c} + \vec{b}\right) = \vec{0}$$

$$\Rightarrow (-x)\overrightarrow{AB} + (-z)\overrightarrow{BC} = \vec{0}$$

$$\Rightarrow \overrightarrow{AB} = \frac{-z}{x}\overrightarrow{BC} \qquad (\because x, y, z \text{ are not all zero, we assume that } x \neq 0)$$

$\Rightarrow \overrightarrow{AB}, \overrightarrow{BC}$ are collinear vectors.

$\Rightarrow A, B, C$ are collinear points.

1.5.8 Coplanar Vectors

A set of vectors which lie on the same plane or whose lines of support are parallel to the same plane are called **coplanar vectors**.

From theorem 1.5.5, we notice that $\vec{a}, \vec{b}, \vec{c}$ are coplanar, if there exist scalars *m* and *n* such that $\vec{c} = m\vec{a} + n\vec{b}$.

A set of vectors which do not lie on the same plane or parallel to the same plane are called **non-coplanar vectors**.

1.5.9 Theorem

If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero, non-coplanar vectors and *m, n* and *p* are three scalars such that $m\vec{a} + n\vec{b} + p\vec{c} = \vec{0}$, then $m = n = p = 0$.

Proof:

Suppose, $m \neq 0$.

Then $m\vec{a} + n\vec{b} + p\vec{c} = \vec{0}$

$$\Rightarrow m\vec{a} = -n\vec{b} - p\vec{c} \quad \Rightarrow \vec{a} = \frac{-n}{m}\vec{b} - \frac{p}{m}\vec{c}.$$

By theorem 1.5.5, the vector $\vec{a} = \frac{-n}{m}\vec{b} - \frac{p}{m}\vec{c}$ lies on the plane of $\vec{b}$ and $\vec{c}$.

i.e. $\vec{a}$ lies on the plane of $\vec{b}$ and $\vec{c}$, a contradiction since $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar.

$\therefore m = 0$.

By a similar argument, we can show that $n = p = 0$

Hence, $m = n = p = 0$.

1.5.10 Theorem

Every vector can be expressed as a linear combination of three non-coplanar vectors.

Proof: Let O be the origin and P be any point in space with $\overrightarrow{OP} = \vec{r}$ (Fig 1.16)

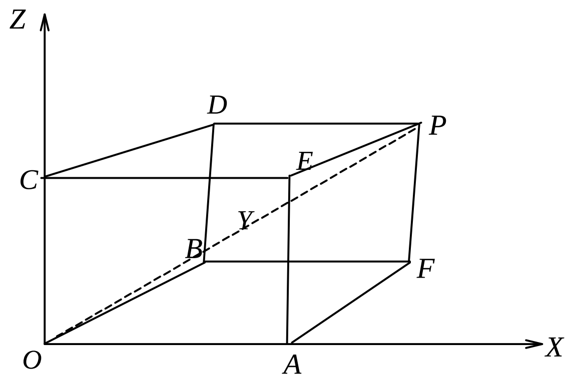

Fig: 1.16

Let $\hat{a}, \hat{b}, \hat{c}$ be unit vectors in three non-coplanar directions OX, OY and OZ respectively. Construct a parallelopiped with OP as diagonal and OA, OB, OC along OX, OY, OZ respectively as edges.

Let $OA = x, OB = y, OC = z$.

Then $\overrightarrow{OA} = x\hat{a}, \overrightarrow{OB} = y\hat{b}, \overrightarrow{OC} = z\hat{c}$.

From Fig. 1.16, $\vec{r} = \overrightarrow{OP} = \overrightarrow{OF} + \overrightarrow{FP}$

$$= \overrightarrow{OA} + \overrightarrow{AF} + \overrightarrow{FP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

$$= x\hat{a} + y\hat{b} + z\hat{c}$$

Thus, any vector can be represented as a linear combination of three non-coplanar vectors.

Uniqueness: If possible, let $\vec{r} = x'\hat{a} + y'\hat{b} + z'\hat{c}$

Then $x\hat{a} + y\hat{b} + z\hat{c} = x'\hat{a} + y'\hat{b} + z'\hat{c}$

$$\Rightarrow (x - x')\hat{a} + (y - y')\hat{b} + (z - z')\hat{c} = \vec{0}$$

$\Rightarrow x - x' = 0,\ y - y' = 0,\ z - z' = 0$ by using theorem 1.5.5

$\Rightarrow x = x',\ y = y',\ z = z'$.

Thus the representation of $\vec{r}$ is unique.

1.5.11 Note

(i) $\vec{r}$ is called the resultant of the vectors $x\hat{a},\ y\hat{b}$ and $z\hat{c}$, which are called components of $\vec{r}$. The numbers *x, y, z* are the cartesian coordinates of *P* w.r.t. the axes *OX, OY* and *OZ.*

(ii) If *OX, OY* and *OZ* are mutually perpendicular, then $\hat{a} = \hat{i}, \hat{b} = \hat{j}$ and $\hat{c} = \hat{k}$. Thus $\vec{r} = x\hat{a} + y\hat{b} + z\hat{c} = x\hat{i} + y\hat{j} + z\hat{k}$ in the position vector of $\vec{r}$ w.r.t .'0'.

In this case $x\hat{i},\ y\hat{j},\ z\hat{k}$ are rectangular components of $\vec{r}$ and *x, y, z* are rectangular cartesian coordinates of *P* as discussed in 1.4.10.

(iii) The above theorem enables us to add two or more vectors by adding their components.

Thus if $\vec{r}_1 = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$ and $\vec{r}_2 = x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$

then $\vec{r}_1 + \vec{r}_2 = (x_1 + x_2)\vec{a} + (y_1 + y_2)\vec{b} + (z_1 + z_2)\vec{c}$.

Similar result applies to subtraction and scalar multiplication.

1.5.12 Example

Show that the points *A, B* and *C* whose position vectors are, $2\hat{i} + 4\hat{j} - \hat{k},\ 4\hat{i} + 5\hat{j} + \hat{k}$ and $3\hat{i} + 6\hat{j} - 3\hat{k}$ form the vertices of a right angled triangle.

Solution:

Let *O* be the origin of reference.

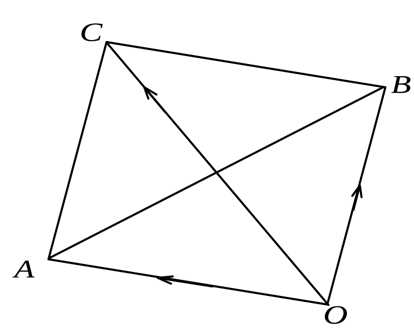

Fig: 1.17

From Fig. 1.17,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \left(4\hat{i} + 5\hat{j} + \hat{k}\right) - \left(2\hat{i} + 4\hat{j} - \hat{k}\right)$$

$$= 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{4+1+4} = 3.$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= \left(3\hat{i} + 6\hat{j} - 3\hat{k}\right) - \left(4\hat{i} + 5\hat{j} + \hat{k}\right)$$

$$= -\hat{i} + \hat{j} - 4\hat{k}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1+1+16} = 3\sqrt{2}.$$

$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$$

$$= \left(2\hat{i} + 4\hat{j} - \hat{k}\right) - \left(3\hat{i} + 6\hat{j} - 3\hat{k}\right)$$

$$= -\hat{i} - 2\hat{j} + 2\hat{k}$$

$$\left|\overrightarrow{CA}\right| = \sqrt{1+4+4} = 3.$$

We see that $BC^2 = 18 = 9 + 9 = AB^2 + CA^2$.

By Pythagoras theorem, *ABC* is a right angled triangle with right angle at *A*.

1.5.13 Example

If $\vec{a}, \vec{b}$ are the vectors determined by the two adjacent sides of a regular hexagon, then find the vectors determined by the other sides taken in order.

Solution:

Let $\vec{a}, \vec{b}$ represent the adjacent sides of $\overrightarrow{AB}, \overrightarrow{BC}$ respectively of the regular hexagon *ABCDEF* (Fig. 1.18).

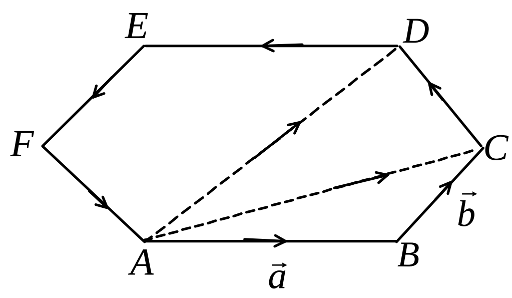

Fig: 1.18

Then $\overline{AC} = \overline{AB} + \overline{BC} = \vec{a} + \vec{b}$

$\overline{AD} = 2\overline{BC} = 2\vec{b}$

$\overline{CD} = \overline{AD} - \overline{AC} = 2\vec{b} - \left(\vec{a} + \vec{b}\right) = \vec{b} - \vec{a}$

$\overline{DE} = -\overline{BA} = -\vec{a}$

$\overline{EF} = \overline{CB} = -\overline{BC} = -\vec{b}$

$\overline{FA} = \overline{DC} = -\overline{CD} = -\left(\vec{b} - \vec{a}\right) = \vec{a} - \vec{b}$.

Hence the othersides of the hexagon are represented by $\vec{b} - \vec{a}, -\vec{a}, -\vec{b}, \vec{a} - \vec{b}$ respectively.

1.5.14 Example

Show that the points whose position vectors are $-2\vec{a} + 3\vec{b} + 5\vec{c}$, $\vec{a} + 2\vec{b} + 3\vec{c}$ and $7\vec{a} - \vec{c}$ are collinear.

Solution:

Let O be the origin of reference and A, B, C be the given points.

$\overline{AB} = \overline{OB} - \overline{OA}$

$= \left(\vec{a} + 2\vec{b} + 3\vec{c}\right) - \left(-2\vec{a} + 3\vec{b} + 5\vec{c}\right)$

$= 3\vec{a} - \vec{b} - 2\vec{c}$

$\overline{AC} = \overline{OC} - \overline{OA}$

$= \left(7\vec{a} - \vec{c}\right) - \left(-2\vec{a} + 3\vec{b} + 5\vec{c}\right) = 9\vec{a} - 3\vec{b} - 6\vec{c} = 3\left(3\vec{a} - \vec{b} - 2\vec{c}\right) = 3\overline{AB}$

$\therefore \overrightarrow{AB}, \overrightarrow{AC}$ are collinear.

$\therefore$ The points *A*, *B*, *C* are collinear.

1.5.15 Example

If $2\hat{i}+4\hat{j}-5\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$ are sides of a parallelogram, then find the unit vectors parallel to the diagonals.

Solution:

Let *ABCD* be a parallelogram with sides $\overrightarrow{AB}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\overrightarrow{BC}=\hat{i}+2\hat{j}+3\hat{k}$.

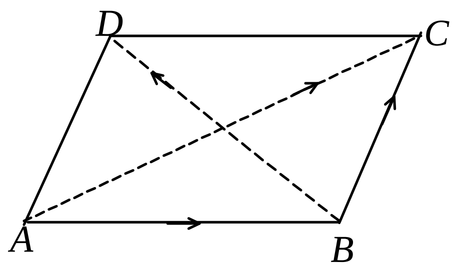

Fig: 1.19

Then from Fig. 1.19, $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$

$$=\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$

$$=3\hat{i}+6\hat{j}-2\hat{k}$$

$$\left|\overrightarrow{AC}\right|=\sqrt{9+36+4}=7.$$

Unit vector along $\overrightarrow{AC}=\dfrac{\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|}=\dfrac{3\hat{i}+6\hat{j}-2\hat{k}}{7}$.

$$\overrightarrow{BD}=\overrightarrow{BC}+\overrightarrow{CD}=\overrightarrow{BC}-\overrightarrow{AB}$$

$$=\left(\hat{i}+2\hat{j}+3\hat{k}\right)-\left(2\hat{i}+4\hat{j}-5\hat{k}\right)=-\hat{i}-2\hat{j}+8\hat{k}$$

$$\left|\overrightarrow{BD}\right|=\sqrt{1+4+64}=\sqrt{69}.$$

Unit vector along $\overrightarrow{BD}=\dfrac{\overrightarrow{BD}}{\left|\overrightarrow{BD}\right|}=\dfrac{-\hat{i}-2\hat{j}+8\hat{k}}{\sqrt{69}}$.

1.6 DOT PRODUCT OF SCALAR PRODUCT OF VECTORS

In previous sections you have learnt the basic operations of addition, subtraction and scalar multiplication of vectors. Here, we discuss about multiplication of vectors. This can be done in two different forms one resulting in a scalar and the other resulting in a vector. The present section focuses on the scalar product of two vectors.

1.6.1 Definition

If $\vec{a}, \vec{b}$ are two vectors, the **dot product or scalar product** of $\vec{a}$ and $\vec{b}$ denoted by $\vec{a}.\vec{b}$ is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta,\ 0 \le \theta \le \pi$, where θ is the smallest, non-negative angle between the vectors $\vec{a}$ and $\vec{b}$.

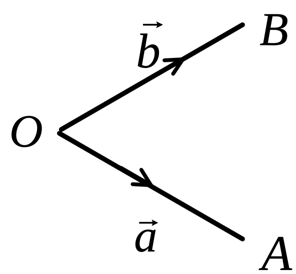

Fig: 1.20

1.6.2 Note

(i) By definition, the scalar product of any vector with zero vector is, a scalar zero.

(ii) The dot product of two vectors is a scalar.

(iii) The scalar product of two vectors is proportional to the length of each vector.

(iv) If θ is an acute angle, $\vec{a}.\vec{b}$ is positive and if θ is an obtuse angle, $\vec{a}.\vec{b}$ is negative.

1.6.3 Properties of Dot Product

(i) The dot product of two vectors is commutative.

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \ = |\vec{b}||\vec{a}|\cos\theta \ = \vec{b}.\vec{a}\ .$$

(ii) If m is a scalar, $m > 0$

$$(m\vec{a}).\vec{b} = |m\vec{a}||\vec{b}|\cos\theta \ = m|\vec{a}||\vec{b}|\cos\theta \ = m\left(\vec{a}.\vec{b}\right).$$

(iii) If *m, n* are scalars,

$$(m\vec{a}).\left(n\vec{b}\right)=m\left(\vec{a}.\left(n\vec{b}\right)\right)$$

$$=m\left(\left(n\vec{b}\right).\vec{a}\right) \quad \text{by (i)}$$

$$=m\left(n\left(\vec{b}.\vec{a}\right)\right) \quad \text{by (ii)}$$

$$=(mn)\left(\vec{b}.\vec{a}\right)=(mn)\left(\vec{a}.\vec{b}\right).$$

1.6.4 Condition for Orthogonality

Let $\vec{a}, \vec{b}$ be two non-zero vectors perpendicular to each other.

Then the angle θ between them is $\frac{\pi}{2}$.

$\therefore \cos\theta = 0$ and $\vec{a}.\vec{b}=|\vec{a}|\left|\vec{b}\right|\cos\theta = 0$.

Conversely, if $\vec{a}, \vec{b}$ are two non-zero vectors,

$\vec{a}.\vec{b}=0 \Rightarrow |\vec{a}|\left|\vec{b}\right|\cos\theta = 0 \Rightarrow \cos\theta = 0 \quad (\because |\vec{a}| \neq 0, \left|\vec{b}\right| \neq 0)$

$\Rightarrow \theta = \frac{\pi}{2}$.

Thus $\vec{a}, \vec{b}$ are orthogonal (perpendicular) to each other.

We know that $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the three mutually perpendicular axes. From above discussion, we have

$\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0$ ...... (i)

Also, by definition of dot product, $\hat{i}.\hat{i}=\left|\hat{i}\right|\left|\hat{i}\right|\cos\theta = 1.1.1 = 1$.

Similarly, $\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$ ...... (ii)

Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ be any vector.

Then $\vec{r}.\hat{i}=x\hat{i}.\hat{i}+y\hat{j}.\hat{i}+z\hat{k}.\hat{i}$

$$= x.1 + y.0 + z.0 \qquad \text{[by using (i) \& (ii)]}$$

$$= x.$$

$$\vec{r}.\hat{j} = x\hat{i}.\hat{j} + y\hat{j}.\hat{j} + z\hat{k}.\hat{j}$$

$$= x.0 + y.1 + z.0 = y.$$

$$\vec{r}.\hat{k} = x\hat{i}.\hat{k} + y\hat{j}.\hat{k} + z\hat{k}.\hat{k}$$

$$= x.0 + y.0 + z.1 = z.$$

Thus, $\vec{r}.\hat{i}, \vec{r}.\hat{j}, \vec{r}.\hat{k}$ are the *x, y, z* components or rectangular components of $\vec{r}$.

$$\therefore \ \vec{r} = \left(\vec{r}.\hat{i}\right)\hat{i} + \left(\vec{r}.\hat{j}\right)\hat{j} + \left(\vec{r}.\hat{k}\right)\hat{k}.$$

1.6.5 Note

If $\vec{a}, \vec{b}$ are parallel vectors, then

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos 0^0 = |\vec{a}||\vec{b}|$, if $\vec{a}, \vec{b}$ have same direction.

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos \pi = -|\vec{a}||\vec{b}|$, if $\vec{a}, \vec{b}$ have opposite direction.

1.6.6 Length of a Vector

If $\vec{a}$ is a non-zero vector, $\vec{a}.\vec{a} = |\vec{a}||\vec{a}|\cos 0 = |\vec{a}|^2$

$\therefore$ Length of $\vec{a} = |\vec{a}| = \sqrt{\vec{a}.\vec{a}}$.

1.6.7 Geometrical interpretation of scalar product

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Draw *BM* perpendicular to *OA* and *AN* perpendicular to *OB* (Fig. 1.21).

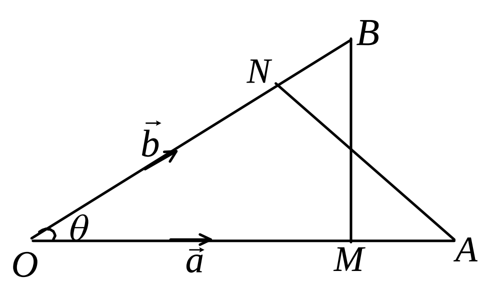

Fig: 1.21

Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

From ΔOMB,

$$\cos\theta = \frac{OM}{OB} = \frac{OM}{\left|\overrightarrow{OB}\right|} = \frac{OM}{\left|\vec{b}\right|}$$

$$\Rightarrow OM = \left|\vec{b}\right|\cos\theta$$

$\therefore$ Projection of $\vec{b}$ on $\vec{a}$ = projection of $\overrightarrow{OB}$ on $\overrightarrow{OA}$

$$= OM = \left|\vec{b}\right|\cos\theta = \frac{\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta}{\left|\vec{a}\right|}$$

$$= \frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|} \qquad \text{...... (1)}$$

Similarly, from ΔONA,

$$\cos\theta = \frac{ON}{OA} = \frac{ON}{\left|\overrightarrow{OA}\right|} = \frac{ON}{\left|\vec{a}\right|}$$

$$\Rightarrow ON = \left|\vec{a}\right|\cos\theta$$

$\therefore$ Projection of $\vec{a}$ on $\vec{b}$ = projection of $\overrightarrow{OA}$ on $\overrightarrow{OB}$

$$= ON = \left|\vec{a}\right|\cos\theta = \frac{\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta}{\left|\vec{b}\right|}$$

$$= \frac{\vec{a}.\vec{b}}{\left|\vec{b}\right|} \qquad \text{...... (2)}$$

From (1), $\vec{a}.\vec{b} = \left|\vec{a}\right|$ (projection of $\vec{b}$ on $\vec{a}$)

= (magnitude of $\vec{a}$) (projection of $\vec{b}$ on $\vec{a}$)

From (2), $\vec{a}.\vec{b} = \left|\vec{b}\right|$ (projection of $\vec{a}$ on $\vec{b}$)

= (magnitude of $\vec{b}$) (projection of $\vec{a}$ on $\vec{b}$)

Thus, geometrically interpreted, the scalar product of two vectors is the product of the magnitude of one vector and the projection of the other in its direction.

1.6.8 Distributive Property of Scalar Product

The scalar product of vectors is distributive over addition of vectors.

For any three vectors $\vec{a}, \vec{b}, \vec{c}$

(i) $\vec{a}.\left(\vec{b}+\vec{c}\right)=\vec{a}.\vec{b}+\vec{a}.\vec{c}$

(ii) $\left(\vec{b}+\vec{c}\right).\vec{a}=\vec{b}.\vec{a}+\vec{c}.\vec{a}$

Proof: Let $\vec{a}, \vec{b}, \vec{c}$ are three vectors represented by $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ respectively (Fig. 1.22)

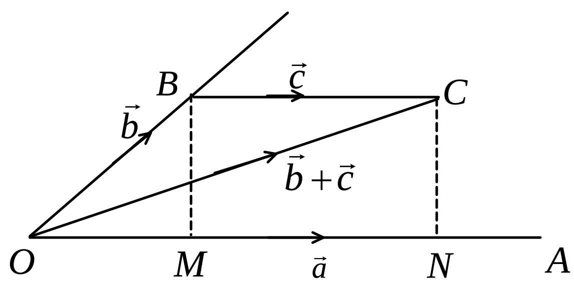

Fig: 1.22

Draw *BM* and *CN* perpendicular to *OA*.

Then *OM, MN, ON* are projections of $\vec{b}, \vec{c}$ and $\vec{b}+\vec{c}$ respectively on $\vec{a}$.

By using 1.6.7,

$\vec{a}.\vec{b}=|\vec{a}|(OM), \vec{a}.\vec{c}=|\vec{a}|(MN)$, and

$\vec{a}.\left(\vec{b}+\vec{c}\right)=|\vec{a}|(ON) \ =|\vec{a}|(OM+MN)$

$$=|\vec{a}|.OM+|\vec{a}|.MN \ =\vec{a}.\vec{b}+\vec{a}.\vec{c}\ .$$

Hence (i) is proved.

Now, $\left(\vec{b}+\vec{c}\right).\vec{a}=\vec{a}.\left(\vec{b}+\vec{c}\right)$ (by commutativity)

$$=\vec{a}.\vec{b}+\vec{a}.\vec{c}$$

$$=\vec{b}.\vec{a}+\vec{c}.\vec{a}$$ (by commutativity)

Hence (ii) is proved.

Using distributive property, the following identities can be obtained.

(i) $\left(\vec{a}+\vec{b}\right)^2 = |\vec{a}|^2 + 2\vec{a}.\vec{b} + \left|\vec{b}\right|^2$

Proof: $\left(\vec{a}+\vec{b}\right)^2 = \left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right)$

$$= \vec{a}.\left(\vec{a}+\vec{b}\right) + \vec{b}.\left(\vec{a}+\vec{b}\right) \ = \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b}$$

$$= |\vec{a}|^2 + \vec{a}.\vec{b} + \vec{a}.\vec{b} + \left|\vec{b}\right|^2 \ = |\vec{a}|^2 + 2\vec{a}.\vec{b} + \left|\vec{b}\right|^2 .$$

Similarly, $\left(\vec{a}-\vec{b}\right)^2 = |\vec{a}|^2 - 2\vec{a}.\vec{b} + \left|\vec{b}\right|^2$

(ii) $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right) = \vec{a}.\left(\vec{a}-\vec{b}\right) + \vec{b}.\left(\vec{a}-\vec{b}\right)$

$$= \vec{a}.\vec{a} - \vec{a}.\vec{b} + \vec{b}.\vec{a} - \vec{b}.\vec{b}$$

$$= |\vec{a}|^2 - \left|\vec{b}\right|^2 \qquad \left(\because \vec{a}.\vec{b} = \vec{b}.\vec{a}\right)$$

(iii) $\frac{1}{4}\left[\left(\vec{a}+\vec{b}\right)^2 - \left(\vec{a}-\vec{b}\right)^2\right]$

$$= \frac{1}{4}\left[\left(|\vec{a}|^2 + 2\vec{a}.\vec{b} + \left|\vec{b}\right|^2\right) - \left(|\vec{a}|^2 - 2\vec{a}.\vec{b} + \left|\vec{b}\right|^2\right)\right]$$

$$= \frac{1}{4}\left(4\vec{a}.\vec{b}\right) \ = \vec{a}.\vec{b} .$$

1.6.9 Dot Product interms of Components

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

Then $\vec{a}.\vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right).\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$

$$= a_1\hat{i}.\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) + a_2\hat{j}.\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) + a_3\hat{k}.\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

$$= a_1b_1\left(\hat{i}.\hat{i}\right) + a_1b_2\left(\hat{i}.\hat{j}\right) + a_1b_3\left(\hat{i}.\hat{k}\right) + a_2b_1\left(\hat{j}.\hat{i}\right) + a_2b_2\left(\hat{j}.\hat{j}\right) + a_2b_3\left(\hat{j}.\hat{k}\right)$$

$$+ a_3b_1\left(\hat{k}.\hat{i}\right) + a_3b_2\left(\hat{k}.\hat{j}\right) + a_3b_3\left(\hat{k}.\hat{k}\right)$$

$$= a_1b_1 + a_2b_2 + a_3b_3 \text{ since } \hat{i}.\hat{i} + \hat{j}.\hat{j} + \hat{k}.\hat{k} = 1 \text{ and } \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0 .$$

Also $\vec{a}.\vec{a}=\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right).\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)$

$$=(a_1)(a_1)+(a_2)(a_2)+(a_3)(a_3)$$

$$=a_1^2+a_2^2+a_3^2$$

$$\Rightarrow |\vec{a}|^2=a_1^2+a_2^2+a_3^2$$

$$\Rightarrow |\vec{a}|=\sqrt{a_1^2+a_2^2+a_3^2}$$

1.6.10 Angle between Vectors

If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$ then $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

$$\therefore \cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \text{ or } \theta=\cos^{-1}\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}.$$

1.6.11 Example

Find the dot product of the vectors $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}+4\hat{j}+5\hat{k}$

Solution: $\vec{a}.\vec{b}=\left(\hat{i}+2\hat{j}+\hat{k}\right).\left(2\hat{i}+4\hat{j}+5\hat{k}\right)$

= (1) (2) + (2) (4) + (1) (5) = 2 + 8 + 5 = 15.

1.6.12 Example

For what value of *x*, the vectors $\vec{a}=2\hat{i}+x\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}-\hat{j}$ are perpendicular?

Solution: If $\vec{a},\vec{b}$ are perpendicular, then $\vec{a}.\vec{b}=0$

$$\Rightarrow \left(2\hat{i}+x\hat{j}+k\right).\left(2\hat{i}-\hat{j}+0\hat{k}\right)=0$$

$$\Rightarrow (2)(2)+(x)(-1)+(1)(0)=0$$

$$\Rightarrow 4-x=0 \;\Rightarrow x=4.$$

1.6.13 Example

Find the angle between $\vec{a}=2\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}=6\hat{i}-3\hat{j}+2\hat{k}$.

Solution: If θ is the angle between the vectors, then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|} = \frac{\left(2\hat{i}+2\hat{j}-\hat{k}\right).\left(6\hat{i}-3\hat{j}+2\hat{k}\right)}{\sqrt{(2)^2+(2)^2+(-1)^2}\sqrt{(6)^2+(-3)^2+(2)^2}}$$

$$= \frac{12-6-2}{\sqrt{4+4+1}\sqrt{36+9+4}} = \frac{4}{(3)(7)} = \frac{4}{21}.$$

$$\therefore \theta = \cos^{-1}\left(\frac{4}{21}\right).$$

1.6.14 Example

If $\vec{a},\vec{b},\vec{c}$ are 3 vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}, |\vec{a}|=3, |\vec{b}|=5, |\vec{c}|=7$ then find the angle between $\vec{a}$ and $\vec{b}$.

Solution: Given $\vec{a}+\vec{b}+\vec{c}=0$

$$\Rightarrow \vec{a}+\vec{b}=-\vec{c}$$

$$\Rightarrow \left(\vec{a}+\vec{b}\right)^2 = (-\vec{c})^2$$

$$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + \left|\vec{b}\right|^2 = (\vec{c})^2 = |\vec{c}|^2$$

$$\Rightarrow 9 + 2|\vec{a}|\left|\vec{b}\right|\cos\theta + 25 = 49$$

$$\Rightarrow 2\times3\times5\cos\theta = 15$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^0$$

$\therefore$ Angle between $\vec{a}$ and $\vec{b}$ is 60^0.

1.6.15 Example

Find the projection of the vector $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$ on the vector $\vec{b}=4\hat{i}-4\hat{j}+7\hat{k}$.

Solution: Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{\left|\vec{b}\right|}$

$$= \frac{\left(\hat{i}-2\hat{j}+\hat{k}\right).\left(4\hat{i}-4\hat{j}+7\hat{k}\right)}{\sqrt{4^2+(-4)^2+(7)^2}} = \frac{4+8+7}{\sqrt{81}} = \frac{19}{9}.$$

1.6.16 Example

Find a unit vector perpendicular to the plane of $\vec{a} = 2\hat{i} - 6\hat{j} - 3\hat{k}$ and $\vec{b} = 4\hat{i} + 3\hat{j} - \hat{k}$.

Solution: Let $\vec{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ be a vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

Then $\vec{a}.\vec{x} = 0$ and $\vec{b}.\vec{x} = 0$

$$\Rightarrow \left(2\hat{i} - 6\hat{j} - 3\hat{k}\right).\left(x_1\hat{i} + x_2\hat{j} + x_3\hat{k}\right) = 0 \text{ and } \left(4\hat{i} + 3\hat{j} - \hat{k}\right).\left(x_1\hat{i} + x_2\hat{j} + x_3\hat{k}\right) = 0$$

$$\Rightarrow 2x_1 - 6x_2 - 3x_3 = 0 \text{ and } 4x_1 + 3x_2 - x_3 = 0$$

$$\Rightarrow \frac{x_1}{6+9} = \frac{x_2}{-12+2} = \frac{x_3}{6+24} \Rightarrow \frac{x_1}{15} = \frac{x_2}{-10} = \frac{x_3}{30} \Rightarrow \frac{x_1}{3} = \frac{x_2}{-2} = \frac{x_3}{6}$$

$\therefore \vec{x} = 3\hat{i} - 2\hat{j} + 6\hat{k}$.

A unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is:

$$\frac{\vec{x}}{|\vec{x}|} = \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{9+4+36}} = \frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}.$$

1.6.17 Application of Dot Product

Dot product can be used to find the workdone by a force.

Let $\overrightarrow{OA} = \vec{F}$ be a force acting on a particle at *O*.

Suppose the displacement of the particle from *O* to *B* is given by $\overrightarrow{OB} = \vec{d}$.

Draw *BM* perpendicular to *OA* from *B* (Fig. 1.23).

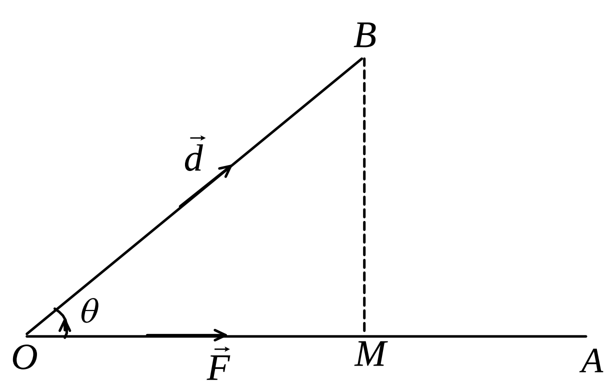

Fig: 1.23

The displacement in the direction of the force $= OM = OB\cos\theta = \left|\vec{d}\right|\cos\theta$.

$\therefore$ work done by the force

= W = (magnitude of force in the direction of motion) (Distance moved)

$= \left|\vec{F}\right|(OM) = \left|\vec{F}\right|\left|\vec{d}\right|\cos\theta = \vec{F}.\vec{d}$.

1.6.18 Example

By applying a force $\vec{F} = 6\hat{j} + 8\hat{k}$, a particle changes its position from $A(1, -1, 2)$ to $B(-1, 1, 2)$. Find the work done by the force.

Solution: Distance covered in moving from A to B is:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \left(-\hat{i} + \hat{j} + 2\hat{k}\right) - \left(\hat{i} - \hat{j} + 2\hat{k}\right)$$

$$= -2\hat{i} + 2\hat{j}$$

Work done = $\vec{F}.\vec{d} = \left(6\hat{j} + 8\hat{k}\right).\left(-2\hat{i} + 2\hat{j}\right)$

= (6) (2) = 12 units.

Check Your Progress:

4. Find a vector on the xy - plane perpendicular to $6\hat{i} + 4\hat{j} + 3\hat{k}$ and equal in length.

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5. Show that the vectors $\hat{i} + 2\hat{j} + \hat{k}$, $\hat{i} + \hat{j} - 3\hat{k}$ and $7\hat{i} - 4\hat{j} + \hat{k}$ are mutually perpendicular.

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1.7 CROSS PRODUCT OR VECTOR PRODUCT OF VECTORS

In this section we learn to multiply two vectors in such a way that the product is again a vector. We also study various properties of this product and their application in geometrical and physical problems.

1.7.1 Definition

The **vector product or cross product** of two vectors $\vec{a}$ and $\vec{b}$ is denoted by and defined as $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n},\ 0\le\theta\le\pi$ where θ is the angle between $\vec{a}$ and $\vec{b}$, and $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

The direction of $\hat{n}$ coincides with that, in which a right handed screw moves when it is rotated from $\vec{a}$ to $\vec{b}$.

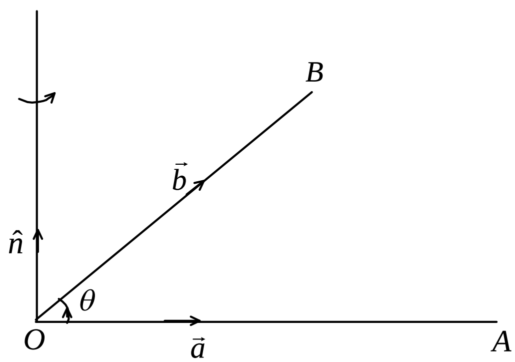

Fig: 1.24

1.7.2 Note

(i) By definition, the cross product of any vector with zero vector is the zero vector.

(ii) The cross product is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

(iii) The cross product of two vectors is proportional to the length of each vector.

1.7.3 Properties of Cross Product

(i) The cross product of two vectors is not commutative.

$$\vec{b}\times\vec{a}=|\vec{b}||\vec{a}|\sin(-\theta)\hat{n}\ =-|\vec{a}||\vec{b}|\sin\theta\,\hat{n}\ =-\left(\vec{a}\times\vec{b}\right).$$

(ii) If $m>0$ is a scalar, then

$$(m\vec{a})\times\vec{b}=|m\vec{a}||\vec{b}|\sin\theta\,\hat{n}\ =m|\vec{a}||\vec{b}|\sin\theta\,\hat{n}\ =m\left(\vec{a}\times\vec{b}\right)$$

Similarly, when $m \le 0$, we can show that

$$\left(m\vec{a}\right)\times\vec{b} = m\left(\vec{a}\times\vec{b}\right)$$

(iii) If *m, n* are scalars, then

$$\left(m\vec{a}\right)\times\left(n\vec{b}\right) = m\left[\vec{a}\times\left(n\vec{b}\right)\right]$$

$$= m\left[-\left(n\vec{b}\times\vec{a}\right)\right] \qquad \text{[by (i)]}$$

$$= -m\left[n\left(\vec{b}\times\vec{a}\right)\right] \qquad \text{[by (ii)]}$$

$$= -\left(mn\right)\left(\vec{b}\times\vec{a}\right) \;\; = mn\left(\vec{a}\times\vec{b}\right)$$

1.7.4 Condition for Parallelism

Let $\vec{a}, \vec{b}$ be two non-zero vectors.

If $\vec{a}, \vec{b}$ are collinear, then $\theta = 0$ and if they are parallel, then $\theta = \pi$.

So $\sin\theta = \sin 0 = 0$ or $\sin\pi = 0$.

$\therefore \vec{a}\times\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|\sin\theta.\hat{n} = \vec{0}$.

Conversely, if $\vec{a}, \vec{b}$ are non-zero vectors and $\vec{a}\times\vec{b} = \vec{0}$,

then $\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta.\hat{n} = \vec{0} \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$ or π.

$\therefore \vec{a}, \vec{b}$ are either collinear or parallel.

In particular, $\vec{a}\times\vec{a} = \vec{0}$.

Also, for the unit vectors $\hat{i}, \hat{j}, \hat{k}$; $\hat{i}\times\hat{i} = \vec{0}$, $\hat{j}\times\hat{j} = \vec{0}$, $\hat{k}\times\hat{k} = \vec{0}$.

For the unit vectors $\hat{i}, \hat{j}, \hat{k}$, we have

$\hat{i}\times\hat{j} = \left|\hat{i}\right|\left|\hat{j}\right|\sin\frac{\pi}{2}\hat{k}$ (since $\hat{i}, \hat{j}, \hat{k}$ are perpendicular to one other, $\theta = \frac{\pi}{2}$ and $\hat{n} = \hat{k}$).

$= 1.1.1.\hat{k} = \hat{k}$.

Also, $\hat{j} \times \hat{i} = -\left(\hat{i} \times \hat{j}\right) = -\hat{k}$.

Similarly, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{j} = -\hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$, $\hat{i} \times \hat{k} = -\hat{j}$.

1.7.5 Distributive Law

The cross product is distributive over the addition of vectors. i.e., for any three vectors $\vec{a}, \vec{b}, \vec{c}$; $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Proof: Let $\vec{p} = \vec{a} \times \left(\vec{b} + \vec{c}\right) - \vec{a} \times \vec{b} - \vec{a} \times \vec{c}$.

Let $\vec{q}$ be any arbitrarily chosen vector.

$$\vec{q}.\vec{p} = \vec{q}.\left[\vec{a} \times \left(\vec{b} + \vec{c}\right) - \vec{a} \times \vec{b} - \vec{a} \times \vec{c}\right]$$

$$= \vec{q}.\left(\vec{a} \times \left(\vec{b} + \vec{c}\right)\right) - \vec{q}\left(\vec{a} \times \vec{b}\right) - \vec{q}\left(\vec{a} \times \vec{c}\right)$$

Here we use the property, $\vec{a}.\left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right).\vec{c}$ which shall be proved later.

$$\therefore \vec{q}.\vec{p} = \left(\vec{q} \times \vec{a}\right).\left(\vec{b} + \vec{c}\right) - \left(\vec{q} \times \vec{a}\right).\vec{b} - \left(\vec{q} \times \vec{a}\right).\vec{c}$$

$$= \left(\vec{q} \times \vec{a}\right).\vec{b} + \left(\vec{q} \times \vec{a}\right).\vec{c} - \left(\vec{q} \times \vec{a}\right).\vec{b} - \left(\vec{q} \times \vec{a}\right).\vec{c}$$

(by using distributive law for scalar product)

$$= \vec{0} \cdot$$

$\Rightarrow \vec{q} = \vec{0}, \vec{p} = \vec{0}$ or $\vec{q}$ is perpendicular to $\vec{p}$.

But, by choice, $\vec{q}$ is arbitrary. Hence we can choose $\vec{q}$ so that $\vec{q} \neq \vec{0}$ and $\vec{q}$ is not perpendicular to $\vec{p}$.

$\therefore \vec{p} = \vec{0}$.

i.e., $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

1.7.6 Cross Product in terms of Components

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

$$\vec{a} \times \vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) \times \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

$$= a_1\hat{i}\times\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)+a_2\hat{j}\times\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)+a_3\hat{k}\times\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)$$

(by using distributive law)

$$= a_1b_1\hat{i}\times\hat{i}+a_1b_2\hat{i}\times\hat{j}+a_1b_3\hat{i}\times\hat{k}+a_2b_1\hat{j}\times\hat{i}+a_2b_2\hat{j}\times\hat{j}+a_2b_3\hat{j}\times\hat{k}$$

$$+a_3b_1\hat{k}\times\hat{i}+a_3b_2\hat{k}\times\hat{j}+a_3b_3\hat{k}\times\hat{k}$$

$$= a_1b_1.\vec{0}+a_1b_2\hat{k}-a_1b_3\hat{j}+a_2b_1\left(-\hat{k}\right)+a_2b_2.\vec{0}+a_2b_3\hat{i}+a_3b_1\hat{j}-a_3b_2\hat{i}+a_3b_3.\vec{0}$$

$$= \left(a_2b_3-a_3b_2\right)\hat{i}-\left(a_1b_3-a_3b_1\right)\hat{j}+\left(a_1b_2-a_2b_1\right)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

1.7.7 Example

If $\vec{a}=-2\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}=3\hat{i}+6\hat{j}+2\hat{k}$, find $\vec{a}\times\vec{b}$.

Solution: $\vec{a}\times\vec{b}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & -1 \\ 3 & 6 & 2 \end{vmatrix}$

$$= \hat{i}\left(4+6\right)-\hat{j}\left(-4+3\right)+\hat{k}\left(-12-6\right)=10\hat{i}+\hat{j}-18\hat{k}.$$

1.7.8 Example

If $\vec{a}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}+\hat{k}$, then find a vector whose magnitude is 5 and perpendicular to both $\vec{a}$ and $\vec{b}$.

Solution: We know that $\vec{a}\times\vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

$$\vec{a}\times\vec{b}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}\left(-1-1\right)-\hat{j}\left(1-1\right)+\hat{k}\left(1+1\right)=-2\hat{i}+2\hat{k}.$$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{\left(-2\right)^2+\left(2\right)^2}=\sqrt{8}=2\sqrt{2}.$$

$\therefore$ A unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is

$$\frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}=\frac{-2\hat{i}+2\hat{k}}{2\sqrt{2}}=\frac{1}{\sqrt{2}}\left(-\hat{i}+\hat{k}\right).$$

$\therefore$ A vector perpendicular $\vec{a}$ and $\vec{b}$ and has magnitude 5 is $\frac{5}{\sqrt{2}}\left(-\hat{i}+\hat{k}\right)$.

Check Your Progress:

6. Find a unit vector perpendicular to both $\vec{a}=2\hat{i}+\hat{j}-3\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$.

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1.7.9 Geometrical Interpretation of Cross Product

Let $\vec{a}, \vec{b}$ be two given vectors. Let $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$ and θ be the angle between $\vec{a}, \vec{b}$. $\hat{n}$ be a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ so that $\vec{a}, \vec{b}$, $\hat{n}$ forms a right handed system (Fig. 1.25).

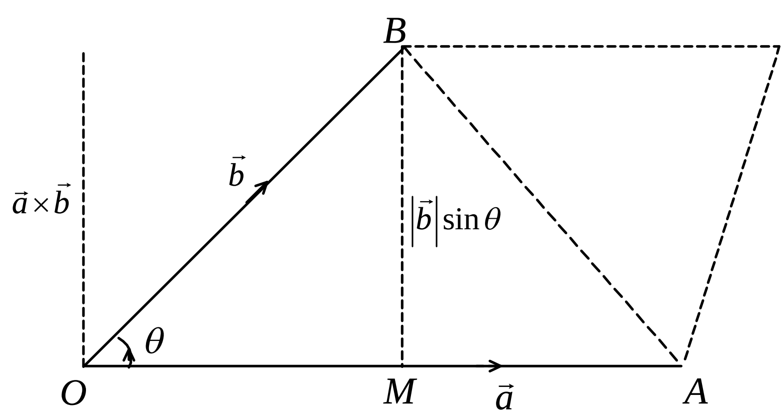

Fig: 1.25

Let *BM* be the perpendicular drawn from *B* to *OA*.

Then, $\sin\theta=\frac{BM}{OB}$

i.e., $BM=OB\sin\theta=\left|\overline{OB}\right|\sin\theta=\left|\vec{b}\right|\sin\theta$

Now $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

$$\left|\vec{a}\times\vec{b}\right|=|\vec{a}||\vec{b}|\sin\theta|\hat{n}|$$

$$=|\vec{a}||\vec{b}|\sin\theta\ \left(\because|\hat{n}|=1\right)$$

$$=2\times\frac{1}{2}|\vec{a}|\left(|\vec{b}|\sin\theta\right)$$

$$=2\left[\frac{1}{2}\times\text{base } OA\times\text{height } BM \text{ of } \Delta AOB\right]$$

= 2 x area of ΔAOB = Area of the parallelogram *OACB*.

Thus, $\left|\vec{a}\times\vec{b}\right|$ gives the area of the parallelogram *OACB* and $\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$ gives the area of the triangle *AOB*.

$\therefore\vec{a}\times\vec{b}$ and $\frac{1}{2}\left(\vec{a}\times\vec{b}\right)$ represent the vector areas of the parallelogram *OACB* and triangle *AOB* whose adjacent sides are $OA=\vec{a}$ and $OB=\vec{b}$.

1.7.10 Example

The position vectors of the vertices of a triangle are $\hat{i}+\hat{j}+2\hat{k}, 2\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-\hat{j}-\hat{k}$. Find the area of the triangle.

Solution: Let *A*, *B*, *C* be the vertices of the triangle where $\overrightarrow{OA}=\hat{i}+\hat{j}+2\hat{k},\ \overrightarrow{OB}=2\hat{i}+2\hat{j}+3\hat{k}$ and $\overrightarrow{OC}=3\hat{i}-\hat{j}-\hat{k}$.

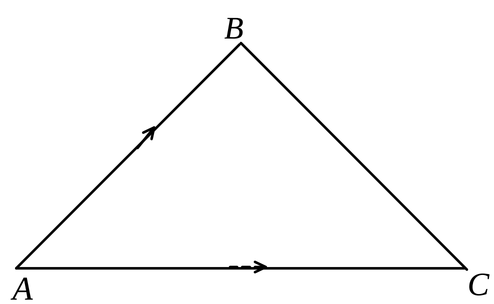

Fig: 1.26

$\therefore\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) - \left(\hat{i} + \hat{j} + 2\hat{k}\right) \;\; = \hat{i} + \hat{j} + \hat{k}$$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$$= \left(3\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} + \hat{j} + 2\hat{k}\right) \;\; = 2\hat{i} - 2\hat{j} - 3\hat{k}$$

$$\overrightarrow{AC} \times \overrightarrow{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}\left(-2+3\right) - \hat{j}\left(2+3\right) + \hat{k}\left(2+2\right) = \hat{i} - 5\hat{j} + 4\hat{k}$$

$$\left|\overrightarrow{AC} \times \overrightarrow{AB}\right| = \sqrt{1+25+16} = \sqrt{42}\ .$$

$\therefore$ Area of the triangle $ABC = \dfrac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|$

$$= \frac{1}{2}\sqrt{42} \text{ sq. units.}$$

1.7.11 Example

Find the area of the parallelogram whose diagonals are $\vec{a} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ and $\vec{b} = 3\hat{i} - 4\hat{j} - \hat{k}$.

Solution: Let $ABCD$ be a parallelogram whose diagonals are $\overrightarrow{AC} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ and $\overrightarrow{DB} = 3\hat{i} - 4\hat{j} - \hat{k}$.

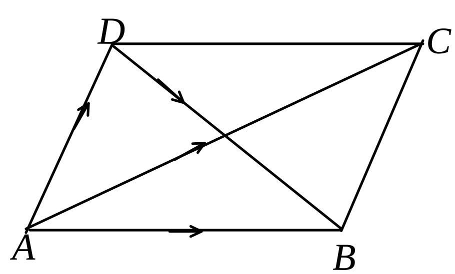

Fig: 1.27

By triangle law of addition,

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC} \text{ and } \overrightarrow{AD}+\overrightarrow{DB}=\overrightarrow{AB}$$

$$\therefore 2\overrightarrow{AB}=\overrightarrow{AB}+\overrightarrow{AB}=\left(\overrightarrow{AC}-\overrightarrow{BC}\right)+\left(\overrightarrow{AD}+\overrightarrow{DB}\right)$$

$$=\overrightarrow{AC}+\overrightarrow{DB}$$

$$=\left(2\hat{i}+3\hat{j}-6\hat{k}\right)+\left(3\hat{i}-4\hat{j}-\hat{k}\right) \ =5\hat{i}-\hat{j}-7\hat{k}.$$

Similarly, $2\overrightarrow{AD}=\overrightarrow{AD}+\overrightarrow{AD}=\overrightarrow{AD}+\overrightarrow{BC}$

$$=\left(\overrightarrow{AB}-\overrightarrow{DB}\right)+\left(\overrightarrow{AC}-\overrightarrow{AB}\right)$$

$$=\overrightarrow{AC}-\overrightarrow{DB}$$

$$=\left(2\hat{i}+3\hat{j}-6\hat{k}\right)-\left(3\hat{i}-4\hat{j}-\hat{k}\right) \ =-\hat{i}+7\hat{j}-5\hat{k}$$

$$\therefore \overrightarrow{AB}=\frac{1}{2}\left(5\hat{i}-\hat{j}-7\hat{k}\right),\overrightarrow{AD}=\frac{1}{2}\left(-\hat{i}+7\hat{j}-5\hat{k}\right).$$

$\therefore$ Vector area of the parallelogram $=\overrightarrow{AB}\times\overrightarrow{AD}$

$$=\frac{1}{2}.\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & -1 & -7\\ -1 & 7 & -5\end{vmatrix}$$

$$=\frac{1}{4}\left[\hat{i}\left(5+49\right)-\hat{j}\left(-25-7\right)+\hat{k}\left(35-1\right)\right]$$

$$=\frac{1}{4}\left[54\hat{i}+32\hat{j}+34\hat{k}\right]$$

$$=\frac{1}{2}\left(27\hat{i}+16\hat{j}+17\hat{k}\right)$$

$\therefore$ Area of the parallelogram $=\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|$

$$=\frac{1}{2}\sqrt{\left(27\right)^2+\left(16\right)^2+\left(17\right)^2} \ =\frac{1}{2}\sqrt{1274}.$$

Check Your Progress:

7. Find the area of the parallelogram whose adjacent sides are $\hat{i}+\hat{j}+7\hat{k}$ and $-\hat{i}-2\hat{j}+4\hat{k}$.

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1.7.12 Physical Interpretation of Cross Product

Let $\vec{F}$ be a force acting on a body at the point *P*. Let $\vec{r}$ be the position vector of *P* w.r.t the point '*O*'. The moment or torque $\vec{m}$ of the force $\vec{F}$ about the point *O* is given by $\vec{m}=\vec{r}\times\vec{F}$. Clearly, $\vec{m}$ is perpendicular to the plane containing '*O*' and the line of $\vec{F}$.

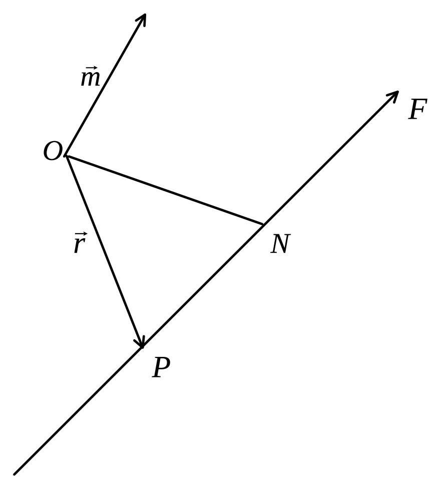

Fig: 1.28

1.7.13 Example

A force given by $\vec{F}=3\hat{i}+2\hat{j}-4\hat{k}$ is applied at the point (1, -1, 2). Find the moment of $\vec{F}$ about the point (2, -1, 3).

Solution: $\vec{r}=\overrightarrow{OP}=2\hat{i}-\hat{j}+3\hat{k}-\left(\hat{i}-\hat{j}+2\hat{k}\right)=\hat{i}+0\hat{j}+\hat{k}$

Moment of $\vec{F}=\vec{r}\times\vec{F}$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & 1\\ 3 & 2 & -4\end{vmatrix}=-2\hat{i}+7\hat{j}+2\hat{k}.$$

1.8 PRODUCT OF THREE VECTORS

The idea of multiplying two vectors can be extended to three or more vectors. In this section we discuss two ways of multiplying given three vectors.

1.8.1 Scalar Triple Product

Let $\vec{a},\vec{b},\vec{c}$ be three given vectors. The scalar triple product of $\vec{a},\vec{b},\vec{c}$ is denoted by $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ and defined as $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\vec{a}.\left(\vec{b}\times\vec{c}\right)$.

This product is known as box product also. Note that it is a scalar quantity.

1.8.2 Properties of Scalar Triple Product

(i) Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$.

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\hat{i}\left(b_2c_3-b_3c_2\right)-\hat{j}\left(b_1c_3-b_3c_1\right)+\hat{k}\left(b_1c_2-b_2c_1\right)$$

$$\left[\vec{a}\quad\vec{b}\quad\vec{c}\right]=\vec{a}.\left(\vec{b}\times\vec{c}\right)$$

$$=\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right).\left(\left(b_2c_3-b_3c_2\right)\hat{i}+\left(b_3c_1-b_1c_3\right)\hat{j}+\left(b_1c_2-b_2c_1\right)\hat{k}\right)$$

$$=a_1\left(b_2c_3-b_3c_2\right)+a_2\left(b_3c_1-b_1c_3\right)+a_3\left(b_1c_2-b_2c_1\right)$$

$$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Similarly, we can show that $\left[\vec{b}\ \vec{c}\ \vec{a}\right]=\begin{vmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix}$ and $\left[\vec{c}\ \vec{a}\ \vec{b}\right]=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

We know that the value of all those determinants is same.

$$\therefore\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\left[\vec{b}\,\vec{c}\,\vec{a}\right]=\left[\vec{c}\,\vec{a}\,\vec{b}\right] \qquad \text{...... (1)}$$

Thus in a scalar triple product, vectors can be changed without changing the cyclic order.

Also, $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=-\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=-\left[\vec{a}\ \vec{c}\ \vec{b}\right].$

$\therefore$ Changing the cyclic order in scalar triple product changes the sign.

(ii) $\left[\hat{i}\quad \hat{j}\quad \hat{k}\right]=\hat{i}.\left(\hat{j}\times\hat{k}\right)=\hat{i}.\left(\hat{i}\right)=1$

$\left[\hat{j}\quad \hat{k}\quad \hat{i}\right]=\hat{j}.\left(\hat{k}\times\hat{i}\right)=\hat{j}.\hat{j}=1$

$\left[\hat{k}\quad \hat{i}\quad \hat{j}\right]=\hat{k}.\left(\hat{i}\times\hat{j}\right)=\hat{k}.\hat{k}=1$

(iii) By using (i),

$\left[\hat{i}\quad \hat{k}\quad \hat{j}\right]=-1$

$\left[\hat{k}\quad \hat{j}\quad \hat{i}\right]=\left[\hat{j}\quad \hat{i}\quad \hat{k}\right]=-1.$

(iv) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then

$$\vec{a}\times\vec{b}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$=\hat{i}\left(a_2b_3-a_3b_2\right)-\hat{j}\left(a_1b_3-a_3b_1\right)+\hat{k}\left(a_1b_2-a_2b_1\right)$$

$$\left(\vec{a}\times\vec{b}\right).\vec{c}=\left[\hat{i}\left(a_2b_3-a_3b_2\right)-\hat{j}\left(a_1b_3-a_3b_1\right)+\hat{k}\left(a_1b_2-a_2b_1\right)\right].\left[c_1\hat{i}+c_2\hat{j}+c_3\hat{k}\right]$$

$$=c_1\left(a_2b_3-a_3b_2\right)-c_2\left(a_1b_3-a_3b_1\right)+c_3\left(a_1b_2-a_2b_1\right)$$

$$=\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=-\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (-)(-)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix}$$

$$\therefore \vec{a}.\left(\vec{b} \times \vec{c}\right) = \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \left(\vec{a} \times \vec{b}\right).\vec{c} .$$

Thus in a scalar triple product, dot and cross can be interchanged. The product is independent of the positions of dot and cross.

(v) If any two vectors in a scalar triple product are equal, then the product is zero.

Suppose $\vec{a} = \vec{b}$.

Then $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \left(\vec{a} \times \vec{b}\right).\vec{c} = \left(\vec{a} \times \vec{a}\right).\vec{c} = 0.\vec{c} = 0$.

1.8.3 Geometrical Meaning of Scalar Triple Product

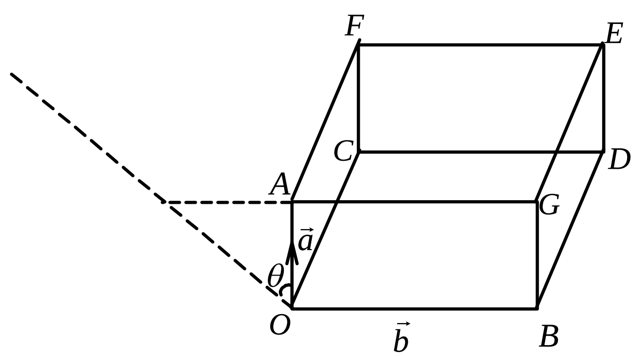

Fig: 1.29

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ form a right handed system. Consider a parallelopiped with *OA, OB, OC* as coterminus edges (Fig. 1.29). Then, $\vec{b} \times \vec{c}$ is perpendicular to the plane of $\vec{b}, \vec{c}$, i.e., $\vec{b} \times \vec{c}$ is perpendicular to the face *OBDC* of the parallelopiped. If θ is the angle between $\vec{a}$ and $\vec{b} \times \vec{c}$, then

$$\vec{a}.\left(\vec{b} \times \vec{c}\right) = |\vec{a}|\left|\vec{b} \times \vec{c}\right|\cos\theta = \left|\vec{b} \times \vec{c}\right||\vec{a}|\cos\theta$$

= (Area of the parallelogram *OBDC*) $\times$ (Height of the parallelopiped)

= Volume of the parallelopiped.

Hence, the scalar triple product $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix}$ represents the volume of the parallelopiped with $\vec{a}, \vec{b}, \vec{c}$ as coterminus edges.

1.8.4 Note

(i) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, then the volume of the parallelopiped is zero.

$$\therefore \left[\vec{a} \quad \vec{b} \quad \vec{c}\right] = 0$$

Thus a necessary and sufficient condition for $\vec{a}, \vec{b}, \vec{c}$ to be coplanar in that

$$\left[\vec{a} \quad \vec{b} \quad \vec{c}\right] = 0 .$$

(ii) If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of the points *A, B, C* w.r.t. *O*, then the volume of the tetrahedron *OABC* is given by $\frac{1}{6}\left[\vec{a} \quad \vec{b} \quad \vec{c}\right]$.

1.8.5 Example

If $\vec{a} = 3\hat{i} - \hat{j} - 2\hat{k}, \vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - 2\hat{k}$ then find $\vec{a}.\left(\vec{b} \times \vec{c}\right)$.

Solution: $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix}$

$$= \hat{i}\left(-2+3\right) - \hat{j}\left(-4+1\right) + \hat{k}\left(6-1\right) = \hat{i} + 3\hat{j} + 5\hat{k} .$$

$$\vec{a}.\left(\vec{b} \times \vec{c}\right) = \left(3\hat{i} - \hat{j} - 2\hat{k}\right).\left(\hat{i} + 3\hat{j} + 5\hat{k}\right)$$

$$= 3 - 3 - 10 = -10.$$

1.8.6 Example

Find the volume of the parallelopiped whose edges are represented by $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}, \vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$.

Solution: Volume of the parallelopiped is given by $\left[\vec{a} \quad \vec{b} \quad \vec{c}\right]$

But by (1.8.2), $\left[\vec{a} \quad \vec{b} \quad \vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\therefore$ Volume of the parallelopiped = $\begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

= 2(4-1) + 3(2+3) + 4(-1-6)

= 6 + 15 - 28 = -7.

The negative sign indicates that the given vectors are in a left handed system.

1.8.7 Example

Find a constant *c* such that the vectors $2\hat{i}-\hat{j}+\hat{k},\ \hat{i}+2\hat{j}-3\hat{k},\ 3\hat{i}+c\hat{j}+5\hat{k}$ are coplanar.

Solution: If three vectors $\vec{a},\vec{b},\vec{c}$ are coplanar, then their scalar triple product is zero.

Let $\vec{a}=2\hat{i}-\hat{j}+\hat{k},\ \vec{b}=\hat{i}+2\hat{j}-3\hat{k},\ \vec{c}=3\hat{i}+c\hat{j}+5\hat{k}$

$$\begin{bmatrix}\vec{a} & \vec{b} & \vec{c}\end{bmatrix}=\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & c & 5 \end{vmatrix}$$

= 2(10+3c) + 1(5+9) + 1(c-6) = 20 + 6c + 14 + c - 6= 7c + 28

$$\begin{bmatrix}\vec{a} & \vec{b} & \vec{c}\end{bmatrix}=0$$

$\Rightarrow 7c+28=0 \Rightarrow c=-4\cdot$

$\therefore$ The given vectors are coplanar if c = -4.

1.8.8 Example

Find the volume of the tetrahedron with vertices *A* (1, 1, -1), *B* (3, -2, 2), *C* (4, 3, 2) and *D* (5, 5, 3).

Solution: Let *ABC* be the base of the tetrahedron and *D* be the 4th vertex (Fig 1.30).

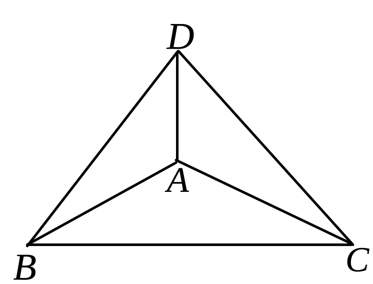

Fig: 1.30

Then $\overrightarrow{DA} = \left(5\hat{i}+5\hat{j}+3\hat{k}\right)-\left(\hat{i}+\hat{j}-\hat{k}\right) = 4\hat{i}+4\hat{j}+4\hat{k}$

$$\overrightarrow{DB} = \left(5\hat{i}+5\hat{j}+3\hat{k}\right)-\left(3\hat{i}-2\hat{j}+2\hat{k}\right) = 2\hat{i}+7\hat{j}+\hat{k}$$

$$\overrightarrow{DC} = \left(5\hat{i}+5\hat{j}+3\hat{k}\right)-\left(4\hat{i}+3\hat{j}+2\hat{k}\right) = \hat{i}+2\hat{j}-\hat{k}$$

Volume of the tetrahedron $= \frac{1}{6}\left[\overrightarrow{DA} \quad \overrightarrow{DB} \quad \overrightarrow{DC}\right]$

$$= \frac{1}{6}\begin{vmatrix} 4 & 4 & 4 \\ 2 & 7 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \frac{1}{6}\left[4(-7-2)-4(-2-1)+4(4-7)\right]$$

$$= \frac{4}{6}[-9+3-3] = -6.$$

1.8.9 Vector Triple Product

Let $\vec{a},\vec{b},\vec{c}$ be three given vectors. The vector triple product of $\vec{a},\vec{b},\vec{c}$ is denoted by $\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ and defined as the cross product of $\vec{a}\times\vec{b}$ with $\vec{c}$.

Note that $\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ is a vector quantity and it represents a vector perpendicular to the plane of $\vec{a}\times\vec{b}$ and $\vec{c}$. But, since $\vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$, $\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$.

1.8.10 Properties of Vector Triple Product

(i) Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

Then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= \hat{i}\left(a_2b_3-a_3b_2\right)-\hat{j}\left(a_1b_3-a_3b_1\right)+\hat{k}\left(a_1b_2-a_2b_1\right)$$

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_2b_3-a_3b_2 & -a_1b_3+a_3b_1 & a_1b_2-a_2b_1\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\hat{i}\left[c_3\left(a_1b_3-a_3b_1\right)-c_2\left(a_1b_2-a_2b_1\right)\right]-\hat{j}\left[c_3\left(a_2b_3-a_3b_2\right)-c_1\left(a_1b_2-a_2b_1\right)\right]$$

$$+\hat{k}\left[c_2\left(a_2b_3-a_3b_2\right)-c_1\left(-a_1b_3+a_3b_1\right)\right]$$

$$=\hat{i}\left(a_1b_3c_3-a_3b_1c_3-a_1b_2c_2+a_2b_1c_2\right)+\hat{j}\left(a_3b_2c_3-a_2b_3c_3+a_1b_2c_1-a_2b_1c_1\right)$$

$$+\hat{k}\left(a_2b_3c_2-a_3b_2c_2+a_1b_3c_2-a_3b_1c_1\right) \qquad \text{...... (1)}$$

$$\left(\vec{a}.\vec{c}\right)\vec{b}=\left[\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right).\left(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}\right)\right]\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)$$

$$=\left(a_1c_1+a_2c_2+a_3c_3\right)\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)$$

$$=\left(a_1b_1c_1+a_2c_2b_1+a_3b_1c_3\right)\hat{i}+\left(a_1b_2c_1+a_2b_2c_2+a_3b_2c_3\right)\hat{j}$$

$$+\left(a_1b_3c_1+a_2b_3c_2+a_3b_3c_3\right)\hat{k} \qquad \text{...... (2)}$$

$$\left(\vec{b}.\vec{c}\right)\vec{a}=\left[\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right).\left(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}\right)\right].\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)$$

$$=\left(b_1c_1+b_2c_2+b_2c_3\right)\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)$$

$$=\left(a_1b_1c_1+a_1b_2c_2+a_1b_3c_3\right)\hat{i}+\left(a_2b_1c_1+a_2b_2c_2+a_2b_3c_3\right)\hat{j}$$

$$+\left(a_3b_1c_1+a_3b_2c_2+a_3b_3c_3\right)\hat{k} \qquad \text{...... (3)}$$

Now (2) - (3) gives:

$$\left(\vec{a}\ .\vec{c}\right)\vec{b}-\left(\vec{b}\ .\vec{c}\right)\vec{a}=\left(a_2c_2b_1+a_3b_1c_3-a_1b_2c_2-a_1b_3c_3\right)\hat{i}$$

$$+\left(a_1b_2c_1+a_3b_2c_3-a_2b_1c_1-a_2b_3c_3\right)\hat{j}$$

$$+\left(a_1b_3c_1+a_2b_3c_2-a_3b_1c_1-a_3b_2c_2\right)\hat{k} \qquad \text{...... (4)}$$

Comparing (1) and (4), we see that

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{b}.\vec{c}\right)\vec{a} \qquad \text{...... (5)}$$

(ii) $\vec{a}\times(\vec{b}\times\vec{c}) = -((\vec{b}\times\vec{c})\times\vec{a})$

$$= -\left[(\vec{b}.\vec{a})\vec{c} - (\vec{c}.\vec{a})\vec{b}\right] \text{ (by using equation (5))}$$

$$= (\vec{c}.\vec{a})\vec{b} - (\vec{b}.\vec{a})\vec{c} \qquad \text{...... (6)}$$

Comparing (5) and (6) we see that

$\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$.

(iii) $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$

Using equation (6),

LHS = $\left\{(\vec{c}.\vec{a})\vec{b} - (\vec{b}.\vec{a})\vec{c}\right\} + \left\{(\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a}\right\} + \left\{(\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}\right\} = \vec{0}$.

1.8.11 Example

If $\vec{a} = \hat{i} - 2\hat{j} - 3\hat{k}, \vec{b} = 3\hat{i} - 2\hat{j} + 4\hat{k}, \vec{c} = 4\hat{i} + \hat{j} - 2\hat{k}$, find $(\vec{a}\times\vec{b})\times\vec{c}$.

Solution: $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

$$= \left[(\hat{i} - 2\hat{j} - 3\hat{k}).(4\hat{i} + \hat{j} - 2\hat{k})\right](3\hat{i} - 2\hat{j} + 4\hat{k})$$

$$- \left[(3\hat{i} - 2\hat{j} + 4\hat{k}).(4\hat{i} + \hat{j} - 2\hat{k})\right](\hat{i} - 2\hat{j} - 3\hat{k})$$

$$= (4 - 2 + 6)(3\hat{i} - 2\hat{j} + 4\hat{k}) - (12 - 2 - 8)(\hat{i} - 2\hat{j} - 3\hat{k})$$

$$= (24\hat{i} - 16\hat{j} + 32\hat{k}) - (2\hat{i} - 4\hat{j} - 6\hat{k}) = 22\hat{i} - 12\hat{j} + 38\hat{k} .$$

Check Your Progress:

8. Let A = (1, 3, -1), B = (0, 1, 6), C = (-1, 3, 1) be 3 points in space. Find the coordinates of point D on the y-axis so that the volume of the tetrahedron $ABCD$ is 10 cubic units.

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9. If $\vec{a} = \hat{i} - 2\hat{j} - 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - 2\hat{k}$ find $(\vec{a} \times \vec{c}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$. Are they equal?

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1.9 SUMMARY

In this unit we have introduced the concept of vector to identify physical quantities having both direction and magnitude. A directed line segment is used to represent a vector graphically. Algebraic operations on vectors are defined. Triangle law and parallelogram law of addition of vectors are explained. Representation of a vector in terms of its components is given. Collinearity and coplanarity of vectors are defined and conditions to obtain them are derived. Dot product of vectors and its application to find angle between vectors is demonstrated. Cross product of vectors and its application to finding the area of a triangle and area of a parallelogram are studied. Vector and scalar triple products are defined. The volume of a parallelopiped and volume of a tetrahedron are calculated with the help of box product. A number of examples are given to illustrate the concepts, theorems, the geometrical and physical interpretations.

1.10 CHECK YOUR PROGRESS - MODEL ANSWERS

1. (a), (b): Energy and electric charge are scalars since there is no direction associated with them.

   (c) Electric current is a vector because it flows in a particular direction.

   (d) vector, (e) scalar, (f) scalar

2. $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

   Magnitude of $\vec{a} = |\vec{a}| = \sqrt{x^2 + y^2 + z^2}$

   $\therefore$ Unit vector in the direction of $\vec{a} = \hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\hat{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\hat{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\hat{k}$$

   Hence components of $\vec{a}$ along *OX, OY, OZ* are:

$$\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

3. Given that $\overrightarrow{OA} = \hat{i} + 3\hat{j} - 7\hat{k}$ and $\overrightarrow{OB} = 5\hat{i} - 2\hat{j} + 4\hat{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(5\hat{i} - 2\hat{j} + 4\hat{k}\right) - \left(\hat{i} + 3\hat{j} - 7\hat{k}\right) = 4\hat{i} - 5\hat{j} + 11\hat{k}.$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(4)^2 + (-5)^2 + (11)^2} = \sqrt{16+25+121} = \sqrt{162}.$$

$\therefore$ A unit vector parallel to $\overrightarrow{AB} = \dfrac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \dfrac{4\hat{i} - 5\hat{j} + 11\hat{k}}{\sqrt{162}}$.

4. Let $\vec{a} = a_1\hat{i} + a_2\hat{j}$ be a vector in *xy*-plane perpendicular to $\vec{b} = 6\hat{i} + 4\hat{j} + 3\hat{k}$.

Then $\vec{a}.\vec{b} = 0 \Rightarrow 6a_1 + 4a_2 = 0$ ... (1)

$$|\vec{a}| = |\vec{b}| \Rightarrow \sqrt{a_1^2 + a_2^2} = \sqrt{36+16+9} = \sqrt{61}$$

$\Rightarrow a_1^2 + a_2^2 = 61$ ... (2)

From (1), $a_1 = \dfrac{-2}{3}a_2$

Substituting a_1 in (2), we get

$$\frac{4}{9}a_2^2 + a_2^2 = 61 \Rightarrow 13a_2^2 = 61 \times 9 \Rightarrow a_2^2 = \frac{61\times 9}{13} \Rightarrow a_2 = \frac{3\sqrt{61}}{\sqrt{13}}$$

$$\therefore a_1 = \frac{-2\sqrt{61}}{\sqrt{13}}.$$

$$\therefore \vec{a} = \frac{\sqrt{61}}{\sqrt{13}}\left(-2\hat{i} + 3\hat{j}\right).$$

5. Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}, \vec{b} = \hat{i} + \hat{j} - 3\hat{k}, \vec{c} = 7\hat{i} - 4\hat{j} + \hat{k}$

$\vec{a}.\vec{b} = 1+2-3 = 0$; $\vec{b}.\vec{c} = 7-4-3 = 0$; $\vec{c}.\vec{a} = 7-8+1 = 0$.

$\therefore \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular.

6. $$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -3\\ 1 & -2 & 1\end{vmatrix}=\hat{i}(1-6)-\hat{j}(2+3)+\hat{k}(-4-1)=-5\hat{i}-5\hat{j}-5\hat{k}.$$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{(-5)^2+(-5)^2+(-5)^2}=\sqrt{25+25+25}=5\sqrt{3}.$$

A unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}=\dfrac{-1}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right)$.

7. Let $\vec{a}=\hat{i}+\hat{j}+7\hat{k}$ and $\vec{b}=-\hat{i}-2\hat{j}+4\hat{k}$

Area of the parallelogram with $\vec{a},\vec{b}$ as adjacent sides = $\left|\vec{a}\times\vec{b}\right|$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 7\\ -1 & -2 & 4\end{vmatrix}=\hat{i}(4+14)-\hat{j}(4+7)+\hat{k}(-2+1)=18\hat{j}-11\hat{j}-\hat{k}.$$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{(18)^2+(-11)^2+(-1)^2}$$

$$=\sqrt{324+121+1}=\sqrt{446}.$$

8. Since D lies on y-axis D is of the form $(0,\alpha,0)$.

$$\overrightarrow{AB}=-\hat{i}-2\hat{j}+7\hat{k}$$

$$\overrightarrow{AC}=-2\hat{i}+0\hat{j}+2\hat{k}$$

$$\overrightarrow{AD}=-\hat{i}+(\alpha-3)\hat{j}+\hat{k}$$

Volume of the tetrahedron = $\frac{1}{6}\left[\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right]$

$$=\frac{1}{6}\begin{vmatrix}-1 & -2 & 7\\ -2 & 0 & 2\\ -1 & \alpha-3 & 1\end{vmatrix}$$

$$=\frac{1}{6}\left[-1\left(0-2(\alpha-3)\right)+2(-2+2)+7\left(-2(\alpha-3)-0\right)\right]$$

$$=\frac{1}{6}[2\alpha-6-14\alpha+42] \;=\frac{1}{6}[36-12\alpha]=6-2\alpha$$

Given that volume of the tetrahedron = 10.

$\Rightarrow 6-2\alpha=10 \Rightarrow 2\alpha=-4 \Rightarrow \alpha=-2\cdot$

$\therefore D=(0,-2,0)$.

9. (i) $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & -3\\ 2 & 1 & -1\end{vmatrix}=\hat{i}(2+3)-\hat{j}(-1+6)+\hat{k}(1+4)=5\hat{i}-5\hat{j}+5\hat{k}$

$(\vec{a}\times\vec{b})\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & -5 & 5\\ 1 & 3 & -2\end{vmatrix}=\hat{i}(10-15)-\hat{j}(-10-5)+\hat{k}(15+5)=-5\hat{i}+15\hat{j}+20\hat{k}$

(ii) $\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -1\\ 1 & 3 & -2\end{vmatrix}=\hat{i}(-2+3)-\hat{j}(-4+1)+\hat{k}(6-1)=\hat{i}+3\hat{j}+5\hat{k}$.

$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & -3\\ 1 & 3 & 5\end{vmatrix}=\hat{i}(-10+9)-\hat{j}(5+3)+\hat{k}(3+2)=-\hat{i}-8\hat{j}+5\hat{k}$.

Clearly, $(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$.

1.11 MODEL EXAMINATION QUESTIONS

1. If $\vec{a}=\hat{i}+3\hat{j}-2\hat{k}$ and $\vec{b}=4\hat{i}-2\hat{j}+4\hat{k}$, find $|3\vec{a}+2\vec{b}|$.

2. If $\vec{a}=3\hat{i}-2\hat{j}+\hat{k}, \vec{b}=2\hat{i}-4\hat{j}-3\hat{k}, \vec{c}=-\hat{i}+2\hat{j}+2\hat{k}$, find a unit vector parallel to $2\vec{a}-3\vec{b}-5\vec{c}\cdot$

3. Show that the points *A, B, C* with position vectors $2\hat{i}+4\hat{j}-\hat{k}, 4\hat{i}+5\hat{j}+\hat{k}$ and $3\hat{i}+6\hat{j}-3\hat{k}$ form a right angled triangle.

4. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then show that the points $-2\vec{a}+3\vec{b}+5\vec{c}$, $\vec{a}+2\vec{b}+3\vec{c}$ and $7\vec{a}-\vec{c}$ are collinear.

5. Show that the four points $4\hat{i}+5\hat{j}+\hat{k}, -\hat{j}-\hat{k}, 3\hat{i}+9\hat{j}+4\hat{k}$ and $4\left(-\hat{i}+\hat{j}+\hat{k}\right)$ are coplanar.

6. The position vectors of *A, B, C, D* are respectively, $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=2\hat{i}+3\hat{j}$, $\vec{c}=3\hat{i}+5\hat{j}-2\hat{k}$ and $\vec{d}=-\hat{j}+\hat{k}$. Evaluate $\overrightarrow{AB}.\overrightarrow{CD}$ and $\overrightarrow{AB}\times\overrightarrow{CD}$.

7. Find the projection of the vector $\vec{a}$ on the vector $\vec{b}$ where $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=-6\hat{i}+2\hat{j}-3\hat{k}$.

8. Show that the vectors $9\hat{i}+\hat{j}-6\hat{k}$ and $4\hat{i}-6\hat{j}+5\hat{k}$ are orthogonal.

9. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}, |\vec{a}|=7, |\vec{b}|=3, |\vec{c}|=5$, find the angle between $\vec{b}$ and $\vec{c}$.

10. *A*(2, -1, 1), *B*(3, 0, 1), *C*(1, -2, 3) are points in space with respect to origin (0, 0, 0). Find a unit vector perpendicular to the plane containing *A, B, C.*

11. If $\vec{a}=3\hat{i}-\hat{j}+2\hat{k}, \vec{b}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+2\hat{k}$ show that $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$.

12. Find the volume of the parallelopiped whose edges are given by the vectors $-3\hat{i}+7\hat{j}+5\hat{k}, 5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$.

13. Find a vector coplanar with $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ and perpendicular to a vector $\vec{c}=\hat{i}+\hat{j}$.

14. Use vector methods to find the volume of the tetrahedron with vertices (0, 0, 0), (-1, 1, 1), (1, -1, 1) and (1, 1, -1).

Answers

1. $\sqrt{150}$

2. $\dfrac{5\hat{i}-2\hat{j}+\hat{k}}{\sqrt{30}}$

6. -18, 0

7. -1

9. 60^0

10. $\pm\frac{\sqrt{2}}{2}\left(\hat{i}-\hat{j}\right)$

12. 304 cubic units

13. $\hat{j}-\hat{k}$

4. $\frac{2}{3}$

UNIT-2: DIFFERENTIATION OF VECTOR FUNCTIONS, CURVES, TANGENTS, ARC LENGTH

Contents

2.0 OBJECTIVES

After studying this unit, you will be able to:

- Understand the concepts of vector and scalar point functions, vector fields and scalar fields.
- Find derivatives, partial derivatives and total derivatives of vector functions.
- Use the methods of vector calculus to study the differential geometry of space curves.

2.1 INTRODUCTION

In the previous chapter we have learnt that physical quantities can be classified as vectors and scalars. In practice we come across physical quantities, whose values are not constant but change with time. For instance, the density of a gas is a scalar quantity which changes from place to place and it is different at different times. The velocity of moving body is a vector quantity which changes from time to time depending on the position of the body. The tangent to a curve takes different directions at different points of the curve. This variable nature of vectors and scalars leads to the definition of vector and scalar functions. In this chapter we formally define these functions and introduce concepts of similar to those of ordinary calculus.

2.2 VECTOR AND SCALAR FUNCTIONS

2.2.1 Definition

For every real number t in some subset I of real numbers, if there corresponds a scalar $F(t)$, then $F(t)$ is called a **scalar function** of t on I.

2.2.2 Example

Let $t \in [0,1]$. The function $F:[0,1] \to R$ defined by $F(t) = 2t^3 + e^t$ is a scalar function of t.

2.2.3 Definition

For each real number 't' in some subset I of real numbers, if there corresponds a unique vector $\vec{F}(t)$, then $\vec{F}(t)$ is called a **vector function** of the scalar variable t on I.

2.2.4 Example

For $t \in [0,1]$, if $\vec{F}(t) = t^2\hat{i} - 2t\hat{j} + e^t\hat{k}$ then $\vec{F}(t)$ is a vector function on [0, 1].

2.2.5 Definition

Let D be a region in space. For each point $(x, y, z) \in D$, if a scalar quantity $\phi(x, y, z)$ is associated, then $\phi(x, y, z)$ is called a **scalar point function**.

2.2.6 Example

(i) For each point $P(x, y, z)$ in space, define $\phi(x, y, z)$ = distance of P from the origin $O = \sqrt{x^2 + y^2 + z^2}$.

Then ϕ is a scalar point function.

(ii) The temperature at any point P of an object occupying a region D is a scalar point function.

2.2.7 Definition

For each point $(x, y, z) \in D$, a region in space, if a vector quantity $\vec{F}(x, y, z)$ is associated, then $\vec{F}(x, y, z)$ is called a **vector point function**.

2.2.8 Example

(i) For each point $P(x,y,z)$ in space, define

$\vec{F}(x,y,z)$ = position of vector of P w.r.t O = $x\hat{i}+y\hat{j}+z\hat{k}$.

Then $\vec{F}$ is a vector point function.

(ii) The gravitational force exerted by Sun on a unit mass is a vector which varies with the position of the mass. This is an example of a vector point function.

2.2.9 Definition

If ϕ is a scalar point function defined in a region *R* of space, then the set of points *P* in the region together with values $\phi(P)$ is called a **scalar field** over *R*.

Some examples of scalar fields are:

(i) Mass distribution in a body

(ii) The electrostatic potential of a system of charges.

2.2.10 Definition

If $\vec{F}$ is a vector point function defined in a region *R* of space, then the set of points *P* in the region together with values $\vec{F}(P)$ is called a **vector field** over *R*.

Some examples of vector fields are:

(i) Velocity of particles in a fluid under flow.

(ii) Velocity of a rotating body.

In general scalar and vector fields are three dimensional, i.e., depend on all three coordinates *x, y, z*, and it is difficult to visualize such fields. However, if the fields depend on two co-ordinates only, then they can be visualized graphically. The following examples illustrate this.

(i) Consider the scalar field $\phi(x,y)=x^2+y^2$. For different values of *k*, we plot $\phi(x,y)=k$. i.e., $x^2+y^2=k$. We observe that these curves are concentric circles centered at (0, 0) as shown below.

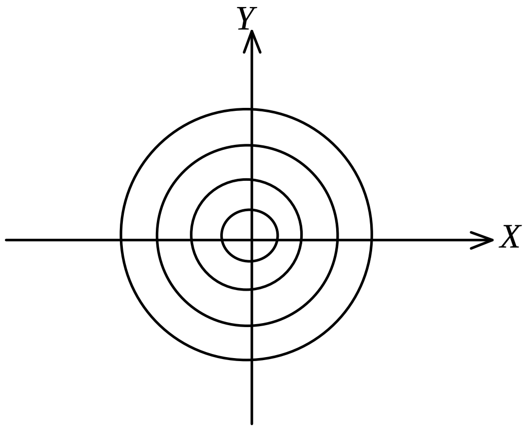

Fig: 2.1

(ii) Consider the vector field $\vec{F}(x,y) = y\hat{i} + x\hat{k}$. When $(x,y) = (1,0)$ $\vec{F}(x,y) = (0,1)$. Corresponding to this we plot a vector of magnitude in y-direction. When $(x,y) = (0,1)$, $\vec{F}(x,y) = (1,0)$. Corresponding to this we plot a vector of magnitude in x-direction. When $(x,y) = (1,1)$, $\vec{F}(x,y) = (1,1)$, Plotting this and considering few other points, we build the vector field shown below.

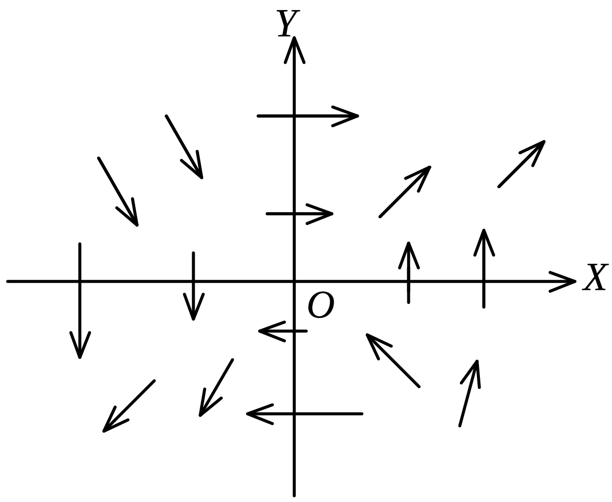

Fig: 2.2

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. Determine which of the following is a vector field?

(i) Density of air in atmosphere.

(ii) Electric current in a circuit.

(iii) $\phi(x,y,z) = x^2y^2 + y^2z^2 + z^2x^2$

(iv) $\vec{F}(x,y,z) = x^2y\hat{i} + 2yz\hat{j} + z^3\hat{k}$.

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2.3 DERIVATIVE OF A VECTOR FUNCTION

Let $\vec{F}(t)$ be a vector of the scalar variable *t* in domain *S*. Let *t* be any point in *S* and $t+\Delta t$ be a point in *S* close to *t*.

Let $\Delta\vec{F}(t) = \vec{F}(t+\Delta t) - \vec{F}(t)$.

If $\lim_{\Delta t \to 0} \frac{\Delta\vec{F}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t}$ exists, then it is called the **derivative of $\vec{\mathbf{F}}$** w.r.t *t* at *t*. It is denoted by $\frac{d\vec{F}}{dt}$ or $\vec{F}'(t)$.

If $\vec{F}'(t)$ exists at all points of *S*, then $\vec{F}$ is said to be **derivable or differentiable** on *S*. We note that $\frac{d\vec{F}}{dt}$ the derivative of $\vec{F}$, is itself a vector function of *t*.

So, we can find the derivative of $\frac{d\vec{F}(t)}{dt}$. This is called **second derivative** of $\vec{F}(t)$, and it is denoted by $\frac{d^2\vec{F}}{dt^2}$ or $\vec{F}''(t)$. Thus $\frac{d^2\vec{F}(t)}{dt^2} = \frac{d\vec{F}'(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{F}'(t+\Delta t) - \vec{F}'(t)}{\Delta t}$.

We can define the higher order derivatives $\frac{d^3\vec{F}}{dt^3}, \frac{d^4\vec{F}}{dt^4}, \ldots\ldots$ in a similar fashion.

If the vector function $\vec{F}(t)$ is expressed in terms of its components, say $\vec{F}(t) = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}$, then $\dfrac{d\vec{F}}{dt} = \dfrac{dF_1(t)}{dt}\hat{i} + \dfrac{dF_2(t)}{dt}\hat{j} + \dfrac{dF_3(t)}{dt}\hat{k}$.

Check Your Progress:

2. Find the components of $\dfrac{d\vec{F}}{dt}$ when $\vec{F}(t) = e^{-t}\hat{i} - \tan t\,\hat{j} + \log\left(t^2+1\right)\hat{k}$.

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2.3.1 Geometrical Interpretation of Derivative of a Vector Function

Let $\vec{r}(t)$ be the position vector of the point $P(x, y, z)$ in space. Then $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, where *x, y, z* are functions of the scalar variable *t*. As *t* changes, the end point *P* of $\vec{r}$ describes a space curve with parametric equations $x(t),\ y(t),\ z(t)$. Let *P, Q* be two near by points on the curve with position vectors $\vec{r}$ and $\vec{r} + \Delta\vec{r}$ corresponding to the values *t* and $t + \Delta t$ of the scalar variable. Then

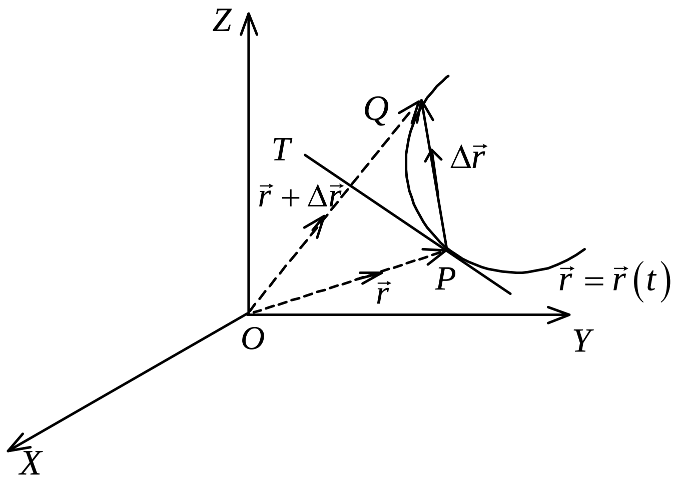

Fig: 2.3

$$\overline{PQ} = \overline{OQ} - \overline{OP} = \vec{r} + \Delta\vec{r} - \vec{r} = \Delta\vec{r}$$

$$\Rightarrow \frac{\Delta\vec{r}}{\Delta t} = \frac{\overline{PQ}}{\Delta t} \Rightarrow \lim_{\Delta t \to 0}\frac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t \to 0}\frac{\overline{PQ}}{\Delta t} = \lim_{Q \to P}\frac{\overline{PQ}}{\Delta t}$$

As $\Delta t \to 0$, $Q \to P$ and the chord *PQ* tends to the tangent *PT* to the curve at *P*.

Thus, $\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$ represents a vector along the tangent to the curve at *P*, in the direction of increasing *t*. $\left|\frac{d\vec{r}}{dt}\right|$ denotes the absolute value of the slope of the tangent at *P*.

2.3.2 Example

Find the unit tangent vector at any point of the curve $x = t^2 + 1,\ y = 4t - 3,\ z = 2t^2 - 6t$. Find the unit tangent when $t = 2$.

Solution: Given that $\vec{r}(t) = (t^2 + 1)\hat{i} + (4t - 3)\hat{j} + (2t^2 - 6t)\hat{k}$ is the curve.

A tangent vector to the curve at any point is

$$\frac{d\vec{r}}{dt} = 2t\hat{i} + 4\hat{j} + (4t - 6)\hat{k}$$

$$\text{Unit tangent vector} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|} = \frac{2t\hat{i} + 4\hat{j} - (4t - 6)\hat{k}}{\sqrt{4t^2 + 16 + (4t - 6)^2}}$$

$$\text{when } t = 2\text{, unit tangent vector} = \frac{4\hat{i} + 4\hat{j} - 2\hat{k}}{\sqrt{16 + 16 + 4}} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}.$$

2.3.3 Physical Interpretation of Derivative of a Vector Function

From Fig. 2.3, we have

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{Q \to P} \frac{\overrightarrow{PQ}}{\Delta t}.$$

If *t* denotes time, then $\Delta \vec{r}$ denotes the displacement and hence $\frac{d\vec{r}}{dt}$ is the rate of change of displacement w.r.t time *t*. Thus, $\frac{d\vec{r}}{dt}$ represents the velocity $\vec{v}$ with which the point *P* moves along the curve.

Similarly, if $\Delta \vec{v}$ is the increment in $\vec{v}$ corresponding to the increment Δt in *t*, then the rate of change of velocity of the point $\vec{v}$ w.r.t time *t* is given by

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}.$$

Thus the acceleration $\vec{a}$ of the point *P* along the curve is given by $\vec{a} = \frac{d^2\vec{r}}{dt^2}$.

2.3.4 Example

A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t denotes time. Determine the velocity and acceleration of the particle at any time t. Find the magnitude of velocity and acceleration when t = 0.

Solution: The position vector of the particle is given by

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k} = e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}\ .$$

Velocity $\vec{v} = \dfrac{d\vec{r}}{dt} = -e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}$

Acceleration $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$

At t = 0, $\vec{v} = -\hat{i} + 6\hat{k}$ and $\vec{a} = \hat{i} - 18\hat{j}$.

Magnitude of velocity at t = 0 is $\sqrt{1+36} = \sqrt{37}$

Magnitude of acceleration at t = 0 is $\sqrt{1+324} = \sqrt{325}$.

2.3.5 Definition

A vector whose direction and magnitude are fixed is called a **constant vector**.

2.3.6 Theorem

A differentiable vector function is constant, if and only if its derivative is zero.

Proof: Let $\vec{F}(t)$ be a differentiable vector function of t.

Suppose $\vec{F}(t)$ is constant. Then $\vec{F}(t+\Delta t) = \vec{F}(t)$

$$\therefore \frac{d\vec{F}}{dt} = \lim_{\Delta t\to 0}\frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\vec{0}}{\Delta t} = \vec{0}\ .$$

$\therefore$ The condition is necessary.

Conversely, suppose that $\dfrac{d\vec{F}}{dt} = \vec{0}$.

Let $\vec{F}(t) = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}$

Then $\dfrac{d\vec{F}}{dt} = \dfrac{dF_1}{dt}\hat{i} + \dfrac{dF_2}{dt}\hat{j} + \dfrac{dF_3}{dt}\hat{k} = \vec{0}$

$$\Rightarrow \frac{dF_1}{dt}=0, \frac{dF_2}{dt}=0, \frac{dF_3}{dt}=0, \text{ since } \hat{i}, \hat{j}, \hat{k} \text{ are non-coplanar vectors.}$$

This means F_1, F_2, F_3 are constant scalar functions.

Hence $\vec{F}(t)$ is a constant vector function.

Check Your Progress:

3. Give an example of a constant vector.

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2.3.7 Rules of Differentiation

Let $\vec{F}(t), \vec{G}(t), \vec{H}(t)$ be three differentiable vector functions of a scalar variable *t*. Let $\phi(t)$ be a differentiable scalar function and $\vec{c}$ be a constant vector. Then the following rules of differentiation which are analogus to those in calculus hold good.

1. $\frac{d\vec{c}}{dt}=\vec{0}$

Proof: Let $\vec{F}(t)=\vec{c}$, a constant vector. Then $\vec{F}(t+\Delta t)=\vec{c}$.

$$\therefore \frac{d\vec{F}}{dt}=\lim_{\Delta t\to 0}\frac{\vec{F}(t+\Delta t)-\vec{F}(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{\vec{c}-\vec{c}}{\Delta t}=\vec{0}$$

$$\therefore \frac{d\vec{c}}{dt}=\vec{0}.$$

2. $\frac{d}{dt}\left(\vec{F}(t)\pm\vec{G}(t)\right)=\frac{d\vec{F}(t)}{dt}\pm\frac{d\vec{G}(t)}{dt}$

Proof: Let $\vec{f}(t)=\vec{F}(t)\pm\vec{G}(t)$.

Then $\vec{f}(t+\Delta t)=\vec{F}(t+\Delta t)\pm\vec{G}(t+\Delta t)$

$$\frac{\vec{f}(t+\Delta t)-\vec{f}(t)}{\Delta t}=\frac{\vec{F}(t+\Delta t)-\vec{F}(t)}{\Delta t}\pm\frac{\vec{G}(t+\Delta t)-\vec{G}(t)}{\Delta t}$$

Taking limit as $\Delta t\to 0$, $\frac{d\vec{f}}{dt}=\frac{d\vec{F}}{dt}\pm\frac{d\vec{G}}{dt}$.

i.e., $\dfrac{d}{dt}\left(\vec{F}(t)\pm\vec{G}(t)\right)=\dfrac{d\vec{F}(t)}{dt}\pm\dfrac{d\vec{G}(t)}{dt}$.

3. $\dfrac{d}{dt}\left(\vec{F}(t).\vec{G}(t)\right)=\vec{F}.\dfrac{d\vec{G}}{dt}+\vec{G}.\dfrac{d\vec{F}}{dt}$

Proof: Let $\vec{f}(t)=\vec{F}(t).\vec{G}(t)$

We note that $\vec{f}(t)$ is a scalar function of *t*.

$$\vec{f}(t+\Delta t)-\vec{f}(t)=\vec{F}(t+\Delta t).\vec{G}(t+\Delta t)-\vec{F}(t).\vec{G}(t)$$

$$=\vec{F}(t+\Delta t).\vec{G}(t+\Delta t)-\vec{F}(t).\vec{G}(t+\Delta t)+\vec{F}(t).\vec{G}(t+\Delta t)-\vec{F}(t).\vec{G}(t)$$

$$=\left(\vec{F}(t+\Delta t)-\vec{F}(t)\right)\vec{G}(t+\Delta t)+\vec{F}(t)\left(\vec{G}(t+\Delta t)-\vec{G}(t)\right)$$

$$\lim_{\Delta t\to 0}\frac{\vec{f}(t+\Delta t)-\vec{f}(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{\vec{F}(t+\Delta t)-\vec{F}(t)}{\Delta t}.\vec{G}(t+\Delta t)+\vec{F}(t).\lim_{\Delta t\to 0}\frac{\vec{G}(t+\Delta t)-\vec{G}(t)}{\Delta t}$$

$$\Rightarrow\frac{d\vec{f}}{dt}=\frac{d\vec{F}}{dt}.\vec{G}(t)+\vec{F}(t).\frac{d\vec{G}(t)}{dt}$$

i.e. $\dfrac{d}{dt}\left(\vec{F}(t).\vec{G}(t)\right)=\vec{F}.\dfrac{d\vec{G}}{dt}+\vec{G}.\dfrac{d\vec{F}}{dt}$.

4. $\dfrac{d}{dt}\left(\vec{F}(t)\times\vec{G}(t)\right)=\vec{F}\times\dfrac{d\vec{G}}{dt}+\vec{G}\times\dfrac{d\vec{F}}{dt}$

5. $\dfrac{d}{dt}\left(\phi\vec{F}\right)=\phi\dfrac{d\vec{F}}{dt}+\dfrac{d\phi}{dt}\vec{F}$

Proof: 4, 5 can be proved in a similar manner.

6. $\dfrac{d}{dt}\left(\vec{F}.\left(\vec{G}\times\vec{H}\right)\right)=\dfrac{d\vec{F}}{dt}.\left(\vec{G}\times\vec{H}\right)+\vec{F}.\left(\dfrac{d\vec{G}}{dt}\times\vec{H}\right)+\vec{F}.\left(\vec{G}\times\dfrac{d\vec{H}}{dt}\right)$

Proof: $\dfrac{d}{dt}\left(\vec{F}.\left(\vec{G}\times\vec{H}\right)\right)=\dfrac{d\vec{F}}{dt}.\left(\vec{G}\times\vec{H}\right)+\vec{F}.\dfrac{d}{dt}\left(\vec{G}\times\vec{H}\right)$ [by rule (3)]

$$=\frac{d\vec{F}}{dt}.\left(\vec{G}\times\vec{H}\right)+\vec{F}.\left(\frac{d\vec{G}}{dt}\times\vec{H}+\vec{G}\times\frac{d\vec{H}}{dt}\right)\text{ [by rule (4)]}$$

$$= \frac{d\vec{F}}{dt}.\left(\vec{G} \times \vec{H}\right) + \vec{F}.\left(\frac{d\vec{G}}{dt} \times \vec{H}\right) + \vec{F}.\left(\vec{G} \times \frac{d\vec{H}}{dt}\right).$$

Thus, we have $\frac{d}{dt}\left[\vec{F}\ \vec{G}\ \vec{H}\right] = \left[\frac{d\vec{F}}{dt}\vec{G}\ \vec{H}\right] + \left[\vec{F}\ \frac{d\vec{G}}{dt}\vec{H}\right] + \left[\vec{F}\ \vec{G}\frac{d\vec{H}}{dt}\right].$

Therefore, $\frac{d}{dt}\left(\vec{F}.\left(\vec{G} \times \vec{H}\right)\right) = \frac{d\vec{F}}{dt}.\left(\vec{G} \times \vec{H}\right) + \vec{F}.\left(\frac{d\vec{G}}{dt} \times \vec{H}\right) + \vec{F}.\left(\vec{G} \times \frac{d\vec{H}}{dt}\right).$

7. $\frac{d}{dt}\left(\vec{F} \times \left(\vec{G} \times \vec{H}\right)\right) = \frac{d\vec{F}}{dt} \times \left(\vec{G} \times \vec{H}\right) + \vec{F} \times \left(\frac{d\vec{G}}{dt} \times \vec{H}\right) + \vec{F} \times \left(\vec{G} \times \frac{d\vec{H}}{dt}\right)$

Proof: It can be proved in a similar manner, by noting that the order of the factors must be maintained.

8. If $\vec{F}$ is a differentiable vector function of a scalar variable *u* and *u* is a differentiable function of a scalar variable *t*, then $\vec{F}$ is a vector function of the scalar variable *t* and

$$\frac{d\vec{F}}{dt} = \frac{d\vec{F}}{du}.\frac{du}{dt}$$

Proof: Let Δt be a small increment in *t*. Corresponding to this, let Δu be an increment in *u* and $\Delta\vec{F}$ be increment in $\vec{F}$.

As $\Delta t \to 0$, we notice that $\Delta u \to 0$ and $\Delta\vec{F} \to 0$.

We have, $\frac{\Delta\vec{F}}{\Delta t} = \frac{\Delta\vec{F}}{\Delta u}.\frac{\Delta u}{\Delta t}$

Taking limit as $\Delta t \to 0$,

$$\lim_{\Delta t \to 0}\frac{\Delta\vec{F}}{\Delta t} = \lim_{\Delta t \to 0}\frac{\Delta\vec{F}}{\Delta u}.\lim_{\Delta t \to 0}\frac{\Delta u}{\Delta t} = \lim_{\Delta u \to 0}\frac{\Delta\vec{F}}{\Delta u}.\lim_{\Delta t \to 0}\frac{\Delta u}{\Delta t}$$

$$\Rightarrow \frac{d\vec{F}}{dt} = \frac{d\vec{F}}{du}.\frac{du}{dt}.$$

2.3.8 Example

If $\vec{F}(t) = t^2\hat{i} - t\,\hat{j} + (2t+1)\hat{k}$ and $\vec{G}(t) = (2t-3)\hat{i} + \hat{j} - t\hat{k}$, then find $\frac{d\vec{F}}{dt}, \frac{d\vec{G}}{dt}$ and verify that $\frac{d}{dt}\left(\vec{F}.\vec{G}\right) = \vec{F}.\frac{d\vec{G}}{dt} + \vec{G}.\frac{d\vec{F}}{dt}$, $\frac{d}{dt}\left(\vec{F} \times \vec{G}\right) = \vec{F} \times \frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt} \times \vec{G}$.

Solution: $\frac{d\vec{F}}{dt} = 2t\hat{i} - \hat{j} + 2\hat{k},\ \frac{d\vec{G}}{dt} = 2\hat{i} - \hat{k}\cdot$

$$\vec{F}.\vec{G} = \left(t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}\right).\left[(2t-3)\hat{i} + \hat{j} - t\hat{k}\right]$$

$$= t^2(2t-3) - t - t(2t+1) = 2t^3 - 5t^2 - 2t\ .$$

$$\frac{d}{dt}\left(\vec{F}.\vec{G}\right) = 6t^2 - 10t - 2 \qquad \text{..... (1)}$$

$$\vec{F}.\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}.\vec{G} = \left(t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}\right).\left(2\hat{i} - \hat{k}\right) + \left(2t\hat{i} - \hat{j} + 2\hat{k}\right).\left((2t-3)\hat{i} + \hat{j} - t\hat{k}\right)$$

$$= 2t^2 - (2t+1) - 1 + 2t(2t-3) - 2t$$

$$= 6t^2 - 10t - 2 \qquad \text{...... (2)}$$

From (1) and (2), $\frac{d}{dt}\left(\vec{F}.\vec{G}\right) = \vec{F}.\frac{d\vec{G}}{dt} + \vec{G}.\frac{d\vec{F}}{dt}$

$$\vec{F} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & 2t+1 \\ 2t-3 & 1 & -t \end{vmatrix}$$

$$= \hat{i}\left(t^2 - 2t - 1\right) - \hat{j}\left[-t^3 - (2t+1)(2t-3)\right] + \hat{k}\left(t^2 + 2t^2 - 3t\right)$$

$$= \left(t^2 - 2t - 1\right)\hat{i} - \hat{j}\left(-t^3 - 4t^2 + 4t + 3\right) + \left(3t^2 - 3t\right)\hat{k}$$

$$\frac{d}{dt}\left(\vec{F} \times \vec{G}\right) = (2t-2)\hat{i} - \left(-3t^2 - 8t + 4\right)\hat{j} + (6t-3)\hat{k}$$

$$= (2t-2)\hat{i} + \left(3t^2 + 8t - 4\right)\hat{j} + (6t-3)\hat{k} \qquad \text{...... (3)}$$

$$\vec{F} \times \frac{d\vec{G}}{dt} + \vec{G} \times \frac{d\vec{F}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & 2t+1 \\ 2 & 0 & -1 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & -1 & 2 \\ 2t-3 & 1 & -t \end{vmatrix}$$

$$= \left[\hat{i}(t) - \hat{j}\left(-t^2 - 4t - 2\right) + \hat{k}(2t)\right] + \left[\hat{i}(t-2) - \hat{j}\left(-2t^2 - 4t + 6\right) + \hat{k}(2t + 2t - 3)\right]$$

$$=\left(2t-2\right)\hat{i}+\left(t^2+4t+2+2t^2+4t-6\right)\hat{j}+\left(2t+4t-3\right)\hat{k}$$

$$=\left(2t-2\right)\hat{i}+\left(3t^2+8t-4\right)\hat{j}+\left(6t-3\right)\hat{k} \qquad \text{...... (4)}$$

From (3) and (4), $\frac{d}{dt}\left(\vec{F}\times\vec{G}\right)=\vec{F}\times\frac{d\vec{G}}{dt}+\vec{G}\times\frac{d\vec{F}}{dt}$.

Hence verified.

We now derive conditions for a vector function to have constant magnitude or constant direction.

2.3.9 Theorem

The necessary and sufficient condition that a vector function $\vec{F}\left(t\right)$ has a constant magnitude in that $\vec{F}.\frac{d\vec{F}}{dt}=0$.

Proof: Suppose $\vec{F}\left(t\right)$ is of constant magnitude.

Then $\left|\vec{F}\left(t\right)\right|^2=\vec{F}\left(t\right).\vec{F}\left(t\right)$ is a constant.

Differentiating w.r.t *t*, using 2.2.8 (3), we get

$$\vec{F}.\frac{d\vec{F}}{dt}+\frac{d\vec{F}}{dt}.\vec{F}\left(0\right)=0$$

$$\Rightarrow \vec{F}.\frac{d\vec{F}}{dt}=0.$$

$\therefore$ The condition is necessary.

Conversely, suppose $\vec{F}.\frac{d\vec{F}}{dt}=0$

Then $\vec{F}.\frac{d\vec{F}}{dt}+\frac{d\vec{F}}{dt}.\vec{F}=0$

$$\Rightarrow \frac{d}{dt}\left(\vec{F}.\vec{F}\right)=0$$

$\Rightarrow \left|\vec{F}\right|^2=\vec{F}.\vec{F}$ is constant by 2.3.7

$\Rightarrow \vec{F}$ has constant magnitude.

Check Your Progress:

4. Give an example of a vector with constant magnitude.

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2.3.10 Theorem

The necessary and sufficient condition that a vector function $\vec{F}(t)$ has a constant direction is that $\vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}$.

Proof: Let $\hat{F}(t)$ denotes the unit vector in the direction of $\vec{F}(t)$ and $f(t)$ be its magnitude so that $\vec{F}(t) = f(t).\hat{F}(t)$.

$$\therefore \frac{d\vec{F}}{dt} = \frac{d}{dt}\left(f(t)\hat{F}(t)\right) = f(t)\frac{d\hat{F}(t)}{dt} + \frac{df(t)}{dt}\hat{F}(t)$$

$$\Rightarrow \vec{F} \times \frac{d\vec{F}}{dt} = f\hat{F} \times \frac{d}{dt}\left(f\hat{F}\right)$$

$$= f\hat{F} \times \left(f\frac{d\hat{F}}{dt}\right) + \left(\frac{df}{dt}\hat{F}\right)$$

$$= f^2\hat{F} \times \frac{d\hat{F}}{dt} + f\frac{df}{dt}\hat{F} \times \hat{F}$$

$$= f^2\hat{F} \times \frac{d\hat{F}}{dt} \qquad \left(\because \hat{F} \times \hat{F}\right) \qquad \text{..... (1)}$$

Now, suppose $\vec{F}$ has constant direction.

Then $\hat{F}$ is a constant vector since its direction and magnitude are constants.

$$\therefore \frac{d\hat{F}}{dt} = \vec{0}.$$

Thus, from (1) we get $\vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}$.

Conversely, suppose $\vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}$.

From (1), $f^2\hat{F}\times\dfrac{d\hat{F}}{dt}=\vec{0}$

$\Rightarrow \hat{F}\times\dfrac{d\hat{F}}{dt}=\vec{0}\ \left(\because f^2\neq 0\right)$ ...... (2)

Also since $\hat{F}$ has constant magnitude by 2.3.10,

we get $\hat{F}.\dfrac{d\hat{F}}{dt}=0$ ...... (3)

From (2) and (3), we have $\dfrac{d\hat{F}}{dt}=\vec{0}$.

$\Rightarrow \hat{F}$ is a constant vector and hence $\vec{F}$ has constant direction.

Check Your Progress:

5. Give an example of a vector function with constant direction.

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We now give some examples to understand the above formulae.

2.3.11 Example

If $\vec{F}(t)=t^2\hat{i}-t\hat{j}+(2t+1)\hat{k}$, find the values of $\dfrac{d\vec{F}}{dt},\ \dfrac{d^2\vec{F}}{dt^2},\ \left|\dfrac{d\vec{F}}{dt}\right|,\ \left|\dfrac{d^2\vec{F}}{dt^2}\right|$ at $t = 0$.

Solution: $\vec{F}(t)=t^2\hat{i}-t\hat{j}+(2t+1)\hat{k}$.

We know that if $\vec{F}=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ then

$$\frac{d\vec{F}}{dt}=\frac{dF_1}{dt}\hat{i}+\frac{dF_2}{dt}\hat{j}+\frac{dF_3}{dt}\hat{k}$$

So, $\dfrac{d\vec{F}}{dt}=2t\hat{i}-\hat{j}+2\hat{k}$.

$$\frac{d^2\vec{F}}{dt^2}=\frac{d}{dt}\left(\frac{d\vec{F}}{dt}\right)=2\hat{i}\qquad (\because\ \hat{i},\ \hat{j} \text{ are constant vectors, their derivatives are zero})$$

When $t = 0$, $\dfrac{d\vec{F}}{dt}=-\hat{j}+2\hat{k}$ and $\dfrac{d^2\vec{F}}{dt^2}=2\hat{i}$.

$$\left|\frac{d\vec{F}}{dt}\right| = \sqrt{(-1)^2 + 2^2} = \sqrt{5} \text{ and } \left|\frac{d^2\vec{F}}{dt^2}\right| = \sqrt{(2)^2} = 2.$$

2.3.12 Example

If $\vec{r} = e^t\left[\vec{a}\cos 2t + \vec{b}\sin 2t\right]$ where $\vec{a}, \vec{b}$ are constant vectors, then show that $\frac{d^2\vec{r}}{dt^2} - 2\frac{d\vec{r}}{dt} + 5\vec{r} = \vec{0}$.

Solution: We have $\vec{r} = e^t\left[\vec{a}\cos 2t + \vec{b}\sin 2t\right]$

Differentiating $\vec{r}$ w.r.t t,

$$\frac{d\vec{r}}{dt} = e^t\left[-2\vec{a}\sin 2t + 2\vec{b}\cos 2t\right] + e^t\left[\vec{a}\cos 2t + \vec{b}\sin 2t\right]$$

$$= e^t\left[\left(-2\vec{a} + \vec{b}\right)\sin 2t + \left(\vec{a} + 2\vec{b}\right)\cos 2t\right]$$

$$\frac{d^2\vec{r}}{dt^2} = e^t\left[2\left(-2\vec{a} + \vec{b}\right)\cos 2t - 2\left(\vec{a} + 2\vec{b}\right)\sin 2t\right] + e^t\left[\left(-2\vec{a} + \vec{b}\right)\sin 2t + \left(\vec{a} + 2\vec{b}\right)\cos 2t\right]$$

$$= e^t\left[\left(-3\vec{a} + 4\vec{b}\right)\cos 2t + \left(-4\vec{a} - 3\vec{b}\right)\sin 2t\right]$$

$$\frac{d^2\vec{r}}{dt^2} - 2\frac{d\vec{r}}{dt} + 5\vec{r}$$

$$= e^t\left[\left(-3\vec{a} + 4\vec{b}\right)\cos 2t + \left(-4\vec{a} - 3\vec{b}\right)\sin 2t\right]$$

$$-2e^t\left[\left(-2\vec{a} + \vec{b}\right)\sin 2t + \left(\vec{a} + 2\vec{b}\right)\cos 2t\right] + 5e^t\left[\vec{a}\cos 2t + \vec{b}\sin 2t\right]$$

$$= e^t\left[\left(-3\vec{a} + 4\vec{b} - 2\vec{a} - 4\vec{b} + 5\vec{a}\right)\cos 2t + \left(-4\vec{a} - 3\vec{b} + 4\vec{a} - 2\vec{b} + 5\vec{b}\right)\sin 2t\right]$$

$$= \vec{0}\cdot$$

2.3.13 Example

If $\vec{F} = \sin t\,\hat{i} + \cos t\,\hat{j} + t\,\hat{k}$, $\vec{G} = \cos t\,\hat{i} - \sin t\,\hat{j} - 3\hat{k}$, $\vec{H} = 2\hat{i} + 3\hat{j} - \hat{k}$, find $\frac{d}{dt}\left(\vec{F} \times \left(\vec{G} \times \vec{H}\right)\right)$ at $t = 0$.

Solution: $\vec{G}\times\vec{H}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \cos t & -\sin t & -3\\ 2 & 3 & -1\end{vmatrix}=(\sin t+9)\hat{i}-\hat{j}(-\cos t+6)+\hat{k}(3\cos t+2\sin t)$

$$\therefore \vec{F}\times\left(\vec{G}\times\vec{H}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \sin t & \cos t & t\\ \sin t+9 & \cos t-6 & 3\cos t+2\sin t\end{vmatrix}$$

$$=\left[3\cos^2 t+2\sin t\cos t-t\cos t+6t\right]\hat{i}-\left[3\sin t.\cos t+2\sin^2 t-t\sin t-9t\right]\hat{j}$$

$$+\left[\sin t.\cos t-6\sin t-\sin t.\cos t-9\cos t\right]\hat{k}$$

$$\therefore \frac{d}{dt}\left(\vec{F}\times\left(\vec{G}\times\vec{H}\right)\right)=\left(-6\cos t.\sin t+2\cos^2 t-2\sin^2 t-\cos t+t\sin t+6\right)\hat{i}$$

$$-\left(-3\sin^2 t+3\cos^2 t+4\sin t.\cos t-\sin t-t\cos t-9\right)\hat{j}+\left(-6\cos t+9\sin t\right)\hat{k}$$

When $t = 0$

$$\frac{d}{dt}\left(\vec{F}\times\left(\vec{G}\times\vec{H}\right)\right)=(2-1+6)\hat{i}-(3-9)\hat{j}-6\hat{k}\ =7\hat{i}+6\hat{j}-6\hat{k}\,.$$

2.4 SPACE CURVES, TANGENT, ARC LENGTH

In section 2.3, we have seen that if $x(t), y(t), z(t)$ are functions of a single variable '*t*', then the point $P(x,y,z)$ describes a curve in space for different values of *t*. In this case, $\vec{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$ is the position vector of *P*, and the locus of *P* describes a space curve with parametric equation $x(t), y(t), z(t)$ where *t* is a parameter.

For example,

(i) If $x(t)=x_1+lt, y(t)=y_1+mt, z(t)=z_1+nt$, then $P(x,y,z)$ describes a straight line passing through the point (x_1,y_1,z_1) and having direction cosines proportional to *l, m, n.*

(ii) If $x(t)=a\cos t,\ y(t)=a\sin t$ and $z(t) = 0$, then $P(x,y,z)$ describes a circle in *xy* - plane with radius '*a*' and center (0, 0, 0).

(iii) If $x(t)=a\cos t, y(t)=a\sin t$ and $z(t)=bt(b\neq 0)$ then the locus of $P(x,y,z)$ is a space curve known as circular helix.

We have learnt in (2.3.1) of section (2.3), that if $\vec{r}(t)$ is the position vector of a point on a space curve, then $\frac{d\vec{r}}{dt}$ represents the tangent to the curve at the point *P*. A space can also be described in terms of arc lengths, measured from a fixed point on the curve to a variable point on it.

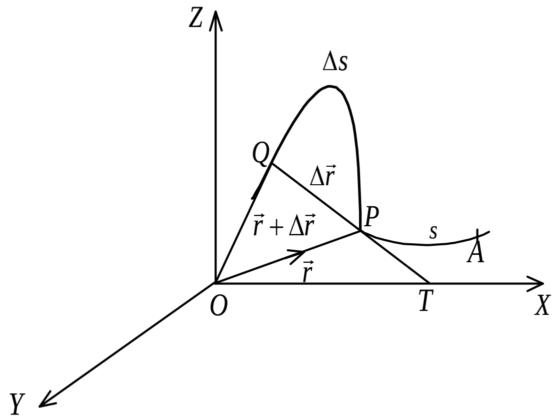

Fig: 2.4

Let $P(x, y, z)$ be any point on the space curve $\vec{r} = \vec{r}(s) = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$, where *s* is arc length *AP* measured from the fixed point *A* on the curve to the point *P*.

Let *Q* be a neighbouring point to *P* and arc $AQ = s + \Delta s$.

Let $\overrightarrow{OP} = \vec{r}$ and $\overrightarrow{OQ} = \vec{r} + \Delta\vec{r}$. Then $\overrightarrow{PQ} = \Delta\vec{r}$ and arc $PQ = \Delta s$.

$$\frac{\Delta\vec{r}}{\Delta s} = \frac{\overrightarrow{PQ}}{arc\,PQ}$$

As $\Delta s \to 0$, *Q* approaches *P* and *PQ* tends to the tangent *PT* at *P*.

Thus, $\frac{d\vec{r}}{dS} = \lim_{\Delta S \to 0} \frac{\Delta\vec{r}}{\Delta s}$,

if it exists, represents the tangent vector to the space curve at *P*.

Differentiating $\vec{r}(s) = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$, w.r.t. *s*,

we get $\frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$

$$\left|\frac{d\vec{r}}{ds}\right| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = \sqrt{\frac{(ds)^2}{(ds)^2}} = 1.$$

Thus, $\frac{d\vec{r}}{ds}$ represents a unit tangent vector to the space curve at *P*. $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ give the direction cosines of the tangent line.

Hence, if the curve is represented by $\vec{r} = \vec{r}(s)$, where *s* is arc length then $\frac{d\vec{r}}{ds}$ is the unit tangent vector at *P*.

Check Your Progress:

6. If a space curve is given by $\vec{r} = \vec{r}(t)$, where *t* is a parameter then

(i) tangent vector to the curve is __________

(ii) unit tangent vector to the curve is __________

____

____

____

We now have some examples to calculate the tangent and unit tangent to a curve at a given point.

2.4.1 Example

Find the unit vector in the direction of the tangent at any point of the curve $\vec{r}(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 4t\,\hat{k}$.

Solution: Tangent vector to the curve at any point is :

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(3\cos t)\hat{i} + \frac{d}{dt}(3\sin t)\,\hat{j} + \frac{d}{dt}(4t)\hat{k}$$

$$= -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}.$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2}$$

$$= \sqrt{9\left(\sin^2 t + \cos^2 t\right) + 16}$$

$$= \sqrt{9+16} = \sqrt{25} = 5.$$

$$\frac{d\vec{r}}{dt}\bigg/\left|\frac{d\vec{r}}{dt}\right| = \frac{-3}{\sqrt{13}}\sin t\,\hat{i} + \frac{3}{\sqrt{13}}\cos t\,\hat{j} + \frac{4}{\sqrt{13}}\hat{k}.$$

2.4.2 Example

Find a unit vector in the direction of the tangent at the point $t = 1$ to the space curve $\vec{r} = (t^2+1)\hat{i} + 4t\,\hat{j} + (2t^2 - 6t + 3)\hat{k}$.

Solution: Tangent vector at any point 't' is:

$$\frac{d\vec{r}}{dt} = 2t\,\hat{i} + 4\,\hat{j} + (4t-6)\,\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4t^2 + 16 + (4t-6)^2}$$

When $t = 1$,

$$\frac{d\vec{r}}{dt} = 2\hat{i} + 4\,\hat{j} - 2\hat{k}\,,\quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{4+16+16} = 6.$$

Unit tangent vector when $t = 1$ is

$$\frac{d\vec{r}}{dt}\bigg/\left|\frac{d\vec{r}}{dt}\right| = \frac{2\hat{i} + 4\,\hat{j} - 2\hat{k}}{6} = \frac{\hat{i} + 2\,\hat{j} - \hat{k}}{3}.$$

2.4.3 Frenet - Secrret Formulae

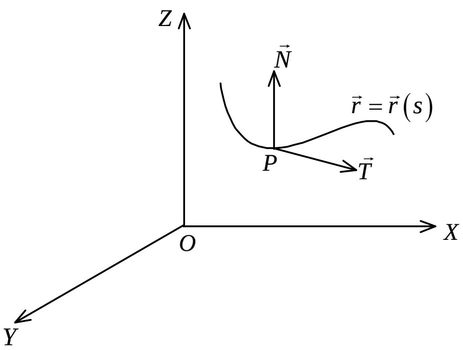

Fig: 2.5

For the space curve $\vec{r} = \vec{r}(s)$, we know that the unit tangent vector is given by $\frac{d\vec{r}}{ds}$.

Let $\vec{T} = \frac{d\vec{r}}{ds}$

Since $\vec{T}$ has constant length *l*,

$$\vec{T}.\vec{T} = \left|\vec{T}\right|^2 = 1$$

$$\Rightarrow \frac{d}{ds}\left(\vec{T}.\vec{T}\right) = 0$$

$$\Rightarrow \vec{T}.\frac{d\vec{T}}{ds} + \frac{d\vec{T}}{ds}.\vec{T} = 0$$

$$\Rightarrow \vec{T}.\frac{d\vec{T}}{ds} = 0\,.$$

$\Rightarrow \dfrac{d\vec{T}}{ds}$ is a vector perpendicular to the tangent vector.

$\Rightarrow$ The rate of change of the unit tangent vector with respect to the arc length is perpendicular to the tangent vector.

We note that $\dfrac{d\vec{T}}{ds}$ is perpendicular to the tangent vector and thus it is in the direction of the normal to the space curve at *P*. If $\vec{N}$ is the unit vector along $\dfrac{d\vec{T}}{ds}$, then $\vec{N}$ is called the **principal normal** at *P* to the curve. Let *k* be a positive real number such that $\dfrac{d\vec{T}}{ds} = k\vec{N}$. The scalar *k* is called the **curvature** of the curve at the point and its reciprocal $\rho = \dfrac{1}{k}$ is called the **radius of curvature** at *P*.

Let $\vec{B}$ be the unit vector perpendicular to both the tangent vector $\vec{T}$ and the principal normal vector $\vec{N}$, such that $\vec{T}\times\vec{N} = \vec{B},\ \vec{N}\times\vec{B} = \vec{T},\ \vec{B}\times\vec{T} = \vec{N}$ (i.e., $\vec{T},\ \vec{N}, \vec{B}$ form a right handed system of unit vectors.

$$\therefore \vec{B}.\vec{T} = 0\,.$$

Differentiating w.r.t. *s*, we get

$$\vec{B}.\frac{d\vec{T}}{ds} + \frac{d\vec{B}}{ds}.\vec{T} = 0 \ \Rightarrow \vec{B}.(k\vec{N}) + \frac{d\vec{B}}{ds}.\vec{T} = 0$$

$$\Rightarrow \frac{d\vec{B}}{ds}.\vec{T} = 0 \qquad \left(\because \vec{B}.\vec{N} = 0\right)$$

$$\Rightarrow \frac{d\vec{B}}{ds} \text{ is perpendicular to } \vec{T}.$$

Thus $\frac{d\vec{B}}{ds}$ is perpendicular to both $\vec{T}$ and $\vec{B}$ and hence it is parallel to $\vec{N}$.

$\therefore$ There exists a scalar τ such that $\frac{d\vec{B}}{ds} = -\tau \vec{N}$. (The negative sign here by convention)

Since $\left|\vec{N}\right| = 1$, we get $\left|\tau\right| = \left|\frac{d\vec{B}}{ds}\right|$

τ is called **torsion** and $\sigma = \frac{1}{\tau}$ is called **radius of torsion**.

Now consider $\vec{N} = \vec{B} \times \vec{T}$. Then

$$\frac{d\vec{N}}{ds} = \frac{d\vec{B}}{ds} \times \vec{T} + \vec{B} \times \frac{d\vec{T}}{ds}$$

$$= \vec{B} \times k\vec{N} - \tau \vec{N} \times \vec{T}$$

$$= k\left(\vec{B} \times \vec{N}\right) - \tau\left(\vec{N} \times \vec{T}\right)$$

$$= -k\vec{T} + \tau \vec{B}$$

$$\therefore \frac{d\vec{N}}{ds} = \tau \vec{B} - k\vec{T}.$$

Frenet - Serret Formulae

We derived the relations

$$\frac{d\vec{T}}{ds} = k\vec{N}, \ \frac{d\vec{B}}{ds} = -T\vec{N}, \ \frac{d\vec{N}}{ds} = \tau \vec{B} - k\vec{T}$$

involving the derivatives of the fundamental vectors $\vec{T}, \vec{N}, \vec{B}$. These equations are called **Frenet - Serret formuale**.

If $\vec{W} = \tau \vec{T} + k\vec{B}$, then $\vec{W} \times \vec{T} = \left(\tau \vec{T} + k\vec{B}\right) \times \vec{T}$

$$= \tau \vec{T} \times \vec{T} + k\vec{B} \times \vec{T} = \tau .0 + k\vec{N} = k\vec{N} = \frac{d\vec{T}}{ds}.$$

Similarly, $\vec{W} \times \hat{B} = \frac{d\vec{B}}{ds}, \vec{W} \times \hat{N} = \frac{d\vec{N}}{ds}$

Here $\vec{W}$ is known as **Darboox vector**.

2.4.4 Example

Find

(i) The unit tangent $\vec{T}$

(ii) The principal normal $\vec{N}$, curvature k and radius of curvature ρ

(iii) the bi-normal $\vec{B}$, torsion τ and radius of torsion σ for the space curve $x = 3\cos t,\ y = 3\sin t, z = 4t$.

Solution: At any point of the curve, the position vector is given by $\vec{r} = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 4t\,\hat{k}$.

$$\therefore \frac{d\vec{r}}{dt} = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\,\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} = 5.$$

(i) $$\frac{d\vec{r}}{ds} = \vec{T} \Rightarrow \left|\frac{d\vec{r}}{ds}\right| = \left|\vec{T}\right| = 1 \Rightarrow \left|\frac{d\vec{r}}{dt}.\frac{dt}{ds}\right| = 1$$

$$\Rightarrow \left|\frac{d\vec{r}}{dt}\right| = \left|\frac{ds}{dt}\right| \Rightarrow \frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$$

$$\therefore \frac{ds}{dt} = 5.$$

$$\therefore \vec{T} = \frac{d\vec{r}}{dt}\Big/\frac{ds}{dt} = \frac{1}{5}\left(-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}\right).$$

(ii) $$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}.\frac{dt}{ds} = \frac{d\hat{T}}{dt}\Big/\frac{ds}{dt}$$

$$= \frac{1}{5}\left[\frac{1}{5}\left(-3\cos t\,\hat{i} - 3\sin t\,\hat{j}\right)\right] = -\frac{3}{25}\left(\cos t\,\hat{i} + \sin t\,\hat{j}\right)$$

$$\frac{d\vec{T}}{ds} = k\vec{N} \Rightarrow k = \left|\frac{d\vec{T}}{ds}\right| \qquad \left(\because \left|\vec{N}\right| = 1\right)$$

$$\Rightarrow k = \frac{3}{25}\sqrt{\cos^2 t + \sin^2 t} \ = \frac{3}{25}$$

$\therefore$ Curvature $= k = \dfrac{3}{25}$ and radius of curvature $= \dfrac{1}{k} = \dfrac{25}{3}$.

Principal normal $\vec{N} = \dfrac{1}{k}\dfrac{d\vec{T}}{ds}$

$$= \frac{25}{3}\left(\frac{-3}{25}\right)\left(\cos t\,\hat{i} + \sin t\,\hat{j}\right) = -\left(\cos t\,\hat{i} + \sin t\,\hat{j}\right)$$

(iii) $\vec{B} = \vec{T} \times \vec{N}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \hat{i}\left(\frac{4}{5}\sin t\right) - \hat{j}\left(\frac{4}{5}\cos t\right) + \hat{k}\left(\frac{3}{5}\sin^2 t + \frac{3}{5}\cos^2 t\right)$$

$$= \frac{1}{5}\left(4\sin t\,\hat{i} + 4\cos t\,\hat{j} + \frac{3}{5}\hat{k}\right)$$

$$\frac{d\vec{B}}{ds} = \frac{d\vec{B}}{ds}\bigg/\frac{ds}{dt}$$

$$= \frac{1}{5}\frac{\left(4\cos t\,\hat{i} - 4\sin t\,\hat{j}\right)}{5} = \frac{4}{25}\left(\cos t\,\hat{i} - \sin t\,\hat{j}\right).$$

$$\frac{d\vec{B}}{ds} = \tau\vec{N}$$

$$\Rightarrow \tau = \left|\frac{d\vec{B}}{ds}\right|$$

$$= \frac{4}{25}\sqrt{\cos^2 t + \sin^2 t} \ = \frac{4}{25}$$

$\therefore$ Torsion $\tau = \dfrac{4}{25}$ and radius of torsion $\sigma = \dfrac{1}{\tau} = \dfrac{25}{4}$.

2.5 PARTIAL DERIVATIVES OF A VECTOR FUNCTION

In section 2.3 we have studied the derivative of a vector function which depends on one scalar variable t. In case a vector function depends on more than one independent scalar variables, we define partial derivatives.

Let $\vec{F} = \vec{F}(x, y, z)$ be a function of three independent scalar variables x, y and z. Then, the first order partial derivative of $\vec{F}(x, y, z)$ w.r.t x is defined as:

$$\overrightarrow{F_x} = \frac{\partial \vec{F}}{\partial x} = \lim_{\Delta x \to 0} \frac{\vec{F}(x + \Delta x, y, z) - \vec{F}(x, y, z)}{\Delta x}$$

provided the limit exists.

Similarly, the partial derivatives of $\vec{F}$ w.r.t y and z are defined as

$$\overrightarrow{F_y} = \frac{\partial \vec{F}}{\partial y} = \lim_{\Delta y \to 0} \frac{\vec{F}(x, y + \Delta y, z) - \vec{F}(x, y, z)}{\Delta y}$$

$$\overrightarrow{F_z} = \frac{\partial \vec{F}}{\partial z} = \lim_{\Delta z \to 0} \frac{\vec{F}(x, y, z + \Delta z) - \vec{F}(x, y, z)}{\Delta z}$$

provided the limits exist.

We note that $\frac{\partial \vec{F}}{\partial x}, \frac{\partial \vec{F}}{\partial y}, \frac{\partial \vec{F}}{\partial z}$ are functions of x, y, z again. Hence, they can be differentiated again w.r.t. x, y, z to find the 2nd order partial derivatives. Thus we have,

$$\frac{\partial^2 \vec{F}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{F}}{\partial x}\right), \ \frac{\partial^2 \vec{F}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{F}}{\partial y}\right), \ \frac{\partial^2 \vec{F}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \vec{F}}{\partial z}\right)$$

$$\frac{\partial^2 \vec{F}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{F}}{\partial y}\right), \ \frac{\partial^2 \vec{F}}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{F}}{\partial x}\right), \ \frac{\partial^2 \vec{F}}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{F}}{\partial z}\right), \ldots \text{ etc.}$$

If $\vec{F}$ has continuous partial derivatives of 2nd order, then

$$\frac{\partial^2 \vec{F}}{\partial x \partial y} = \frac{\partial^2 \vec{F}}{\partial y \partial x}, \ \frac{\partial^2 \vec{F}}{\partial y \partial z} = \frac{\partial^2 \vec{F}}{\partial z \partial y}, \ \ldots..$$

The 3rd order and higher order partial derivatives are defined in a similar manner.

i.e. $\frac{\partial^3 \vec{F}}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \vec{F}}{\partial x^2}\right), \ \frac{\partial^3 \vec{F}}{\partial x \partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \vec{F}}{\partial z^2}\right), \ \ldots..$

Let $\vec{F}, \vec{G}, \phi$ be the functions of more than one scalar variable. Then the following rules hold good.

1. $\dfrac{\partial}{\partial x}\left(\phi\vec{F}\right) = \dfrac{\partial\phi}{\partial x}\vec{F} + \phi\dfrac{\partial\vec{F}}{\partial x}$

2. If k is a constant, $\dfrac{\partial}{\partial x}\left(k\vec{F}\right) = k\dfrac{\partial\vec{F}}{\partial x}$

3. If $\vec{c}$ is a constant vector, then $\dfrac{\partial}{\partial x}\left(\phi\vec{c}\right) = \vec{c}\dfrac{\partial\phi}{\partial x}$

4. $\dfrac{\partial}{\partial x}\left(\vec{F} \pm \vec{G}\right) = \dfrac{\partial\vec{F}}{\partial x} \pm \dfrac{\partial\vec{G}}{\partial x}$

5. $\dfrac{\partial}{\partial x}\left(\vec{F}.\vec{G}\right) = \dfrac{\partial\vec{F}}{\partial x}.\vec{G} + \vec{F}.\dfrac{\partial\vec{G}}{\partial x}$

6. $\dfrac{\partial}{\partial x}\left(\vec{F}\times\vec{G}\right) = \dfrac{\partial\vec{F}}{\partial x}\times\vec{G} + \vec{F}\times\dfrac{\partial\vec{G}}{\partial x}$

7. If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ then, $\dfrac{\partial\vec{F}}{\partial x} = \dfrac{\partial F_1}{\partial x}\hat{i} + \dfrac{\partial F_2}{\partial x}\hat{j} + \dfrac{\partial F_3}{\partial x}\hat{k}$.

Also, as in calculus of real variables, the differentials of $\vec{F}$ are given by

$$d\vec{F} = \frac{\partial\vec{F}}{\partial x}dx + \frac{\partial\vec{F}}{\partial y}dy + \frac{\partial\vec{F}}{\partial z}dz$$

$d\vec{F} = dF_1\,\hat{i} + dF_2\,\hat{j} + dF_3\,\hat{k}$, when $\vec{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$.

We shall now look at some examples to illustrate the above formulae.

2.5.1 Example

If $\vec{F} = \cos xy\,\hat{i} + \left(3xy - 2x^2\right)\hat{j} - \left(3x + 2y\right)\hat{k}$ find $\dfrac{\partial^2\vec{F}}{\partial y^2}$ and $\dfrac{\partial^2\vec{F}}{\partial x\partial y}$.

Solution: $\dfrac{\partial\vec{F}}{\partial y} = \dfrac{\partial}{\partial y}\left(\cos xy\right)\hat{i} + \dfrac{\partial}{\partial y}\left(3xy - 2x^2\right)\hat{j} - \dfrac{\partial}{\partial y}\left(3x + 2y\right)\hat{k}$

Treating x as a constant, while differentiating w.r.t. y,

$$\frac{\partial\vec{F}}{\partial y} = \left(-\sin xy\right).x\,\hat{i} - \left(3x\right)\hat{j} - \left(2\right)\hat{k} \qquad \text{....... (1)}$$

Differentiating (1) w.r.t. y again,

$$\frac{\partial^2 \vec{F}}{\partial y^2} = \frac{\partial}{\partial y}\left(-x\sin xy\right)\hat{i} - \frac{\partial}{\partial y}\left(3x\right)\hat{j} - \frac{\partial}{\partial y}\left(2\right)\hat{k}$$

$$= -x\left(\cos xy\right).x\hat{i} - 0.\hat{j} - 0.\hat{k} = -x^2\cos xy .$$

Differentiating (1) w.r.t x, we get

$$\frac{\partial^2 \vec{F}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-x\sin xy\right)\hat{i} - \frac{\partial}{\partial x}\left(3x\right)\hat{j} - \frac{\partial}{\partial x}\left(2\right)\hat{k}$$

$$= \left(-x\left(\cos xy\right).y - \sin xy\right)\hat{i} - 3\hat{j} \quad = -\left(xy\cos xy + \sin xy\right)\hat{i} - 3\hat{j}$$

2.5.2 Example

If $\vec{F} = 2\hat{i} - \hat{j} - 2\hat{k}$ and $\phi = x^2yz + 4xz^2$, find $\vec{F}.\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$ at (1, -2, 1).

Solution: Given $\phi = x^2yz + 4xz^2$.

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2, \ \frac{\partial\phi}{\partial y} = x^2z, \ \frac{\partial\phi}{\partial z} = x^2y + 8xz$$

At (1, -2, 1),

$$\frac{\partial\phi}{\partial x} = -4 + 4 = 0, \quad \frac{\partial\phi}{\partial y} = 1, \quad \frac{\partial\phi}{\partial z} = -2 + 8 = 6 .$$

$\therefore \vec{F}.\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$ at (1, -2, 1) is

$$\left(2\hat{i} - \hat{j} - 2\hat{k}\right).\left(\hat{j} + 6\hat{k}\right) \quad = -1 - 12 = -13 .$$

2.5.3 Example

If $\vec{F} = xyz\hat{i} + xz^2\hat{j} - y^3\hat{k}$ and $\vec{G} = x^3\hat{i} - xyz\hat{j} + x^2z\hat{k}$, evaluate $\frac{\partial^2\vec{F}}{\partial y^2} \times \frac{\partial^2\vec{G}}{\partial x^2}$ at (1, 1, 0).

Solution: $\vec{F} = xyz\hat{i} + xz^2\hat{j} - y^3\hat{k}$, $\vec{G} = x^3\hat{i} - xyz\hat{j} + x^2z\hat{k}$.

$$\frac{\partial\vec{F}}{\partial y} = xz\hat{i} - 3y^2\hat{k}, \ \frac{\partial^2\vec{F}}{\partial y^2} = -6y\hat{k}$$

at (1, 1, 0) $\dfrac{\partial^2\vec{F}}{\partial y^2} = -6\hat{k}$.

$$\frac{\partial\vec{G}}{\partial x} = 3x^2\hat{i} - yz\,\hat{j} + 2xz\,\hat{k}\,,\quad \frac{\partial^2\vec{G}}{\partial x^2} = 6x\,\hat{i} + 2z\,\hat{k}.$$

At (1, 1, 0), $\dfrac{\partial^2\vec{G}}{\partial x^2} = 6\hat{i}$.

$$\therefore \frac{\partial^2\vec{F}}{\partial y^2}\times\frac{\partial^2\vec{G}}{\partial x^2} \text{ at } (1,1,0) = -6\hat{k}\times 6\hat{i} = -36\left(-\hat{j}\right) = 36\,\hat{j}.$$

2.5.4 Example

If $\phi(x,y,z) = xy^2z$ and $\vec{F} = xz\,\hat{i} - xy^2\,\hat{j} + yz^2\hat{k}$, evaluate $\dfrac{\partial^3}{\partial x^2\partial z}\left(\phi\vec{F}\right)$ at (2, -1, 1).

Solution: $\phi\vec{F} = xy^2z\left(xz\,\hat{i} - xy^2\,\hat{j} + yz^2\hat{k}\right)$

$$= x^2y^2z^2\hat{i} - x^2y^4z\,\hat{j} + xy^3z^3\hat{k}.$$

$$\frac{\partial}{\partial z}\left(\phi\vec{F}\right) = 2x^2y^2z\,\hat{i} - x^2y^4\,\hat{j} + 3xy^3z^2\hat{k}$$

$$\frac{\partial^2}{\partial x\partial z}\left(\phi\vec{F}\right) = 4xy^2z\,\hat{i} - 2xy^4\,\hat{j} + 3y^3z^2\hat{k}$$

$$\frac{\partial^3}{\partial x^2\partial z}\left(\phi\vec{F}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2\phi\vec{F}}{\partial x\partial z}\right) = 4y^2z\,\hat{i} - 2y^4\,\hat{j}.$$

At (2, -1, 1) $\dfrac{\partial^3\left(\phi\vec{F}\right)}{\partial x^2\partial z} = 4\hat{i} - 2\,\hat{j}$.

2.5.5 Example

Show that the function $\vec{F}(x,y) = e^{-\lambda x}\left(\vec{a}\sin xy + \vec{b}\cos xy\right)$ satisfies the partial differential equation $\dfrac{\partial^2\vec{F}}{\partial x^2} + \dfrac{\partial^2\vec{F}}{\partial y^2} = \vec{0}$.

Solution: Given $\vec{F}(x,y) = e^{-\lambda x}\left(\vec{a}\sin\lambda y + \vec{b}\cos\lambda y\right)$

$$\frac{\partial \vec{F}}{\partial x} = -\lambda e^{-\lambda x}\left(\vec{a}\sin\lambda y + \vec{b}\cos\lambda y\right) = -\lambda\vec{F} \quad \text{...... (1)}$$

$$\frac{\partial^2 \vec{F}}{\partial x^2} = \frac{\partial}{\partial x}\left(-\lambda\vec{F}\right) = -\lambda\frac{\partial \vec{F}}{\partial x} = -\lambda\left(-\lambda\vec{F}\right) \quad \text{(From (1))}$$

$$= \lambda^2\vec{F} \quad \text{...... (2)}$$

$$\frac{\partial \vec{F}}{\partial y} = e^{-\lambda x}\left(\lambda\vec{a}\cos\lambda y - \lambda\vec{b}\sin\lambda y\right) = \lambda e^{-\lambda x}\left(\vec{a}\cos\lambda y - \vec{b}\sin\lambda y\right)$$

$$\frac{\partial^2 \vec{F}}{\partial y^2} = \lambda e^{-\lambda x}\left(-\lambda\vec{a}\sin\lambda y - \lambda\vec{b}\cos\lambda y\right) = -\lambda^2 e^{-\lambda x}\left(\vec{a}\sin\lambda y + \vec{b}\cos\lambda y\right)$$

$$= -\lambda^2\vec{F} \quad \text{...... (3)}$$

Adding (2) and (3),

$$\frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} = \lambda^2\vec{F} - \lambda^2\vec{F} = \vec{0}\,.$$

2.6 SUMMARY

In this unit you have learned vector functions of a scalar variable of *t* and extended the concept of derivative from calculus to derivative of vector function. Formulae for finding derivatives of dot product, cross product, triple product, sum, difference, scalar multiple of vector functions are derived. Necessary and sufficient conditions for a vector function to have constant direction or magnitude are derived. Application of derivatives to find tangent vectors to space curves and to find velocity and acceleration of moving bodies is explained with examples. As many problems that occur in Mechanics, Fluid dynamics, Electricity and magnetism involve scalar and vector quantities that depend on position as well as time, vector fields and scalar fields are introduced with illustrations. Detailed description of application of differential geometry is omitted here. Interested readers may refer to books on vector calculus for further exploration.

2.7 CHECK YOUR PROGRESS - MODEL ANSWERS

1. (i) Scalar field (ii) Vector field

   (iii) Scalar field (iv) Vector field

2. $\vec{F}(t) = e^{-t}\hat{i} - \tan t\,\hat{j} + \log\left(t^2+1\right)\hat{k}$

$$\frac{d\vec{F}}{dt} = -e^{-t}\hat{i} - \sec^2 t\,\hat{j} + \frac{1}{t^2+1}.2t\,\hat{k}$$

$\therefore$ Components of $\dfrac{d\vec{F}}{dt}$ are: $-e^{-t}, -\sec^2 t, \dfrac{2t}{t^2+1}$.

3. For any three scalar a, b, c the vector $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ is a constant vector.

For example, $\vec{r} = \hat{i} - \hat{j} + 2\hat{k}$ is a constant vector.

4. Let $\vec{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + 2\hat{k}$

Then $|\vec{r}| = \sqrt{\cos^2 t + \sin^2 t + 4} = \sqrt{1+4} = \sqrt{5}$.

$\therefore$ $\vec{r}$ has constant magnitude.

5. A vector parallel to any given line has constant direction.

Let A,B be two points a line with position vectors $\vec{a}, \vec{b}$ respectively. Then $\vec{b} - \vec{a} = \overrightarrow{AB}$ has direction from A to B. For any scalar t, let $\vec{r} = t\left(\vec{b} - \vec{a}\right)$, $t > 0$.

Then $\vec{r}$ is a vector whose direction is same as that of $\overrightarrow{AB}$ and hence constant.

6. $\dfrac{d\vec{r}}{dt}$ and unit tangent vector is: $\dfrac{d\vec{r}}{dt} \Big/ \left|\dfrac{d\vec{r}}{dt}\right|$.

2.8 MODEL EXAMINATION QUESTIONS

1. If $\vec{F} = (1-\cos t)\hat{i} + (t - \sin t)\hat{j}$, find $\dfrac{d\vec{F}}{dt}, \dfrac{d^2\vec{F}}{dt^2}$.

2. If $\vec{r} = (t+1)\hat{i} + \left(t^2+t+1\right)\hat{j} + \left(t^3+t^2+t+1\right)\hat{k}$, then find $\left|\dfrac{d\vec{r}}{dt}\right|$ and $\left|\dfrac{d^2\vec{r}}{dt^2}\right|$ at t = 2.

3. If $\vec{a}, \vec{b}$ are constant vectors and $\vec{r} = e^{nt}\vec{a} + e^{-nt}\vec{b}$, show that $\dfrac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$.

4. If $\vec{F} = t^3\hat{i} - 2t\,\hat{j} + (2t+1)\hat{k}$ and $\vec{G} = (3-2t)\hat{i} + t\,\hat{j} - \hat{k}$, then find $\dfrac{d}{dt}\left(\vec{F}.\vec{G}\right)$ and $\dfrac{d}{dt}\left(\vec{F}\times\vec{G}\right)$ when t = -1.

5. If $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + at\tan t\,\hat{k}$, find $\left[\dfrac{d\vec{r}}{dt} \quad \dfrac{d^2\vec{r}}{dt^2} \quad \dfrac{d^3\vec{r}}{dt^3}\right]$ and $\left|\dfrac{d\vec{r}}{dt}\times\dfrac{d^2\vec{r}}{dt^2}\right|$.

6. If $\vec{F}(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$, then verify theorem 2.3.6.

7. If $\vec{F}(t) = e^{t}\left(\hat{i} + \hat{j} + \hat{k}\right)$ then verify theorem 2.3.6.

8. A particle moves along a curve whose parametric equations are $x = e^{-t},\ y = 2\cos 3t,\ z = 2\sin 3t$, where t is the time. Determine its velocity and acceleration at time t.

9. Suppose a particle P moves along a curve whose parametric equations are $x = t - \sin t,\ y = 1 - \cos t, z = 4\sin\left(\dfrac{t}{2}\right)$, where t is time.

(a) Find the velocity and acceleration at time $t = 0$.

(b) Find the magnitude of velocity and acceleration at time $t = 0$.

10. Given that $x = t, y = t^2, z = \dfrac{2}{3}t^3$ in a space curve, determine

(a) the tangent vector at any point of the curve.

(b) the unit tangent vector to the space curve at any point on the curve.

(c) the unit tangent vector to the curve when $t = 1$.

11. Find the unit tangent at any point of the curve $\vec{r} = 2\log t\,\hat{i} + 4t\,\hat{j} + \left(2t^2 + 1\right)\hat{k}$.

12. Find the components of acceleration when a particle moves along the curve $x = 1 - 4t + t^3,\ y = 2 + 4t + t^2,\ z = 3 + 8t^2 - 3t^3$ at $t = 2$.

13. If $\vec{F} = \cos xy\,\hat{i} + \left(3xy - 2x^2\right)\hat{j} - (3x + 2y)\hat{k}$, find $\dfrac{\partial \vec{F}}{\partial x}, \dfrac{\partial \vec{F}}{\partial y}, \dfrac{\partial^2 \vec{F}}{\partial x^2}, \dfrac{\partial^2 \vec{F}}{\partial y^2}, \dfrac{\partial^2 \vec{F}}{\partial x \partial y}$.

14. If $\vec{F} = 2x^2\hat{i} - 3yz\,\hat{j} + xz^2\hat{k},\ \phi = 2z - x^3 y$, find $\vec{F} \times \left(\hat{i}\dfrac{\partial \phi}{\partial x} + \hat{j}\dfrac{\partial \phi}{\partial y} + \hat{k}\dfrac{\partial \phi}{\partial z}\right)$ at (1, -1, 1).

15. If $\vec{F} = \left(2x^2 y - x^4\right)\hat{i} + \left(e^{xy} - y\sin x\right)\hat{j} + x^2\cos y\,\hat{k}$, verify that $\dfrac{\partial^2 \vec{F}}{\partial x \partial y} = \dfrac{\partial^2 \vec{F}}{\partial y \partial x}$.

16. If $\vec{F} = xyz\,\hat{i} + xz\,\hat{j} - y^2\hat{k}$ and $\vec{G} = x^2\hat{i} - xy^2\hat{j} + xz^3\hat{k}$. Evaluate $\dfrac{\partial^2 \vec{F}}{\partial x \partial y}, \dfrac{\partial^2 \vec{G}}{\partial x \partial y}$ at the origin.

Answers

1. $\sin t\,\hat{i} + (1-\cos t)\,\hat{j},\ \cos t\,\hat{i} + \sin t\,\hat{j}$

2. $\sqrt{315}, 10\sqrt{2}$

4. $-5, 5\hat{i} + 10\hat{j} + 17\hat{k}$

5. $a^3 \tan\theta,\ a^2 \sec\theta$

8. Velocity: $-e^{-t}\hat{i} - 6\sin 3t\hat{j} + 6\cos 3t\,\hat{k}$

   Acceleration: $e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$

9. (a) $2\hat{k},\ \hat{j}$ (b) 2, 1

10. (a) $\hat{i} + 2t\,\hat{j} + 2t^2\hat{k}$ (b) $\dfrac{\hat{i} + 2t\,\hat{j} + 2t^2\hat{k}}{\sqrt{1+4t^2+4t^4}}$ (c) $\dfrac{\hat{i} + 2\,\hat{j} + 2\,\hat{k}}{3}$

11. $\dfrac{\left(\dfrac{2}{t}\hat{i} + 4\,\hat{j} + 4t\,\hat{k}\right)}{\left(\dfrac{4}{t^2} + 16 + 16t^2\right)}$

12. 12, 4, -20

13. $\dfrac{\partial \vec{F}}{\partial x} = -6\sin xy\,\hat{i} + (3y - 4x)\,\hat{j} - 3\hat{k}$

    $\dfrac{\partial \vec{F}}{\partial y} = -x\sin xy\,\hat{i} + 3x\,\hat{j} - 2\hat{k}$

    $\dfrac{\partial^2 \vec{F}}{\partial x^2} = -y^2\cos xy\,\hat{i} - 4\hat{j}$

    $\dfrac{\partial^2 \vec{F}}{\partial y^2} = -x^2\cos xy\,\hat{i}$

    $\dfrac{\partial^2 \vec{F}}{\partial x\,\partial y} = -x^2\cos xy\,\hat{i} + 3\,\hat{j}$

14. 11

16. 0, 0

UNIT - 3 : GRADIENT, DIVERGENCE, CURL OPERATORS AND THEIR ALGEBRA

Contents

3.0 OBJECTIVES

After studying this unit, you will be able to:

- Understand the meaning of differential operators.
- Apply gradient, divergence and curl on scalar point and vector point functions.
- Understand the geometrical and physical significance of these operators.

3.1 INTRODUCTION

In this unit we learn about differential operators and their application to problems in physics and geometry. Differential operators are those, which when applied on vector or scalar point functions lead to new point functions. These operators are widely used in hydrodynamics and electrostatics. We introduce the operators gradient, divergence and curl and study their properties. Identities involving these operators will be discussed.

3.2 GRADIENT OF A SCALAR POINT FUNCTION

Let $\phi(x, y, z)$ be a scalar point function defined for every (x, y, z) in some region of space and it is differentiable. The **gradient of** ϕ denoted by grad ϕ or $\nabla\phi$ is defined as

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

We notice that $\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$

Here $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ is a vector differential operator. It is known as del or nabla. ∇ applied on a scalar point function ϕ yield a vector point function. We note that $\nabla\phi$ or grad ϕ is a vector point function and $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$ are its components.

Properties of Gradient

We now see some results that indicate the properties of gradient.

3.2.1 Theorem

A scalar point function $\phi(x, y, z)$ is a constant if and only if $\nabla\phi = \vec{0}$.

Proof: Let $\phi(x, y, z) = k$, a constant.

Then $\frac{\partial\phi}{\partial x} = 0, \frac{\partial\phi}{\partial y} = 0, \frac{\partial\phi}{\partial z} = 0.$

$$\therefore \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \hat{i}.0 + \hat{j}.0 + \hat{k}.0 = \vec{0}.$$

Conversely, suppose $\nabla\phi = \vec{0}$.

i.e., $\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \vec{0}$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 0, \frac{\partial\phi}{\partial y} = 0, \frac{\partial\phi}{\partial z} = 0$$

$\Rightarrow$ ϕ is independent of x, y, z.

$\Rightarrow \phi(x, y, z)$ is a constant.

3.2.2 Theorem

Let $\phi(x,y,z)$ and $\psi(x,y,z)$ be scalar point functions. Then $\nabla(\phi+\psi)=\nabla\phi+\nabla\psi$.

Proof: $\nabla(\phi+\psi)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(\phi+\psi)$

$$=\hat{i}\frac{\partial}{\partial x}(\phi+\psi)+\hat{j}\frac{\partial}{\partial y}(\phi+\psi)+\hat{k}\frac{\partial}{\partial z}(\phi+\psi)$$

$$=\hat{i}\left(\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial x}\right)+\hat{j}\left(\frac{\partial\phi}{\partial y}+\frac{\partial\psi}{\partial y}\right)+\hat{k}\left(\frac{\partial\phi}{\partial z}+\frac{\partial\psi}{\partial z}\right)$$

$$=\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)+\left(\hat{i}\frac{\partial\psi}{\partial x}+\hat{j}\frac{\partial\psi}{\partial y}+\hat{k}\frac{\partial\psi}{\partial z}\right)$$

$$=\nabla\phi+\nabla\psi\ .$$

3.2.3 Theorem

Let c be a constant. Then $\nabla(c\phi)=c\nabla\phi$.

Proof: $\nabla(c\phi)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(c\phi)$

$$=\hat{i}\frac{\partial}{\partial x}(c\phi)+\hat{j}\frac{\partial}{\partial y}(c\phi)+\hat{k}\frac{\partial}{\partial z}(c\phi)$$

$$=\hat{i}\,c\frac{\partial\phi}{\partial x}+\hat{j}\,c\frac{\partial\phi}{\partial y}+\hat{k}\,c\frac{\partial\phi}{\partial z}$$

$$=c\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$=c\nabla\phi\ .$$

3.2.4 Note

Combining theorems (3.2.2) and (3.2.3), for constants *a*, *b* and scalar point functions $\phi(x,y,z)$, $\psi(x,y,z)$, we have $\nabla(a\phi+b\psi)=a\nabla\phi+b\nabla\psi$. This is called linear property of the gradient.

We now look at some examples to calculate the gradient.

3.2.5 Example

If $\phi(x,y,z) = x^3 + y^3 + z^3 + 3xyz$, then find $\nabla\phi$.

Solution: $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$= \hat{i}\frac{\partial}{\partial x}\left(x^3 + y^3 + z^3 + 3xyz\right) + \hat{j}\frac{\partial}{\partial y}\left(x^3 + y^3 + z^3 + 3xyz\right) + \hat{k}\frac{\partial}{\partial z}\left(x^3 + y^3 + z^3 + 3xyz\right)$$

$$= \hat{i}\left(3x^2 + 3yz\right) + \hat{j}\left(3y^2 + 3xz\right) + \hat{k}\left(3z^2 + 3xy\right)$$

$$= 3\left[\left(x^2 + yz\right)\hat{i} + \left(y^2 + xz\right)\hat{j} + \left(z^2 + xy\right)\hat{k}\right].$$

3.2.6 Example

If $\phi(x,y,z) = 3x^2y - y^3z^2$, find grad ϕ at the point (1, -2, -1).

Solution: Given that $\phi(x,y,z) = 3x^2y - y^3z^2$.

Differentiating w.r.t. *x, y, z* respectively,

$$\frac{\partial\phi}{\partial x} = 6x, \frac{\partial\phi}{\partial y} = 3x^2 - 3y^2z^2, \frac{\partial\phi}{\partial z} = -2y^3z .$$

$$\therefore \text{ grad } \phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= 6x\hat{i} + 3\left(x^2 - y^2z^2\right)\hat{j} - 2y^3z\hat{k}$$

At (1, -2, -1), grad $\phi = 6\hat{i} + 3(1-4)\hat{j} - 2(-2)^3(-1)\hat{k} = 6\hat{i} - 9\hat{j} - 16\hat{k}$.

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. $\nabla\left(x^3 + y^3 + z^3 - 3xyz\right) =$ ________

____

____

____

3.2.7 Example

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$, then prove that

(i) $\nabla \log|\vec{r}| = \frac{1}{r^2}\vec{r}$ (ii) $\nabla r^n = n.r^{n-2}\vec{r}$

Solution: Given that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow r^2 = x^2 + y^2 + z^2$.

Differentiating w.r.t. x, y, z respectively,

$$2r\frac{\partial r}{\partial x} = 2x,\ 2r\frac{\partial r}{\partial y} = 2y,\ 2r\frac{\partial r}{\partial z} = 2z$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r} \qquad \text{...... (1)}$$

Now, (i) $\nabla \log|\vec{r}| = \nabla \log r$

$$= \hat{i}\frac{\partial}{\partial x}(\log r) + \hat{j}\frac{\partial}{\partial y}(\log r) + \hat{k}\frac{\partial}{\partial z}(\log r)$$

$$= \hat{i}\frac{1}{r}\frac{\partial r}{\partial x} + \hat{j}\frac{1}{r}\frac{\partial r}{\partial y} + \hat{k}\frac{1}{r}\frac{\partial r}{\partial z}$$

$$= \hat{i}\frac{1}{r}.\frac{x}{r} + \hat{j}\frac{1}{r}.\frac{y}{r} + \hat{k}\frac{1}{r}.\frac{z}{r}$$

$$= \frac{1}{r^2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \frac{\vec{r}}{r^2}.$$

(ii) $\nabla r^n = \hat{i}\frac{\partial}{\partial x}r^n + \hat{j}\frac{\partial}{\partial y}r^n + \hat{k}\frac{\partial}{\partial z}r^n$

$$= \hat{i}\,n r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\ n r^{n-1}\frac{\partial r}{\partial y} + \hat{k}\ n r^{n-1}\frac{\partial r}{\partial z}$$

$$= n r^{n-1}\frac{x}{r}\hat{i} + n r^{n-1}\frac{y}{r}\hat{j} + n r^{n-1}\frac{z}{r}\hat{k}$$

$$= n r^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = n r^{n-2}\vec{r}.$$

3.2.8 Example

If $\phi(x,y)=\log_e\sqrt{x^2+y^2}$ and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then prove that

$$\text{grad } \phi=\frac{\vec{r}-(\vec{r}.\hat{k})\hat{k}}{\left[\vec{r}-(\vec{r}.\hat{k})\hat{k}\right].\left[\vec{r}-(\vec{r}.\hat{k})\hat{k}\right]}.$$

Solution: Given that $\phi(x,y)=\log_e\sqrt{x^2+y^2}$ and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

$$\text{grad } \phi=\nabla\phi$$

$$=\nabla\log\sqrt{x^2+y^2}=\nabla\log\left(x^2+y^2\right)^{\frac{1}{2}}$$

$$=\nabla\frac{1}{2}\log\left(x^2+y^2\right)=\frac{1}{2}\nabla\log\left(x^2+y^2\right)$$

$$=\frac{1}{2}\left[\hat{i}\frac{\partial}{\partial x}\left(x^2+y^2\right)+\hat{j}\frac{\partial}{\partial y}\left(x^2+y^2\right)+\hat{k}\frac{\partial}{\partial z}\left(x^2+y^2\right)\right]$$

$$=\frac{1}{2}\left[\frac{2x}{x^2+y^2}\hat{i}+\frac{2y}{x^2+y^2}\hat{j}+0\hat{k}\right]$$

$$=\frac{x\hat{i}+y\hat{j}}{x^2+y^2}$$

$$=\frac{x\hat{i}+y\hat{j}}{(x\hat{i}+y\hat{j}).(x\hat{i}+y\hat{j})} \qquad \text{...... (1)}$$

Since $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $\vec{r}-z\hat{k}=x\hat{i}+y\hat{j}$

also $\vec{r}.\hat{k}=\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\hat{k}$

$$=x\left(\hat{i}.\hat{k}\right)+y\left(\hat{j}.\hat{k}\right)+z\left(\hat{k}.\hat{k}\right)$$

$$=x.0+y.0+z.1=z.$$

$$\therefore x\hat{i}+y\hat{j}=\vec{r}-z\hat{k}=\vec{r}-(\vec{r}.\hat{k})\hat{k} \qquad \text{...... (2)}$$

From (1) and (2),

$$\text{grad } \phi = \frac{\vec{r}-\left(\vec{r}.\hat{k}\right)\hat{k}}{\left[\vec{r}-\left(\vec{r}.\hat{k}\right)\hat{k}\right].\left[\vec{r}-\left(\vec{r}.\hat{k}\right)\hat{k}\right]}.$$

3.2.9 Example

If $f = x+y+z, g = x^2+y^2+z^2, h = xy+yz+zx$, prove that

grad f. [grad $g \times$ grad h] = $\vec{0}$.

Solution: Given $f = x+y+z$.

Differentiating f w.r.t. x, y, z partially, we get

$$\frac{\partial f}{\partial x}=1, \frac{\partial f}{\partial y}=1, \frac{\partial f}{\partial z}=1,$$

$$\therefore \text{ grad } f = \hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z} = \hat{i}+\hat{j}+\hat{k}.$$

Given $g = x^2+y^2+z^2$.

Differentiating g w.r.t. x, y, z partially, we get

$$\frac{\partial g}{\partial x}=2x, \frac{\partial g}{\partial y}=2y, \frac{\partial g}{\partial z}=2z.$$

$$\therefore \text{ grad } g = \hat{i}\frac{\partial g}{\partial x}+\hat{j}\frac{\partial g}{\partial y}+\hat{k}\frac{\partial g}{\partial z} = 2x\hat{i}+2y\hat{j}+2z\hat{k}.$$

Given $h = xy+yz+zx$.

Differentiating h w.r.t. x, y, z partially, we get

$$\frac{\partial h}{\partial x}=y+z, \frac{\partial h}{\partial y}=x+z, \frac{\partial h}{\partial z}=y+x.$$

$$\therefore \text{ grad } h = \hat{i}\frac{\partial h}{\partial x}+\hat{j}\frac{\partial h}{\partial y}+\hat{k}\frac{\partial h}{\partial z} = (y+z)\hat{i}+(x+z)\hat{j}+(x+y)\hat{k}.$$

grad f. [grad g x grad h] = [grad f, grad g, grad h]

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 1\left[2y(x+y) - 2z(x+z)\right] - 1\left[2x(x+y) - 2z(y+z)\right] + 1\left[2x(x+z) - 2y(y+z)\right]$$

$$= 2xy + 2y^2 - 2xz - 2z^2 - 2x^2 - 2xy - 2yz - 2z^2 + 2x^2 + 2xz - 2y^2 - 2yz = 0 .$$

We now introduce the idea of directional derivative and see how gradient can be used to find this.

3.3 DIRECTIONAL DERIVATIVE

3.3.1 Definition: (Level Surface)

Let $\phi(x, y, z)$ be a continuous scalar point function defined in a region *R* of space. If $\phi(x, y, z)$ has the same value at each point of the surface drawn through any point $P(x, y, z)$ of the region *R*, then such a surface is called a **level surface**. It is denoted by $\phi(x, y, z) = c$. Here *c* is a parameter. It is a constant for a particular level surface, but has different values for different level surfaces.

3.3.2 Theorem

$\nabla\phi$ is a vector normal to the level surface $\phi(x, y, z) = c$ (*c* is a constant).

Proof: Let $P(x, y, z)$ and $Q(x+\Delta x, y+\Delta y, z+\Delta z)$ be two points on the level surface.

Then, the position vector of $P = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and

the position vector of $Q = \vec{r} + \Delta\vec{r}$

$\therefore \overrightarrow{PQ} = \Delta\vec{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}$ ...... (1)

$$= \left[(x+\Delta x)\hat{i} + (y+\Delta y)\hat{j} + (z+\Delta z)\hat{k}\right]$$

As $Q \to P$, the straight line *PQ* becomes a tangent to the surface at *P*.

Taking limit as $Q \to P$, from (1), we get

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ ...... (2)

which lies along the tangent at *P*.

$$\therefore \nabla Q.d\vec{r} = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \right).\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \right)$$

$$= \frac{\partial \phi}{\partial x}\partial x + \frac{\partial \phi}{\partial y}\partial y + \frac{\partial \phi}{\partial z}\partial z = d\phi \qquad \text{...... (3)}$$

Since $\phi(x, y, z)$ is a level surface, the values of ϕ at P and Q are both equal to c.

So $d\phi = 0$.

Thus, we get $\nabla \phi.d\vec{r} = 0$.

Hence $\nabla \phi$ is a vector perpendicular to $d\vec{r}$, which is along the tangent line.

Thus, $\nabla \phi$ is a vector normal to the level surface $\phi(x, y, z) = c$.

3.3.3 Definition: (Directional Derivative of a Scalar Point Function)

Let $\phi(x, y, z)$ be a scalar point function defined and continuous on a region R in space. Let $\hat{a}$ be a given union vector and $P(x, y, z)$ be any point in R. Let Q be a point on the line through P, drawn in the direction of the unit vector $\hat{a}$. If $\lim\limits_{Q \to P} \frac{\phi(Q) - \phi(P)}{PQ}$ exists, then it is called the **directional derivative of** ϕ in the direction of the unit vector $\hat{a}$.

3.3.4 Remark:

In the above definition, let $P(x, y, z)$ and $Q(x + \Delta x, y + \Delta y, z + \Delta z)$. Let s denotes the distance of P from some fixed point on the line through P, in the direction of $\hat{a}$ and $PQ = \Delta s$.

Then the directional derivative of ϕ in the direction of $\hat{a} = \lim\limits_{Q \to P} \frac{\phi(Q) - \phi(P)}{PQ}$

$$= \lim_{\Delta s \to 0} \frac{\phi(x + \Delta x, y + \Delta y, z + \Delta x) - \phi(x, y, z)}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s} = \frac{d\phi}{ds}.$$

Thus, the directional derivative represents the rate of change of ϕ w.r.t distance s in the direction of $\hat{a}$.

3.3.5 Theorem: (Calculation of Directional Derivative)

The directional derivative of a scalar point function ϕ in the direction of the unit vector $\hat{a}$ is given by $\frac{d\phi}{ds} = \nabla\phi.\hat{a}$.

Proof: Let s be the distance of $P(x, y, z)$ from a fixed point on the line through P in the direction of $\hat{a}$.

Let x, y, z be functions of the parameter s.

Now, the position vector of $P = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\hat{a} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$$

$$\nabla\phi.\hat{a} = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right).\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right)$$

$$= \frac{\partial\phi}{\partial s}\hat{i} + \frac{\partial\phi}{\partial s}\hat{j} + \frac{\partial\phi}{\partial s}\hat{k} = \frac{d\phi}{ds}.$$

= The directional derivative of ϕ in the direction of $\hat{a}$.

3.3.6 Note

(i) Let $\phi(x, y, z) = c$ be a level surface. Let $\hat{a}$ be a unit vector along the tangent line to the surface at a given point $P(x, y, z)$ on it. From (3.3.2), $\nabla\phi$ is a vector normal to the surface. So, $\nabla\phi$ is a vector normal to $\hat{a}$ also.

$\therefore \nabla\phi.\hat{a} = 0$.

Hence, the directional derivative of a scalar point function in the direction of the tangent line is zero.

(ii) In theorem (3.3.5), taking $\hat{a} = \hat{i}, \hat{j}$ and $\hat{k}$ respectively.

The directional derivative of ϕ along X - axis $= (\nabla\phi.\hat{i})$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).\hat{i} = \frac{\partial\phi}{\partial x}.$$

Similarly, the directional derivative of ϕ along Y - axis $= (\nabla\phi.\hat{j}) = \frac{\partial\phi}{\partial y}$

and the directional derivative of ϕ along Z - axis $=\left(\nabla\phi.\hat{k}\right)=\frac{\partial\phi}{\partial z}$.

3.3.7 Theorem

Let $\hat{n}$ be a unit vector normal to the level surface $\phi(x,y,z)=c$, at any point $P(x,y,z)$ in the direction of *s* increasing where *s* is the distance of *P* from any fixed point in the direction of $\hat{n}$. Then $\nabla\phi=\frac{d\phi}{ds}\hat{n}$.

Proof: From (3.3.2), we know that $\nabla\phi$ is a vector normal to the level surface $\phi(x,y,z)=c$.

Thus $\nabla\phi$ and $\hat{n}$ are both normal to the surface.

$\therefore$ There exists a scalar k such that $\nabla\phi=k.\hat{n}$ ...... (1)

$$\therefore \nabla\phi.\hat{n}=\frac{d\phi}{ds}\hat{n}.\hat{n}$$

From (3.3.5),

$$=\frac{d\phi}{ds}(\hat{n}.\hat{n})=\frac{d\phi}{ds} \quad (\because \hat{n} \text{ is a unit vector, } \hat{n}.\hat{n}=1) \qquad \text{...... (2)}$$

Also, $\nabla\phi.\hat{n}=k\,\hat{n}.\hat{n}=k$ ...... (3)

From (1), (2) and (3),

$$\nabla\phi=k\,\hat{n}=\frac{d\phi}{ds}\hat{n}.$$

3.3.8 Note

From the above theorems, it is clear that directional derivative of ϕ in the direction of a unit vector $\hat{a}$ is $\nabla\phi.\hat{a}=\frac{d\phi}{ds}\hat{n}.\hat{a}$

$=\frac{d\phi}{ds}|\hat{n}||\hat{a}|\cos\theta$, where θ is the angle between $\hat{a}$ and $\hat{n}$.

$$=\left(\frac{d\phi}{ds}\right)\cos\theta.$$

Since $-1\le\cos\theta\le 1$ and the maximum value of $\cos\theta=1$, occurs when $\theta=0^0$, the directional derivative is maximum when $\hat{a}$ is also in the direction of $\hat{n}$.

i.e., directional derivative is maximum along the normal and its magnitude is $\frac{d\phi}{ds} = |\nabla\phi|$.

Thus, $\nabla\phi$ is a vector in the direction of maximum directional derivative and $|\nabla\phi|$ gives the maximum value of directional derivative.

Check Your Progress:

2. A vector perpendicular to any vector that lies on the plane $x+y+z=5$ is

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3.3.9 Example

Find the gradient and unit normal to the surface $x^2+y-z=4$ at (2, 0, 0).

Solution: Let $\phi(x,y,z) = x^2+y-z$.

Gradient = $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = 2x\hat{i} + \hat{j} - \hat{k}$.

At (2, 0, 0), $\nabla\phi = 4\hat{i} + \hat{j} - \hat{k}$.

$|\nabla\phi| = \sqrt{4^2+1^2+(-1)^2} = \sqrt{18} = 3\sqrt{2}$.

Unit normal to the surface at (2, 0, 0) is $\frac{\nabla\phi}{|\nabla\phi|} = \frac{4\hat{i}+\hat{j}-\hat{k}}{3\sqrt{2}}$.

3.3.10 Example

Find the angle between the surfaces $xy^2z = 3x+z^2$ and $3x^2-y^2+2z=1$ at (1, -2, 1).

Solution: Let $\phi_1(x,y,z) = 3x - xy^2z + z^2$ and $\phi_2(x,y,z) = 3x^2 - y^2 + 2z$.

$$\nabla\phi_1 = \hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}$$

$$= \left(3 - y^2z\right)\hat{i} + \left(-2xyz\right)\hat{j} + \left(-xy^2+2z\right)\hat{k}.$$

At (1, -2, 1), $\nabla\phi_1 = -\hat{i} + 4\hat{j} - 2\hat{k}$

$\therefore |\nabla\phi_1| = \sqrt{1+16+4} = \sqrt{21}.$

$$\nabla\phi_2 = \hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z}$$

$$= 6x\hat{i} - 2y\hat{j} + 2\hat{k}.$$

At (1, -2, 1), $\nabla\phi_2 = 6\hat{i} + 4\hat{j} + 2\hat{k}$

$$\therefore |\nabla\phi_2| = \sqrt{36+16+4} = \sqrt{56}.$$

Unit normal to the surface $\phi_1(x,y,z) = 0$ is

$$\hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{-\hat{i}+4\hat{j}-2\hat{k}}{\sqrt{21}}.$$

Unit normal to the surface $\phi_2(x,y,z)$ = 1 is

$$\hat{n}_2 = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{56}}.$$

We know that angle between surfaces is angle between their normals. Thus if θ is the angle between the surfaces, then

$$\cos\theta = \hat{n}_1.\hat{n}_2 = \frac{-\hat{i}+4\hat{j}-2\hat{k}}{\sqrt{21}}.\frac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{56}}$$

$$= \frac{-6+16-4}{\sqrt{21}\sqrt{56}} = \frac{6}{7\sqrt{24}} = \frac{6}{7\times 2\sqrt{6}} = \frac{3}{7\sqrt{16}}$$

$$\therefore \theta = \cos^{-1}\frac{3}{7\sqrt{16}}.$$

3.3.11 Example

Find the directional derivative of the scalar field $2x + y + z^2$ in the direction of the vector (1, 1, 1) and evaluate this at the origin.

Solution: Let $\phi(x,y,z) = 2x + y + z^2$.

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = 2\hat{i} + \hat{j} + 2z\hat{k}.$$

Given vector = $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

Unit vector in the direction of $\vec{a} = \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{1+1+1}} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$.

Directional derivative in the direction $\hat{a} = \nabla\phi.\hat{a}$

$$= \left(2\hat{i} + \hat{j} + 2z\hat{k}\right).\frac{\left(\hat{i}+\hat{j}+\hat{k}\right)}{\sqrt{3}} = \frac{2+1+2z}{\sqrt{3}} = \frac{3+2z}{\sqrt{3}} .$$

At the origin, directional derivative in the direction of $\hat{a} = \frac{3}{\sqrt{3}} = \sqrt{3}$.

3.3.12 Example

Find the directional derivative of the function $\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}$ at the point (3, 1, 2) in the direction of the vector $yz\hat{i} + zx\hat{j} + xy\hat{k}$.

Solution: Let $\phi(x,y,z) = \left(x^2+y^2+z^2\right)^{\frac{-1}{2}}$

Then $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$= \hat{i}\left[-\frac{1}{2}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}.2x\right] + \hat{j}\left[-\frac{1}{2}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}.2y\right]$$

$$+\hat{k}\left[-\frac{1}{2}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}.2z\right]$$

$$= -\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} .$$

At (3, 1, 2), $\nabla\phi = \frac{-\left(3\hat{i}+\hat{j}+2\hat{k}\right)}{\left(9+1+4\right)^{\frac{3}{2}}} = \frac{-\left(3\hat{i}+\hat{j}+2\hat{k}\right)}{14\sqrt{14}}$.

Given direction: $\vec{a} = yz\hat{k} + zx\hat{j} + xy\hat{k}$.

Unit vector in the direction of $\vec{a} = \hat{a} = \dfrac{yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}}{\sqrt{y^2z^2 + z^2x^2 + x^2y^2}}$

At (3, 1, 2), $\hat{a} = \dfrac{2\hat{i} + 6\hat{j} + 3\hat{k}}{\sqrt{4+36+9}} = \dfrac{2\hat{i} + 6\hat{j} + 3\hat{k}}{7}$.

$\therefore$ Directional derivative of ϕ in the direction of $\hat{a} = \nabla\phi.\hat{a}$

$$= -\frac{\left(3\hat{i} + \hat{j} + 2\hat{k}\right)}{14\sqrt{14}} \times \frac{2\hat{i} + 6\hat{j} + 3\hat{k}}{7} = -\frac{6+6+6}{98\sqrt{14}} = \frac{-18}{98\sqrt{14}} = \frac{-9}{49\sqrt{14}}.$$

3.3.13 Example

Find the constants *m* and *n* such that the surface $mx^2 - 2nyz = (m+4)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, -1, 2).

Solution: The point (1, -1, 2) lies on both the surfaces.

$\therefore m + 4n = m + 4 \Rightarrow 4n = 4 \Rightarrow n = 1$.

Let $\phi_1(x, y, z) = mx^2 - 2nyz - (m+4)x$

Since $n = 1$, $\phi_1(x, y, z) = mx^2 - 2yz - (m+4)x$.

$\nabla\phi_1 = \left[2mx - (m+4)\right]\hat{i} + \left[-2z\right]\hat{j} + \left[-2y\right]\hat{k}$

At (1, -1, 2), $\nabla\phi_1 = (2m - m - 4)\hat{i} - 4\hat{j} + 2\hat{k} = (m-4)\hat{i} - 4\hat{j} + 2\hat{k}$.

Let $\phi_2(x, y, z) = 4x^2y + z^3 - 4$

$\nabla\phi_2 = 8xy\,\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

At (1, -1, 2), $\nabla\phi_2 = -8\hat{i} + 4\hat{j} + 12\hat{k}$.

If the surfaces are orthogonal, then their normals are perpendicular to each other.

$\therefore \nabla\phi_1.\nabla\phi_2 = 0$.

$\Rightarrow \left((m-4)\hat{i} - 4\hat{j} + 2\hat{k}\right).\left(-8\hat{i} + 4\hat{j} + 12\hat{k}\right) = 0$

$\Rightarrow -8(m-4) - 16 + 24 = 0$

$$\Rightarrow -8(m-4)+8=0 \Rightarrow -m+4+1=0 \Rightarrow m=5\cdot$$

Hence $m = 5$ and $n = 1$.

3.3.14 Example

Find the direction in which the directional derivative of $\phi(x,y)=\frac{(x^2+y^2)}{xy}$ at (1, 1) is zero.

Solution: Given $\phi(x,y)=\frac{(x^2+y^2)}{xy}$.

$$\frac{\partial\phi}{\partial x}=\frac{xy(2x)-(x^2+y^2)y}{x^2y^2}$$

$$=\frac{2x^2y-x^2y-y^3}{x^2y^2}=\frac{x^2y-y^3}{x^2y^2}$$

$$\frac{\partial\phi}{\partial y}=\frac{xy(2y)-(x^2+y^2)x}{x^2y^2}$$

$$=\frac{2xy^2-x^3-xy^2}{x^2y^2}=\frac{xy^2-x^3}{x^2y^2}$$

$$\nabla\phi=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}=\frac{x^2y-y^3}{x^2y^2}\hat{i}+\frac{xy^2-x^3}{x^2y^2}\hat{j}$$

At (1, 1), $\nabla\phi=0\hat{i}+0\hat{j}=\vec{0}$.

If $\hat{a}$ is any unit vector then $\nabla\phi.\hat{a}=\vec{0}.\hat{a}=0$.

$\therefore$ The directional derivative of ϕ at (1, 1) is zero in any direction.

3.3.15 Example

Find the greatest value of the directional derivative of the scalar point function $2x^2-y-z^4$ at the point (2, -1, 1).

Solution: Let $\phi(x,y,z)=2x^2-y-z^4$.

$$\nabla\phi = 4x\hat{i} - \hat{j} - 4z^3\hat{k}$$

At (2, -1, 1), $\nabla\phi = 8\hat{i} - \hat{j} - 4\hat{k}$.

Greatest value of directional derivative $= |\nabla\phi|$

$$= \sqrt{64+1+16} = \sqrt{81} = 9.$$

3.3.16 Example

In what direction from the point (2, 3, -1) is the directional derivative of $\phi = x^2y^3z^4$ maximum? Find the value of this maximum derivative.

Solution: Given $\phi(x, y, z) = x^2y^3z^4$.

$$\nabla\phi = \hat{i}\left(2xy^3z^4\right) + \hat{j}\left(3x^2y^2z^4\right) + \hat{k}\left(4x^2y^3z^3\right)$$

At (2, 3, -1), $\nabla\phi = 108\hat{i} + 108\hat{j} - 108\hat{k} \ = 108\left(\hat{i} + \hat{j} - \hat{k}\right)$

The directional derivative is maximum in the direction of $\nabla\phi$. i.e., along $108\left(\hat{i} + \hat{j} - \hat{k}\right)$.

Maximum value of this derivative $= |\nabla\phi| = 108\sqrt{1+1+1} = 108\sqrt{3}$.

3.3.17 Physical Applications of Gradient

The gradient of a scalar point function has many important applications. As we have seen above, it can be used to find the normals to surfaces and obtain rates of change of functions in any direction. If a vector field $\vec{F}$ is related to a scalar field ϕ by $\vec{F} = \nabla\phi$ then we say $\vec{F}$ is conservative. In this case the corresponding scalar field ϕ is called the potential for $\vec{F}$. Gradient is useful in finding the potential functions. We give below two physical contexts where gradient plays an important role.

(a) Let P denotes the pressure within a gas. Then the force $\vec{F}$ acting on any element of volume δv is given by $\vec{F} = -\nabla P\,\delta v$.

(b) Suppose a material with constant thermal conductivity k has a variable temperature $T(\vec{r})$. Because of temperature variation, heat flows from hot regions to the cold regions. The heat flux q is a vector quantity given by $q = -k\nabla T$.

3.4 DIVERGENCE OF A VECTOR POINT FUNCTION

3.4.1 Definition

Let $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector point function defined and differentiable at each point of a region R in space. The **divergence of** $\vec{F}$ is denoted by $div\,\vec{F}$ or $\nabla.\vec{F}$ and defined as $\text{div}\,\vec{F} = \nabla.\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

3.4.2 Note

(i) Divergence of a vector point function $\vec{F}$ is a scalar point function.

(ii) The operator used in defining divergence can be written as

$$\nabla = \hat{i}.\frac{\partial}{\partial x} + \hat{j}.\frac{\partial}{\partial y} + \hat{k}.\frac{\partial}{\partial z}$$

(iii) $\nabla.\vec{F} \neq \vec{F}.\nabla$

3.4.3 Theorem

(i) Let $\vec{F}$ be a differentiable vector point function and C be a constant.

Then $\text{div}\left(C\vec{F}\right) = \nabla.\left(C\vec{F}\right) = C\nabla.\vec{F} = C\,\text{div}\,\vec{F}$

(ii) If $\vec{C}$ is a constant vector then $\text{div}\,\vec{C} = 0$.

Proof:

(i) Let $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

Then $C\vec{F} = CF_1\hat{i} + CF_2\hat{j} + CF_3\hat{k}$

$\text{div}\left(C\vec{F}\right) = \nabla.C\vec{F}$

$$= \frac{\partial(CF_1)}{\partial x} + \frac{\partial(CF_2)}{\partial y} + \frac{\partial(CF_3)}{\partial z}$$

$$= C\frac{\partial F_1}{\partial x} + C\frac{\partial F_2}{\partial y} + C\frac{\partial F_3}{\partial z}$$

$$= C\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

$$= C\,\text{div}\,\vec{F}\ .$$

(ii) Let $\vec{C} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$, where C_1, C_2, C_3 are constants.

Then $\text{div}\,\vec{C} = \frac{\partial C_1}{\partial x} + \frac{\partial C_2}{\partial y} + \frac{\partial C_3}{\partial z} = 0 + 0 + 0 = 0\cdot$

3.4.4 Definition

A vector $\vec{F}$ is said to be solenoidal if $\text{div}\,\vec{F} = 0$.

3.4.5 Example

Evaluate $\text{div}\left[\left(x^2 - y^2\right)\hat{i} + 2xy\,\hat{j} + \left(y^2 - 2xy\right)\hat{k}\right]$.

Solution: Let $\vec{F}(x, y, z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$= \left(x^2 - y^2\right)\hat{i} + 2xy\,\hat{j} + \left(y^2 - 2xy\right)\hat{k}\ .$$

Then $F_1 = x^2 - y^2,\ F_2 = 2xy,\ F_3 = y^2 - 2xy$.

$$\frac{\partial F_1}{\partial x} = 2x,\ \frac{\partial F_2}{\partial y} = 2x,\ \frac{\partial F_3}{\partial z} = 0$$

$$\therefore \text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x + 2x + 0 = 4x.$$

3.4.6 Example

If $u = x^2y^3z^4$, find div grad u i.e., $\nabla.\left(\nabla u\right)$.

Solution: By definition of gradient,

$$\nabla u = \hat{i}\frac{\partial}{\partial x}\left(x^2y^3z^4\right) + \hat{j}\frac{\partial}{\partial y}\left(x^2y^3z^4\right) + \hat{k}\frac{\partial}{\partial z}\left(x^2y^3z^4\right)$$

$$= 2xy^3z^4\hat{i} + 3x^2y^2z^4\,\hat{j} + 4x^2y^3z^3\hat{k}\ .$$

By definition of divergence,

$$\text{div (grad } u) = \nabla.(\nabla u) = \frac{\partial}{\partial x}\left(2xy^3z^4\right) + \frac{\partial}{\partial y}\left(3x^2y^2z^4\right) + \frac{\partial}{\partial z}\left(4x^2y^3z^3\right)$$

$$= 2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^2 .$$

3.4.7 Example

Show that the vector $\vec{F} = 3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

Solution: By definition of divergence,

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(3y^4z^2\right) + \frac{\partial}{\partial y}\left(4x^3z^2\right) + \frac{\partial}{\partial z}\left(-3x^2y^2\right)$$

$$= 0 + 0 + 0 = 0.$$

Since div $\vec{F} = 0$, $\vec{F}$ is solenoidal.

Check Your Progress:

3. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find div $\vec{r}$.

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4. Show that $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ is solenoidal.

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3.4.8 Theorem

If $\vec{F}, \vec{G}$ are differentiable vector point functions, then $\nabla.\left(\vec{F} + \vec{G}\right) = \nabla.\vec{F} + \nabla.\vec{G}$.

Proof: Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\vec{G} = G_1\hat{i} + G_2\hat{j} + G_3\hat{k}$ be differentiable vector point functions.

Then $\vec{F} + \vec{G} = \left(F_1 + G_1\right)\hat{i} + \left(F_2 + G_2\right)\hat{j} + \left(F_3 + G_3\right)\hat{k}$.

By definition of divergence,

$$\nabla.\left(\vec{F}+\vec{G}\right)=\frac{\partial}{\partial x}\left(F_1+G_1\right)+\frac{\partial}{\partial y}\left(F_2+G_2\right)+\frac{\partial}{\partial z}\left(F_3+G_3\right)$$

$$=\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)+\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}+\frac{\partial G_3}{\partial z}\right)$$

$$=\nabla.\vec{F}+\nabla.\vec{G}\cdot$$

3.4.9 Note

If a, b are constants and $\vec{F},\vec{G}$ are differentiable vector point functions, then by theorem (3.4.8), we have $\nabla.\left(a\vec{F}+b\vec{G}\right)=\nabla.\left(a\vec{F}\right)+\nabla.\left(b\vec{G}\right)$. Using theorem 3.4.3 (i), we get

$$\nabla.\left(a\vec{F}+b\vec{G}\right)=\nabla.\left(a\vec{F}\right)+\nabla.\left(b\vec{G}\right)=a\nabla.\vec{F}+b\nabla.\vec{G}\,.$$

This is known as linearity property of divergence.

3.4.10 Theorem

Let $\vec{F}(x,y,z)$ be a vector point function and $\phi(x,y,z)$ be a scalar point function. Then $\nabla.\left(\phi\vec{F}\right)=\nabla\phi.\vec{F}+\phi\left(\nabla.\vec{F}\right)$.

Proof: Let $\vec{F}(x,y,z)=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ be a vector point function.

Then $\phi\vec{F}=\phi F_1\hat{i}+\phi F_2\hat{j}+\phi F_3\hat{k}$.

By definition of divergence,

$$\nabla.\left(\phi\vec{F}\right)=\frac{\partial}{\partial x}(\phi F_1)+\frac{\partial}{\partial y}(\phi F_2)+\frac{\partial}{\partial z}(\phi F_3)$$

$$=\left(\frac{\partial\phi}{\partial x}F_1+\phi\frac{\partial F_1}{\partial x}\right)+\left(\frac{\partial\phi}{\partial y}F_2+\phi\frac{\partial F_2}{\partial y}\right)+\left(\frac{\partial\phi}{\partial z}F_3+\phi\frac{\partial F_3}{\partial z}\right)$$

$$=\left(\frac{\partial\phi}{\partial x}F_1+\frac{\partial\phi}{\partial y}F_2+\frac{\partial\phi}{\partial z}F_3\right)+\left(\phi\frac{\partial F_1}{\partial x}+\phi\frac{\partial F_2}{\partial y}+\phi\frac{\partial F_3}{\partial z}\right)$$

$$=\left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right).\left(F_1\hat{i}+F_2\hat{j}+F_3\hat{k}\right)+\phi\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)$$

$$=\nabla\phi.\vec{F}+\phi\left(\nabla.\vec{F}\right).$$

3.4.11 Example

Prove that $\nabla\left(\frac{1}{r}\vec{r}\right)=\frac{2}{r}$ where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $|\vec{r}|=r$.

Solution: Given $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $r=|\vec{r}|=\sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x}=\frac{1}{2\sqrt{x^2+y^2+z^2}}.2x=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}.$$

In theorem 3.4.10, taking $\phi=\frac{1}{r}$ and $\vec{F}=\vec{r}$, we get

$$\nabla\left(\frac{1}{r}\vec{r}\right)=\nabla\left(\frac{1}{r}\right).\vec{r}+\frac{1}{r}(\nabla.\vec{r}) \qquad \text{... (1)}$$

$$\nabla\left(\frac{1}{r}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$=\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$=-\frac{1}{r^2}\frac{\partial r}{\partial x}\hat{i}-\frac{1}{r^2}\frac{\partial r}{\partial y}\hat{j}-\frac{1}{r^2}\frac{\partial r}{\partial z}\hat{k}$$

$$=-\frac{1}{r^2}\left[\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right]$$

$$=-\frac{1}{r^3}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$\nabla\left(\frac{1}{r}\right).\vec{r}=-\frac{1}{r^3}\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$=-\frac{1}{r^3}\left(x^2+y^2+z^2\right)=-\frac{r^2}{r^3}=-\frac{1}{r}.$$

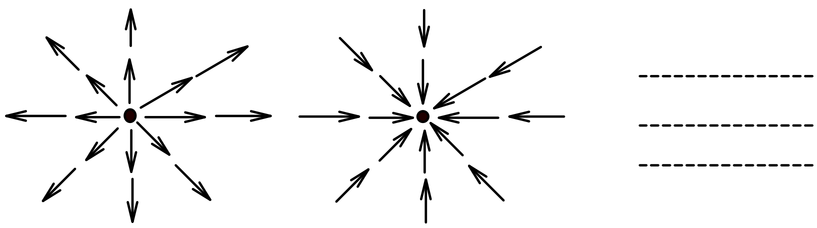
$$\nabla.\vec{r}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)$$

= 1 + 1 + 1= 3.

Substituting in (1), $\nabla\left(\frac{1}{r}.\vec{r}\right)=-\frac{1}{r}+\frac{1}{r}.3=\frac{2}{r}$.

3.4.12 Physical Interpretation of Divergence

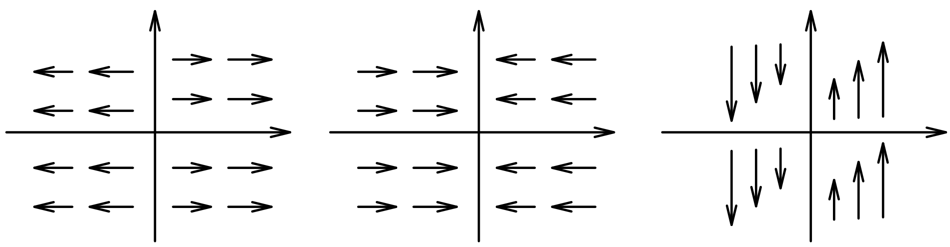
The divergence of a vector point function $\vec{F}$ at a point is a measure of how much $\vec{F}$ expands or stretches at that point. A point where divergence is positive is known as a **source** and a point where divergence is negative is known as a **sink**.



(a) Source (b) Sink (c) div$\vec{F}=0$

Fig: 3.1

For instance, consider the vector field given by $\vec{F}(x,y,z)=(x,0,0)=x\hat{i}+0\hat{j}+0\hat{k}$. For convenience, suppose that this vector field represents the motion of a gas. Fig 3.2 (a) represents the expansion of a gas and divergence of $\vec{F}$ is 1 everywhere. Fig 3.2(b) represents the vector field $-\vec{F}=(-x,0,0)=-x\hat{i}+0\hat{j}+0\hat{k}$. This shows contraction of a gas and its divergence is -1. Fig 3.2 (c) represents the vector field $\vec{G}=(0,x,0)=0\hat{i}+x\hat{j}+0\hat{k}$. This vector field neither expands nor contracts and its divergence is zero.



$(a)\ \vec{F}=(x,0,0)$ $(b)\ -\vec{F}=(-x,0,0)$ $(c)\ \vec{G}=(0,x,0)$

Fig: 3.2

3.5 CURL OF A VECTOR POINT FUNCTION

3.5.1 Definition

If $\vec{F}(x,y,z)=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ is a differentiable vector point function, then the **curl of** $\vec{F}$, denoted by curl $\vec{F}$ or $\nabla\times\vec{F}$ is defined as,

$$curl\,\vec{F}=\nabla\times\vec{F}$$

$$=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\times\left(F_1\hat{i}+F_2\hat{j}+F_3\hat{k}\right)$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}=\hat{i}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+\hat{j}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+\hat{k}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right).$$

3.5.2 Example

If $\vec{F}(x,y,z)=xz^3\hat{i}-2x^2yz\,\hat{j}+2yz^4\hat{k}$, find curl $\vec{F}$ at the point (1, -1, 1).

Solution: By the definition of $\text{curl}\,\vec{F}$,

$$\text{curl}\,\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ xz^3 & -2x^2yz & 2yz^4\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}\left(2yz^4\right)+\frac{\partial}{\partial z}\left(2x^2yz\right)\right]-\hat{j}\left[\frac{\partial}{\partial x}\left(2yz^4\right)-\frac{\partial}{\partial z}\left(xz^3\right)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}\left(-2x^2yz\right)-\frac{\partial}{\partial y}\left(xz^3\right)\right]$$

$$=\hat{i}\left[2z^4+2x^2y\right]-\hat{j}\left[0-3xz^2\right]+\hat{k}\left[-4xyz-0\right]$$

$$=\left(2z^4+2x^2y\right)\hat{i}+3xz^2\,\hat{j}+4xyz\,\hat{k}\,.$$

At (1, -1, 1), $\text{curl}\,\vec{F}=\left(2+2(-1)\hat{i}+3\,\hat{j}-4(-1)\hat{k}\right)=3\,\hat{j}+4\hat{k}$.

Check Your Progress:

5. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find curl $\vec{r}$.

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3.5.3 Definition

If $\vec{F}(x, y, z)$ is a vector point function and $\nabla \times \vec{F} = \vec{0}$, then $\vec{F}$ is said to be irrotational.

3.5.4 Example

Show that the vector point function

$\vec{F}(x, y, z) = \left(y^2 - z^2 + 3yz - 2x\right)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is irrotational.

Solution: Given $\vec{F} = \left(y^2 - z^2 + 3yz - 2x\right)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}\left(y^2 - z^2 + 3yz - 2x\right)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}\left(y^2 - z^2 + 3yz - 2x\right)\right]$$

$$= \hat{i}[3x - 3x] - \hat{j}[3y - 2z - (-2z + 3y)] + \hat{k}[3z + 2y - (2y + 3z)]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}.$$

$\therefore \vec{F}$ is irrotational.

Check Your Progress:

6. Check whether the function $\vec{F}(x,y,z) = y\hat{i} + x\hat{j} + z\hat{k}$ is irrotational.

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3.5.5 Definition

If $\nabla \times \vec{F} = \vec{0}$, then we can find a scalar point function ϕ such that $\vec{F} = \nabla\phi$. In this case the $\vec{F}$ is called a **conservative vector field** and ϕ is called the **scalar potential**.

3.5.6 Example

Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative field. Find the scalar potential function ϕ.

Solution: Given $\vec{F}(x,y,z) = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy + z^3)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3)\right]$$

$$= \hat{i}[2x - 2x] - \hat{j}\left[3z^2 - 3z^2\right] + \hat{k}[2x - 2x] = \vec{0}.$$

$\therefore \vec{F}$ is a conservative field.

So, there exists a scalar point function ϕ such that $\vec{F} = \nabla\phi$.

i.e., $(2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k} = \dfrac{\partial\phi}{\partial x}\hat{i} + \dfrac{\partial\phi}{\partial y}\hat{j} + \dfrac{\partial\phi}{\partial z}\hat{k}$.

$$\Rightarrow \frac{\partial\phi}{\partial x} = 2xy + z^3,\ \frac{\partial\phi}{\partial y} = x^2,\ \frac{\partial\phi}{\partial z} = 3xz^2 .$$

On integration, we get

$$\phi = \frac{2x^2}{2}y + xz^3 + h_1(y,z),\ \ \phi = x^2 y + h_2(x,z),\ \phi = 3x\frac{z^3}{3} + h_3(x,y)$$

$$\Rightarrow \phi = x^2 y + xz^3(y,z),\ \phi = x^2 y + h_2(x,z),\ \phi = xz^3 + h_3(x,y)$$

$\Rightarrow \phi(x,y,z) = x^2 y + xz^3$ is the required potential function.

3.5.7 Theorem

If $\vec{F}$ is a differentiable vector point function, ϕ is a differentiable scalar point function, then $\nabla\times(\phi\vec{F}) = \nabla\phi\times\vec{F} + \phi(\nabla\times\vec{F})$.

Proof: Let $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Then $\phi\vec{F} = \phi F_1\hat{i} + \phi F_2\hat{j} + \phi F_3\hat{k}$.

$$\nabla\times(\phi\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi F_1 & \phi F_2 & \phi F_3 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(\phi F_3) - \frac{\partial}{\partial z}(\phi F_2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(\phi F_3) - \frac{\partial}{\partial z}(\phi F_1)\right] + \hat{k}\left[\frac{\partial}{\partial x}(\phi F_2) - \frac{\partial}{\partial y}(\phi F_1)\right]$$

$$= \hat{i}\left[\frac{\partial\phi}{\partial y}F_3 + \phi\frac{\partial F_3}{\partial y} - \frac{\partial\phi}{\partial z}F_2 - \phi\frac{\partial F_2}{\partial z}\right] - \hat{j}\left[\frac{\partial\phi}{\partial x}F_3 + \phi\frac{\partial F_3}{\partial x} - \frac{\partial\phi}{\partial z}F_1 - \phi\frac{\partial F_1}{\partial z}\right]$$

$$+\hat{k}\left[\frac{\partial\phi}{\partial x}F_2 + \phi\frac{\partial F_2}{\partial x} - \frac{\partial\phi}{\partial y}F_1 - \phi\frac{\partial F_1}{\partial y}\right]$$

$$= \hat{i}\left(\frac{\partial\phi}{\partial y}F_3 - \frac{\partial\phi}{\partial z}F_2\right) - \hat{j}\left(\frac{\partial\phi}{\partial x}F_3 - \frac{\partial\phi}{\partial z}F_1\right) + \hat{k}\left(\frac{\partial\phi}{\partial x}F_2 - \frac{\partial\phi}{\partial y}F_1\right)$$

$$+\phi\left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\right] \qquad \text{...... (1)}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} + \phi\left(\nabla \times \vec{F}\right) \qquad \text{(using 3.5.1)}$$

$$= \left(\nabla \phi \times \vec{F}\right) + \phi\left(\nabla \times \vec{F}\right).$$

3.5.8 Theorem

If $\vec{F}, \vec{G}$ are differentiable vector point functions then $\nabla \times \left(\vec{F} + \vec{G}\right) = \nabla \times \vec{F} + \nabla \times \vec{G}$.

Proof: Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\vec{G} = G_1\hat{i} + G_2\hat{j} + G_3\hat{k}$.

Then $\vec{F} + \vec{G} = \left(F_1 + G_1\right)\hat{i} + \left(F_2 + G_2\right)\hat{j} + \left(F_3 + G_3\right)\hat{k}$

$$\nabla \times \left(\vec{F} + \vec{G}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 + G_1 & F_2 + G_2 & F_3 + G_3 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(F_3 + G_3\right) - \frac{\partial}{\partial z}\left(F_2 + G_2\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(F_3 + G_3\right) - \frac{\partial}{\partial z}\left(F_1 + G_1\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(F_2 + G_2\right) - \frac{\partial}{\partial y}\left(F_1 + G_1\right)\right]$$

$$= \hat{i}\left\{\frac{\partial F_3}{\partial y} + \frac{\partial G_3}{\partial y} - \frac{\partial F_2}{\partial z} - \frac{\partial G_2}{\partial z}\right\} - \hat{j}\left\{\frac{\partial F_3}{\partial x} + \frac{\partial G_3}{\partial x} - \frac{\partial F_1}{\partial z} - \frac{\partial G_1}{\partial z}\right\}$$

$$+ \hat{k}\left\{\frac{\partial F_2}{\partial x} + \frac{\partial G_2}{\partial x} - \frac{\partial F_1}{\partial y} - \frac{\partial G_1}{\partial y}\right\}$$

$$= \left[\hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$+ \left[\hat{i}\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right) - \hat{j}\left(\frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial z}\right) + \hat{k}\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\right]$$

$$= \nabla \times \vec{F} + \nabla \times \vec{G}.$$

3.5.9 Note

(i) If C is a constant, and $\vec{F}$ is a vector point function then, $\nabla \times \left(C\vec{F}\right) = C\left(\nabla \times \vec{F}\right)$.

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

$$\nabla \times \left(C\vec{F}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ CF_1 & CF_2 & CF_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(CF_3) - \frac{\partial}{\partial z}(CF_2)\right) - \hat{j}\left(\frac{\partial}{\partial x}(CF_3) - \frac{\partial}{\partial z}(CF_1)\right) + \hat{k}\left(\frac{\partial}{\partial x}(CF_2) - \frac{\partial}{\partial y}(CF_1)\right)$$

$$= \hat{i}\left[C\frac{\partial F_3}{\partial y} - C\frac{\partial F_2}{\partial z}\right] - \hat{j}\left[C\frac{\partial F_3}{\partial x} - C\frac{\partial F_1}{\partial z}\right] + \hat{k}\left[C\frac{\partial F_2}{\partial x} - C\frac{\partial F_1}{\partial y}\right]$$

$$= C\left[\hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$= C\left(\nabla \times \vec{F}\right)$ by definition of curl $\vec{F}$.

(ii) If a, b are constants and $\vec{F}, \vec{G}$ are vector point functions, then

$$\nabla \times \left(a\vec{F} + b\vec{G}\right) = a\left(\nabla \times \vec{F}\right) + b\left(\nabla \times \vec{G}\right).$$

This can be proved using the above note.

$\nabla \times \left(a\vec{F} + b\vec{G}\right) = \nabla \times \left(a\vec{F}\right) + \nabla \times \left(b\vec{G}\right)$ [using theorem (3.5.8)]

$= a\left(\nabla \times \vec{F}\right) + b\left(\nabla \times \vec{G}\right)$ [using note (3.5.9 (i))]

Check Your Progress:

7. Prove Note (i) of (3.5.9) by using theorem (3.5.7).

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3.5.10 Physical interpretation of Curl

Let us consider the rotation of a rigid body. Let 'O' be the origin of reference and suppose a rigid body is rotating about an axis through 'O' with a uniform angular velocity $\vec{w}$. Let $P(x, y, z)$ be any point on this body with position vector $\vec{r}$. Since $\vec{w}$ is constant for all points of the body, $\vec{w}$ is a constant vector. So, we can take $\vec{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$, where w_1, w_2, w_3 are constants. Also, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

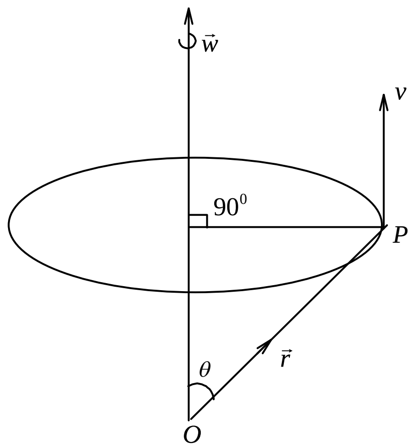

Fig: 3.3

$$\therefore \text{ Velocity of } P = \vec{v} = \vec{w}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(w_2 z - w_3 y\right) - \hat{j}\left(w_1 z - w_3 x\right) + \hat{k}\left(w_1 y - w_2 x\right)$$

$$\text{curl}\,\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(w_1 y - w_2 x\right) - \frac{\partial}{\partial z}\left(w_3 x - w_1 z\right)\right] - \hat{j}\left(\frac{\partial}{\partial x}\left(w_1 y - w_2 x\right) - \frac{\partial}{\partial z}\left(w_2 z - w_3 y\right)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}\left(w_3 x - w_1 z\right) - \frac{\partial}{\partial y}\left(w_2 z - w_3 y\right)\right)$$

$$= \hat{i}\left(w_1 + w_1\right) - \hat{j}\left(-w_2 - w_2\right) + \hat{k}\left(w_3 + w_3\right)$$

$$= 2\left(w_1\hat{i} + w_2\hat{j} + w_3\hat{k}\right) = 2\vec{w}.$$

Thus, the curl of the velocity of any particle on a rigid body is equal to twice the angular velocity of the body.

3.6 VECTOR IDENTITIES

While defining gradient, divergence and curl, we have seen the differential operators $\nabla.$ and $\nabla\times$.

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z},$$

$$\nabla. = \hat{i}.\frac{\partial}{\partial x} + \hat{j}.\frac{\partial}{\partial y} + \hat{k}.\frac{\partial}{\partial z},$$

and $$\nabla\times = \hat{i}\times\frac{\partial}{\partial x} + \hat{j}\times\frac{\partial}{\partial y} + \hat{k}\times\frac{\partial}{\partial z}.$$

We note that the operator ∇ can be applied only on scalar point functions where as (∇ .) and ($\nabla.f$) can be applied on vector point functions only. We now prove some identities involving these operators.

Let $\phi(x,y,z), \psi(x,y,z)$ be differentiable scalar point functions and $\vec{F}(x,y,z), \vec{G}(x,y,z)$ be differentiable vector point functions. Then the following identities hold good.

3.6.1 $\text{grad}(\phi\psi) = \phi\,\text{grad}\psi + \psi\,\text{grad}\phi$ (or) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$.

Proof: $$\nabla(\phi\psi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi\psi)$$

$$= \hat{i}\frac{\partial}{\partial x}(\phi\psi) + \hat{j}\frac{\partial}{\partial y}(\phi\psi) + \hat{k}\frac{\partial}{\partial z}(\phi\psi)$$

$$= \hat{i}\left(\phi\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial x}\psi\right) + \hat{j}\left(\phi\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial y}\psi\right) + \hat{k}\left(\phi\frac{\partial\psi}{\partial z} + \frac{\partial\phi}{\partial z}\psi\right)$$

$$= \phi\left(\hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}\right) + \psi\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\nabla\psi + \psi\nabla\phi.$$

3.6.2 $\nabla\left(\frac{\phi}{\psi}\right)=\frac{\psi\,\nabla\phi-\phi\nabla\psi}{\psi^2}$.

Proof: $\nabla\left(\frac{\phi}{\psi}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\phi}{\psi}\right)$

$$=\hat{i}\frac{\partial}{\partial x}\left(\frac{\phi}{\psi}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{\phi}{\psi}\right)+\hat{k}\frac{\partial}{\partial z}\left(\frac{\phi}{\psi}\right)$$

$$=\hat{i}\left[\frac{\psi\frac{\partial\phi}{\partial x}-\phi\frac{\partial\psi}{\partial x}}{\psi^2}\right]+\hat{j}\left[\frac{\psi\frac{\partial\phi}{\partial y}-\phi\frac{\partial\psi}{\partial y}}{\psi^2}\right]+\hat{k}\left[\frac{\psi\frac{\partial\phi}{\partial z}-\phi\frac{\partial\psi}{\partial z}}{\psi^2}\right]$$

$$=\frac{1}{\psi^2}\left[\psi\left\{\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right\}-\phi\left\{\hat{i}\frac{\partial\psi}{\partial x}+\hat{j}\frac{\partial\psi}{\partial y}+\hat{k}\frac{\partial\psi}{\partial z}\right\}\right]$$

$$=\frac{1}{\psi^2}\left[\psi\,\nabla\phi-\phi\nabla\psi\right]=\frac{\psi\,\nabla\phi-\phi\nabla\psi}{\psi^2}\,.$$

3.6.3 $\text{div}\left(\vec{F}\times\vec{G}\right)=\vec{G}.$ curl $\vec{F}-\vec{F}$. curl $\vec{G}$ (or) $\nabla.\left(\vec{F}\times\vec{G}\right)=\vec{G}.\nabla\times\vec{F}\text{-}\vec{F}.\nabla\times\vec{G}$.

Proof: $\nabla.\left(\vec{F}\times\vec{G}\right)=\left(\hat{i}.\frac{\partial}{\partial x}+\hat{j}.\frac{\partial}{\partial y}+\hat{k}.\frac{\partial}{\partial z}\right)\left(\vec{F}\times\vec{G}\right)$

$$=\hat{i}.\frac{\partial}{\partial x}\left(\vec{F}\times\vec{G}\right)+\hat{j}.\frac{\partial}{\partial y}\left(\vec{F}\times\vec{G}\right)+\hat{z}.\frac{\partial}{\partial z}\left(\vec{F}\times\vec{G}\right)$$

$$=i.\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}+\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)+\hat{j}.\left(\frac{\partial\vec{F}}{\partial y}\times\vec{G}+\vec{F}\times\frac{\partial\vec{G}}{\partial y}\right)+\hat{k}.\left(\frac{\partial\vec{F}}{\partial z}\times\vec{G}+\vec{F}\times\frac{\partial\vec{G}}{\partial z}\right)$$

$$=\left[\hat{i}.\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right)+\hat{j}.\left(\frac{\partial\vec{F}}{\partial y}\times\vec{G}\right)+\hat{k}.\left(\frac{\partial\vec{F}}{\partial z}\times\vec{G}\right)\right]$$

$$+\left[\hat{i}.\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)+\hat{j}.\left(\vec{F}\times\frac{\partial\vec{G}}{\partial y}\right)+\hat{k}.\left(\vec{F}\times\frac{\partial\vec{G}}{\partial z}\right)\right]$$

$$=\left[\left(\hat{i}\times\frac{\partial\vec{F}}{\partial x}\right).\vec{G}+\left(\hat{j}\times\frac{\partial\vec{F}}{\partial y}\right).\vec{G}+\left(\hat{k}\times\frac{\partial\vec{F}}{\partial z}\right).\vec{G}\right]$$

$$-\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}\right).\vec{F}-\left(\hat{j}\times\frac{\partial\vec{G}}{\partial y}\right).\vec{F}-\left(\hat{k}\times\frac{\partial\vec{G}}{\partial z}\right).\vec{F}$$

Since $\vec{a}.\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right).\vec{c}$ and $\vec{a}.\left(\vec{b}\times\vec{c}\right)=-\left(\vec{a}\times\vec{c}\right).\vec{b}$

$$=\left(\hat{i}\times\frac{\partial\vec{F}}{\partial x}+\hat{j}\times\frac{\partial\vec{F}}{\partial y}+\hat{k}\times\frac{\partial\vec{F}}{\partial z}\right).\vec{G}-\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}+\hat{j}\times\frac{\partial\vec{G}}{\partial y}+\hat{k}\times\frac{\partial\vec{G}}{\partial z}\right).\vec{F}$$

$$=\left(\nabla\times\vec{F}\right).\vec{G}-\left(\nabla\times\vec{G}\right).\vec{F}$$

$$=\vec{G}.\left(\nabla\times\vec{F}\right)-\vec{F}.\left(\nabla\times\vec{G}\right)\qquad\left(\because\vec{a}.\vec{b}=\vec{b}.\vec{a}\right).$$

3.6.4 $\operatorname{curl}\left(\vec{F}\times\vec{G}\right)=\vec{F}\,div\vec{G}-\vec{G}\,div\vec{F}+\left(\vec{G}.\nabla\right)\vec{F}-\left(\vec{F}.\nabla\right)\vec{G}$

(or) $\nabla\times\left(\vec{F}\times\vec{G}\right)=\vec{F}\left(\nabla.\vec{G}\right)-\vec{G}\left(\nabla.\vec{F}\right)+\left(\vec{G}.\nabla\right)\vec{F}-\left(\vec{F}.\nabla\right)\vec{G}$.

Proof: $curl\left(\vec{F}\times\vec{G}\right)=\nabla\times\left(\vec{F}\times\vec{G}\right)$

$$\nabla\times\left(\vec{F}\times\vec{G}\right)=\left(\hat{i}\times\frac{\partial}{\partial x}+\hat{j}\times\frac{\partial}{\partial y}+\hat{k}\times\frac{\partial}{\partial z}\right)\left(\vec{F}\times\vec{G}\right)$$

$$=\sum\hat{i}\times\frac{\partial}{\partial x}\left(\vec{F}\times\vec{G}\right)$$

$$=\sum\hat{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}+\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)$$

$$=\sum\hat{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right)+\sum\hat{i}\times\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)$$

$$=\sum\left[\left(\hat{i}.\vec{G}\right)\frac{\partial\vec{F}}{\partial x}+\left(\hat{i}.\frac{\partial\vec{F}}{\partial x}\right)\vec{G}\right]-\sum\left[\left(\hat{i}.\vec{F}\right)\frac{\partial\vec{G}}{\partial x}+\left(\hat{i}.\frac{\partial\vec{G}}{\partial x}\right)\vec{F}\right]$$

$$\left[\because\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}\right]$$

$$=\sum\left(\hat{i}.\vec{G}\right)\frac{\partial\vec{F}}{\partial x}-\sum\left(\hat{i}.\frac{\partial\vec{F}}{\partial x}\right)\vec{G}-\sum\left(\hat{i}.\vec{F}\right)\frac{\partial\vec{G}}{\partial x}+\sum\left(\hat{i}.\frac{\partial\vec{G}}{\partial x}\right)\vec{F}$$

$$=\sum\left(\vec{G}.\hat{i}\right)\frac{\partial\vec{F}}{\partial x}-\left(\sum\hat{i}.\frac{\partial\vec{F}}{\partial x}\right)\vec{G}-\sum\left(\vec{F}.\hat{i}\right)\frac{\partial\vec{G}}{\partial x}+\left(\sum\hat{i}.\frac{\partial\vec{G}}{\partial x}\right)\vec{F}$$

$$=\left(\vec{G}.\nabla\right)\vec{F}-\left(\nabla.\vec{F}\right)\vec{G}-\left(\vec{F}.\nabla\right)\vec{G}+\left(\nabla.\vec{G}\right)\vec{F}$$

$$=\left(\nabla.\vec{G}\right)\vec{F}-\left(\nabla.\vec{F}\right)\vec{G}+\left(\vec{G}.\nabla\right)\vec{F}-\left(\vec{F}.\nabla\right)\vec{G}\ .$$

3.6.5 $\quad Grad\left(\vec{F}.\vec{G}\right)=\vec{F}\times curl\,\vec{G}+\vec{G}\times curl\,\vec{F}+\left(\vec{F}.\nabla\right)\vec{G}+\left(\vec{G}.\nabla\right)\vec{F}$

(or) $\nabla\left(\vec{F}.\vec{G}\right)=\vec{F}\times\left(\nabla\times\vec{G}\right)+\vec{G}\times\left(\nabla\times\vec{F}\right)+\left(\vec{F}.\nabla\right)\vec{G}+\left(\vec{G}.\nabla\right)\vec{F}$.

Proof: $\nabla\left(\vec{F}.\vec{G}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\vec{F}.\vec{G}\right)$

$$=\sum\hat{i}\frac{\partial}{\partial x}\left(\vec{F}.\vec{G}\right)\ =\sum\hat{i}\left(\frac{\partial\vec{F}}{\partial x}.\vec{G}+\vec{F}.\frac{\partial\vec{G}}{\partial x}\right)$$

$$=\sum\hat{i}\left(\frac{\partial\vec{F}}{\partial x}.\vec{G}\right)+\sum\hat{i}\left(\vec{F}.\frac{\partial\vec{G}}{\partial x}\right) \qquad \text{...... (1)}$$

$\vec{G}\times\text{curl}\,\vec{F}=\vec{G}\times\left(\nabla\times\vec{F}\right)$

$$=\vec{G}\times\left(\sum\hat{i}\times\frac{\partial\vec{F}}{\partial x}\right)=\sum\vec{G}\times\left(\hat{i}\times\frac{\partial\vec{F}}{\partial x}\right)$$

$$=\sum\left[\left(\vec{G}.\frac{\partial\vec{F}}{\partial x}\right)\hat{i}-\left(\vec{G}.\hat{i}\right)\frac{\partial\vec{F}}{\partial x}\right]$$

$$=\sum\hat{i}\left(\vec{G}.\frac{\partial\vec{F}}{\partial x}\right)-\sum\left(\vec{G}.\hat{i}\right)\frac{\partial\vec{F}}{\partial x}$$

$$=\sum\hat{i}\left(\frac{\partial\vec{F}}{\partial x}.\vec{G}\right)-\sum\left(\vec{G}.\hat{i}\right)\frac{\partial\vec{F}}{\partial x}$$

$$\Rightarrow\sum\hat{i}\left(\frac{\partial\vec{F}}{\partial x}.\vec{G}\right)=\vec{G}\times\text{curl}\,\vec{F}+\sum\left(\vec{G}.\hat{i}\right)\frac{\partial\vec{F}}{\partial x}$$

$$=\vec{G}\times\text{curl}\,\vec{F}+\left(\vec{G}.\nabla\right)\vec{F} \qquad \text{...... (2)}$$

Interchanging $\vec{F}$ and $\vec{G}$, we get

$$\sum \hat{i}\left(\vec{F}.\frac{\partial \vec{G}}{\partial x}\right) = \sum \hat{i}\left(\frac{\partial \vec{G}}{\partial x}.\vec{F}\right) = \vec{F} \times \text{curl}\,\vec{G} + \left(\vec{F}.\nabla\right)\vec{G} \qquad \dots (3)$$

Using (2) and (3) in (1), we get

$$\nabla\left(\vec{F}.\vec{G}\right) = \vec{G} \times \text{curl } \vec{F} + \left(\vec{G}.\nabla\right)\vec{F} + \vec{F} \times \text{curl}\,\vec{G} + \left(\vec{F}.\nabla\right)\vec{G}$$

$$= \vec{F} \times \left(\nabla \times \vec{G}\right) + \vec{G} \times \left(\nabla \times \vec{F}\right) + \left(\vec{F}.\nabla\right)\vec{G} + \left(\vec{G}.\nabla\right)\vec{F} .$$

3.6.6 curl grad $\phi = \vec{0}$ (or) $\nabla \times \left(\nabla\phi\right) = \vec{0}$.

Proof: curl grad ϕ = curl $\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right] - \hat{j}\left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right] + \hat{k}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right]$$

$$= \hat{i}.0 + \hat{j}.0 + \hat{k}.0 \;\; = \vec{0} .$$

3.6.7 div curl $\vec{F} = 0$ (or) $\nabla.\left(\nabla \times \vec{F}\right) = 0$.

Proof: Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

Then, curl $\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$.

div (curl $\vec{F}$) $= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= 0 \left(\because \frac{\partial^2 F_3}{\partial x \partial y} = \frac{\partial^2 F_3}{\partial y \partial x}\ etc \right)$$

3.6.8 The Laplacian Operator

So far we have studied three differential operators namely, gradient, divergence and curl. Now, we study another important operator called the **Laplacian operator**. This is useful in the study of sound waves, electricity and magnetism.

3.6.9 Definition:

The Laplacian operator is denoted by ∇^2 and defined as $\nabla^2 = \nabla.\nabla$

i.e., $\nabla^2 = \vec{\nabla}.\vec{\nabla}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right).\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

If $\phi(x, y, z)$ is a scalar point function, then

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}.$$

If $\vec{F}(x, y, z)$ is a vector point function, then

$$\nabla^2 \vec{F} = \frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} + \frac{\partial^2 \vec{F}}{\partial z^2}.$$

3.6.10 Note

(i) The Laplacian operator can be applied on both scalar and vector point functions.

(ii) The Laplacian operator transforms a scalar point function into another scalar point function and a vector point function into another vector point function.

(iii) The equation $\nabla^2 \phi = 0$ is called Laplace's equation.

(iv) If ϕ is a scalar point function, div(grad ϕ) = $\nabla.(\nabla \phi) = \nabla^2 \phi$.

3.6.11 Example

Find $\nabla^2 \phi$ if $\phi(x, y, z) = 3x^2 z - y^2 z^3 + 4x^3 y + 2x - 3y - 5$.

Solution: Given that $\phi(x,y,z)=3x^2z-y^2z^3+4x^3y+2x-3y-5$.

$$\frac{\partial\phi}{\partial x}=6xz+12x^2y+2 \Rightarrow \frac{\partial^2\phi}{\partial x^2}=6z+24xy.$$

$$\frac{\partial\phi}{\partial y}=-2yz^3+4x^3-3 \Rightarrow \frac{\partial^2\phi}{\partial y^2}=-2z^3.$$

$$\frac{\partial\phi}{\partial z}=3x^2-3y^2z^2 \Rightarrow \frac{\partial^2\phi}{\partial z^2}=-6yz^2.$$

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

$$=6z+24xy-2z^3-6yz^2.$$

3.6.12 Example

If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $r=|\vec{r}|$, find $\nabla^2\left(\frac{\vec{r}}{r^2}\right)$.

Solution: Given that $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $r=|\vec{r}|=\sqrt{x^2+y^2+z^2}$.

Let $\phi=\frac{\vec{r}}{r^2}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}$

$$\frac{\partial\phi}{\partial x}=\frac{\left(x^2+y^2+z^2\right)\hat{i}-\left(x\hat{i}+y\hat{j}+z\hat{k}\right).2x}{\left(x^2+y^2+z^2\right)^2}$$

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\left(x^2+y^2+z^2\right)^2\left[2x\hat{i}-\left\{\left(x\hat{i}+y\hat{j}+z\hat{k}\right).2+2x.\hat{i}\right\}\right]}{\left(x^2+y^2+z^2\right)^4}$$

$$-\frac{\left[\left(x^2+y^2+z^2\right)\hat{i}-\left(x\hat{i}+y\hat{j}+z\hat{k}\right).2x\right]2\left(x^2+y^2+z^2\right).2x}{\left(x^2+y^2+z^2\right)^4}$$

$$=-\frac{2\left(x\hat{i}+y\hat{j}+z\hat{k}\right)}{\left(x^2+y^2+z^2\right)^2}-\frac{4x\hat{i}}{\left(x^2+y^2+z^2\right)^2}+\frac{8x^2\left(x\hat{i}+y\hat{j}+z\hat{k}\right)}{\left(x^2+y^2+z^2\right)^3}$$

$$= \frac{-2\vec{r}}{r^4} - \frac{4x\hat{i}}{r^4} + \frac{8x^2\,\vec{r}}{r^6} \qquad \text{...... (1)}$$

Similarly, $$\frac{\partial^2\phi}{\partial y^2} = \frac{-2\vec{r}}{r^4} - \frac{4y\,\hat{j}}{r^4} + \frac{8y^2\vec{r}}{r^6} \qquad \text{...... (2)}$$

and $$\frac{\partial^2\phi}{\partial z^2} = \frac{-2\vec{r}}{r^4} - \frac{4z\,\hat{k}}{r^4} + \frac{8z^2\vec{r}}{r^6} \qquad \text{...... (3)}$$

Adding (1), (2) and (3), we get

$$\therefore \nabla^2\phi = \frac{-6\vec{r}}{r^4} - \frac{4}{r^4}\left(x\hat{i} + y\,\hat{j} + z\,\hat{k}\right) + \frac{8\vec{r}}{r^6}\left(x^2 + y^2 + z^2\right)$$

$$= -\frac{6\vec{r}}{r^4} - \frac{4\vec{r}}{r^4} + \frac{8\vec{r}\,r^2}{r^6} = \frac{-10\vec{r}}{r^4} + \frac{8\,\vec{r}}{r^4} = -\frac{2\vec{r}}{r^4}\,.$$

3.6.13 curl curl $\vec{F}$ = grad div $\vec{F}$ - $\nabla^2\vec{F}$, where ∇^2 is the Laplacian operator.

Proof: Let $\vec{F} = F_1\hat{i} + F_2\,\hat{j} + F_3\hat{k}$. Then

$$\text{curl } \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$\therefore \text{ curl curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z} & \dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x} & \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\right]$$

$$= \hat{i}\left[\frac{\partial^2 F_2}{\partial y\,\partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial z\,\partial x}\right] + \hat{j}\left[\frac{\partial^2 F_3}{\partial z\,\partial y} - \frac{\partial^2 F_2}{\partial z^2} - \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_1}{\partial x\,\partial y}\right]$$

$$+\hat{k}\left[\frac{\partial^2 F_1}{\partial x\,\partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y\,\partial z}\right]$$

$$= \hat{i}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y\,\partial x} + \frac{\partial^2 F_3}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial x^2} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right]$$

$$+ \hat{j}\left[\frac{\partial^2 F_1}{\partial x\,\partial y} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z\,\partial y} - \frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_2}{\partial y^2} - \frac{\partial^2 F_2}{\partial z^2}\right]$$

$$+ \hat{k}\left[\frac{\partial^2 F_1}{\partial x\,\partial z} + \frac{\partial^2 F_2}{\partial y\,\partial z} + \frac{\partial^2 F_3}{\partial z^2} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} - \frac{\partial^2 F_3}{\partial z^2}\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)F_1\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)F_2\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)F_3\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(div\,\vec{F}\right) - \nabla^2 F_1\right] + \hat{j}\left[\frac{\partial}{\partial y}\left(div\,\vec{F}\right) - \nabla^2 F_2\right] + \hat{k}\left[\frac{\partial}{\partial z}\left(div\,\vec{F}\right) - \nabla^2 F_3\right]$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(div\,\vec{F}\right) - \left[\nabla^2\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)\right]$$

$$= \nabla\left(div\,\vec{F}\right) - \nabla^2\vec{F} \;= \text{grad (div } \vec{F}\text{)} - \nabla^2\vec{F}\,.$$

3.7 SUMMARY

In this unit, the differential operators gradient, divergence and curl are defined. Their properties such as linearity are discussed. Application of gradient to find normals to level surfaces and directional derivatives are studied. Physical application of divergence and application of curl to identify irrotational vectors and there by finding scalar potential functions is explained. Relation between these operators is established as vector identities. A number of examples are given to understand the concepts clear.

3.8 CHECK YOUR PROGRESS - MODEL ANSWERS

1. $\nabla\left(x^3+y^3+z^3-3xyz\right)$

$$=\hat{i}\frac{\partial}{\partial x}\left(x^3+y^3+z^3-3xyz\right)+\hat{j}\frac{\partial}{\partial y}\left(x^3+y^3+z^3-3xyz\right)+\hat{k}\frac{\partial}{\partial z}\left(x^3+y^3+z^3-3xyz\right)$$

$$=\hat{i}\left(3x^2-3yz\right)+\hat{j}\left(3y^2-3xz\right)+\hat{k}\left(3z^2-3xy\right)$$

$$=3\left[\left(x^2-yz\right)\hat{i}+\left(y^2-xz\right)\hat{j}+\left(z^2-xy\right)\hat{k}\right].$$

2. Let $\phi(x,y,z)=x+y+z$. Then $\nabla\phi=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}=\hat{i}+\hat{j}+\hat{k}$.

Since $\nabla\phi$ is a vector perpendicular to the plane, a vector perpendicular to any vector that lies on the plane is: $\nabla\phi=\hat{i}+\hat{j}+\hat{k}$.

3. $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\text{div } \vec{r}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

$$=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3.$$

4. $\vec{F}=yz\hat{i}+zx\hat{j}+xy\hat{k}$.

$$\text{div } \vec{F}=\frac{\partial}{\partial x}(yz)+\frac{\partial}{\partial y}(zx)+\frac{\partial}{\partial z}(xy)=0+0+0=0.$$

$\therefore \vec{F}$ is solenoidal.

5. $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\text{curl } \vec{r}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \hat{i}.0 + \hat{j}.0 + \hat{k}.0 = 0.$$

6. $\vec{F}(x, y, z) = y\hat{i} + x\hat{j} + 2\hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 2 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(1-1) = \vec{0}$$

$\therefore \vec{F}$ is irrotational.

7. By theorem (3.5.7), $\nabla \times (\phi\vec{F}) = \nabla\phi \times \vec{F} + \phi(\nabla \times \vec{F})$

Taking $\phi = c$, a constant

$$\nabla \times (c\vec{F}) = \nabla c \times \vec{F} + c(\nabla \times \vec{F})$$

Since c is a constant by theorem (3.2.2), $\nabla c = \vec{0}$.

$$\nabla \times (c\vec{F}) = \vec{0} \times \vec{F} + c(\nabla \times \vec{F}) = c(\nabla \times \vec{F}).$$

3.9 MODEL EXAMINATION QUESTIONS

1. Find grad ϕ, where $\phi(x, y, z) = x^3 y e^z$.

2. Find grad f, when $f(x, y, z) = x^3 - y^3 + xz^2$ at the point (1, -1, 2).

3. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$ then find ∇r.

4. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, and $\vec{a}, \vec{b}$ are constant vectors, then show that

   (i) $\nabla\phi(r) = \frac{\phi'(r)}{r}\vec{r}$ (ii) $\nabla\phi(r) \times \vec{r} = \vec{0}$.

5. Find the directional derivative of $x^2 + y^2 + z^2$ along the vector $\hat{j} - \hat{k}$ at the point (1, -1, 2).

6. Find the directional derivative of $xy^2 + yz^2 + zx^2$ along the direction of the coordinate axes at the point (1, -2, 5).

7. Find the greatest rate of increase of the function $\phi(x,y,z) = xyz^2$ at the point (1, 0, 3).

8. If the surfaces $5x^2 - 2byz = 9x$ and $4x^2y + z^3 = 4$ are orthogonal, then find *b*.

9. Find the unit vector normal to the surface $3x^2 + 4y = z$ at the point (1, 1, 7).

10. If $\vec{F}(x,y,z) = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$, find div $\vec{F}$.

11. If $\phi(x,y,z) = 2x^3y^2z^4$, find $\nabla.(\nabla\phi)$.

12. Prove that the vector point function $\vec{F}(x,y,z) = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal.

13. Find $\nabla \times \vec{F}$, where $\vec{F}(x,y,z) = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$.

14. If $\vec{F}(x,y,z) = xy^2\hat{i} + 2x^2yz\,\hat{j} - 3yz^2\hat{k}$, find div $\vec{F}$ and curl $\vec{F}$. Also evaluate them at the point (1, -1, 1).

15. If $\vec{F} = (x+y+1)\hat{i} + \hat{j} + (-x-y)\hat{k}$, then prove that $\vec{F}.curl\,\vec{F} = 0$.

16. For $\vec{F} = x^2y\,\hat{i} + xz\,\hat{j} + 2yz\,\hat{k}$, verify that div curl $\vec{F} = 0$.

17. Verify that curl grad ϕ is zero, where $\phi(x,y,z) = x^2y + 2xy + z^2$.

18. Show that the vector $\vec{F}(x,y,z) = (\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x-y)\hat{k}$ is irrotational.

19. If $\vec{F} = (2x - yz)\hat{i} + (2y - zx)\hat{j} + (2z - xy)\hat{k}$, show that $\vec{F}$ is irrotational and find scalar potential function.

20. Show that the function $\phi = x^2 - y^2$ satisfies Laplace's equation.

21. If $\phi = 3x^2y, \psi = xz^2 - 2y$, evaluate $\nabla(\nabla\phi.\nabla\psi)$.

Answers

1. $\left(2y\,\hat{i} + x\,\hat{j} + xy\,\hat{k}\right)xe^z$

2. $7\hat{i} - 3\hat{j} + 4\hat{k}$

3. $\dfrac{\vec{r}}{r} = \hat{r}$

5. $-3\sqrt{2}$

6. 14, 21, -19

7. 9

8. 1

9. $\frac{1}{\sqrt{53}}\left(6\hat{i}+4\hat{j}-\hat{k}\right)$

10. $2y\left(x+1\right)$

11. $4xz^2\left(3y^2z^2+x^2z^2+6x^2y^2\right)$

13. $2\left(x+z\right)\hat{i}-\left(2z+x^2\right)\hat{k}$

14. curl $\vec{F}=-\left(3z^2+2x^2y\right)\hat{i}+2xy\left(2z-1\right)\hat{k}$, div $\vec{F}=y^2+2x^2z-6yz$.

At (1, -1, 1), div $\vec{F}=9$, curl $\vec{F}=-\hat{i}-2\hat{k}$.

19. $x^2+y^2+z^2-xyz$

21. $\left(6yz^2-12x\right)\hat{i}+6xz^2\,\hat{j}+12xyz\,\hat{k}$

BLOCK - II : MULTIPLE INTEGRALS

The Cartesian coordinate system provides a straight forward way to describe the location of points in space. However, some surfaces, can be difficult to model with equations based on the Cartesian system. Polar coordinates often provide a useful alternative system for describing the location of a point in the describing the location of a point in the plane, particularly incases involving circles.

When we expand the traditional Cartesian coordinate system from two dimensions to three dimensions, we simply add a new axis to model the third dimension. Starting with polar coordinates, we can follow the same process to create a new three dimensional coordinate system called the cylindrical coordinate system. In this way, cylindrical coordinates provide a natural extension of polar coordinates to three dimensions. In the cylindrical coordinate system, location of a point in space is described using two distances (r and z) and an angle measure θ. In the spherical coordinate system, we again use an ordered triple to describe the location of a point in space. In this case, the triple describes one distance and two angles. In unit 4, we discuss how to transform rectangular coordinates to cylindrical spherical and vice-versa.

The multiple integral is an integral of a function of more than one real variable, for example $f\left(x,y\right)$ or $f\left(x,y,z\right)$. Integrals of a function of two variables over a region in R^2 are called double integrals, and integrals of a function of three variables over a region of R^3 are called triple integrals. In unit - 5, we will evaluate double integrals by changing them into polar coordinates and by changing the order of integration. In unit - 6, we will learn how to evaluate triple integrals. We will also find the area enclosed by plane curve and volume of region by using double and triple integrals.

This block includes the following units:

Unit - 4 : Transformations, Polar, Spherical Polar, Cylindrical Polar Coordinates

Unit - 5 : Double Integrals, Change of Order of Integration

Unit - 6 : Triple Integrals, Applications of Multiple Integrals

UNIT-4: TRANSFORMATIONS, POLAR, SPHERICAL POLAR, CYLINDRICAL POLAR COORDINATES

Contents

4.0 OBJECTIVES

After studying this unit, you will be able to:

- Convert coordinates from cartesian to polar coordinates and back.
- Convert coordinates from cartesian to spherical polar coordinates and back.
- Convert coordinates from cartesian to cylindrical polar coordinates and back.

4.1 INTRODUCTION

The usual cartesian system can be quite difficult to use in certain situations. Some of the most common situations when cartesian coordinates are difficult to employ involve those in which circular, cylindrical or spherical symmetry is present. For these situations it is often more convenient to use a different coordinate system. We will present polar coordinates in two dimensional and cylindrical and spherical coordinates in three dimensions.

4.2 POLAR COORDINATES

To define polar coordinates, we fix an origin O called the pole and an initial ray from O. Then each point P can be located by assigning to it an ordered pair (r,θ) where r gives the distance from O to P and θ is the angle between initial ray and OP.

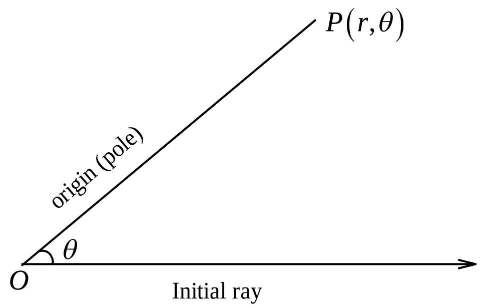

Fig: 4.1

We call (r,θ) the polar coordinates of P. Suppose that P has Cartesian coordinates (x,y). Then the relation between two coordinate systems is given by the following formula. Polar coordinates to cartesian coordinates

$$x = r\cos\theta,\ y = r\sin\theta\ .$$

Cartesian coordinates to polar coordinates

$$r = \sqrt{x^2 + y^2},\ \tan\theta = \frac{y}{x}$$

$$0 \le \theta < \pi \ \text{ if } \ y > 0;\ 2\pi \le \theta < \pi \ \text{ if } \ y \le 0\ .$$

Note that the function $\tan\theta$ has period π and the principal value for inverse tangent function is $-\frac{\pi}{2} < arc\tan\frac{y}{x} < \frac{\pi}{2}$.

So the angle should be determined by

$$\theta = \begin{cases} arc\tan\frac{y}{x}, & if\ \ x > 0 \\ arc\tan\frac{y}{x} + \pi, & if\ x < 0 \\ \frac{\pi}{2}, & if\ \ x = 0,\ y > 0 \\ \frac{-\pi}{2}, & if\ x = 0, y < 0 \end{cases}.$$

4.2.1 Example

Find the Cartesian coordinates of P whose polar coordinates are $\left(2, \frac{\pi}{3}\right)$.

Solution: Given that $r = 2, \theta = \frac{\pi}{3}$.

The Cartesian coordinates (x, y) is given by $x = r\cos\theta,\ y = r\sin\theta$

i.e., $x = 2\cos\frac{\pi}{3} = 2.\frac{1}{2} = 1$

$$y = 2\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$\therefore$ Cartesian coordinate is $\left(1, \sqrt{3}\right)$.

4.2.2 Example

Find the polar coordinates of $P(-1, -1)$.

Solution: The given point is (-1, -1).

i.e., x = -1, y = -1.

$$r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{-1} = 1 \Rightarrow \theta = \frac{\pi}{4}$$

Since $x < 0$, $\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

$\therefore \left(\sqrt{2}, \frac{5\pi}{4}\right)$ is the polar coordinates of (-1, -1).

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. Convert $\left(-4, \frac{2\pi}{3}\right)$ into cartesian coordinates.

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2. Convert (1, 0) into polar coordinates.

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3. Convert $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ into polar coordinates.

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Now we look at how to convert the equation in cartesian coordinates into an equation in polar coordinates.

4.2.3 Example

What is the equation of the circle of radius 2 centred at the origin in polar coordinates?

Solution: In cartesian coordinates the equation of the circle of radius 2 centred at origin is

$$x^2 + y^2 = 4.$$

Put $x = r\cos\theta = 2\cos\theta$, $y = r\sin\theta = 2\sin\theta$.

$$x^2 + y^2 = 4 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2.$$

4.2.4 Example

Convert $x^2 - y^2 = 4$ into a polar equation.

Solution: Put $x = r\cos\theta,\ y = r\sin\theta$.

$$\begin{aligned} x^2 - y^2 = 4 \quad &\Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 4 \\ &\Rightarrow r^2\left(\cos^2\theta - \sin^2\theta\right) = 4 \\ &\Rightarrow r^2\cos 2\theta = 4\cdot \end{aligned}$$

This is an equation of Hyperbola in polar coordinates.

Check Your Progress:

4. Convert $y^2 = 4x$ into polar form.

5. Convert $(x+2)^2 + (y-4)^2 = 16$.

6. Convert $r\sin\theta = r\cos\theta + 4$ into equivalent cartesian equation.

7. Convert $r = 4\tan\theta\sec\theta$ into its equivalent cartesian equation.

4.3 SPHERICAL POLAR COORDINATES

This is a three dimensional coordinate system. The coordinates of a point *P* are given by the ordered triad (ρ, θ, ϕ), where

ρ = The distance from the origin to *P*, we assume $\rho \geq 0$.

θ = θ has the same meaning as in polar coordinates.

ϕ is the angle between positive *z*- axis and the line from the origin to *P*.

We restrict ϕ to $0 \leq \phi \leq \pi$.

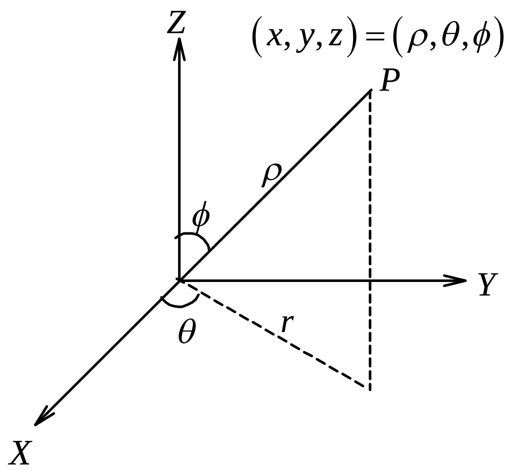

Fig: 4.2

Now we know the formulae to derive relationship between cartesian and spherical coordinates.

Using the triangle one can can make with the *z*- axis and the line from the origin through *P*, we have

$$z = \rho \cos\phi$$

$$r = \rho \sin\phi$$

and $z^2 + r^2 = \rho^2$.

Using the triangle one can make with the x - axis and line from the origin to the projection of *P* on the *xy* - plane, we have

$$x = r\cos\theta$$

$$y = r\sin\theta$$

and $\quad x^2 + y^2 = r^2$

$\therefore$ we have $\quad x = \rho\sin\phi\cos\theta$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi$$

$$\rho^2 = x^2 + y^2 + z^2 .$$

4.3.1 Example

Convert $\left(-1, 1, -\sqrt{2}\right)$ from cartesian coordinate system to spherical polar coordinate system.

Solution: $x=-1,\ y=1,\ z=-\sqrt{2}$.

$$\rho^2=(-1)^2+(1)^2+\left(-\sqrt{2}\right)^2=4$$

$$\therefore \rho=2.$$

From $z=\rho\cos\phi$, we have

$$\cos\phi=\frac{-\sqrt{2}}{2}=\frac{-1}{\sqrt{2}}.$$

$$\therefore \cos\phi=-\cos\frac{\pi}{4}=\cos\frac{3\pi}{4}$$

$$\Rightarrow \pi=\frac{3\pi}{4}.$$

To find θ, we use the equation involving *x* or *y*. If consider *y*, then

$$\sin\theta=\frac{y}{\rho\sin\phi}$$

$$=\frac{1}{2\sin\frac{3\pi}{4}}=\frac{1}{2}.\sqrt{2}=\frac{1}{\sqrt{2}}.$$

i.e. $\theta=\frac{\pi}{4}$ or $\frac{3\pi}{4}$.

However, the projection of ρ on *xy* - plane is in the 2nd quadrant, we must have $\theta=\frac{3\pi}{4}$.

Thus the spherical co-ordinates are:

$$\left(2, \frac{3\pi}{4}, \frac{3\pi}{4}\right).$$

4.3.2 Example

Find the spherical equation for the hyperboloid of two sheets $x^2 - y^2 - z^2 = 1$.

Solution: By direct substitution, we obtain

$$(\rho \sin\phi \cos\theta)^2 - (\rho \sin\phi \sin\theta)^2 - (\rho \cos\phi)^2 = 1$$

$$\Rightarrow \rho^2 \left(\sin^2\phi \cos^2\theta - \sin^2\phi \sin^2\theta - \cos^2\phi\right) = 1$$

4.3.3 Example

Find a rectangular equation for the spherical equation $\rho = 2\sin\phi \sin\theta$.

Soltuion: We have to eliminate the spherical variables $\rho,\ \phi,\ \theta$ and replace them with $x,\ y, z$.

$$\rho = 2\sin\phi \sin\theta$$

$$\Rightarrow \rho^2 = 2\rho \sin\phi \sin\theta$$

From conversion formula, we have

$$x^2 + y^2 + z^2 = 2y$$

$$\Rightarrow x^2 + y^2 - 2y + 1 + z^2 = 1$$

$$\Rightarrow x^2 + (y-1)^2 + z^2 = 1$$

which is a sphere with centre (0, 1, 0) and radius 1.

Check Your Progress:

8. Convert the point $\left(4, \frac{\pi}{4}, \frac{\pi}{6}\right)$ from spherical to rectangular coordinates.

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4.4 CYLINDRICAL POLAR COORDINATES

In the cylindrical coordiante system, a point P in space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P on to the xy - palne and z is the directed distance from the xy - plane to P.

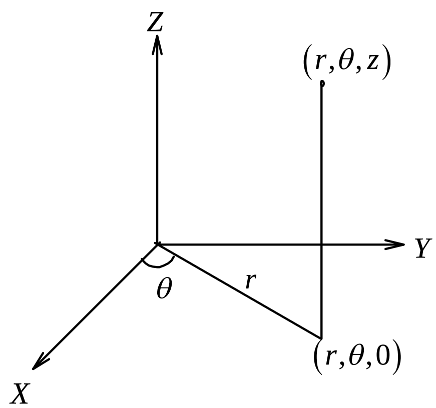

Fig: 4.3

A point expressed in cylindrical coordiantes

To convert the point from cylindrical coordination to rectangular coordinates we use the relations $x = r\cos\theta,\ y = r\sin\theta, z = z$.

To convert from rectangular coordinates to cylindrical coordinates we use the relations

$$r = \sqrt{x^2 + y^2},\ \tan\theta = \frac{y}{x},\ z = z.$$

4.4.1 Example

Convert the point $\left(2, \frac{4\pi}{3}, 8\right)$ from cylindrical to rectangular coordinates.

Solution: Since $r = 2,\ \theta = \frac{4\pi}{3},\ z = 8$;

$$x = r\cos\theta \ = 2\cos\frac{4\pi}{3} = 2\left(\frac{-1}{2}\right) = -1$$

and $y = r\sin\theta = 2\sin\frac{4\pi}{3} = 2\left(\frac{-\sqrt{3}}{2}\right) = -\sqrt{3}$

Thus the point in rectangular coordinates is $\left(-1, -\sqrt{3}, 8\right)$.

4.4.2 Example

Convert the point $\left(\sqrt{3}, 1, 4\right)$ to cylindrical coordinates.

Solution: Since $x = \sqrt{3},\ y = 1,\ z = 4$;

$$r = \sqrt{x^2 + y^2} = \sqrt{3+1} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$z = 4$$

Thus the given point in cylindrical coordinates is $\left(2, \frac{\pi}{6}, 4\right)$.

4.4.3 Example

Convert $\left(\sqrt{6}, \frac{\pi}{4}, \sqrt{2}\right)$ from cylindrical to spherical coordinates.

Solution: Let us first note that θ is the same in both coordinate systems.

Since $r = \sqrt{6},\ z = \sqrt{2}$; we have $\rho^2 = z^2 + r^2 = 2 + 6 = 8$.

$$\Rightarrow \rho = 2\sqrt{2}$$

Now, $z = \rho\cos\phi \Rightarrow \sqrt{2} = 2\sqrt{2}\cos\phi$

$$\Rightarrow \cos\phi = \frac{1}{2} \text{ or } \phi = \frac{\pi}{3}.$$

Hence spherical coordinates are $\left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3}\right)$.

4.4.4 Example

Find the cylindrical equation for the ellipsoid $4x^2 + 4y^2 + z^2 = 1$.

Solution: $4x^2 + 4y^2 + z^2 = 1 \Rightarrow 4r^2 + z^2 = 1$.

4.4.5 Example

Find the cylindrical equation for the ellipsoid $x^2 + 4y^2 + z^2 = 1$.

Solution: $x^2 + 4y^2 + z^2 = 1$

$$\Rightarrow r^2 \cos^2\theta + 4r^2 \sin^2\theta + z^2 = 1 \Rightarrow r^2 + 3r^2 \sin^2\theta = 1 \cdot$$

Check Your Progress:

9. Convert $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 5\right)$ to cylindrical coordinates.

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10. Find the rectangular coordinates of the cylindrical coordinates $\left(2, \frac{2\pi}{3}, 1\right)$.

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11. Find the cylindrical coordinates of the point (3, -3, -7).

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4.5 JACOBIANS

4.5.1 Definition

If *u* and *v* are the functions of two independent variables *x* and *y* then the determinant

$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the **Jacobian** of *u* and *v* with respect to *x* and *y*.

It is denoted by $\frac{\partial(u,v)}{\partial(x, y)}$ or $J(u,v)$.

4.5.2 Definition

If u, v and w are the functions of three independent variables x, y and z then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$ is called the **Jacobian** of u, v and w with represent to x, y and z.

It is denoted by $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ or $J(u, v, w)$.

Similarly if $u_1, u_2, ..., u_n$ are the n functions of independent variables $x_1, x_2, ..., x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \text{..........} & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \text{.......} & \frac{\partial u_2}{\partial x_n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} \text{......} & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the **Jacobian** of $u_1, u_2, ..., u_n$ with respect to the variables $x_1, x_2, ..., x_n$.

It is denoted by $\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)}$ or $J(u_1, u_2, ..., u_n)$.

4.5.3 Example

If $x = r\cos\theta,\ y = r\sin\theta$, then find $\frac{\partial(x, y)}{\partial(r, \theta)}$ and $\frac{\partial(r, \theta)}{\partial(x, y)}$.

Solution: $\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = 1.$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix}$$

$x = r\cos\theta, y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$.

Differentiating partially w.r.t x and y we obtain

$$2r.\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r.\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\tan\theta = \frac{y}{x} \Rightarrow \sec^2\theta.\frac{\partial\theta}{\partial x} = -\frac{y}{x^2}$$

$$\Rightarrow \frac{\partial\theta}{\partial x} = -\frac{y}{x^2\sec^2\theta} = -\frac{y}{r^2\cos^2\theta\sec^2\theta} = -\frac{y}{r^2}.$$

$$\tan\theta = \frac{y}{x} \Rightarrow \sec^2\theta\frac{\partial\theta}{\partial y} = \frac{1}{x}$$

$$\Rightarrow \frac{\partial\theta}{\partial y} = \frac{1}{x\sec^2\theta} = \frac{\cos^2\theta}{x} = \frac{x^2}{r^2}.\frac{1}{x} = \frac{x}{r^2}.$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{x}{r} & \dfrac{y}{r} \\ \dfrac{-y}{r^2} & \dfrac{x}{r^2} \end{vmatrix} = \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}.$$

4.5.4 Example

If $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$, then show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

Solution: $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \end{vmatrix}$

$$x = r\sin\theta\cos\phi$$

$$\Rightarrow \frac{\partial x}{\partial r} = \sin\theta\cos\phi\,,\ \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi,\ \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi\,.$$

$$y = r\sin\theta\sin\phi$$

$$\Rightarrow \frac{\partial y}{\partial r} = \sin\theta\sin\phi,\ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi,\ \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi\,.$$

$$z = r\cos\theta$$

$$\Rightarrow \frac{\partial z}{\partial r} = \cos\theta, \frac{\partial z}{\partial \theta} = -r\sin\theta,\ \frac{\partial z}{\partial \phi} = 0\,.$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\left(r^2\sin^2\theta\cos\phi\right) - r\cos\theta\cos\phi\left(-r\sin\theta\cos\theta\cos\phi\right)$$

$$-r\sin\theta\sin\phi\left(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi\right)$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\cos^2\theta\cos^2\phi\sin\theta + r^2\sin^3\theta\sin^2\phi$$

$$+ r^2\sin\theta\cos^2\theta\sin^2\phi$$

$$= r^2\sin^3\theta\left(\cos^2\phi + \sin^2\phi\right) + r^2\cos^2\theta\sin\theta\left(\cos^2\phi + \sin^2\phi\right)$$

$$= r^2\sin^3\theta + r^2\cos^2\theta\sin\theta = r^2\sin\theta\left(\sin^2\theta + \cos^2\theta\right) = r^2\sin\theta\,\cdot$$

Check Your Progress:

12. If $u = e^x \sin y,\ v = e^x \cos y$, find the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$.

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4.6 WORKED OUT EXERCISES

4.6.1 Exericse

The cartesian coordinates of a point are (2, -6). Determine a set of polar coordinates for that point.

Solution: Let us first determine *r*.

$$r = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (-6)^2} = \sqrt{40}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{6}{2}\right) = \tan^{-1}(-3)$$

$\therefore$ Cartesian coordinates are $\left(2\sqrt{10},\ \tan^{-1}(-3)\right) = \left(2\sqrt{10},\ -1.2490\right)$.

4.6.2 Exercise

Convert the cylindrical coordinates $\left(3, \frac{\pi}{3}, -4\right)$ to cartesian coordinates.

Solution: $x = r\cos\theta = 3\cos\frac{\pi}{3} = \frac{3}{2}$

$$y = r\sin\theta = 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$$

$z = -4$

$\therefore$ Cartesian coordinates are $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, -4\right)$.

4.6.3 Exercise

Convert the cylindrical coordinates $\left(4, \frac{2\pi}{3}, -2\right)$ into rectangular coordinates.

Solution: The cylindrical coordinates are $\left(4, \frac{2\pi}{3}, -2\right)$.

i.e., $x = r\cos\theta = 4\cos\frac{2\pi}{3} = -2$

$y = r\sin\theta = 4\sin\frac{2\pi}{3} = 2\sqrt{3}$

z = -2

$\therefore$ The rectangular coordinates are $\left(-2,\ 2\sqrt{3}, -2\right)$.

4.6.4 Exercise

Convert the rectangular coordinates (1, -3, 5) to cylindrical coordinates.

Solution: We know that $r^2 = x^2 + y^2$.

$\Rightarrow r = \pm\sqrt{x^2 + y^2} = \pm\sqrt{1+(-3)^2} = \pm\sqrt{10}$.

We choose the positive square root, so $r = \sqrt{10}$.

Now we apply the formula to find θ

In this case y is negative and x is positive which means that we must select the value of θ between $\frac{3\pi}{2}$ and 2π.

$\tan\theta = \frac{y}{x} = \frac{-3}{1} \Rightarrow \theta = \tan^{-1}(-3)$

and z = 5.

Therefore that point with rectangular coordiantes (1, -3, 5) has the cylindrical coordinates $\left(\sqrt{10}, \tan^{-1}(-3), 5\right)$.

4.6.5 Exercise

Convert the spherical coordinates $\left(8, \frac{\pi}{3}, \frac{\pi}{6}\right)$ into rectangular and cylindrical coordinates.

Solution: Given $x = r\sin\phi\cos\theta$

$$= 8\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right) = 8.\frac{1}{2}.\frac{1}{2} = 2.$$

$y = r\sin\phi\sin\theta$

$$= 8\sin\left(\frac{\pi}{6}\right).\sin\left(\frac{\pi}{3}\right) = 8.\frac{1}{2}.\frac{\sqrt{3}}{2} = 2\sqrt{3}.$$

$z = r\cos\phi$

$$= 8\cos\left(\frac{\pi}{6}\right) = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}.$$

$\therefore$ The point with spherical coordinates $\left(8, \frac{\pi}{3}, \frac{\pi}{6}\right)$ has rectangular coordinates $\left(2, 2\sqrt{3}, 4\sqrt{3}\right)$.

Now we find in cylindrical coordinates

$$r = \rho\sin\phi = 8\sin\frac{\pi}{6} = 4$$

$$\theta = \frac{\pi}{3}$$

$$z = \rho\cos\phi = 8\cos\frac{\pi}{6} = 4\sqrt{3}.$$

Thus, the cylindrical coordinates for the point are $\left(4, \frac{\pi}{3}, 4\sqrt{3}\right)$.

4.6.6 Exercise

Convert the cartesian coordiantes (-2, 2, 3) to cylindrical coordinates.

Solution: $r^2 = \sqrt{x^2 + y^2}$

$$= \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{2}{-2} = -1$$

$$\therefore \theta = \frac{3\pi}{4}$$

$z = 3$

$\therefore$ The cylindrical coordinates are $\left(2\sqrt{2}, \frac{3\pi}{4}, 3\right)$.

4.6.7 Exercise

Convert the cartesian coordinates $\left(2\sqrt{3}, 6, -4\right)$ to spherical coordinates.

Solution: $\rho = \sqrt{x^2 + y^2 + z^2}$

$$= \sqrt{\left(2\sqrt{3}\right)^2 + (6)^2 + (-4)^2}$$

$$= \sqrt{12 + 36 + 16} = \sqrt{64} = 8$$

$$\tan\theta = \frac{y}{x} = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

i.e. $\theta = \frac{\pi}{3}$.

$$\cos\phi = \frac{z}{\rho} = \frac{-4}{8} = \frac{-1}{2}$$

i.e., $\cos\phi = -\cos 60^0$

$$\therefore \phi = \frac{2\pi}{3}$$

Hence spherical coordinates are $\left(8, \frac{\pi}{3}, \frac{2\pi}{3}\right)$.

4.6.8 Exercise

If $x = u(1+v)$, $y = v(1+u)$, find $J(x,y)$.

Solution: $J(x,y) = \dfrac{\partial(x,y)}{\partial(u,v)}$

We have $x = u(1+v)$ and $y = v(1+u)$.

$\dfrac{\partial x}{\partial u} = 1+v,\ \dfrac{\partial x}{\partial v} = v$ and $\dfrac{\partial y}{\partial v} = 1+u$, $\dfrac{\partial y}{\partial u} = v$.

$$J(x,y) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - vu = 1 + v + u + vu - vu = 1 + u + v \cdot$$

4.7 SUMMARY

In this unit we have discussed some important coordinate systems like polar cylindrical and spherical and their trans formations. In the cylindrical coordinate system, a point in space is represented by the ordered triple (r,θ,z), where (r,θ) represents the polar coordinates of the points projection in the *xy* - plane and *z* represents the points projection on to the *z* - axis.

To convert a point from cartesian to cylindrical coordinates, use the equations $r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$ and $z = z$.

In the spherical coordinate system a point *P* in space is represented by ordered triple (ρ,θ,ϕ) where ρ is the distance between ρ and the origin. θ is the same angle used to describe the location in cylindrical coordinates, and ϕ is the angle formed by positive *z* - axis and the line segment *OP* where *O* is origin, $0 \le \phi \le \pi$.

To convert a point from spherical coordinates to cartesian coordinates use $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$.

To convert a point from cartesian coordinates to spherical coordinates use the equations

$$\rho^2 = x^2 + y^2 + z^2,\ \tan\theta = \frac{y}{x},\ \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

To convert a ponit from spherical coordinates to cylindrical coordinates use $r = \rho \sin\phi,\ \theta = \theta$ and $z = \rho\cos\phi$.

To convert a point from cylindrical coordinates to spherical coordinates use the equations

$\rho = \sqrt{r^2 + z^2},\ \theta = \theta$ and $\pi = \cos^{-1}\left(\dfrac{z}{\sqrt{r^2+z^2}}\right)$.

4.8 CHECK YOUR PROGRESS - MODEL ANSWERS

1. $\left(2, -2\sqrt{3}\right)$

2. $(1, 0)$

3. $\left(1, \dfrac{\pi}{4}\right)$

4. $r = \cot\theta \operatorname{cosec}\theta$

5. $r^2 + 4r\cos\theta - 8r\sin\theta + 4 = 0$

6. $y = x + 4$

7. $x^2 = 4y$

8. $(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$

9. $\left(1, \dfrac{\pi}{3}, 5\right)$

10. $\left(-1, \sqrt{3}, 1\right)$

11. $\left(\sqrt{18}, -\dfrac{\pi}{4}, -7\right)$

12. $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix} = -e^x\left(\sin^2 y + \cos^2 y\right) = -e^x$.

4.9 MODEL EXAMINATION QUESTIONS

1. Explain polar spherical and cylindrical coordinate systems with example.

2. Convert the cartesian coordinates $\left(-1,1,-\sqrt{2}\right)$ to spherical coordinates.

3. Find the spherical equation for the hyperboloid of two sheets $x^2 - y^2 - z^2 = 1$.

4. Convert the cylindrical coordinates $\left(\sqrt{6}, \frac{\pi}{4}, \sqrt{2}\right)$ to spherical coordinates.

5. If $x = r\cos\theta, y = r\sin\theta$ then find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

Answers:

2. $\left(2, \frac{3\pi}{4}, \frac{3\pi}{4}\right)$

3. $\rho^2\left(\sin^2\phi\cos^2\theta - \sin^2\phi\sin^2\theta - \cos^2\phi\right) = 1$

4. $\left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3}\right)$

5. r

UNIT - 5 : DOUBLE INTEGRALS, CHANGE OF ORDER OF INTEGRATION

Contents

5.0 OBJECTIVES

After studying this unit, you will be able to:

- Find the limits of integration for a given region.
- Evaluate double integrals.
- Change the order of integration for a given double integrals.

5.1 INTRODUCTION

The concept of a definite integral $\int_a^b f(x)dx$ is physically the area under a curve $y = f(x)$, the x - axis and the lines $x = a$ and $x = b$. It is defined as the limit of the sum $f(x)_1 \delta x_1 + f(x_2)\delta x_2 + ... + f(x_n)\delta x_n$. When $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, ..., \delta x_n$ tends to zero.

Here $\delta x_1, \delta x_2, ..., \delta x_n$ are n subdivisions into which the range of integration has been divided and $x_1, x_2, ..., x_n$ are the values of x lying respectively in the 1st, 2nd, ..., nth sub intervals.

A double integral is the counter part of the above definition in two dimensions.

Let $f(x,y)$ be a single valued and bounded function of two independent variables x and y defined in a closed region A in xy - plane. Let A be divided into n elementary areas $\delta A_1, \delta A_2, ..., \delta A_n$.

Consider the sum

$$f(x_1,y_1)\delta A_1 + f(x_2,y_2)\delta A_2 + ... + f(x_n,y_n)\delta A_n = \sum_{r=1}^{n} f(x_r,y_r)\delta A_r \qquad ...... (1)$$

Then the limit of the sum (1) if exists as $n \to \infty$ and each sub elementary area approaches to zero, is termed as double integral of $f(x,y)$ over the region A and expressed as

$$\iint_A f(x,y)\, dA$$

i.e., $\displaystyle\iint_A f(x,y)\, dA = \lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r,y_r)\delta A_r$.

5.2 EVALUATION OF DOUBLE INTEGRALS

Double integrals over a region R may be evaluated by two successive integrations as follows.

Suppose that R can be described by in equalities of the form $a \le x \le b$, $y_1(x) \le y_2(x)$ so that $y = y_1(x)$, $y = y_2(x)$ represent the boundary of R. Then

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\, dy\, dx = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x,y)\, dy \right] dx .$$

The integral with in the square brackets is evaluated first by integrating the integrand partially w.r.t. y treating x as constant.

Then the expression there in after integration turns out to be a function of x which when integrated w.r.t. x between the limits $x = a$ and $x = b$ gives the value of the double integral.

Similarly, if R can be described by inequalities of the form $a \le y \le b$, $x_1(y) \le x \le x_2(y)$. Then we obtain

$$\iint_R f(x,y)\, dx\, dy = \int_a^b \left[\int_{x_1(y)}^{x_2(y)} f(x,y)\, dx \right] dy$$

We now integrate over x treating y as constant and then the resulting function of y from a to b.

If all the limits of integration are constants, then the double integral can be evaluated in either way i.e., first we integrate w.r.t. x and then w.r.t. y or we first integrate w.r.t. y and later w.r.t. x.

5.2.1 Example

Evaluate $\int_0^2 \int_0^x y\, dy\, dx$.

Solution: $$\int_0^2 \int_0^x y\, dy\, dx = \int_{x=0}^{2} \left[\int_{y=0}^{x} y\, dy \right] dx$$

$$= \int_{x=0}^{2} \left[\frac{y^2}{2} \right]_0^x dx = \int_{x=0}^{2} \frac{x^2}{2} dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{6} = \frac{4}{3}.$$

5.2.2 Example

Evaluate $\int_2^3 \int_2^4 x^2 y^3\, dx\, dy$.

Solution: $$\int_{x=2}^{3} \int_{y=2}^{4} x^2 y^3\, dx\, dy = \int_{x=2}^{3} x^2 \left[\frac{y^4}{4} \right]_2^4 dx$$

$$= \frac{1}{4}\left(4^4 - 2^4\right) \int_2^3 x^2\, dx$$

$$= 60 \left[\frac{x^3}{3} \right]_2^3 = 20\ (27\text{-}8) = 380.$$

5.2.3 Example

Evaluate $\iint_R y\, dx\, dy$ where R is region bounded by y - axis and the curve $y = x^2$ and the line $x + y = 2$ in the first quadrant.

Solution: Given curves are $x=0,\ y=x^2,\ x+y=2$.

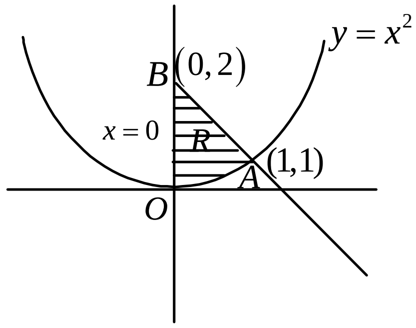

Fig: 5.1

Solving we get,

$x+y=2 \Rightarrow x^2+x-2=0$

$\Rightarrow (x+2)(x-1)=0 \ \Rightarrow x=-2$ or 1.

$x=1 \Rightarrow y=1$.

$x=0 \Rightarrow y=0$. (from $y=x^2$).

Also $x=0 \Rightarrow y=2$ (from $x+y=2$).

$\therefore$ The given curves intersect at the point $O(0, 0)$, $A(1, 1)$ and $B(0, 2)$.

Hence $\iint_R y\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{2-x} y\,dy\,dx$

$$=\frac{1}{2}\int_{x=0}^{1}\left[y^2\right]_{y=x^2}^{2-x} = \frac{1}{2}\int_0^1\left[(2-x)^2-x^4\right]dx$$

$$=\frac{1}{2}\left[\frac{-(2-x)^3}{3}-\frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left[\left(-\frac{1}{3}-\frac{1}{5}\right)-\left(-\frac{8}{3}-0\right)\right]=\frac{16}{15}.$$

5.2.4 Example

Evaluate $\iint_R xy(x+y)\,dx\,dy$ over the region R bounded by $y=x^2$ and $y=x$.

Solution: The given curves $y=x^2$, $y=x$ intersect at (0, 0) and (1, 1).

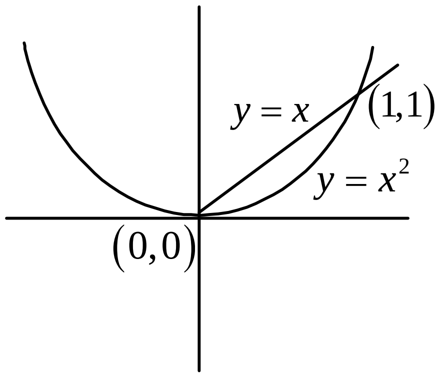

Fig: 5.2

$$\iint_R xy(x+y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} \left(xy(xy+y)\,dy\right)dx$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} \left[\left(x^2y + xy^2\right)dy\right]dx$$

$$= \int_{x=0}^{1}\left[x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_{x^2}^{x} dx = \int_{x=0}^{1}\left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right)dx$$

$$= \int_{x=0}^{1}\left(\frac{5}{6}x^4 - \frac{1}{2}x^6 - \frac{1}{3}x^7\right)dx = \left[\frac{5}{6}\frac{x^5}{5} - \frac{1}{2}\frac{x^7}{7} - \frac{1}{3}\frac{x^8}{8}\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{9}{168} = \frac{3}{56}.$$

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. Evaluate $\int_{x=0}^{2}\int_{y=0}^{3} xy\,dx\,dy$.

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2. Evaluate $\int\limits_{y=0}^{a} \int\limits_{x=0}^{b} \left(x^2 + y^2\right) dx\, dy$.

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5.3 DOUBLE INTEGRALS IN POLAR COORDINATES

A double integral can be evaluated interms of polar coordinates also. Some times the evaluation of a double or triple integral with its present form may not be simple to evaluate. By choice of an appropriate coordinate system, which we discussed in unit 4, a given integral can be transformed into a simpler integral involving the new variables.

Let $x = f(u,v)$ and $y = g(u,v)$ be the relations between (x,y) with (u,v) of the new coordinate system. Then

$$\iint\limits_R F(x,y)\, dx\, dy = \iint\limits_R F(f,g)|J|\, du\, dv, \text{ where } J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

which is called the Jacobian of the co-ordinate transformation.

Change of variables from Cartesian to polar coordinates:

In this case, we have $u = r,\ v = \theta$ and $x = r\cos\theta,\ y = r\sin\theta$ and

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$$\therefore \iint\limits_R F(x,y)\, dx\, dy = \iint\limits_R F(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta.$$

This corresponds to

$$\iint F(r,\theta)\, dA = \int\limits_{\theta=\theta_1}^{\theta_2} \int\limits_{r=f_1(\theta)}^{f_2(\theta)} F(r,\theta)\, r\ dr\ d\theta$$

5.3.1 Example

Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$.

Solution: $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta = \int_{\theta=0}^{\pi}\left[\int_{r=0}^{a\sin\theta} r\,dr\right]d\theta$

$$= \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta \;= \frac{a^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{\pi a^2}{4}.$$

5.3.2 Example

Evaluate $\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta$

Solution: $\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\left[\int_{r=a(1-\cos\theta)}^{a} r^2\,dr\right]d\theta$

$$= \int_{\theta=0}^{\pi/2}\left[\frac{r^3}{3}\right]_{a(1-\cos\theta)}^{a} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\left[a^3 - a^3(1-\cos\theta)^3\right]d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\left[1-(1-\cos\theta)^3\right]d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\left[1-\left(1-3\cos\theta+3\cos^2\theta-\cos^3\theta\right)\right]d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\left(3\cos\theta-3\cos^2\theta+\cos^3\theta\right)d\theta$$

$$=\frac{a^3}{3}\left[3(\sin\theta)_0^{\pi/2}-3\int_0^{\pi/2}\cos^2\theta\,d\theta+\int_0^{\pi/2}\cos^3\theta\,d\theta\right]$$

$$=\frac{a^3}{3}\left[3-3.\frac{1}{2}.\frac{\pi}{2}+\frac{2}{3.1}\right]=\frac{a^3}{3}\left[3-\frac{3\pi}{4}+\frac{2}{3}\right]=\frac{a^3}{36}[44-9\pi].$$

5.3.3 Example

Evaluate the inetegral $\int_0^a\int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$ by transforming into polar coordinates.

Solution: The region of integration is given by $y=0,\ y=\sqrt{a^2-x^2}$, x = 0 and x = a.

i.e. $y=0,\ x^2+y^2=a^2,\ x=0$ and x = a.

Changing into polar coordinates by putting $x=r\cos\theta,\ y=r\sin\theta$, we have

$x^2+y^2=r^2$ and $dx\,dy=r\,dr\,d\theta$.

The limits for r : 0 to a and θ: 0 to $\frac{\pi}{2}$.

$$\therefore\int_0^a\int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}(r\sin\theta).r.r\,dr\,d\theta\ .$$

$$=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^3\sin\theta\,dr\,d\theta=\int_{\theta=0}^{\pi/2}\left[\frac{r^4}{4}\right]_0^a\sin\theta\,d\theta$$

$$=\frac{a^4}{4}\int_0^{\pi/2}\sin\theta\,d\theta=\frac{a^4}{4}[-\cos\theta]_0^{\pi/2}=\frac{a^4}{4}.$$

Check Your Progress:

3. Evaluate the double integral of $\int_0^a\int_0^{\sqrt{a^2-y^2}}\left(x^2+y^2\right)dy\,dx$.

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5.4 CHANGE OF ORDER OF INTEGRATION

The concept of change of order of integration evolved to help in handling typical integrals occurring in evaluation of double integrals.

When the limits of given integral $\int_{a}^{b} \int_{y=\phi(x)}^{y=\pi(x)} f(x,y)\,dy\,dx$ are clearly drawn and the region of integration is identified, then we can well change the order of integration by performing integration first w.r.t. x as a function of y and then w.r.t. y from c to d.

Mathematically expressed as $I = \int_{c}^{d} \int_{x=\phi(y)}^{x=\psi(y)} f(x,y)\,dx\,dy$.

Sometimes the given region may be split into two or three parts for defining new limits for each region in the changed order.

5.4.1 Example

Evaluate the integral $\int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} y^2\,dy\,dx$ by changing the order of integration.

Solution: In the above integral, y on vertical strip varies as a function of x and then the strip slides between $x = 0$ to $x = 1$.

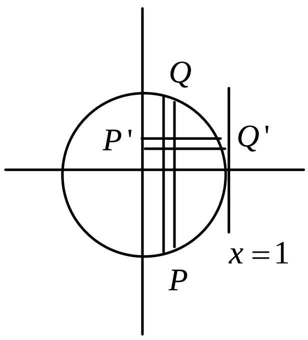

Fig: 5.3

Here $y = 0$ is the x - axis and $y = \sqrt{1-x^2}$

i.e., $x^2 + y^2 = 1$ is the circle.

In the changed order, we get the curve $x = 0$ and $x = \sqrt{1-y^2}$ and finally the strip slides between $y = 0$ to $y = 1$.

Now, $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx = \int_0^1 y^2 \left(\int_0^{\sqrt{1-y^2}} dx \right) dy$

$$= \int_0^1 y^2 \, [x]_0^{\sqrt{1-y^2}} \, dy = \int_0^1 y^2 \sqrt{1-y^2} \, dy .$$

Put $y = \sin\theta$ so that $dy = \cos\theta \, d\theta$ and θ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ The given integral is

$$\int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta = \frac{(2-1)(2-1)}{4.2} \cdot \frac{\pi}{2} = \frac{\pi}{16} .$$

$$\therefore \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{(m-1)(m-3)......(n-1)(n-3)}{(m+n)(m+n-2)......} \times \frac{\pi}{2}$$

only if *m, n* are both positive even integers.

5.4.2 Example

By changing the order of integration, evaluate $\int_0^1 \int_1^{2-x} xy \; dx \, dy$.

Solution: Consider $\int_0^1 \int_1^{2-x} xy \; dx \, dy$.

Let us write this in the standard form $\int_{x=0}^{1} \int_{y=0}^{2-x} xy \; dx \, dy$.

In the above first by fixing *x*, we have to vary *y* from 1 to 2-*x*. Then *x* has to be varied from 0 to 1.

This implies that the region bounded by the lines $x = 0,\ y = 1$ and $x + y = 2$.

Now, we shall change the order of integration.

Let us fix *y*. For a fixed *y* in the region *x* varied from 0 to $2 - y$. Then to be in the region *y* has to vary from 1 to 2.

After changing the order of integration, the given integral is $\int_{y=1}^{2}\int_{x=0}^{2-y} xy \, dx\, dy$.

$$\Rightarrow \int_{y=1}^{2}\left[\int_{x=0}^{2-y} x\, dx\right] y\, dy = \int_{y=1}^{2} \frac{(2-y)^2}{2}.y\, dy$$

$$= \frac{1}{2}\int_{y=1}^{2}\left(4y-4y^2+y^3\right)dy$$

$$= \frac{1}{2}\left[4.\frac{y^2}{2}-4.\frac{y^3}{3}+\frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left[\left(8-\frac{32}{3}+4\right)-\left(2-\frac{4}{3}+\frac{1}{4}\right)\right] = \frac{5}{24}.$$

5.4.3 Example

Change the order of integration and evaluate $\int_0^b \int_0^{\frac{a\sqrt{b^2-y^2}}{b}} xy \, dx\, dy$.

Solution: Here the integration is first w.r.t. *x*, then w.r.t. *y*.

The limits of integration are $x = 0$ and $x = \frac{a}{b}\sqrt{b^2-y^2}$.

Squaring, $b^2x^2 = a^2\left(b^2-y^2\right)$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

On changing the order of integration, given integral can be written as

$$\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\, dy\, dx = \int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_0^{\frac{b\sqrt{a^2-x^2}}{a}} dx$$

$$= \int_0^a \frac{x}{2}\left[\frac{b^2}{a^2}\left(a^2 - x^2\right)\right] dx$$

$$= \frac{b^2}{2}\left[\frac{x^2}{2}\right]_0^a - \frac{b^2}{2a^2}\left[\frac{x^4}{4}\right]_0^a$$

$$= \frac{b^2a^2}{4} - \frac{b^2a^2}{8} = \frac{b^2a^2}{8}.$$

5.4.4 Example

Change the order of integration in $\int\limits_0^{\pi/2} \int\limits_0^{2a\cos\theta} f(r,\theta)\, dr\, d\theta$.

Solution: The given integration is $\int\limits_0^{\pi/2} \int\limits_0^{2a\cos\theta} f(r,\theta)\, dr\, d\theta$.

For a fixed θ, r varies from 0 to $2a\cos\theta$.

Let us draw the curve $r = 2a\cos\theta$.

If $\theta = 0$ then $r = 2a\cos\theta \Rightarrow r = 2a$.

As θ increases to $\frac{\pi}{2}$, $r \to 0$.

Using $\cos\theta = \frac{x}{r}$, we get $r = 2a\cos\theta = \frac{2ax}{r}$

i.e., $r^2 = 2ax \Rightarrow x^2 + y^2 - 2ax = 0$.

Hence $r = 2a\cos\theta$ is a circle with centre at (a, 0) and radius a.

To change the order of integration, let us fix $r^2 = k$, a constant.

For a fixed r, θ varies from 0 to $\cos^{-1}\left(\frac{r}{2a}\right)$.

Hence the given integral after changing the order of integration is

$$\int\limits_{r=0}^{2a} \int\limits_{\theta=0}^{\cos^{-1}\left(\frac{r}{2a}\right)} f(r,\theta)\, d\theta\, dr .$$

5.5 WORKED OUT EXERCISES

5.5.1 Exercise

Evaluate $\int_0^3 \int_1^2 xy(1+x+y)\,dy\,dx$.

Solution: Here all the four limits are constants. So the double integral can be evaluated either way.

$$\int_0^3 \int_1^2 xy(1+x+y)\,dy\,dx = \int_0^3 \left[\int_1^2 \left(xy + x^2 y + xy^2 \right) dy \right] dx$$

$$= \int_0^3 \left[x.\frac{y^2}{2} + x^2.\frac{y^2}{2} + x.\frac{y^3}{3} \right]_{y=1}^{2} dx$$

$$= \int_0^3 \left[\left(2x + 2x^2 + \frac{8}{3}x \right) - \left(\frac{x}{2} + \frac{x^2}{2} + \frac{x}{3} \right) \right] dx$$

$$= \int_0^3 \left[\left(\frac{14}{3}x + 2x^2 \right) - \left(\frac{5x}{6} + \frac{x^2}{2} \right) \right] dx$$

$$= \int_0^3 \left(\frac{23}{6}x + \frac{3}{2}x^2 \right) dx = \frac{123}{4}.$$

5.5.2 Exercise

Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x- axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

Solution: Consider the figure and identify the region.

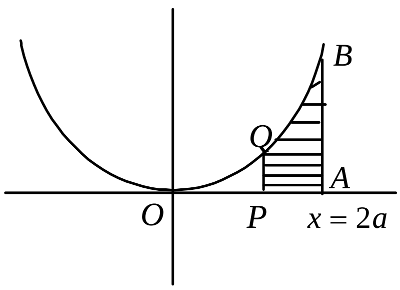

Fig: 5.4

Let us consider a fixed *x*.

Now for this fixed *x*, *y* varies from 0 to $\frac{x^2}{4a}$.

To be in the region, we have to vary *x* from 0 to 2*a*.

Hence $\iint_R xy\,dx\,dy = \int_{x=0}^{2a} x\left[\int_{y=0}^{\frac{x^2}{4a}} y\,dy\right]dx$

$$= \int_{x=0}^{2a}\left[\frac{y^2}{2}\right]_0^{\frac{x^2}{4a}} x\ dx = \int_{x=0}^{2a}\frac{x^4}{32a^2}x\ dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx \quad = \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$= \frac{1}{32a^2}.\frac{64a^6}{6} \quad = \frac{a^4}{3}.$$

5.5.3 Exercise

$\iint_R \left(x^2+y^2\right)dx\,dy$ in positive quadrant for which $x+y\le 1$.

Solution: $\iint_R \left(x^2+y^2\right)dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x}\left(x^2+y^2\right)dx\,dy$

$$= \int_{x=0}^{1}\left[x^2y+\frac{y^3}{3}\right]_0^{1-x} dx \ = \int_{x=0}^{1}\left[x^2(1-x)+\frac{1}{3}(1-x)^3\right]dx$$

$$= \int_{x=0}^{1}\left(x^2-x^3+\frac{1}{3}(1-x)^3\right)dx = \left[\frac{x^3}{3}-\frac{x^4}{4}-\frac{1}{12}(1-x)^4\right]_{x=0}^{1}$$

$$= \frac{1}{3}-\frac{1}{4}-0+\frac{1}{12} = \frac{1}{6}.$$

5.5.4 Exercise

Evaluate $\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dx\,dy$ by changing into polar coordinates. Hence show that $\int_0^\infty e^{-x^2}dx=\dfrac{\sqrt{\pi}}{2}$.

Solution: Since both x and y vary from 0 to ∞, the region of integration is the first quadrant of the xy - plane.

Changing into polar coordinates by putting $x=r\cos\theta,\ y=r\sin\theta$; we have

$dx\,dy=r\,dr\,d\theta$ and $x^2+y^2=r^2$.

$$\therefore \text{ Given integral } = \int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

Put $r^2=t,\ 2r\,dr=dt$. Also $r=0\Rightarrow t=0$, $r=\infty\Rightarrow t=\infty$.

$$=\frac{1}{2}\int_{\theta=0}^{\pi/2}\left[\int_{t=0}^{\infty}e^{-t}dt\right]d\theta=-\frac{1}{2}\int_0^{\pi/2}\left[e^{-t}\right]_{t=0}^{\infty}d\theta$$

$$=-\frac{1}{2}\int_0^{\pi/2}\left(0-1\right)d\theta=-\frac{1}{2}\left[-\theta\right]_0^{\pi/2}=\frac{\pi}{4}.$$

$$\text{Also } \int_0^\infty e^{-x^2}dx\times\int_0^\infty e^{-y^2}dy=\left[\int_0^\infty e^{-x^2}dx\right]^2$$

$$\therefore \int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}.$$

5.5.5 Exercise

Transform the following to cartesian form and evaluate $\int_0^\pi\int_0^a r^3\sin\theta\cos\theta\,dr\,d\theta$.

Solution: The region of integration is $r=0,\ r=a$ and $\theta=0,\ \theta=\pi$.

Since θ varies from 0 to π, the region of integration is the semi circle

$x^2+y^2=a^2$.

Let $x = r\cos\theta,\ y = r\sin\theta$. Then $dx\,dy = r\,dr\,d\theta$.

$$\therefore \int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=0}^{a} (r\sin\theta)(r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} x\,y\,dx\,dy$$

$$= \int_{x=-a}^{a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=-a}^{a} x\left(a^2 - x^2\right) = 0 \text{ [}\because\text{ The integrand is odd function.]}$$

5.5.6 Exercise

Evaluate $\iint_D (x+y)^2\,dx\,dy$, where D is the parallelogram bounded by the lines $x + y = 0$, $x + y = 1,\ 2x - y = 0$ and $2x - y = 3$.

Solution: Let $u = x + y$, $v = 2x - y$.

In these new variables the region D is described by $0 \le u \le 1,\ 0 \le v \le 3$.

Now the Jacobian J is given by $J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

Now $u = x + y,\ v = 2x - y$, then $x = \frac{1}{3}(u+v),\ y = \frac{2}{3}u - \frac{1}{3}v$.

$$\therefore J = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}.$$

So $|J| = \frac{1}{3}$.

$$\therefore \iint_D (x+y)^2\, dx\, dy = \int_0^3\int_0^1 u^2 \frac{1}{3} du\, dv$$

$$= \frac{1}{3}\int_0^3 \left[\frac{u^3}{3}\right]_0^1 d\theta \quad = \frac{1}{9}\int_0^3 1 dv = \frac{3}{9} = \frac{1}{3}.$$

5.5.7 Exercise

Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\, dr\; d\theta$.

Solution: The given integral $= \int_{\theta=0}^{\pi}\left[\int_{r=0}^{a(1+\cos\theta)} r\, dr\right] d\theta$

$$= \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{\theta=0}^{\pi} a^2(1+\cos\theta)^2\, d\theta$$

$$= \frac{1}{2}.a^2\int_0^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta \quad = \frac{a^2}{2}\int_0^{\pi} 4\cos^4\frac{\theta}{2} d\theta$$

$$= 2a^2\int_0^{\pi}\cos^4\frac{\theta}{2} d\theta.$$

Put $\frac{\theta}{2} = t,\ d\theta = 2dt$. Also

$\theta = 0 \Rightarrow t = 0$, $\theta = \pi \Rightarrow t = \frac{\pi}{2}$.

$$= 2a^2\int_0^{\pi/2}\cos^4 t.2dt = 4a^2\int_0^{\pi/2}\cos^4 t\, dt$$

$$= 4a^2.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{3\pi a^2}{4}.$$

5.5.8 Exercise

Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Solution: $\iint r^3\, dr\, d\theta = \int\limits_{\theta=0}^{\pi} \int\limits_{r=2\sin\theta}^{4\sin\theta} r^3 dr\, d\theta$

$$= \int_0^{\pi} \left[\int\limits_{r=2\sin\theta}^{4\sin\theta} r^3 dr \right] d\theta = \int_0^{\pi} \left[\frac{r^4}{4} \right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi} \left(256\sin^4\theta - 16\sin^4\theta \right) d\theta$$

$$= 60\int_0^{\pi} \sin^4\theta\, d\theta \;\; = 60 \times 2 \times \int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$\left(\therefore \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx, \; if \;\; f(2a - x) = f(x) \right)$$

$$= 120 \times \frac{3}{4} \times \frac{1}{2} . \frac{\pi}{2} \; = \frac{45\pi}{2} .$$

5.5.9 Exercise

Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the cardioid $r = a(1 - \cos\theta)$ above the initial line.

Solution: To integrate first with respect to *r*, the limits are from *r* = 0 to $r = a(1-\cos\theta)$ and θ varies from 0 to π .

$$\iint\limits_R r\sin\theta\, dr\, d\theta = \int_0^{\pi} \sin\theta \left[\int_0^{a(1-\cos\theta)} r\, dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2} \right]_0^{a(1-\cos\theta)} d\theta \; = \int_0^{\pi} \frac{\sin\theta \; a^2 (1 - \cos\theta)^2}{2} d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} (1 - \cos\theta)^2 .\sin\theta\, d\theta \; = \frac{a^2}{2} \left[\frac{(1-\cos\theta)^3}{3} \right]_0^{\pi}$$

$$= \frac{a^2}{2} . \frac{8}{3} \; = \frac{4}{3} a^2 .$$

5.5.10 Exercise

Evaluate $\iint r^3 dr\, d\theta$, over the area included between the circles $r = 2a\cos\theta$ and $r = 2b\cos\theta$ $(b < a)$.

Solution: Given $r = 2a\cos\theta$

$$\Rightarrow r^2 = 2ar\cos\theta$$

$$\Rightarrow x^2 + y^2 = 2ax$$

i.e. $(x-a)^2 + (y-0)^2 = a^2$

i.e., this curve represents the circle with centre $(a, 0)$ and radius a.

Similarly, $r = 2b\cos\theta$ represents the circle with centre $(b, 0)$ and radius b.

Limits: r varies from $2b\cos\theta$ to $2a\cos\theta$ and θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

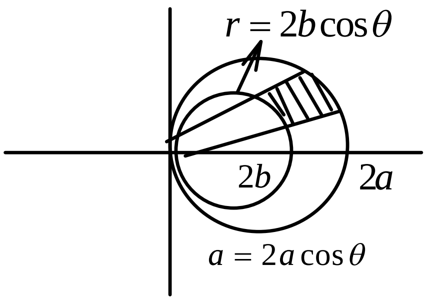

Fig: 5.5

$$\iint r^3\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2b\cos\theta}^{2a\cos\theta} r^3\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_{2b\cos\theta}^{2a\cos\theta} d\theta = \frac{1}{4}\left(a^4 - b^4\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 16\cos^4\theta\, d\theta$$

$$= 4\left(a^4 - b^4\right).2\int_{0}^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 8\left(a^4 - b^4\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$=8\left(a^4-b^4\right).\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}=\frac{3}{2}\pi\left(a^4-b^4\right).$$

Particular case: When $r=2\cos\theta$ and $r=4\cos\theta$ i.e., when $a=2$ and $b=1$ then

$$I=\frac{3}{2}\pi\left(a^4-b^4\right)=\frac{3}{2}\pi\left(2^4-1^4\right)=\frac{45\pi}{2} \text{ units.}$$

5.6 SUMMARY

To evaluate a double integral we do it in stages, starting from the inside and working out, using our knowledge of the methods for single integrals.

To evaluate $\iint f(x,y)\,dx\,dy$ we proceed as follows.

(i) Work out the limits of integration if they are not already known.

(ii) Work out the inner integral for y.

(iii) Work out the outer integral.

5.7 CHECK YOUR PROGRESS - MODEL ANSWERS

1. 9

2. $\frac{ab}{3}\left(a^2+b^2\right)$

3. $\frac{\pi a^4}{8}$

5.8 MODEL EXAMINATION QUESTIONS

1. Evaluate $\iint\left(x^2+y^2\right)dx\,dy$ in positive quadrant for which $x+y\leq 1$.

2. Evaluate $\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dx\,dy$ by changing into polar coordinates.

3. Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r=2\sin\theta$ and $r=4\sin\theta$.

4. Evaluate $\int\limits_{y=0}^{2}\int\limits_{x=0}^{3} xy\,dx\,dy$.

5. Evaluate $\int\limits_{0}^{1}\int\limits_{x}^{\sqrt{x}} \left(x^2+y^2\right)dx\,dy$.

6. Evaluate $\iint\limits_R xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$.

7. Evaluate $\iint\limits_R r^2\sin\theta\,dr\,d\theta$ where R is the semi circle $r=2a\cos\theta$ above initial line.

8. Evaluate $\int\limits_{0}^{\pi}\int\limits_{0}^{a\sin\theta} r\,dr\,d\theta$.

Answer:

1. $\frac{1}{6}$

2. $\frac{\pi}{4}$

3. $\frac{45\pi}{2}$

4. $\frac{4}{3}$

5. $\frac{3}{35}$

6. $\frac{a^4}{8}$

7. $\frac{2a^2}{3}$

8. $\frac{\pi a^2}{4}$

UNIT - 6 : TRIPLE INTEGRALS AND APPLICATIONS OF MULTIPLE INTEGRALS

Contents

6.0 OBJECTIVES

After studying this unit, you will be able to:

- Evaluate triple integral by changing variables.
- Apply double and triple integrals to find area and volume of the given region.

6.1 INTRODUCTION

Let $f(x,y,z)$ be a single valued function of the independent variables in x, y, z in finite region V. Divide the region V into n sub regions $\delta V_1, \delta V_2, ..., \delta V_n$. Let p be any point on the boundary or inside V.

Form the sum $S_n = f(x_1, y_1, z_1)\delta V_1 + f(x_2, y_2, z_2)\delta V_2 + ... + f(x_n, y_n, z_n)\delta V_n$

$$= \sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r \qquad ...... (1)$$

when n tends to infinity, the limit of a sum (1) tends to zero is called the triple integral of the function $f(x,y,z)$ over region V and is denoted by $\iiint_V f(x,y,z)dV$.

6.2 EVALUATION OF TRIPLE INTEGRAL

The triple integral is evaluated as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f\left(x,y,z\right)dx\,dy\,dz$$

where the limits of z are z_1, z_2 which are either constants or functions of x and y. The y limits y_1, y_2 are either constants or functions of x and the x limits x_1, x_2 are constants. The above multiple integral is evaluated as follows:

First $f\left(x,y,z\right)$ is in integrated w.r.t z between the limits z_1 and z_2 keeping y, x fixed. The resulting expression is integrated w.r.t y between the limits y_1 and y_2 keeping x constant. The result is finally integrated w.r.t x from x_1 to x_2.

i.e., $$\int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f\left(x,y,z\right)dz\right]dy\right]dx$$

6.2.1 Example

Evaluate $\int_0^1\int_1^2\int_2^3 x\,yz\,dx\,dy\,dz$.

Solution: Let $I = \int_0^1\int_1^2\int_2^3 x\,yz\,dx\,dy\,dz$

$$I = \int_0^1\int_1^2 yz\left[\frac{x^2}{2}\right]_2^3 dy\,dz = \int_0^1\int_1^2\left(\frac{9}{2}-\frac{4}{2}\right)yz\,dy\,dz$$

$$= \frac{5}{2}\int_0^1\int_1^2 yz\,dz = \frac{5}{2}\int_0^1\left[\frac{y^2}{2}\right]_1^2 z\,dz$$

$$= \frac{5}{2}\int_0^1\left(\frac{4}{2}-\frac{1}{2}\right)z\,dz = \frac{15}{4}\int_0^1 z\,dz$$

$$= \frac{15}{4}\left[\frac{z^2}{2}\right]_0^1 = \frac{15}{8}.$$

6.2.2 Example

Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\,dy\,dx\,dz$.

Solution: We have the given integral

$$I = \int_0^1 \int_{y^2}^1 \int_0^{1-x} x\,dy\,dx\,dz = \int_0^1 \int_{y^2}^1 x\left[z\right]_0^{1-x} dy\,dx$$

$$= \int_0^1 \int_{y^2}^1 x(1-x)\,dy\,dx = \int_0^1 \int_{y^2}^1 \left(x - x^2\right) dy\,dx$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{y^2}^1 dy = \int_0^1 \left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2}y^4 - \frac{1}{3}y^6\right)\right] dy$$

$$= \int_0^1 \left(\frac{1}{6} - \frac{1}{2}y^4 + \frac{1}{3}y^6\right) dy = \left[\frac{1}{6}y - \frac{1}{2}\frac{y^5}{5} + \frac{1}{3}\frac{y^7}{7}\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{10} + \frac{1}{21} = \frac{4}{35}.$$

6.2.3 Example

Evaluate $\iiint_V \left(x^2 + y^2 + z^2\right) dx\,dy\,dz$, where V is the volume of the cube bounded by the coordinate planes and $x = y = z = a$ planes.

Solution: Here the limits of x, y, z varies from 0 to a.

$$\iiint_V \left(x^2 + y^2 + z^2\right) dx\,dy\,dz = \int_0^a \int_0^a \int_0^a \left(x^2 + y^2 + z^2\right) dx\,dy\,dz$$

$$= \int_0^a \int_0^a \left(x^2 z + y^2 z + \frac{z^3}{3}\right)_{z=0}^a dx\,dy = \int_0^a \int_0^a \left(x^2 a + y^2 a + \frac{a^3}{3}\right) dx\,dy$$

$$= \int_0^a \left(x^2 a y + a\frac{y^3}{3} + \frac{a^3}{3}y\right)_{y=0}^a dx = \int_0^a \left(x^2 a^2 + \frac{a^4}{3} + \frac{a^4}{3}\right) dx$$

$$= \left[a^2 . \frac{x^3}{3} + \frac{a^4}{3}x + \frac{a^4}{3}x\right]_0^a = \frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3} = a^5.$$

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. Evaluate $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2} x^2 yz\,dx\,dy\,dz$.

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6.3 CHANGE OF VARIABLE IN TRIPLE INTEGRAL

Let the functions $x=\phi_1(u,v,w)$, $y=\phi_2(u,v,w)$ and $z=\phi_3(u,v,w)$ be the transformations from cartesian coordinates to the curvilinear coordinates u,v,w.

The Jacobian for this transformation is given by

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Then $\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(\phi_1,\phi_2,\phi_3)|J|\,du\,dv\,dw$

where V' is the corresponding domain in the curvilinear coordinates u,v,w.

6.3.1 Change of Variables from Cartesian to Spherical Polar - Coordinate System

The relations between the Cartesian coordinates x,y,z and spherical polar coordinates ρ,θ,ϕ $[u=\rho, v=\theta, w=\phi]$ are given by

$$x=\rho\sin\theta\cos\phi$$

$$y=\rho\sin\theta\sin\phi$$

$$z=\rho\sin\theta$$

and $dx\,dy\,dz=|J|\,d\rho\;d\theta\;d\phi$

where $J=\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\theta & -\rho\sin\theta & 0 \end{vmatrix}$$

$$= \rho^2 \sin\theta$$

$$\therefore \iiint_V f(x, y, z) dx\, dy\, dz$$

$$= \iiint_{V'} f(\rho\sin\theta\cos\phi,\ \rho\sin\theta\sin\phi,\ \rho\cos\theta)\rho^2 \sin\theta\ d\rho\ d\theta\ d\phi\ .$$

The region V in (x, y, z) is to be covered by the limits of ρ, θ, ϕ and is denoted as V'.

6.3.2 Change of Variables from Cartesian to Cylindrical Coordinate System

The relations between the cartesian coordinates x, y, z and the cylindrical coordinates ρ, θ, z are given by

$$x = \rho\cos\theta,\ y = \rho\sin\theta, z = z$$

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \theta, z)}$$

$$= \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

$$\therefore \iiint_V f(x, y, z) dx\, dy\, dz = \iiint_{V'} f(\rho\cos\theta, \rho\sin\theta, z)\rho\ d\rho\ d\theta\ dz\ .$$

6.3.3 Example

Evaluate $\iiint \left(x^2 + y^2 + z^2\right) dx\,dy\,dz$ taken over the volume enclosed by the sphere $x^2 + y^2 + z^2 = 1$, by transforming into spherical polar coordinates.

Solution: Converting the given integral into spherical polar coordinates by putting

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$.

We have $J = r^2\sin\theta$ and $x^2 + y^2 + z^2 = r^2$.

$$\iiint \left(x^2 + y^2 + z^2\right) dx\,dy\,dz = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r^2\left(r^2\sin\theta\right) dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^4 \sin\theta\, dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{r^5}{5}\right]_0^1 \sin\theta\,d\theta\,d\phi$$

$$= \frac{1}{5}\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi} d\phi = \frac{2}{5}\int_0^{2\pi} d\phi = \frac{4\pi}{5}.$$

6.4 APPLICATIONS OF MULTIPLE INTEGRALS

I. Area Enclosed by a Plane Curve

Consider the area enclosed by the curves $f = f(x),\ y = g(x),\ x = a,\ x = b$ in xy plane.

The area of the region R bounded by the given curves is given by

$$\iint_R dx\,dy \text{ or } \iint_R dy\,dx = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)} dy\,dx.$$

If the region is represented through polar coordinates, then the area is given by $\iint_R r\,dr\,d\theta$.

6.4.1 Example

Find the area of the region bounded by the parabolas

$y^2 = 4ax$ and $x^2 = 4ay$.

Solution: Given curves are

$y^2 = 4ax$ ...... (1)

$x^2 = 4ay$ ...... (2)

We solve (1) and (2) to find the points of intersection.

$x^2 = 4ay$

$\Rightarrow x^4 = 16a^2 y^2$

$\Rightarrow x^4 = 16a^2.4ax \Rightarrow x^4 = 64a^3 x$

$\Rightarrow x\left(x^3 - 64a^3\right) = 0 \Rightarrow x = 0$ or $x = 4a$

when $x = 0,\ y = 0$ and when $x = 4a,\ y = 4a$.

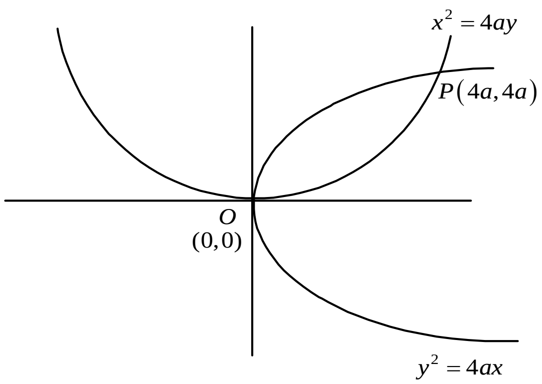

Fig: 6.1

Hence the given two parabolas intersect at (0, 0) and $(4a, 4a)$.

$\therefore$ Area $A = \iint\limits_R dx\,dy$.

The region R can be covered by varying x from the upper curve $x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$ while y varies from 0 to $4a$.

Thus $A = \int\limits_{y=0}^{4a} \int\limits_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int\limits_{0}^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$

$$= \int\limits_{0}^{4a} \left(2\sqrt{a}\sqrt{y} - \frac{y^2}{4a} \right) dy = \left[2\sqrt{a}\,\frac{y^{\frac{3}{2}}}{3/2} - \frac{y^3}{12a} \right]_0^{4a} = \frac{16a^2}{3} \text{ sq. units.}$$

Check Your Progress:

2. Find the area enclosed by the parabolas $x^2 = y$ and $y^2 = x$.

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II. Volume as a Double Integral

Let $z = f(x, y)$ be a surface above *xy* plane.

Let its projection on the *xy* plane be the area *S*. Suppose we divide the entire *S* into elementary rectangular of area $\delta x \delta y$ by drawing lines parallel to *x* and *y* axes.

Consider a representative of elementary at rectangle as base and erect a prism having its length parallel to *z* axis.

The volume of the prism between *xy* plane and the given surface $z = f(x, y)$, $\delta v = z\delta x \delta y = f(x, y)\delta x \delta y$.

The volume of the solid cylinder with *S* as base, bounded by the given surface with generators parallel to the *z* - axis is equal to $\iint z\,dx\,dy$ or $\iint f(x, y)\,dx\,dy$ integrated over the region *S*.

If the base area is represented through polar coordinates, then the required volume will be $\iint f(r, \theta)\,r\,dr\,d\theta$.

6.4.2 Example

Find the volume of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate planes.

Solution: Given $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\Rightarrow z = f(x, y) = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

If $f(x, y)$ is a continuous and single valued function over the region *R*, in the *xy*- plane then $z = f(x, y)$ is the equation of the surface.

Let *C* be the closed curve that is the boundary of *R*. Using *R* as a base, construct a cylinder having elements parallel to *z* - axis. This cylinder intersects $z = f(x, y)$ in a curve τ, whose projection on the *xy* plane is *C*.

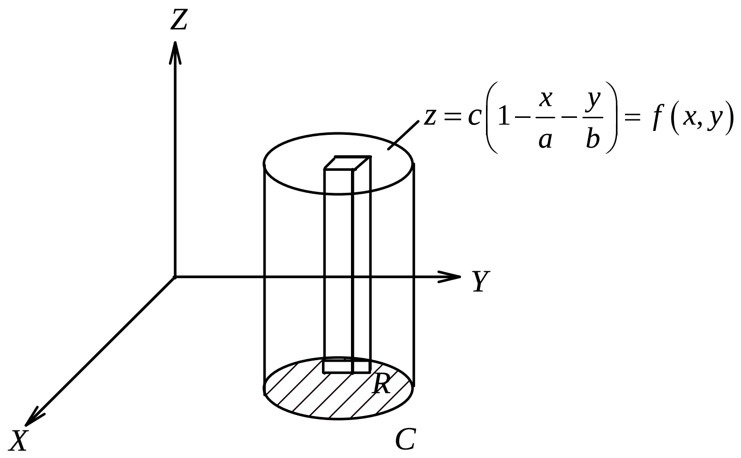

Fig: 6.2

The equation of the surface under which the region whose volume is required may be written in the form

$$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right).$$

Hence the volume of the region $= \iint_R z\,dx\,dy = \iint_R c\left(1 - \frac{x}{a} - \frac{y}{b}\right)dx\,dy$.

The equation of the intersection of given surface with *xy* - plane is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

The limits for *y* are from *y* = 0 to the line $y = b\left(1 - \frac{x}{a}\right)$ and *x* is from 0 to *a*.

$$\therefore V=\int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dx\,dy$$

$$=c\int_{x=0}^{a}\left[y-\frac{xy}{a}-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)}dx \;=c\int_0^a b\left[\frac{1}{2}-\frac{x}{a}+\frac{x^2}{2a^2}\right]dx$$

$$=bc\left[\frac{1}{2}x-\frac{x^2}{2a}+\frac{x^3}{6a^2}\right]_0^a \;=bc\left[\frac{a}{2}-\frac{a^2}{2a}+\frac{a^3}{6a^2}\right] \;=\frac{abc}{6}.$$

6.4.3 Example

Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.

Solution: Clearly $z=4-y$ is to be integrated over the circle $x^2+y^2=4$ in the xy - plane.

Hence x varies from $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$ and y varies from -2 to 2.

Hence the desired volume $V=\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} z\,dx\,dy$

$$=2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}(4-y)\,dy\,dx=2\int_{-2}^{2}(4-y)\left[\int_0^{\sqrt{4-y^2}}dx\right]dy$$

$$=2\int_{-2}^{2}(4-y)\sqrt{4-y^2}\,dy \;=2\int_{-2}^{2}4\sqrt{4-y^2}\,dy-\int_{-2}^{2}y\sqrt{4-y^2}\,dy$$

$$=8\int_{-2}^{2}\sqrt{4-y^2}\,dy-0 \text{ [since the integrand is an odd function]}$$

$$=8\left[\frac{y}{2}\sqrt{4-y^2}+\frac{4}{2}\sin^{-1}\frac{y}{2}\right]_{-2}^{2} \;=16\pi.$$

III. Volume as Triple Integral

Suppose a three dimensional solid is cut into elemental rectangular parallelopipeds by drawing planes parallel to the coordinate planes.

The volume of an elemental parallelopiped δv is $\delta x \delta y \delta z$.

Hence the total volume of the solid is $\iiint_V dv = \iiint_V dx\,dy\,dz$ where the integration is carried over the entire volume.

6.4.4 Example

Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Solution: Given cylinders are

$$x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}$$

$$x^2 + z^2 = a^2 \Rightarrow z^2 = a^2 - x^2 \text{ or } z = \pm\sqrt{a^2 - x^2}\,.$$

Required volume can be covered as follows:

limits of z: from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$

limits of y : from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$

limits of x: from $-a$ to a

$$\therefore V = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx \;= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2}}\, dy\,dx \;= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dy\,dx$$

$$= 8\int_0^a \sqrt{a^2 - x^2}\,[y]_0^{\sqrt{a^2-x^2}}\, dx \;= 8\int_0^a \left(a^2 - x^2\right) dx$$

$$= 8\left[a^3 - \frac{a^3}{3}\right] = \frac{16a^3}{3} \text{ cubic units.}$$

6.4.5 Example

Using spherical polar coordinates find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution: Introducing spherical polar coordinates

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

We have $dx\ dy\ dz = r^2\sin\theta\, dr\, d\theta\, d\phi$.

Using this transformation, the given region of integration becomes

$$\{(r,\theta,\phi): 0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi\}.$$

$\therefore$ The required volume $V = \iiint dx\, dy\, dz$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^a \sin\theta\, dr\, d\phi \quad = \frac{a^3}{3}\int_0^{2\pi}[-\cos\theta]_0^{\pi}\, d\phi$$

$$= \frac{2a^3}{3}\int_0^{2\pi} d\phi \ = \frac{4\pi a^3}{3}.$$

6.4.6 Example

Using cylindrical coordinates, find the volume of the cylinder with base radius '*a*' and height *h*.

Solution: The region of integration is bounded by $x^2 + y^2 \le a^2,\ 0 \le z \le h$.

Required volume $= \iiint dx\, dy\, dz$

$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r\, dr\, d\theta\, dz \ = \int_0^h\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^a d\theta\, dz$$

$$= \frac{a^2}{2}\int_0^h[\theta]_0^{2\pi}\, dz = \frac{2\pi a^2}{2}\int_0^h dz \ = \pi a^2 h.$$

6.5 WORKED OUT EXERCISES

6.5.1 Exercise

Evaluate the triple integral $\int_0^1\int_y^1\int_0^{1-x} x\,dz\,dx\,dy$.

Solution: Given integral $I=\int_0^1\int_y^1\int_0^{1-x} x\,dz\,dx\,dy$

$$=\int_0^1\int_y^1 x(z)_0^{1-x}\,dx\,dy \;=\int_0^1\int_y^1 x(1-x)\,dx\,dy$$

$$=\int_0^1\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_y^1 dy \;=\int_0^1\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{y^2}{2}-\frac{y^3}{3}\right)dy$$

$$=\int_0^1\left(\frac{1}{6}+\frac{y^3}{3}-\frac{y^2}{2}\right)dy \;=\left[\frac{1}{6}y+\frac{y^4}{12}-\frac{y^3}{6}\right]_0^1$$

$$=\frac{1}{6}+\frac{1}{12}-\frac{1}{6}=\frac{1}{12}.$$

6.5.2 Exercise

Evaluate $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$, where V is the region of space bounded by $x=0, x=1,\ y=0,\ y=2,\ z=0,\ z=3$.

Solution: $\iiint_V (xy+yz+zx)\,dx\,dy\,dz=\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3}(xy+yz+zx)\,dx\,dy\,dz$

$$=\int_0^1\int_0^2\left[\frac{x^2}{2}y+xyz+\frac{zx^2}{2}\right]_0^1 dy\,dz \;=\int_{z=0}^{3}\int_{y=0}^{2}\left(\frac{y}{2}+yz+\frac{z}{2}\right)dy\,dz$$

$$=\int_{z=0}^{3}\left[\frac{y^2}{4}+\frac{y^2}{2}z+\frac{yz}{2}\right]_0^2 dz \;=\int_{z=0}^{3}(1+2z+z)\,dz$$

$$=\int_0^3(1+3z)\,dz \;=\left(z+\frac{3}{2}z^2\right)_0^3=3+\frac{27}{2}=\frac{33}{2}.$$

6.5.3 Exercise

Using double integration determine the area of the region bounded by the curves $y^2 = 4ax,\ x + y = 3a$ and $y = 0$.

Solution: Given curves are $y^2 = 4ax,\ x + y = 3a$ and $y = 0$.

Solving these equations we get the points of intersectiON of curves

$(3a - x)^2 = 4ax \Rightarrow 9a^2 - 6ax + x^2 - 4ax = 0$

$\Rightarrow x^2 - 10ax + 9a^2 = 0 \Rightarrow (x-a)(x-9a) = 0 \Rightarrow x = a$ or $9a$.

Substituting $x = a$ in $x + y = 3a$ we get $y = 2a$.

$\therefore y^2 = 4ax$ and $x + y = 3a$ intersect at $(a, 2a)$ and $x + y = 3a$ and $y = 0$ meet at $A(3a, 0)$.

Hence the required area $A = \iint_R dx\,dy$

$$= \iint_{R_1} dx\,dy + \iint_{R_2} dx\,dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{4ax}} dy\,dx + \int_{x=a}^{3a} \int_{y=0}^{3a-x} dy\,dx$$

$$= \int_{x=0}^{a} [y]_0^{\sqrt{4ax}}\,dx + \int_{a}^{3a} [y]_0^{3a-x}\,dx$$

$$= 2\sqrt{a}\int_0^a \sqrt{x}\,dx + \int_a^{3a} (3a - x)\,dx$$

$$= 2\sqrt{a}.\frac{2}{3}\left(x^{3/2}\right)_0^a + \left[\frac{(3a-x)^2}{-2}\right]_a^{3a}$$

$$= \frac{4\sqrt{a}}{3}a^{3/2} - \frac{1}{2}\left[0 - 4a^2\right]$$

$$= \frac{4a^2}{3} + 2a^2 = \frac{10}{3}a^2 \text{ sq. units.}$$

6.5.4 Exercise

Find the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Solution: Given cylinders are

$$x^2+y^2=a^2 \qquad \text{... (1)}$$

$$x^2+z^2=a^2 \qquad \text{... (2)}$$

The section of the cylinder $x^2+z^2=a^2$ with *xy* plane *z* = 0 is the circle $x^2+y^2=a^2$.

From (2) we have $z=\pm\sqrt{a^2-x^2}$.

$\therefore$ Each of the surface is symmetrical about the *xy* - plane.

Hence the required volume = $2\iint\limits_R z\,dx\,dy$, where $z=\sqrt{a^2-x^2}$

and *R* is the region bounded by the circle (1) on the *xy* - plane i.e.,

$-a\le x\le a$ and $-\sqrt{a^2-x^2}\le y\le\sqrt{a^2-x^2}$

$$\text{Required volume } = 2\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z\,dx\,dy$$

$$=2\int_{-a}^{a} dx\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2}\,dy = 2\int_{-a}^{a}\sqrt{a^2-x^2}\,\left[y\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dx$$

$$=4\int_{x=-a}^{a}\sqrt{a^2-x^2}.\sqrt{a^2-x^2}\,dx = 4\int_{x=-a}^{a}\left(a^2-x^2\right)dx$$

$$=8\int_0^a\left(a^2-x^2\right)dx=\frac{16a^3}{3}.$$

6.5.5 Exercise

Find the volume bounded by the *xy* - plane, the cylinder $x^2+y^2=1$ and the plane $2x+3y+4z=12$.

Solution: On the plane $2x+3y+4z=12$ we have

$4z = 12 - 2x - 3y$ or $z = \frac{1}{4}(12 - 2x - 3y)$.

For a fixed (x, y) on the *xy* - plane *z* varies from 0 to $\frac{1}{4}(12 - 2x - 3y)$.

(x, y) varies with in the circular region $A = x^2 + y^2 \le 1$.

Hence the required volume $= \iint_A \int_{z=0}^{\frac{1}{4}(12-2x-3y)} z\,dx\,dy$

$$= \iint_A \frac{1}{4}(12 - 2x - 3y)\,dx\,dy$$

Let us take $x = r\cos\theta,\ y = r\sin\theta$ where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

$dx\,dy = r\,dr\,d\theta$.

Hence the required volume $= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{1}{4}(12 - 2r\cos\theta - 3r\sin\theta)\,r\,dr\,d\theta$

$$= \int_0^{2\pi} \frac{1}{4}\left[12\frac{r^2}{2} - \frac{2r^3}{3}\cos\theta - \frac{3r^3}{3}\sin\theta\right]_0^1 d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{4}\left[6 - \frac{2}{3}\cos\theta - \sin\theta\right] d\theta$$

$$= \frac{1}{4}\left[6\theta - \frac{2}{3}\sin\theta + \cos\theta\right]_0^{2\pi}$$

$$= \frac{1}{4}.12\pi = 3\pi \text{ cubic units.}$$

6.5.6 Exercise

Find by double integration the area between the curves $y = x^2 - 6x + 3$ and $y = 2x - 9$.

Solution: $y = x^2 - 6x + 3 + 6 - 6 = x^2 - 6x + 9 - 6 = (x-3)^2 - 6$

$$\Rightarrow (x-3)^2 = y + 6 \qquad \text{...... (1)}$$

which is the equation of a parabola with vertical axis and vertex $A(3,-6)$

The parabola meets y - axis (x = 0) at the point $B(0,3)$ and

x - axis at $(x-3)^2 - 6 = 0 \Rightarrow x = 3 \pm \sqrt{6}$

i.e. at point $C\left(3-\sqrt{6},0\right)$, $D\left(3+\sqrt{6},0\right)$

$y = 2x - 9$ ...... (2)

which is a straight line which cuts x - axis at the point $\left(\frac{9}{2},0\right)$ and y - axis at the point (0, -9).

The curves (1) and (2) intersect each other at the points given by

$2x - 9 = x^2 - 6x + 3 \Rightarrow x^2 - 8x + 12 = 0$

$\Rightarrow (x-2)(x-6) = 0 \Rightarrow x = 2$ or $x = 6$.

$x = 2 \Rightarrow y = 2x - 9 = -5$

$x = 6 \Rightarrow y = 2x - 9 = 3$.

$\therefore$ The points of intersection of (1) and (2) are given by (2, -5), (6, 3).

The required area A is given by

$$A = \iint_R dx\,dy = \int_{x=2}^{6}\left[\int_{y=x^2-6x+3}^{y=2x-9} 1\,dy\right]dx$$

$$= \int_{x=2}^{6} [y]_{x^2-6x+3}^{2x-9}\,dx = \int_{x=2}^{6}\left(2x-9-\left(x^2-6x+3\right)\right)dx$$

$$= \int_2^6\left(-x^2+8x-12\right)dx = \left[\frac{-x^3}{3} + 8.\frac{x^2}{2} - 12x\right]_2^6$$

$$= (-72+144-72) - \left(\frac{-8}{3} + 16 - 24\right) = \frac{32}{3} \text{ sq. units.}$$

6.5.7 Exercise

Find the volume with in the cylinder $x^2 + y^2 = 4x$ cut by the cylinder $z^2 = 4x$.

Solution: Required volume V is given by

$V = \iiint_R dx\,dy\,dz$, where R is the region bounded by the cylinders

(i) $x^2 + y^2 = 4x$ which is in xy - plane i.e., $z = 0$

(ii) $z^2 = 4x$ which is in xz plane i.e., $y = 0$

Points of integration:

On $z^2 = 4x,\ y = 0$ substituting in (i), we get

$x^2 = 4x \Rightarrow x(x-4) = 0 \Rightarrow x = 0$ or $x = 4$.

$z^2 = 4x \Rightarrow z = \pm 2\sqrt{x}$

$x^2 + y^2 = 4x \Rightarrow y = \pm\sqrt{4x - x^2}$ and $0 \le x \le 4$.

$$\therefore V = \int_{x=0}^{4} \int_{y=\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} \int_{z=-2\sqrt{x}}^{2\sqrt{x}} dz\,dy\,dx = \int_{x=0}^{4} \int_{y=-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} (z)_{-2\sqrt{x}}^{2\sqrt{x}}\,dy\,dx$$

$$= \int_{x=0}^{4} \int_{y=-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} 4\sqrt{x}\,dy\,dx = \int_{x=0}^{4} 4\sqrt{x}\,[6]_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}}\,dx$$

$$= 8\int_0^4 \sqrt{x}\sqrt{4x - x^2}\,dx = 8\int_0^4 x\sqrt{4-x}\,dx.$$

Put $x = 4\sin^2\theta,\ dx = 8\sin\theta\cos\theta\,d\theta$

If $x = 0$, then $\theta = 0$

$x = 4$, then $\sin^2\theta = 1 \Rightarrow \theta = \dfrac{\pi}{2}$.

$$\therefore V = 8\int_0^{\pi/2} 4\sin^2\theta.\ 2\cos\theta.\ 8\sin\theta\ \cos\theta\ d\theta$$

$$= 512 \int_0^{\pi/2} \sin^3\theta \cos^2\theta \, d\theta$$

$$= 512 \times \frac{2\times 1}{5\times 3\times 1} = \frac{1024}{15} \text{ cubic units.}$$

6.5.8 Exercise

Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: Let *OABC* be the positive octant of the given ellipsoid which is bounded by the planes $x=0,\ y=0,\ z=0$ and the surface *ABC*.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

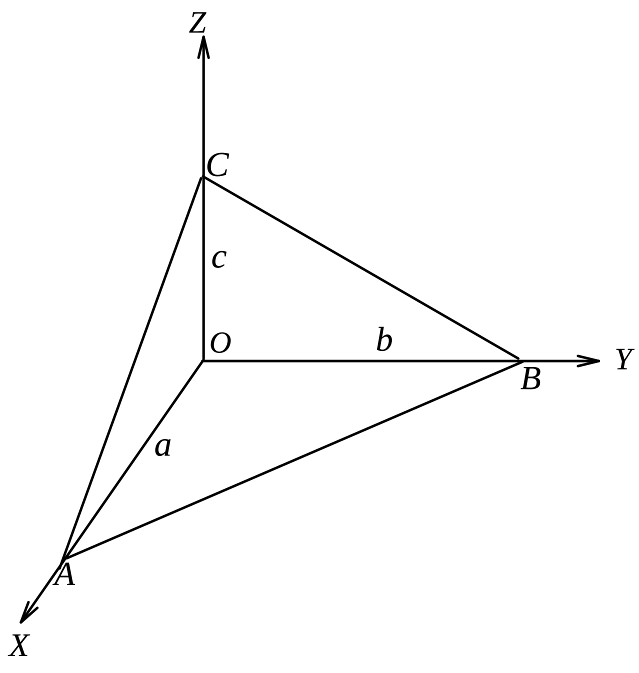

Fig: 6.3

Divide this region *R* into rectangular parallelopipeds of volume $\delta x \delta y \delta z$.

Let $P(x, y, z)$ be any point.

$\therefore$ The required volume $= 8\iiint\limits_R dx\,dy\,dz$

Limits: z varies from 0 to $c\sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}}$

y varies from 0 to $b\sqrt{1-\dfrac{x^2}{a^2}}$

and x varies from 0 to a.

Hence the volume of the ellipsoid is

$$8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\,dy\,dz = 8\int_{0}^{a}\int_{0}^{b\sqrt{1-\frac{x^2}{a^2}}}\left[z\right]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\ dz$$

$$= 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dz$$

$$= \frac{8c}{b}\int_0^a dx\int_0^t \sqrt{t^2-y^2}\ dy\text{, where } t = b\sqrt{1-\frac{x^2}{a^2}}$$

$$= \frac{8c}{b}\int_0^a dx\left[\frac{y\sqrt{t^2-y^2}}{2}+\frac{t^2}{2}\sin^{-1}\frac{y}{t}\right]_0^t$$

$$= \frac{8c}{b}\int_0^a \frac{b^2}{2}\left(1-\frac{x^2}{a^2}\right)\frac{\pi}{2}dx \quad = 2\pi bc\int_0^a\left(\frac{1-x^2}{a^2}\right)dx$$

$$= 2\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a = 2\pi bc\left[a-\frac{a^3}{3a^2}\right]$$

$$= \frac{4\pi abc}{3} \text{ cubic units.}$$

6.6 SUMMARY

In this unit, we learnt how to evaluate a triple integral by finding limits of integration. We also evaluated the triple integral by changing into cylindrical and spherical coordinates by using Jacobians. Double integrals are very useful for finding the area of a region bounded by curves of functions. We have seen how a double integral can be used to find the volume of a solid bounded by a region. Triple integrals are mainly used to calculate the volume of a three dimensional solid.

6.7 CHECK YOUR PROGRESS - MODEL ANSWERS

1. 1

2. $\frac{1}{3}$ sq. units

6.8 MODEL EXAMINATION QUESTIONS

1. Evaluate $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} x\,y\,z\,dx\,dy\,dz$.

2. Find the area of the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

3. Using the double integration determine the area of the region bounded by the curves $y^2 = 4ax,\ x + y = 3a$ and $y = 0$.

4. Find the volume bounded by the elliptic paraboloids $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

5. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.

6. Evaluate $\int_0^1 \int_0^1 \int_0^1 \left(x^2 + y^2 + z^2\right) dx\,dy\,dz$.

7. Evaluate the triple integral $\iiint xy^2 z\ dx\,dy\,dz$ taken through the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

8. Evaluate $\iiint z^2\,dx\,dy\,dz$ taken over the volume bounded by the surfaces $x^2 + y^2 = a^2, x^2 + y^2 = z$ and $z = 0$.

9. Evaluate $\iiint\limits_R (x+y+z)\,dx\,dy\,dz$ where R is the region bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$.

10. Using cylindrical coordinates, find the volume of the cylinder with base radius a and height h.

Answer:

1. $\frac{1}{720}$

2. $\frac{16a^2}{3}$

3. $\frac{10a^2}{3}$

4. $8\sqrt{2}\pi$

5. $18(\pi - 2)$ sq. units

6. 1

7. $\frac{a^7}{105}$

8. $\frac{\pi a^8}{12}$

9. $\frac{3}{2}$

10. $\pi a^2 h$

BLOCK - III : VECTOR INTEGRATION

In the block - I you have learnt about the differentiation of vectors and arrived at the important concepts of directional derivatives, gradient, divergence and curl. This block is totally dedicated to vector integration. In the three units of this block, we will discuss about the evaluation of line, surface and volume integrals over vector fields. In geometry, curvilinear coordinates are a coordinate system for Euclidean space in which the coordinate lines may be curved. These coordinates may be derived from a set of Cartesian coordinates by using a transformation which is a bijection. i.e. we can convert a point given in a Cartesian coordinate system to its curvilinear coordinates and back.

This block includes the following units:

Unit -7: Line Integrals and Surface Integrals

Unit - 8: Volume Integrals and Applications of Vector Integration

Unit - 9: Curvilinear Coordinates

UNIT -7 : LINE INTEGRALS AND SURFACE INTEGRALS

Contents

7.0 OBJECTIVES

After studying this unit, you will be able to:

- Evaluate a given line integral over a curve *C* by parameterizing *C*.
- Find a potential function *f* for a given conservative vector field $\vec{F}$, such that $\vec{F} = \Delta f$
- Evaluate a surface integral of a function $f(x, y)$ of two variables over a given surface.

7.1 INTRODUCTION

When a curve $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k},\ a \le t \le b$, passes through the domain of a function $f(x, y, z)$ in space, the values of *f* along the curve are given by composite function $f(x(t), y(t), z(t))$. If we integrate this composite function along the curve from $t = a$ to $t = b$, then it is called the line integral of *f*. When we study physical phenomena that are represented by vectors, we replace integrals over closed intervals by integrals over paths through vector fields. We use such integrals to find the work done in moving an object along a path against a variable force or to find the work done by a vector field in moving an object along a path through the field.

A vector field on a domain in the plane or in space is a function that assigns a vector to each point in the domain.

We represent it as $\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$

This vector field is continuous if the component functions *f, g, h* are continuous. Differentiable if *f, g, h* are differentiable and so on. In this unit we discuss about line integrals and surface integrals. A line integral is an integral where the function to be integrated along a curve and a surface integral is an integral of a function of two variables over a given surface.

We will also discuss about line integrals as independence of path and as conservative force fields. We will evaluate surface integrals by taking the projection of given surface in coordinate planes.

7.2 INTEGRATION OF A VECTOR

Integration is the inverse operation of differentiation. Let $\vec{F}(t)$ be a differentiable vector function of a scalar variable *t* and let $\frac{d}{dt}\left[\vec{F}(t)\right] = \vec{f}(t)$. Then $\int \vec{f}(t)\,dt = \vec{F}(t)$.

$\vec{F}(t)$ is called the primitive of $\vec{f}(t)$.

The set of all primitives of $\vec{f}(t)$ i.e., $\int \vec{f}(t)dt + c$, where *c* is any arbitrary constant vector is called the indefinite integral of $\vec{f}(t)$.

If $\vec{f}(t) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k} = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$

then $\int \vec{f}(t)\,dt = \hat{i}\int f_1(t)\,dt + \hat{j}\int f_2(t)\,dt + \hat{k}\int f_3(t)\,dt + c$.

7.2.1 Definition: (Definite Integral)

Let $\int \vec{f}(t)\,dt = \vec{f}(t)$. Then $\vec{F}(b) - \vec{F}(a)$ is called the **definite integral** of $\vec{F}(t)$ between $t = a$ and $t = b$. This is denoted by $\int_a^b \vec{f}(t)\,dt = \vec{F}(b) - \vec{F}(a)$.

Also if $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$

then $\int_a^b \vec{f}(t)\,dt = \hat{i}\int_a^b f_1(t)\,dt + \hat{j}\int_a^b f_2(t)\,dt + \hat{k}\int_a^b f_3(t)\,dt$.

7.2.2 Example

If $\vec{F}(t) = (t - t^2)\hat{i} + 2t^3\hat{j} - 3\hat{k}$, find $\int_1^2 \vec{F}(t)\,dt$.

Solution: $$\int_1^2 \vec{F}(t)\,dt = \hat{i}\int_1^2 (t - t^2)\,dt + 2\hat{j}\int_1^2 t^3\,dt - 3\hat{k}\int_1^2 dt$$

$$= \hat{i}\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_1^2 + 2\hat{j}\left[\frac{t^4}{4}\right]_1^2 - 3\hat{k}[t]_1^2$$

$$= -\frac{5}{6}\hat{i} + \frac{15}{2}\hat{j} - 3\hat{k}.$$

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. Suppose $\vec{R}(u) = 3\hat{i} + (u^3 + 4u^7)\hat{j} + 4\hat{k}$, find (a) $\int \vec{R}(u)\,du$, (b) $\int_1^2 \vec{R}(u)\,du$.

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7.3 LINE INTEGRALS

Let C be a curve in space and A be the initial point and B be the terminal point of the curve C. When the direction along C oriented from A to B is positive then the direction B to A is called the negative direction. If the two points coincide, then the curve C is called a **closed curve**.

7.3.1 Definition

A curve $\vec{r} = \vec{F}(t)$ is called a **smooth curve** if $\vec{F}(t)$ is continuously differentiable. A curve $\vec{C}$ is said to be **piecewise smooth** if it consists of a finite number of smooth curves.

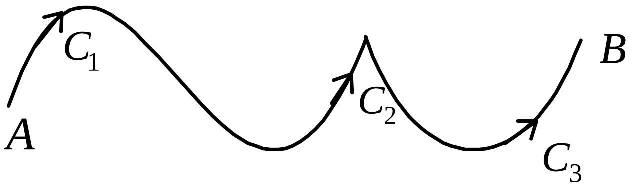

Fig: 7.1

The above curve is composed of three smooth curves C_1, C_2 and C_3.

7.3.2 Line Integrals Notation

Let $\vec{r} = \vec{f}(t)$ be a smooth curve *C* joining the points *A* and *B*.

Let *ds* be the differential of arc length at $P \in C$.

Then $\dfrac{d\vec{r}}{ds} = \vec{T}$ is the unit vector along the tangent to the curve *C* at *P*.

Let $\vec{F}(r)$ be a vector point function and continuous along *C*. The components of $\vec{F}(r)$ along the tangent at *P* is $\vec{F}(r).\vec{T}$.

The integral $\int \vec{F}.\vec{T}\, ds$ taken along the curve *C* is called the **line integral** of $\vec{F}$ along *C*. This is written as

$$\int_A^B \vec{F}.\vec{T}\, ds = \int_C \left(\vec{F}.\frac{d\vec{r}}{ds} \right) ds = \int_C \vec{F}.d\vec{r}$$

This is also called the **tangential line integral** of $\vec{F}$ along *C*.

The another type of line integrals are $\int_C \vec{F} \times d\vec{r}$ and $\int_C \phi\, d\vec{r}$, where $\vec{F}$ is a continuous vector and ϕ is continuous scalar function.

7.3.3 Circulation

Let *C* be a simple closed curve. The line integral of $\vec{F}$ along *C* is called the **circulation** of $\vec{F}$ around the closed curve *C* (since *C* is a closed curve).

$$\oint_C \vec{F}.d\vec{r} = \oint_C F_1 dx + F_2 dy + F_3 dz \, .$$

Cartesian form:

Let $\vec{F}(\vec{r}) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Then $\oint_C \vec{F}.d\vec{r} = \oint_C \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$

$$= \oint_C F_1dx + F_2dy + F_3dz$$

is the line integral in cartesian form.

If *x*, *y*, *z* are functions of *t*, then

$$\oint_C \vec{F}.d\vec{r} = \oint_C F_1dx + F_2dy + F_3dz = \oint_C \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right).dt\,.$$

7.3.4 Note

If $\vec{F}$ is a force acting on a particle at a point *P* whose position vector is $\vec{r}$ on a curve *C*, then $\int_C \vec{F}.d\vec{r}$ represents physically the total work done in moving the particle along *C*.

7.3.5 Example

Evaluate $\int\left(\vec{A}\times\frac{d^2\vec{A}}{dt^2}\right)dt$.

Solution: $\frac{d}{dt}\left(\vec{A}\times\frac{d\vec{A}}{dt}\right) = \vec{A}\times\frac{d^2\vec{A}}{dt} + \frac{d\vec{A}}{dt}\times\frac{d\vec{A}}{dt}$

$$= \vec{A}\times\frac{d^2\vec{A}}{dt} \qquad \left[\because \frac{d\vec{A}}{dt}\times\frac{d\vec{A}}{dt} = \vec{0}\right]$$

$$\therefore \int\frac{d}{dt}\left(\vec{A}\times\frac{d\vec{A}}{dt}\right) = \int\left(\vec{A}\times\frac{d^2\vec{A}}{dt^2}\right)dt$$

Hence $\int \vec{A}\times\frac{d^2\vec{A}}{dt^2}dt = \vec{A}\times\frac{d\vec{A}}{dt} + \vec{C}$.

7.3.6 Example

If $\vec{F}(t)=5t^2\hat{i}+t\,\hat{j}-t^3\hat{k}$ find $\int_1^2\left(\vec{F}\times\frac{d^2\vec{F}}{dt^2}\right)dt$.

Solution: $\int_1^2\left(\vec{F}\times\frac{d^2\vec{F}}{dt^2}\right)dt=\left[\vec{F}\times\frac{d\vec{F}}{dt}\right]_1^2$

Given $\vec{F}=5t^2\hat{i}+t\,\hat{j}-t^3\hat{k}$ and then $\frac{d\vec{F}}{dt}=10t\,\hat{i}+\hat{j}-3t^2\hat{k}$.

$$\vec{F}\times\frac{d\vec{F}}{dt}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5t^2 & t & -t^3\\ 10t & 1 & -3t^2\end{vmatrix}$$

$$=\hat{i}\left(-3t^3+t^3\right)+\hat{j}\left(-10t^4+15t^4\right)+\hat{k}\left(5t^2-10t^2\right)$$

$$=-2t^3\hat{i}+5t^4\hat{j}-5t^2\hat{k}.$$

$$\left[\vec{F}\times\frac{d\vec{F}}{dt}\right]_1^2=\left[-2t^3\hat{i}+5t^4\hat{j}-5t^2\hat{k}\right]_1^2=-14\hat{i}+75\,\hat{j}-15\hat{k}.$$

7.3.7 Example

Suppose $\vec{F}=-3x^2\hat{i}+5xy\,\hat{j}$, evaluate $\int_C\vec{F}.d\vec{r}$, where C is the curve $y=2x^2$ from (0, 0) to (1, 2) in the xy - plane.

Solution: Since the integration is performed in the xy - plane (z = 0), we take

$\vec{r}=x\,\hat{i}+y\,\hat{j}$. So that $d\vec{r}=dx\,\hat{i}+dy\,\hat{j}$.

Now $\int_C\vec{F}.d\vec{r}=\int_C\left(-3x^2\hat{i}+5xy\,\hat{j}\right)\left(dx\,\hat{i}+dy\,\hat{j}\right)$

$$=\int_C-3x^2dx+5xy\,dy.$$

First method:

Let $x = t$ in $y = 2x^2$.

Then the parametric equations of C are $x = t$, $y = 2t^2$.

Points (0, 0) and (1, 2) corresponds to $t = 0$ and $t = 1$ respectively. Then

$$\int_C \vec{F}.d\vec{r} = \int_{t=0}^{1} -3t^2 dt + 5t\left(2t^2\right) d\left(2t^2\right)$$

$$= \int_0^1 \left(-3t^2 + 40t^4\right) = \left[-t^3 + 8t^5\right]_0^1 = 7.$$

Second method:

Substitute $y = 2x^2$ where x varies from 0 to 1.

$$\int_C \vec{F}.\, d\vec{r} = \int_{x=0}^{1} -3x^2 dx + 5x\left(2x^2\right) d\left(2x^2\right) dx$$

$$= \int_{x=0}^{1} \left(-3x^2 + 40x^4\right) = 7.$$

7.3.8 Example

Evaluate the line integral $\int_C x^2 y\, dx + \left(x - z\right) dy + xyz\, dz$, where C is the arc of the parabola $y = x^2$ in the plane $z = 2$ from A (0, 0, 2) to B (1, 1, 2).

Solution:

Since on C, $y = x^2$ and $z = 2$, we have $\frac{dy}{dx} = 2x$ and $dz = 0$.

It follows that on C, the integral of the last term in the given integrand is zero.

$$\int_C x^2 y\, dx + \left(x - z\right) dy + xyz\, dz = \int_0^1 x^2.x^2\, dx + \left(x - 2\right).2x\, dx$$

$$= \int_0^1 x^4 dx + \left(2x^2 - 4x\right) dx = \int_0^1 \left(x^4 + 2x^2 - 4x\right) dx$$

$$= \left[\frac{x^5}{5} + 2.\frac{x^3}{3} - 4.\frac{x^2}{2} \right]_0^1 \quad = \frac{1}{5} + \frac{2}{3} - 2 = -\frac{17}{15}.$$

7.3.9 Example

Evaluate $\int_C \left[\left(x^2 - y\right)dx + \left(y^2 - z\right)dy + \left(z^2 - x\right)dz \right]$, where C is the curve from origin O to the point A (1, 1, 1).

(a) along the straight line OA

(b) along the curve $x = t, y = t^2,\ z = t^3, 0 \le t \le 1$.

Solution: (a) Equation of line OA is $x = y = z$.

For path OA,

$$\int_C \left(x^2 - y\right)dx + \left(y^2 - z\right)dy + \left(z^2 - x\right)dz \ = \int_0^1 \left(x^2 - x\right)dx + \left(x^2 - x\right)dx + \left(x^2 - x\right)dx$$

$$\left(\because x = y = z \Rightarrow dx = dy = dz\right)$$

$$= 3\int_0^1 \left(x^2 - x\right)dx \ = 3\left[\frac{1}{3} - \frac{1}{2}\right] = -\frac{1}{2}.$$

(b) Along the curve $x = t, y = t^2, z = t^3$

$dx = dt, dy = 2t\,dt,\ dz = 3t^2 dt$.

$$\int_C \left(x^2 - y\right)dx + \left(y^2 - z\right)dy + \left(z^2 - x\right)dz$$

$$= \int_0^1 \left(t^2 - t^2\right)dt + \left(t^4 - t^3\right)2t\,dt + \left(t^6 - t\right)3t^2 dt$$

$$= \int_0^1 \left(2t^5 - 2t^4 + 3t^8 - 3t^3\right)dt \ = 2\left[\frac{t^6}{6} - \frac{t^5}{5}\right]_0^1 + 3\left[\frac{t^9}{9} - \frac{t^4}{4}\right]_0^1$$

$$= 2\left(\frac{1}{6} - \frac{1}{5}\right) + 3\left(\frac{1}{9} - \frac{1}{4}\right) \ = -\frac{29}{60}.$$

7.3.10 Example

Suppose $\phi = 2xyz^2, \vec{F} = 2x\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$. Evaluate the integrals (i) $\int_C \phi\, d\vec{r}$ and (ii) $\int_C \vec{F} \times d\vec{r}$.

Solution: Evaluation of (i) $\int_c \phi\, d\vec{r}$

Along C, $\phi = 2xyz^2 = 2\left(t^2\right)\left(2t\right)\left(t^3\right)^2 = 4t^9$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$

and $d\vec{r} = \left(2t\hat{i} + 2\hat{j} + 3t^2\hat{k}\right)dt$.

$$\therefore \int \phi\, d\vec{r} = \int_0^1 4t^9\left(2t\hat{i} + 2\hat{j} + 3t^2\hat{k}\right)dt$$

$$= \hat{i}\int_0^1 8t^{10}dt + \hat{j}\int_0^1 8t^9 dt + \hat{k}\int_0^1 12t^{11}dt = \frac{8}{11}\hat{i} + \frac{4}{5}\hat{j} + \hat{k}.$$

(ii) Along C, we have $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$

$$= \left(t^2\right)\left(2t\right)\hat{i} - t^3\hat{j} + t^4\hat{k} = 2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}.$$

$$\vec{F} \times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} dt$$

$$= \left[\left(-3t^5 - 2t^4\right)\hat{i} + \left(-4t^5\right)\hat{j} + \left(4t^3 + 2t^4\right)\hat{k}\right]dt$$

$$\int_C \vec{F} \times d\vec{r} = \hat{i}\int_0^1\left(-3t^5 - 2t^4\right)dt + \hat{j}\int_0^1\left(-4t^5\right)dt + \hat{k}\int_0^1\left(4t^3 + 2t^4\right)dt$$

$$= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}.$$

7.3.11 Example

Suppose a force field is given by $\vec{F} = (2x - y + z)\hat{i} + \left(x + y - z^2\right)\hat{j} + (3x - 2y + 4z)\hat{k}$. Find the work done in moving a particle once around circle c in the xy - plane with its centre at the origin and a radius of 3 units.

Solution: In the xy plane $z = 0$.

$\vec{F} = (2x - y)\hat{i} + (x + y)\hat{j} + (3x - 2y)\hat{k}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$.

The work done is

$$\int_C \vec{F}.d\vec{r} = \int_C \left[(2x - y)\hat{i} + (x + y)\hat{j} + (3x - 2y)\hat{k}\right].\left(dx\,\hat{i} + dy\,\hat{j}\right)$$

$$= \int_C (2x - y)\,dx + (x + y)\,dy .$$

Choose the parametric equations of the circle as $x = 3\cos t, y = 3\sin t$ where t varies from 0 to 2π . Now

$$\int_C \vec{F}.d\vec{r} = \int_0^{2\pi} 2(3\cos t - 3\sin t)(-3\sin t)\,dt + (3\cos t + 3\sin t)(3\cos t)\,dt$$

$$= \int_0^{2\pi} 9 - 9\sin t\cos t\,dt \;\; = \int_0^{2\pi} \left(9 - \frac{9}{2}\sin 2t\right)dt$$

$$= \left[9t - \frac{9}{2}\left(-\frac{\cos 2t}{2}\right)\right]_0^{2\pi} = 18\pi \,.$$

Check Your Progress:

2. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ and C is given by $\vec{r} = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$, $0 \le t \le 1$.

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7.4 PATH INDEPENDENCE AND CONSERVATIVE FIELDS

A region *D* is connect *d* if any two points of *D* can be joined by a broken line of finitely many linear segments all of which belong to *D*.

Let $\vec{F}$ be a vector field with components F_1, F_2, F_3 which are continuous throughout some connected region *D*. Consider two points *A* and *B* in *D*.

Suppose that *C* is many piece wise smooth curve joining *A* and *B* given by

$$x = x(t), y = y(t), z = z(t),\ t_1 \le t \le t_2 .$$

If there exists a differentiable function *f* such that

$$\vec{F} = \nabla f \quad = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

then along *C*, $f = f = \left(x(t), y(t), z(t)\right)$ is a function of *t* and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt} + \frac{\partial f}{\partial z}.\frac{dz}{dt}$$

$$= \nabla f.\left(\hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}\right) = \nabla f.\frac{d\vec{r}}{dt}$$

Thus we have $\vec{F}.d\vec{r} = \nabla f.d\vec{r}$

$$= \left(\nabla f.\frac{d\vec{r}}{dt}\right)dt = \left(\frac{df}{dt}\right)dt .$$

Now integrating $\vec{F}.d\vec{r}$ along *C* from *A* to *B*, we get

$$\int_C \vec{F}.d\vec{r} = \int_{t_1}^{t_2} \frac{df}{dt}dt$$

$$= \int_{t_1}^{t_2} d\left[f\left(x(t), y(t), z(t)\right)\right]$$

$$= \left[f\left(x(t), y(t), z(t)\right)\right]_{t_1}^{t_2}$$

$$= f\left[\left(x(t_2), y(t_2), z(t_2)\right)\right] - f\left[\left(x(t_1), y(t_1), z(t_1)\right)\right]$$

$$\Rightarrow \int_C \vec{F}.d\vec{r} = \int_A^B \nabla f.d\vec{r} = f(B) - f(A).$$

The value of the integral $f(B) - f(A)$ does not depend on the path *C* at all.

This result is analogue of the fundamental theorem of integral calculus $\int_a^b f'(x)\,dx = f(b) - f(a)$.

The only difference is that we have $\nabla f.d\vec{r}$ in place of $f'(x)\,dx$.

So we can define a function *f* by the rule

$$\int f(x', y', z') = \int_A^{(x',y',z')} \vec{F}.d\vec{r} \qquad \text{...... (1)}$$

Then it will also be true that $\nabla f = \vec{F}$.

This result $\nabla f = \vec{F}$ is also true when the right hand side of relation (1) is path independent. Thus *a* necessary and sufficient condition for the integral $\int_A^B \vec{F}.d\vec{r}$ to be independent of the path joining the points *A* and *B* in some connected region *D* is that there exists a differentiable function *f* such that

$$\vec{F} = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}.$$

Then $\int_A^B \vec{F}.d\vec{r} = F(B) - F(A)\cdot$

When $\vec{F}$ is a force such that the work integral from *A* to *B* is the same for all the paths, the field is said to be conservative.

A force field $\vec{F}$ is conservative if and only if it is a gradient field i.e. $\vec{F} = \nabla f$.

7.4.1 Theorem

A force field $\vec{F}$ is a conservative field if and only if $\nabla \times \vec{F} = \vec{0}$, i.e. $\vec{F}$ is irrotational.

Proof: If $\vec{F}$ is a conservative field, then $\vec{F} = \nabla\phi$.

$\therefore$ curl $\vec{F}$ = curl (grad ϕ) = $\vec{0}$

Conversely suppose that $\nabla \times \vec{F} = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

$$\Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \text{ and } \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

We prove that $\vec{F}$ is conservative field i.e. $\vec{F} = \nabla \phi$.

The work done in moving a particle from $(x_1, y_{1,} z_1)$ to (x, y, z) in the force field $\vec{F}$ is

$$\int_C F_1(x, y, z)\,dx + F_2(x, y, z)\,dy + F_3(x, y, z)\,dz$$

where C is a path joining (x_1, y_1, z_1) and (x, y, z).

Let us choose as a particular path the straight line segments from (x_1, y_1, z_1) to (x, y_1, z_1) to (x, y, z_1) to (x, y, z) and call $\phi(x, y, z)$ the work done along this particular path.

$$\phi(x, y, z) = \int_{x_1}^{x} F_1(x, y_1, z_1)\,dx + \int_{y_1}^{y} F_2(x, y, z_1)\,dy + \int_{z_1}^{z} F_3(x, y, z)\,dz.$$

Now it follows that $\dfrac{\partial \phi}{\partial z} = F_3(x, y, z)$

$$\frac{\partial \phi}{\partial y} = F_2(x, y, z_1) + \int_{z_1}^{z} \frac{\partial F_3}{\partial y}(x, y, z)\,dz$$

$$= F_2(x, y, z_1) + \int_{z_1}^{z} \frac{\partial F_2}{\partial z}(x, y, z)\,dz \quad = F_2(x, y, z_1) + \left[F_2(x, y, z)\right]_{z_1}^{z}$$

$$= F_2(x, y, z_1) + F_2(x, y, z) - F_2(x, y, z_1) = F_2(x, y, z).$$

$$\frac{\partial \phi}{\partial x} = F_1\left(x_1, y_1, z_1\right) + \int_{y_1}^{y} \frac{\partial F_2}{\partial x}\left(x, y, z_1\right) dy + \int_{z_1}^{z} \frac{\partial F_3}{\partial x}\left(x, y, z\right) dz$$

$$= F_1\left(x, y_1, z_1\right) + \int_{y_1}^{y} \frac{\partial F_1}{\partial y}\left(x, y, z_1\right) dy + \int_{z_1}^{z} \frac{\partial F_1}{\partial z}\left(x, y, z\right) dz$$

$$= F_1\left(x, y_1, z_1\right) + \left[F_1\left(x, y, z_1\right)\right]_{y_1}^{y_2} + \left[F_1\left(x, y, z\right)\right]_{z_1}^{z}$$

$$= F_1\left(x, y_1, z_1\right) + F_1\left(x, y, z_1\right) - F_1\left(x, y_1, z_1\right) + F_1\left(x, y, z\right) - F_1\left(x, y, z_1\right)$$

$$= F_1\left(x, y, z\right)$$

Then $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} \; = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} \; = \nabla \phi$.

Hence a necessary and sufficient condition that a field $\vec{F}$ to be conservative is that curl $\vec{F} = \vec{0}$.

7.4.2 Example

Show that $\vec{F} = \left(2xy + z^3\right)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential. Hence find the work done in moving an object in this field from (1, -2, 1) to (3, 4, 1).

solution: From above theorem the given field is conservative if curl $\vec{F} = \vec{0}$.

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ zxy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \hat{i}\left(0\right) - \hat{j}\left(3z^2 - 3z^2\right) + \hat{k}\left(2x - 2x\right) = \vec{0}.$$

Thus $\vec{F}$ is a conservative force field.

Now $\vec{F} = \nabla \phi \; = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$$= \left(2xy + z^3\right)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$$

$$\therefore \frac{\partial\phi}{\partial x} = 2xy + z^3, \ \frac{\partial\phi}{\partial y} = x^2, \ \frac{\partial\phi}{\partial z} = 3xz^2 .$$

Integrating, we get

$$\phi = x^2 y + xz^3 + f(y,z)$$

$$\phi = x^2 y + g(x,z)$$

$$\phi = xz^3 + h(x,y)$$

We choose $f(y,z) = 0, h(x,y) = x^2 y, \ g(x,z) = xz^3$

$$\therefore \phi = x^2 y + xz^3 + \text{constant}$$

Work done $= \left[\phi(x,y,z)\right]_{(1,-2,1)}^{(3,1,4)} = \phi(3,1,4) - \phi(1,-2,1)$

$= [9 + 192] - [-2 + 1] = 202.$

7.4.3 Example

Show that the vector field $\vec{F}$ is given by $\vec{F} = (y + \sin z)\hat{i} + x\hat{j} + x\cos z\hat{k}$ is conservative. Find its scalar potential.

Solution: We have

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + \sin z & x & x\cos z \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(\cos z - \cos z) + \hat{k}(1-1) = \vec{0} \cdot$$

Hence $\vec{F}$ is conservative.

Let $\vec{F} = \nabla\phi \ = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = y + \sin z \Rightarrow \phi = xy + x\sin z + f(y,z)$$

$$\frac{\partial\phi}{\partial y} = x \Rightarrow \phi = xy + g(x,z)$$

$$\frac{\partial\phi}{\partial z} = x\cos z \Rightarrow \phi = x\sin z + h(x, y)$$

The above equations represent ϕ.

These agree if we choose $f(y, z) = 0,\ g(x, z) = x\sin z,\ h(x, y) = xy.$

$\therefore \phi = xy + x\sin z + c$.

Check Your Progress:

3. Evaluate $\int_C yz\,dx + (xz+1)\,dy + xy\,dz$, where C is any path from (1, 0, 0) to (2, 1, 4).

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7.5 SURFACE INTEGRALS

In mathematics, a surface integral is a generalization of multiple integrals to integration over surfaces. Any integral which is to be evaluated over a surface is called a **surface integral.**

Suppose S is a surface of finite area and $f(x, y, z)$ is a single valued function of position defined over S. Subdivide the area S into n elements of areas $\delta S_1, \delta S_2, \ldots\ldots, \delta S_n$.

In each part δS_k we choose an arbitrary point $P_k(x_k, y_k, z_k)$. We define $f(P_k) = f(x_k, y_k, z_k)$

If $\lim_{n\to\infty} \sum_{k=1}^{n} f(P_k)\delta S_k$ exists, then it is called the surface integral of $f(x, y, z)$ over S and is denoted by $\iint_S f(x, y, z)\,dS.$

7.5.1 Definition: (Flux)

Suppose S is a piece wise smooth surface and $\vec{F}(x, y, z)$ is a vector function of position defined and continuous over S. Let P be any point on the surface S and let $\vec{n}$ be the unit vector at P in the direction of outward drawn normal to S at P. Then $\vec{F}.\vec{n}$ is the normal component of $\vec{F}$ at P. The integral $\iint_S \vec{F}.\vec{n}\,dS$ is called the **flux** of $\vec{F}$ over S.

7.5.2 Cartesian form of the Surface Integral

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, where F_1, F_2, F_3 are continuous and differentiable functions of *x, y* and *z*.

Let the outward drawn normal to the surface *S* at *P* makes angles α, β, γ with the positive directions of *x, y* and *z* axes respectively.

If *l, m, n* are the direction cosines of this outward drawn normal, then

$$\vec{n} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k} \;\; = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

$$\vec{F}.\vec{n} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$$

$$\therefore \iint_S \vec{F}.\vec{n}dS = \iint_S \left(F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma\right)dS$$

$$= \iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy\,.$$

Since $dS\cos\alpha$, $dS\cos\beta$ and $dS\cos\gamma$ are the projections of *S* on yz, zx and *xy* planes. If dx, dy, dz are the differentials along the axes then

$$dS\cos\alpha = dy\,dz,\; dS\cos\beta = dz\,dx,\; dS\cos\gamma = dx\,dy\,.$$

7.5.3 Evaluation of Surface Integrals

In order to evaluate surface integrals it is convenient to express them as double integrals taken over the orthogonal projection of the surface *S* on one of the coordinate planes. But this is possible only if any line perpendicular to the coordinate plane chosen meets the surface *S* in not more than one point. If the surface *S* does not satisfy this condition, then it can be subdivided into surfaces which satisfy this condition.

Suppose that the surface *S* is such that any line perpendicular to *yz* plane meets *S* in not more than one point. Then the equation of *S* can be written in the form $x = f\left(y, z\right)$.

Let *R* be the orthogonal projection of *S* on *yz* - plane. If α is the acute angle made by normal $\vec{n}$ at $P\left(x, y, z\right)$ with *x* - axis. Then

$\cos\alpha\, dS = dy\,dz$, where dS is the small element.

$$\therefore dS = \frac{dy\,dz}{\cos\alpha} = \frac{dy\,dz}{\left|n.\hat{i}\right|}$$, where $\hat{i}$ is unit vector along *x* - axis.

$$\iint_S \vec{F}.\vec{n}\,dS = \iint_R \vec{F}.\vec{n}\;\frac{dy\,dz}{\left|\vec{n}.\hat{i}\right|}$$

which can be evaluated with the help of a double integral integrated over *R*.

Similarly, $\iint\limits_S \vec{F}.\vec{n}\,dS = \iint\limits_R \vec{F}.\vec{n}\,\frac{dx\,dy}{\left|\vec{n}.\hat{k}\right|} = \iint\limits_R \vec{F}.\vec{n}\,\frac{dz\;dx}{\left|\vec{n}.\,\hat{j}\right|}$.

7.5.4 Example

Evaluate $\iint\limits_S \vec{F}.\vec{n}\,dS$, where $\vec{F} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}$ and *S* is the part of the plane $2x + 3y + 6z = 12$ in the first octant.

Solution: Let *R* be the projection of *S* on *xy* - plane. Then

$$\iint\limits_S \vec{F}.\vec{n}\,dS = \iint\limits_R \vec{F}.\vec{n}\,\frac{dx\,dy}{\vec{n}.\hat{k}}$$

To obtain $\vec{n}$, find $\nabla(2x + 3y + 6z) = 2\hat{i} + 3\hat{j} + 6\hat{k}$ since we know that gradient is normal to the given surface.

$$\therefore \vec{n} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right).$$

$$\vec{n}.\hat{k} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right).\hat{k} = \frac{6}{7}.$$

$$\frac{dx\,dy}{\left|\vec{n}.\vec{k}\right|} = \frac{7}{6}\,dx\,dy$$

$$\vec{F}.\vec{n} = \left(18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}\right).\frac{\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)}{7} = \frac{36z - 36 + 18y}{7}$$

Now, $2x + 3y + 6z = 12 \;\Rightarrow z = \frac{1}{6}(12 - 2x - 3y)$

$$\vec{F}.\vec{n} = \frac{36z - 36 + 18y}{7}$$

$$= \frac{36.\frac{1}{6}(12 - 2x - 3y) - 36 + 18y}{7}$$

$$= \frac{72 - 12x - 18y - 36 + 18y}{7} = \frac{36 - 12x}{7}.$$

$$\iint_S \vec{F}.\vec{n}\, dS = \iint_R \vec{F}.\vec{n}\frac{dx\,dy}{|\vec{n}.\vec{k}|}$$

$$= \iint_R \frac{36-12x}{7}.\frac{7}{6}dx\,dy \;= \iint_R (6-2x)\,dx\,dy.$$

Now we evaluate this double integral by considering y from 0 to $\frac{12-2x}{3}$ and x from 0 to 6.

$$\iint_R (6-2x)\,dx\,dy = \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6-2x)\,dx\,dy$$

$$= \int_{x=0}^{6}\left[6y-2xy\right]_0^{\frac{12-2x}{3}} dx\,dy \;= \int_{x=0}^{6}\left[24-12x+\frac{4x^2}{3}\right]dx$$

$$= \left[24x-12.\frac{x^2}{2}+\frac{4}{3}.\frac{x^3}{3}\right]_0^6 = 24.$$

7.5.5 Example

Evaluate $\iint_S \vec{F}.\vec{n}\,dS$, where $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ and S is the part of the surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant.

Solution: $\vec{n} = \dfrac{\nabla\left(x^2+y^2+z^2\right)}{\left|\nabla\left(x^2+y^2+z^2\right)\right|} \;= \dfrac{2x\,\hat{i}+2y\,\hat{j}+2z\,\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$

$$= \frac{2\left(x\,\hat{i}+y\,\hat{j}+z\,\hat{k}\right)}{2\sqrt{x^2+y^2+z^2}} \;= x\,\hat{i}+y\,\hat{j}+z\,\hat{k} \text{ since } x^2+y^2+z^2=1.$$

Let R be the projection of S in xy plane. So we have $x^2+y^2=1$ and $z = 0$.

$$\vec{F}.\vec{n} = \left(yz\,\hat{i}+zx\,\hat{j}+xy\,\hat{k}\right)\left(x\,\hat{i}+y\,\hat{j}+z\,\hat{k}\right) \;= xyz+xyz+xyz = 3xyz.$$

$$\vec{n}.\hat{k} = \left(x\,\hat{i}+y\,\hat{j}+z\,\hat{k}\right).\hat{k} = z.$$

$$\iint_S \vec{F}.\vec{n}\,dS = \iint_R 3xyz.\frac{dx\,dy}{z} \;= \iint_R xy\,dx\,dy$$

$$= 3\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (r\cos\theta)(r\sin\theta)\,r\,d\theta\,dr = 3\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^3 \sin\theta\cos\theta\;d\theta\;dr$$

$$= 3\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 \sin\theta\;\cos\theta\;d\theta = \frac{3}{4}\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta$$

$$= \frac{3}{8}\int_0^{\pi/2} \sin 2\theta\,d\theta = \frac{3}{8}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} = \frac{3}{8}.$$

Check Your Progress:

4. Evaluate $\iint_S \vec{F}.\vec{n}\,dS$, where $\vec{F} = z\,\hat{i} + x\,\hat{j} - 3y^2 z\,\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

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7.6 WORKED OUT EXERCISES

7.6.1 Exercise

Evaluate $\int_C \vec{F}.d\vec{r}$, where $\vec{F} = 3x^2\hat{i} + (2xz - y)\,\hat{j} + z\,\hat{k}$ along the straight line C from (0, 0, 0) to (2, 1, 3).

Solution: The equation of the line joining (0, 0, 0) and (2, 1, 3) is $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$.

Then along the line C,

$$x = 2t,\; y = t,\; z = 3t$$

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} = 2t\,\hat{i} + t\,\hat{j} + 3t\,\hat{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = 2\,\hat{i} + \hat{j} + 3\hat{k}$$

$$\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$$

$$= 3(2t)^2\hat{i} + (2(2t)3t - t)\hat{j} + 3t\hat{k}$$

$$= 12t^2\hat{i} + (12t^2 - t)\hat{j} + 3t\hat{k}$$

$$\vec{F}.\frac{d\vec{r}}{dt} = \left(12t^2\hat{i} + (12t^2 - t)\hat{j} + 3t\hat{k}\right).\left(2\hat{i} + \hat{j} + 3\hat{k}\right)$$

$$= 24t^2 + 12t^2 - t + 9t = 36t^2 + 8t$$

$$\int_C \vec{F}.d\vec{r} = \int_{t=0}^{1}\left(\vec{F}.\frac{d\vec{r}}{dt}\right)dt$$

$$= \int_0^1 \left(36t^2 + 8t\right)dt = \left[12t^3 + 4t^2\right]_0^1 = 16.$$

7.6.2 Exercise

Evaluate $\int_C \vec{F}.d\vec{r}$, where $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ and C is given by $x = \cos t,\ y = \sin t, z = t, 0 \le t \le \pi$.

Solution: $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$.

$$x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t$$

$$y = \sin t \Rightarrow \frac{dy}{dt} = \cos t$$

$$z = t \Rightarrow \frac{dz}{dt} = 1$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}.$$

$$\int_C \vec{F}.d\vec{r} = \int_{t=0}^{\pi}\left(\cos^2 t\,\hat{i} + \sin^2 t\,\hat{j} + t^2\hat{k}\right).\left(-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}\right)dt$$

$$= \int_{t=0}^{\pi}\left[-\cos^2 t\sin t\,dt + \sin^2 t\cos t\,dt + t^2 dt\right]$$

$$= \left[\frac{\cos^3 t}{t} + \frac{\sin^3 t}{3} + \frac{t^3}{3} \right]_0^{\pi} = -\frac{1}{3} - \frac{1}{3} + \frac{\pi^3}{3} = \frac{\pi^3 - 2}{3}.$$

7.6.3 Exercise

Use the line integral to compute work done by force $\vec{F} = (2y+3)\hat{i} + xz\,\hat{j} + (yz - x)\hat{k}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2, y = t, z = t^3$.

Solution: Work done by force $\vec{F} = \int_C \vec{F}.d\vec{r}$

$$\vec{F} = (2y+3)\hat{i} + xz\,\hat{j} + (yz - x)\hat{k}$$

$$x = 2t^2 \Rightarrow dx = 4t\,dt$$

$$y = t \Rightarrow dy = dt$$

$$z = t^3 \Rightarrow dz = 3t^2 dt$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} = 4t\,dt\,\hat{i} + dt + 3t^2 dt\,\hat{k}.$$

$$\int_C \vec{F}.d\vec{r} = \int_{t=0}^{1} \left[(2t+3)4t + 2t^5 + \left(t^4 - 2t^2\right)3t^2 \right] dt$$

$$= \int_0^1 \left(8t^2 + 12t + 2t^5 + 3t^6 - 6t^4\right) dt$$

$$= \left[8.\frac{t^3}{3} + 12.\frac{t^2}{2} + 2.\frac{t^6}{6} + 3.\frac{t^7}{7} - 6.\frac{t^5}{5} \right]_0^1$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = \frac{288}{35}.$$

7.6.4 Exersice

Find the work done in moving a particle once around an ellipse C in the xy - plane if the ellipse has centre at the origin with semi-major axis 4 and semi minor axis 3 where $\vec{F}$ is given by $\vec{F} = (3x - 4y + 2z)\hat{i} + \left(4x + 2y - 3z^2\right)\hat{j} + \left(2xz - 4y^2 + z^3\right)\hat{k}$.

Solution: Here path of integration C is the ellipse whose equation is $\frac{x^2}{4^2}+\frac{y^2}{3^2}=1$.

Parametric equations of ellipse are $x=4\cos t, y=3\sin t$ and t varies from 0 to 2π .

Since C is a curve in xy - plane, we have z = 0.

$$\therefore \vec{F}=(3x-4y)\hat{i}+(4x+2y)\hat{j}-4y^2\hat{k}$$

$$d\vec{r}=dx\hat{i}+dy\hat{j}$$

$$\vec{F}.d\vec{r}=(3x-4y)dx+(4x+2y)dy$$

$$x=4\cos t \Rightarrow dx=-4\sin t\,dt$$

$$y=3\sin t \Rightarrow dy=3\cos t\,dt$$

$$\int_C \vec{F}.d\vec{r}=\int_0^{2\pi}(12\cos t-12\sin t)(-4\sin t)\,dt+(16\cos t+6\sin t)3\cos t\,dt$$

$$=\int_0^{2\pi}\left(-48\cos t\sin t+48\sin^2 t+48\cos^2 t+18\cos t\sin t\right)dt$$

$$=\int_0^{2\pi}(48-30\sin t\cos t)\,dt \;=\int_0^{2\pi}(48-15\sin 2t)\,dt$$

$$=\left[48t+\frac{15}{2}\cos 2t\right]_0^{2\pi}=96\pi .$$

7.6.5 Exercise

Find the circulation of $\vec{F}$ around the curve C, where $\vec{F}=\left(2x+y^2\right)\hat{i}+(3y-4x)\hat{j}$ and C is the curve $y=x^2$ from (0, 0) to (1, 1) and the curve $y^2=x$ from (1, 1) to (0, 0).

Solution: $\int_C \vec{F}.d\vec{r}=\int_C\left(2x+y^2\right)dx+(3y-4x)dy$

We have evaluate the given integral along two curves C_1 and C_2.

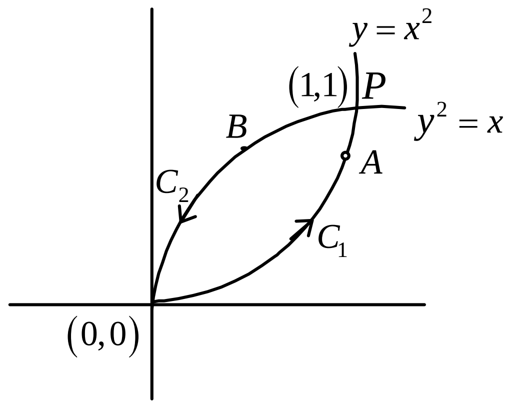

Fig: 7.2

For C_1, $y = x^2 \Rightarrow dy = 2x\,dx$ and x varies from 0 to 1.

For C_2, $x = y^2 \Rightarrow dx = 2y\,dy$ and y varies from 1 to 0.

$$\int_{C_1} \vec{F}.d\vec{r} = \int_0^1 \left(2x + x^4\right)dx + \left(3x^2 - 4x\right)2x\,dx$$

$$= \int_0^1 \left(2x + x^4 + 6x^3 - 8x^2\right)dx$$

$$= \left[2.\frac{x^2}{2} + \frac{x^5}{5} + 6.\frac{x^4}{4} - 8.\frac{x^3}{3}\right]_0^1$$

$$= 1 + \frac{1}{5} + \frac{3}{2} - \frac{8}{3} = \frac{1}{30}\,.$$

$$\int_{C_2} \vec{F}.d\vec{r} = \int_1^0 \left(2y^2 + y^2\right)2y\,dy + \left(3y - 4y^2\right)dy$$

$$= \int_1^0 \left(3y - 4y^2 + 6y^3\right)dy$$

$$= \left[3\frac{y^2}{2} - 4\frac{y^3}{3} + 6\frac{y^4}{4}\right]_1^0$$

$$= -\left[\frac{3}{2} - \frac{4}{3} + \frac{6}{4}\right] = -\frac{5}{3}.$$

$$\int_C \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r} = \frac{1}{30} - \frac{5}{3} = -\frac{49}{30}.$$

7.6.6 Exercise

Determine whether $\vec{F} = \left(y^2 \cos x + z^3\right)\hat{i} + \left(2y \sin x - 4\right)\hat{j} + \left(3xz^2 + 2\right)\hat{k}$ is a vector field. If so find the scalar potential ϕ. Also compute the work done in moving the particle from (0, 1, -1) to $\left(\frac{\pi}{2}, -1, 2\right)$.

solution: $\vec{F}$ is a conservative vector field if curl $\vec{F} = \vec{0}$.

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y \sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= \hat{i}\left[0 - 0\right] - \hat{j}\left[3z^2 - 3z^2\right] + \hat{k}\left(2y \cos x - 2y \cos x\right) = \vec{0}.$$

$\therefore \vec{F}$ is a conservative field.

Let $\vec{F} = \nabla \phi$.

$$\Rightarrow \left(y^2 \cos x + z^3\right)\hat{i} + \left(2y \sin x - 4\right)\hat{j} + \left(3xz^2 + 2\right)\hat{k} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\therefore \frac{\partial \phi}{\partial x} = y^2 \cos x + z^3, \quad \frac{\partial \phi}{\partial y} = 2y \sin x - 4, \quad \frac{\partial \phi}{\partial z} = 3xz^2 + 2.$$

Integrating partially w.r.t *x*, *y* and *z*, we get

$$\phi = y^2 \sin x + xz^3 + f\left(y, z\right) \qquad \text{... (1)}$$

$$\phi = y^2 \sin x - 4y + g\left(z, x\right) \qquad \text{... (2)}$$

$$\phi = xz^3 + 2z + h\left(x, y\right) \qquad \text{... (3)}$$

We get $\phi = xz^3 + y^2 \sin x - 4y + 2z$

work done in a conservative field $= \phi(B) - \phi(A)$

$$= \phi\left(\frac{\pi}{2}, -1, 2\right) - \phi(0,1,-1) \; = 4\pi + 15 .$$

7.6.7 Exercise

Find the flux of $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ outward through the portion of the cylinder $x^2 + y^2 = a^2$ in the first octant and below the plane $z = h$.

Solution:

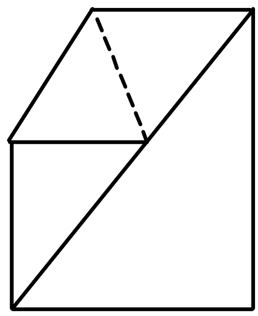

Fig: 7.3

$$\vec{n} = \frac{\nabla\left(x^2 + y^2\right)}{\left|\nabla\left(x^2 + y^2\right)\right|} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\hat{i} + y\hat{j}}{a} \qquad \left(\because x^2 + y^2 = a^2\right)$$

This is the outward normal to the circle $x^2 + y^2 = a^2$ in the xy - plane. $\vec{n}$ has no z - component since it is horizontal.

We use cylindrical coordinates to parametrize the cylinder

$x = a\cos\theta,\ y = a\sin\theta, z = z$, where $0 \le \theta \le \frac{\pi}{2},\ 0 \le z \le h$.

$dS = dz.a\,d\theta$

$$\iint_S \vec{F}.\vec{n}\,dS = \iint_S \frac{xz + xy}{a}.a\,dz\,d\theta$$

$$= \iint_S \left(az\cos\theta + a^2 \sin\theta \cos\theta \right) dz\, d\theta$$

$$= \int_0^{\pi/2} \int_0^h \left(az\cos\theta + a^2 \sin\theta \cos\theta \right) dz\, d\theta$$

$$= \int_0^{\pi/2} \left[a\cos\theta . \frac{z^2}{2} + a^2 \sin\theta \cos\theta . z \right]_0^h d\theta$$

$$= \int_0^{\pi/2} \left(\frac{ah^2}{2} \cos\theta + \frac{a^2 h}{2} . \sin 2\theta \right) d\theta$$

$$= \left[\frac{ah^2}{2} \sin\theta - \frac{a^2 h}{2} \frac{\cos 2\theta}{2} \right]_0^{\pi/2} = \frac{ah^2}{2} + \frac{a^2 h}{4} + \frac{a^2 h}{4}$$

$$= \frac{ah^2}{2} + \frac{a^2 h}{2} = \frac{ah}{2}\left(a + h \right).$$

7.6.8 Exercise

Find the flux of $\vec{F} = xz\,\hat{i} + yz\,\hat{j} + z^2 \hat{k}$ outward through that part of the sphere $x^2 + y^2 + z^2 = a^2$ lying in the first octant.

Solution: $\vec{n} = \dfrac{\nabla\left(x^2 + y^2 + z^2 \right)}{\left| \nabla\left(x^2 + y^2 + z^2 \right) \right|}$

$$= \frac{2x\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\hat{i} + y\,\hat{j} + z\,\hat{k}}{a} \qquad (\because\ x^2 + y^2 + z^2 = a^2)$$

To do the integration we use spherical coordinates $\rho,\ \phi,\ \theta$. On the surface of the sphere $\rho = a$, so the coordinates are the two angles ϕ and θ.

The area element dS is easily found using the volume element.

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = dS.d\rho$$

$$\therefore dS = a^2 \sin\phi\, d\phi\, d\theta \qquad [\because \rho = a]$$

$$x = a\sin\phi\cos\theta, y = a\sin\phi\sin\theta, z = a\cos\phi\ .$$

$$\vec{F}.\vec{n}\,dS = \frac{1}{a}\left(x^2z + y^2z + z^2z\right)a^2 \sin\phi\, d\phi\, d\theta$$

$$\iint_S \vec{F}.\vec{n}\,dS = a^4 \int_0^{\pi/2}\int_0^{\pi/2} \cos\phi \sin\phi\, d\phi\, d\theta$$

$$= a^4.\frac{\pi}{2}.\frac{1}{2} = \frac{\pi a^4}{4}.$$

7.6.9 Exercise

Evaluate $\iint_S \vec{F}.\vec{n}\,dS$, where $\vec{F} = y\,\hat{i} + 2x\,\hat{j} - z\,\hat{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut of by $z = 4$.

Solution: A vector normal to the surface S is given by

$$\nabla(2x + y) = 2\hat{i} + \hat{j}.$$

Unit vector $\vec{n} = \dfrac{\nabla(2x+y)}{\left|\nabla(2x+y)\right|}$

$$= \frac{2\hat{i} + \hat{j}}{\sqrt{4+1}} = \frac{1}{\sqrt{5}}\left(2\hat{i} + \hat{j}\right)$$

$$\iint_S \vec{F}.\vec{n}\,dS = \iint_R \vec{F}.\vec{n}\,\frac{dx\,dz}{\left|\vec{n}.\hat{j}\right|}.$$

where R is the projection of S on xz plane. Here we can't take projection on xy -plane since S is perpendicular to xy - plane.

$$\vec{F}.\vec{n} = \left(y\,\hat{i} + 2x\,\hat{j} - z\,\hat{k}\right).\frac{1}{\sqrt{5}}\left(2\hat{i} + \hat{j}\right) = \frac{2y + 2x}{\sqrt{5}}$$

$$\vec{n}.\hat{j} = \frac{1}{\sqrt{5}}\left(2\hat{i} + \hat{j}\right).\hat{j} = \frac{1}{\sqrt{5}}$$

$$\iint_S \vec{F}.\vec{n}\,dS = \iint_R \frac{1}{\sqrt{5}}(2x + 2y).\sqrt{5}\,dx\,dz$$

$$= 2\iint_R (y + x)\,dx\,dz \quad = 2\iint_R (6 - 2x + x)\,dx\,dz \quad (\because y = 6 - 2x \text{ on } S)$$

$$= 2\int_{z=0}^{4}\int_{x=0}^{3}(6-x)\,dx\,dz = 2\int_{x=0}^{3}(6z-xz)_0^4\,dx$$

$$= 2\int_0^3 (24-4x)\,dx = 2\left[24x-2x^2\right]_0^3 = 2[72-18] = 108.$$

7.6.10 Exercise

If $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$. Evaluate $\iint_S \vec{F}.\vec{n}\,dS$.

Solution: $\iint_S \vec{F}.\vec{n}\,dS = \iint_R \vec{F}.\vec{n}\dfrac{dy\,dz}{|\vec{n}.\hat{i}|}$

where R is the projection of S in yz - plane.

$$dS = \frac{dy\,dz}{|n.\hat{i}|}$$

$$S = y^2 - 8x.$$

$$\vec{n} = \frac{\nabla\left(y^2-8x\right)}{\left|\nabla\left(y^2-8x\right)\right|} = \frac{-8\hat{i}+2y\,\hat{j}}{\sqrt{64+4y^2}} = \frac{-4\hat{i}+4\hat{j}}{\sqrt{16+y^2}}.$$

$$\therefore \vec{F}.\vec{n} = \frac{-8y-yz}{\sqrt{16+y^2}}$$

$$\vec{n}.\hat{i} = \frac{-4}{\sqrt{16+y^2}}$$

$$\iint_S \vec{F}.\vec{n}\,dS = \iint_S \frac{-8y-yz}{\sqrt{16+y^2}}\,\frac{\sqrt{16+y^2}}{-4}\,dy\,dz$$

$$= \frac{1}{4}\int_{y=0}^{4}\int_{z=0}^{6}(8y+yz)\,dy\,dz$$

$$=\frac{1}{4}\int_{y=0}^{4}\left[8yz+y\frac{z^2}{2}\right]_0^6 dy$$

$$=\frac{1}{4}\int_0^4(48y+18y)\,dy=\frac{1}{4}\int_0^4 66\,y\,dy$$

$$=\frac{66}{4}\left[\frac{y^2}{2}\right]_0^4=\frac{66}{4}.\frac{16}{2}=132.$$

7.7 SUMMARY

In this unit we have discussed the following points.

(i) If $\vec{F}(t)$ is a vector function of a scalar variable t and $\frac{d}{dt}\left(\vec{F}(t)\right)=\vec{f}(t)$.

Then $\int \vec{f}(t)\,dt=\vec{F}(t)+\vec{c}$, where c is a constant vector.

(ii) A line integral is evaluated by reducing it into a definite integral of single variable, using parametric representation of the path of integration C.

(iii) If $\vec{F}$ represents a force moving along a curve C, $\vec{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$, $a\le t\le b$ from a point A to a point B in space, then $\int_C \vec{F}.d\vec{r}$ represents the work done by the force $\vec{F}$ in moving a particle from point A to point B along the curve C.

(iv) A force field is conservative if its work integral is independent of path, but depends only on the end points of the path.

(v) A force field is conservative if and only if it is a gradient field.

(vi) We also defined a surface integral as an integral which is evaluated over a surface. Consider a surface S and a point P on it. Let $\vec{F}$ be a vector function of x, y, z defined and continuous over S. If $\hat{\vec{n}}$ is the unit out ward normal to the surface S at P, then the integral of the normal component of $\vec{F}$ at P over the surface S is called the surface integral written as $\iint_S \vec{F}.\vec{n}\,dS$, where dS is the small element area.

7.8 CHECK YOUR PROGRESS - MODEL ANSWERS

1. (a) $(3u)\hat{i} + \left[\frac{1}{4}u^4 + \frac{1}{2}u^8\right]\hat{j} + \frac{u^2}{2}\hat{k} + c$

   (b) $3\hat{i} + \frac{525}{4}\hat{j} + \frac{3}{2}\hat{k}$

2. 1

3. 9

4. 90

7.9 MODEL EXAMINATION QUESTIONS

1. Evaluate the line integral $\int_C x^2 y\,dx + (x-z)\,dy + xyz\,dz$, where C is the arc of the parabola $y = x^2$ in the plane $z = 2$ from A (0, 0, 2) to B (1, 1, 2).

2. Show that a force field $\vec{F}$ is a conservative field iff $\nabla \times \vec{F} = \vec{0}$.

3. Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential.

4. Evaluate $\iint_S \vec{F}.\vec{n}\,dS$, where $\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the part of the plane $2x + 3y + 6z = 12$ in the first octant.

5. If $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$. Then evaluate $\iint_S \vec{F}.\vec{n}\,dS$.

6. If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$. Evaluate $\int_C \vec{F}.dr$ along the curve $y = x^3$ in the xy - plane from (1, 1) to (2, 8).

7. Evaluate $\int_C \vec{F}.dr$ where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and C is given by $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$, $0 \le t \le 1$.

8. If $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$, evaluate $\int_C \vec{F} \times d\vec{r}$, where C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to 1.

9. If $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ evaluate $\iint_S \vec{F}.\vec{n}\,dS$ where S is the surface of the cube bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1, z = 0,\ z = 1$.

Answers:

1. 12

3. $\phi = x^2 y + xz^3$

4. 24

5. 132

6. 35

7. 1

8. $-\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}$

9. $\frac{3}{2}$

UNIT - 8 : VOLUME INTEGRALS AND APPLICATIONS OF VECTOR INTEGRATION

Contents

8.0 OBJECTIVES

After studying this unit, you will be able to:

- Define volume integral.
- Evaluate volume integrals
- Solve problems on vector integration.

8.1 INTRODUCTION

In Unit-7, the concept of a line integral was introduced and we defined and evaluated the double integrals, in which the integrand is a function of two variables. Also a natural generalisation of double integrals was provided interms of a surface integrals, where though the integrand may be a function of three variables, but it is defined on a surface only it is still evaluated as a double integral. However, there are many physical and geometrical situations, where the integrand may be a function defined in a region in space and integration may have to be carried out with respect to all the three variables involved. This gives rise to triple integrals. At triple integral is again a generalisation of double integral.

8.2 VOLUME INTEGRALS

8.2.1 Definition

Suppose V is the volume bounded by a surface S. Divide the volume V into sub-volumes $\delta V_1,\ \delta V_2, ...,\ \delta V_n$. In each δV_i, choose an arbitrary point P_i whose coordinates are $\left(x_i, y_i, z_i\right)$. Let f be a single valued function defined over V. Form the sum $\sum f\left(P_i\right)\ \delta V_i$, where $f\left(P_i\right) = f\left(x_i, y_i, z_i\right)$

Now let us take the limit of the sum as $n \to \infty$, then the limit if it exists is called the volume integral of f over V and is denoted as $\iiint\limits_V f\,dV$.

If $\vec{F}$ is a vector point function defined in the given region of volume V then the vector volume integral of $\vec{F}$ over V is $\iiint\limits_V f\,dV$.

8.2.2 Example

If $\phi = 45x^2 y$ and V is the closed region bounded by the planes $4x + 2y + z = 8,\ x = 0,\ y = 0,\ z = 0$, then evaluate $\iiint\limits_V \phi\,dV$.

Solution: Limits of integration are $0 \le x \le 2, 0 \le y \le 4 - 2x,\ 0 \le z \le 8 - 4x - 2y$.

$$\iiint\limits_V \phi\,dV = \int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45x^2 y\,dz\,dy\,dx$$

$$= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2 y\left[z\right]_0^{8-4x-2y} dy\ dx$$

$$= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2 y\left(8 - 4x - 2y\right) dy\,dx$$

$$= 45\int_{x=0}^{2} x^2\left[\left(8-4x\right)\frac{y^2}{2} - 2.\frac{y^3}{3}\right]_0^{4-2x} dx$$

$$= 45\int_{x=0}^{2}\left[\left(4-2x\right)\left(4-2x\right)^2 - \frac{2}{3}\left(4-2x\right)^3\right]x^2\ dx$$

$$=45\int_0^2 \frac{(4-2x)^3 x^2}{3}dx$$

$$=15\int_0^2 x^2\left[64-3\times 4\times 2x(4-2x)-8x^3\right]dx$$

$$=15\int_0^2 \left(64x^2-96x^3+48x^4-8x^5\right)dx$$

$$=15\left[64\frac{x^3}{3}-96\times\frac{x^4}{4}+48\frac{x^5}{5}-8\frac{x^6}{6}\right]_0^2$$

$$=15\left[\frac{8}{3}\times 64-24\times 16+\frac{48}{5}\times 32-\frac{4}{3}.64\right]$$

= 2560 - 5760 + 4608 - 1280

= 7168 - 7040 = 128.

8.2.3 Example

If $\vec{F}=2xz\,\hat{i}-x\,\hat{j}+y^2\hat{k}$, evaluate $\iiint_V \vec{F}\,dV$, where V is the region bounded by the surfaces $x=0,\ x=2,\ y=0,\ y=6, z=x^2, z=4$.

Solution: Given $\vec{F}=2xz\,\hat{i}-x\,\hat{j}+y^2\hat{k}$.

volume integral = $\iiint_V \vec{F}\,dV$ $=\iiint_V \left(2xz\,\hat{i}-x\,\hat{j}+y^2\hat{k}\right)dx\ dy\,dz$.

$$=\hat{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xz\,dx\ dy\ dz-\hat{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\ dy\ dz+\hat{k}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dx\ dy\ dz$$

$$=\hat{i}\int_{x=0}^{2}\int_{y=0}^{6} 2x\left[\frac{z^2}{2}\right]_{x^2}^{4} dx\ dy-\hat{j}\int_{x=0}^{2}\int_{y=0}^{6} x\left[z\right]_{x^2}^{4} dx\ dy+\hat{k}\int_{x=0}^{2}\int_{y=0}^{6} y^2\left[z\right]_{x^2}^{4} dx\ dy$$

$$=\hat{i}\int_{x=0}^{2} x\left(16-x^4\right)dx\int_{y=0}^{6} y\,dy-\hat{j}\int_{x=0}^{2} x\left(4-x^2\right)dx\int_{y=0}^{6} dy+\hat{k}\int_{x=0}^{2}\left(4-x^2\right)dx\int_{y=0}^{6} y^2\,dy$$

$$=\hat{i}\left[8x^2-\frac{x^6}{6}\right]_0^2 6-\hat{j}\left[2x^2-\frac{x^4}{4}\right]_0^2 6+\hat{k}\left[4x-\frac{x^3}{3}\right]_0^2 \frac{(6)^3}{3}$$

$$=128\hat{i}-24\hat{j}+384\hat{k}.$$

8.2.4 Example

If $\vec{F}=\left(2x^2-3z\right)\hat{i}-2xy\,\hat{j}-4x\,\hat{k}$ evaluate $\iiint_V \nabla.\vec{F}\,dV$ and $\iiint_V curl\ \vec{F}.dV$, where V is the closed region bounded by $x=0, y=0, z=0, 2x+2y+z=4$.

Solution: $\vec{F}=\left(2x^2-3z\right)\hat{i}-2xy\,\hat{j}-4x\,\hat{k}$

$$\nabla.\vec{F}=\hat{i}.\frac{\partial\vec{F}}{\partial x}+\hat{j}.\frac{\partial\vec{F}}{\partial y}+\hat{k}.\frac{\partial\vec{F}}{\partial z}\ =4x-2x=2x.$$

$$\iiint_V \nabla.\vec{F}\,dV\ =\ \iiint_v 2x\,dx\,dy\,dz.$$

The limits are $z=0$ to $4-2x-2y$;

$y=0$ to $\dfrac{4-2x}{2}=2-x$;

$x=0$ to $\dfrac{4}{2}=2$.

$$\iiint_V 2x\,dx\,dy\,dz=2\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} x\,dx\,dy\,dz$$

$$=2\int_{x=0}^{2}\int_{y=0}^{2-x} x\left[z\right]_0^{4-2x-2y}dx\,dy$$

$$=2\int_{x=0}^{2}\int_{y=0}^{2-x} x\left(4-2x-2y\right)dx\,dy$$

$$=2\int_{x=0}^{2}\int_{y=0}^{2-x}\left(4x-2x^2-2xy\right)dx\,dy$$

$$= 4\int_{x=0}^{2}\int_{y=0}^{2-x}\left(2x - x^2 - xy\right)dx\,dy$$

$$= 4\int_{x=0}^{2}\left[\left(2x - x^2\right)y - x.\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= 4\int_{x=0}^{2}\left[\left(2x - x^2\right)(2-x) - x\frac{(2-x)^2}{2}\right]dx$$

$$= \int_0^2\left(2x^3 - 8x^2 + 8x\right)dx = \left[\frac{x^4}{2} - \frac{8x^3}{3} + 4x^2\right]_0^2 = \frac{8}{3}.$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(3-4) + \hat{k}(-2y-0) = \hat{j} - 2y\hat{k}.$$

$$\therefore \int_V \nabla\times\vec{F}\,dV = \iiint_V\left(\hat{j} - 2y\,\hat{k}\right)dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y}\left(\hat{j} - 2y\hat{k}\right)dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\left(\hat{j} - 2y\hat{k}\right)(4 - 2x - 2y)\,dx\,dy$$

$$= \hat{j}\int_{x=0}^{2}\left[(4-2x)y - y^2\right]_0^{2-x} dx - \hat{k}\int_{x=0}^{2}\left[(4-2x)y^2 - 4\frac{y^3}{3}\right]_0^{2-x} dx$$

$$= \hat{j}\int_0^2 (2-x)^2\,dx - \hat{k}\int_0^2 \frac{2}{3}(2-x)^3\,dx$$

$$= \hat{j}\left[\frac{(2-x)^3}{3}\right]_0^2 - \frac{2\hat{k}}{3}\left[\frac{(2-x)^4}{4}\right]_0^2$$

$$= \frac{8}{3}\left(\hat{j} - \hat{k}\right).$$

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. Evaluate $\int_V (2x+y)\,dV$, where V is the closed region bounded by the cylinder $z = 4 - x^2$ and the plane $x = 0,\ y = 0,\ y = 2,\ z = 0$.

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2. Find the volume of the solid in the first octant bounded by the graphs of $z = 1 - y^2, y = 2x$ and $x = 3$.

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8.3 APPLICATIONS OF VECTOR INTEGRATION

In this section, we will study the applications of vector integration. Given acceleration we can find velocity and distance using integration. Line integral gives the work done by a force in moving a particle along given curve C between two given points. We can find scalar potential using line integrals and their path independence. Surface integrals are used to find flux of a vector field across S.

8.3.1 Example

The acceleration of a particle at any time $t \geq 0$ is given by

$\vec{a} = \frac{d\vec{v}}{dt} = (25\cos 2t)\hat{i} + (16\sin 2t)\hat{j} + 9t\hat{k}$. Then find $\vec{r}$ and $\vec{v}$ at any time given velocity $\vec{v}$ and the displacement $\vec{r}$ are the zero vectors at $t = 0$.

Solution: Given $\vec{a} = (25\cos 2t)\hat{i} + (16\sin 2t)\hat{j} + 9t\hat{k}$

$$\vec{v} = \hat{i}\int 25\cos 2t\,dt + \hat{j}\int 16\sin 2t\,dt + \hat{k}\int 9t\,dt$$

$$= \frac{25}{2}\sin 2t\,\hat{i} - 8\cos 2t\,\hat{j} + \frac{9}{2}t^2\hat{k} + c_1$$

Putting $\vec{v} = \vec{0}$ when $t = 0$ we find

$$\vec{0} = 0\hat{i} - 8\hat{j} + 0\hat{k} + c_1 \Rightarrow c_1 = 8\hat{j}$$

Then $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{25}{2}\sin 2t\,\hat{i} + (8 - 8\cos 2t)\,\hat{j} + \dfrac{9}{2}t^2\,\hat{k}$

Integrating, we get

$$\vec{r} = \hat{i}\int \frac{25}{2}\sin 2t\,dt + \hat{j}\int (8 - 8\cos 2t)\,dt + \hat{k}\int \frac{9}{2}t^2\,dt$$

$$= -\frac{25}{4}\cos 2t\,\hat{i} + (8t - 4\sin 2t)\,\hat{j} + \frac{3}{2}t^3\,\hat{k}\,.$$

Putting $\vec{r} = \vec{0}$ when $t = 0$ we get

$$\vec{0} = -\frac{25}{4}\hat{i} + c_2 \text{ and } c_2 = \frac{25}{4}\hat{i}$$

Then $\vec{r} = \left(\dfrac{25}{4} - \dfrac{25}{4}\cos 2t\right)\hat{i} + (8 + 4\sin 2t)\,\hat{j} + \left(\dfrac{3}{2}t^3\right)\hat{k}\,.$

8.3.2 Example

If $\dfrac{d^2\vec{f}}{dt^2} = 6t\,\hat{i} - 12t^2\,\hat{j} + 4\cos t\,\hat{k}$ find $\vec{f}$ given that $\dfrac{d\vec{f}}{dt} = -\hat{i} - 3\hat{k}$ and $\vec{f} = 2\hat{i} + \hat{j}$ at $t = 0$.

Solution: Given that $\dfrac{d^2\vec{f}}{dt^2} = 6t\,\hat{i} - 12t^2\,\hat{j} + 4\cos t\,\hat{k}\,.$ ...... (1)

Integrating w.r.t. t, we get

$$\int \frac{d^2\vec{f}}{dt^2}\,dt = \int \left(6t\,\hat{i} - 12t^2\,\hat{j} + 4\cos t\,\hat{k}\right)dt$$

$$\Rightarrow \frac{d\vec{f}}{dt} = 3t^2\hat{i} - 4t^3\,\hat{j} + 4\sin t\,\hat{k} + c_1 \qquad \text{...... (2)}$$

Integrating (2) w.r.t. t, we get

$$\int \frac{d\vec{f}}{dt} = \int \left(3t^2\hat{i} - 4t^3\hat{j} + 4\sin t\,\hat{k} + c_1\right) dt$$

$$= t^3\hat{i} - t^4\hat{j} - 4\cos t\,\hat{k} + c_1 t + c_2 \qquad \text{...... (3)}$$

Given that $\frac{d\vec{f}}{dt} = -\hat{i} - 3\hat{k}$ at $t = 0$ ...... (4)

$\vec{f} = 2\hat{i} + \hat{j}$

$\therefore c_1 = -\hat{i} - 3\hat{k}$

and $-4\hat{k} + c_2 = 2\hat{i} + \hat{j} \Rightarrow c_2 = 2\hat{i} + \hat{j} + 4\hat{k}$.

Putting the values of c_1 and c_2 in (3), we get

$$\vec{f} = t^3.\hat{i} - t^4.\hat{j} - 4\cos t\,\hat{k} + \left(-\hat{i} - 3\hat{k}\right)t + \left(2\hat{i} + \hat{j} + 4\hat{k}\right)$$

$$= \left(t^3 - t + 2\right)\hat{i} + \left(1 - t^4\right)\hat{j} - \left(4\cos t + 3t - 4\right)\hat{k}.$$

Check Your Progress:

3. Given $\vec{F} = 3\sin t\,\hat{i} - \cos t\,\hat{j} + (2-t)\hat{k}$ evaluate $\int_0^{\pi} \vec{F}\,dt$.

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(A) Applications of Line integrals:

Work:

If $\vec{F}$ represents the force acting on a particle moving along an arc PQ then the work done during the small displacement $d\vec{r}$ is equal to $\vec{F}.d\vec{r}$. Therefore, the total work done by $\vec{F}$ during the displacement from P to Q is given by the line integral $\int_P^Q \vec{F}.d\vec{r}$.

8.3.3 Example

Find the total work done in moving a particle in the force field given by $\vec{F} = z\hat{i} + z\,\hat{j} + x\hat{k}$ along the helix c given by $x = \cos t, y = \sin t, z = t$ from $t = 0$ to $\frac{\pi}{2}$.

Solution: $\vec{F} = z\hat{i} + z\,\hat{j} + x\hat{k} \;\; = t\hat{i} + t\,\hat{j} + \cos t\,\hat{k}$.

$x = \cos t, y = \sin t, z = t$

$\Rightarrow dx = -\sin t\,dt,\;\; dy = \cos t\,dt,\;\; dz = dt$.

$$\int_C \vec{F}.d\vec{r} = \int_C z\,dx + z\,dy + x\,dz$$

$$= \int_0^{\pi/2} \left[t\left(-\sin t\right)dt + t.\cos t\,dt + \cos t\,dt\right]$$

$$= \int_0^{\pi/2} \left[-t\sin t\,dt + \left(t+1\right)\cos t\,dt\right]$$

$$= \int_0^{\pi/2} -t\sin t\,dt + \int_0^{\pi/2} \left(t+1\right)\cos t\,dt$$

$$\int_0^{\pi/2} -t\sin t\,dt = -\left[t\cos t - \sin t\right]_0^{\pi/2} = -1$$

$$\int_0^{\pi/2} \left(t+1\right)\cos t dt = \left[\left(t+1\right)\left(\sin t\right)\right]_0^{\pi/2} - \int_0^{\pi/2} \sin t\,dt$$

$$= \frac{\pi}{2} + 1 + \left[\cos t\right]_0^{\pi/2} = \frac{\pi}{2}.$$

Hence the total work done is $\frac{\pi}{2} - 1.$.

8.3.4 Example

Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + \left(2xz - y\right)\hat{j} + z\,\hat{k}$ along

(a) the straight line from (0, 0, 0) to (2, 1, 3).

(b) the curve defined by $x^2 = 4y, 3x^3 = 8z$ from $x = 0$ to $x = 2$.

Solution: Let $\vec{r} = x\hat{i} + y\,\hat{j} + z\,\hat{k}$. Then $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$.

$$\int_C \vec{F}.d\vec{r} = \int_C \left[3x^2dx + (2xz - y)dy + z\,dz\right]$$

(a) The equation of the straight line from (0, 0, 0) to (2, 1, 3) is given by

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t.$$

$\therefore x = 2t,\ y = t,\ z = 3t$ and $dx = 2\,dt,\ dy = dt, dz = 3dt$.

t varies from 0 to 1.

$\therefore$ Work done $\int_C \vec{F}.d\vec{r}$

$$= \int_0^1 \left[3(2t)^2\, 2dt + \left((4t)(3t) - t\right)dt + 3t.3dt\right]$$

$$= \int_0^1 \left(36t^2 + 8t\right)dt \ = \left[36.\frac{t^3}{3} + 8.\frac{t^2}{2}\right]_0^1$$

= 12 + 4 = 16.

(b) Let $x = t$ in $x^2 = 4y,\ 3x^3 = 8z$.

Then the parametric equations of C are

$x = t, y = \frac{t^2}{4}, z = \frac{3t^3}{8}$ and t varies from 0 to 2.

$\therefore$ Work done = $\int_C \vec{F}.d\vec{r}$

$$= \int_0^2 3t^2dt + \left(2t\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right)\frac{d}{dt}\left(\frac{t^2}{4}\right) + \frac{3t^3}{8}d\left(\frac{3t^3}{8}\right)$$

$$= \int_0^2 \left(3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right)dt \ = \left[t^3 - \frac{t^4}{32} + \frac{17}{128}t^6\right]_0^2 = 16.$$

8.3.5 Example

If $\vec{F} = x^2\hat{i} + xy\,\hat{j}$, evaluate $\int\limits_C \vec{F}.d\vec{r}$ from (0, 0) to (1, 1) along

(i) the line $y = x$, (ii) the parabola $y^2 = x$.

Solution: $\vec{F} = x^2\hat{i} + xy\,\hat{j}$

$$\vec{r} = x\hat{i} + y\,\hat{j} \Rightarrow d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\vec{F}.d\vec{r} = x^2\,dx + xy\,dy$$

(i) Along the line $y = x$ we have $dy = dx,\ 0 \le x \le 1$

$$\int\limits_C \vec{F}.dr = \int\limits_{x=0}^{1} \left(x^2 + x^2\right)dx = \int\limits_0^1 2x^2 dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}.$$

(ii) Along $y^2 = x$ we have $2y\,dy = dx,\ 0 \le y \le 1$

$$\int\limits_C \vec{F}.dr = \int\limits_{y=0}^{1} \left(2y^5 + y^3\right)dy = \left[2.\frac{y^6}{6} + \frac{y^4}{4}\right]_0^1 = \frac{7}{12}.$$

8.3.6 Example

Use the line integral, compute work done by a force $\vec{F} = (2y+3)\hat{i} + xz\,\hat{j} + (yz - x)\hat{k}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2, y = t,\ z = t^3$.

Solution: Work done = $\int\limits_C \vec{F}.d\vec{r}$

$$\vec{F} = (2y+3)\hat{i} + xz\,\hat{j} + (yz - x)\hat{k}$$

$$x = 2t^2 \Rightarrow dx = 4t\,dt$$

$$y = t \Rightarrow dy = dt$$

$$z = t^3 \Rightarrow dz = 3t^2 dt$$

t varies from 0 to 1.

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$= 4t\,dt\,\hat{i} + dt\,\hat{j} + 3t^2 dt\,\hat{k}$$

$$\oint_C \vec{F}\,.d\vec{r} = \int F_1\,dx + F_2 dy + F_3 dz$$

$$= \int \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$

$$= \int_0^1 \left[(2t+3)4t + 2t^5 + \left(t^4 - 2t^2\right)3t^2 \right] dt$$

$$= \int_{t=0}^{1} \left[12t + 8t^2 - 6t^4 + 2t^5 + 3t^6 \right] dt$$

$$= \left[6t^2 + \frac{8}{3}t^3 - \frac{6}{5}t^5 + \frac{1}{3}t^6 + \frac{3}{7}t^7 \right]_0^1$$

$$= \frac{288}{35}.$$

8.3.7 Example

Find the work done in moving a particle once around an ellipse C in the xy - plane if the ellipse has the centre at the origin with semi major axis 4 and semi minor axis 3 and if the force field is given by $\vec{F} = (3x - 4y + 2z)\hat{i} + \left(4x + 2y - 3z^2\right)\hat{j} + \left(2xz - 4y^2 + z^3\right)\hat{k}$.

Solution: Here path of integrate on C is the ellipse whose equation is

$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$ and its parametric equations are $x = 4\cos t, y = 3\sin t$.

Also t varies from 0 to 2π.

Since C is a curve in xy- plane we have $z = 0$.

$$\therefore \vec{F} = (3x - 4y)\hat{i} + (4x + 2y)\hat{j}, \quad \vec{r} = x\hat{i} + y\hat{j}$$

$$\therefore d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\vec{F}.d\vec{r} = \left[(3x-4y)\hat{i} + (4x+2y)\hat{j}\right].\left[dx\,\hat{i} + dy\,\hat{j}\right]$$

$$= (3x-4y)\,dx + (4x+2y)\,dy$$

$$x = 4\cos t \Rightarrow dx = -4\sin t\,dt$$

$$y = 3\sin t \Rightarrow dy = 3\cos t\,dt$$

As t varies from 0 to 2π we have

$$\int_C \vec{F}.d\vec{r} = \int_0^{2\pi} \left[(12\cos t - 12\sin t)(-4\sin t)\,dt + (16\cos t + 6\sin t)3\cos t\,dt\right]$$

$$= \int_0^{2\pi} (48 - 30\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} (48 - 15\sin 2t)\,dt \qquad (\because \sin 2t = 2\sin t\cos t)$$

$$= \left[48t + \frac{15}{2}\cos 2t\right]_0^{2\pi} = 96\,\pi\,.$$

8.3.8 Example

If $\vec{F} = 3x^2yz^2\,\hat{i} + x^3z^2\,\hat{j} + 2x^3yz\,\hat{k}$, then show that $\int_C \vec{F}.d\vec{r}$ is independent of path. Hence evaluate the integral when C is any path joining (0, 0, 0) and (1, 2, 3).

Solution: Curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2yz^2 & x^3z^2 & 2x^3yz \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(2x^2yz\right) - \frac{\partial}{\partial z}\left(x^3z^2\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(2x^3yz\right) - \frac{\partial}{\partial z}\left(3x^2yz^2\right)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}\left(x^3z^2\right) - \frac{\partial}{\partial z}\left(3x^2yz^2\right)\right]$$

$$= \hat{i}\left(2x^3z - 2x^3z\right) + \hat{j}\left(6x^2yz - 6x^2yz\right) + \hat{k}\left(3x^2z^2 - 3x^2z^2\right) = \vec{0}$$

$\therefore \int_C \vec{F}.d\vec{r}$ is independent of the path of integration.

$$\vec{F} = grad\,\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\therefore \frac{\partial \phi}{\partial x} = 3x^2yz^2,\ \frac{\partial \phi}{\partial y} = x^3z^2,\ \frac{\partial \phi}{\partial z} = 2x^3yz\ .$$

Integrating, we get $\phi = x^3yz^2$.

$$\therefore \int_C \vec{F}.d\vec{r} = \int_C d\phi = [\phi]_{(0,0,0)}^{(1,2,3)} = \left[x^3yz^2\right]_{(0,0,0)}^{(1,2,3)} = 18.$$

Check Your Progress:

4. Find the work done by the vector field $\vec{F}(x,y,z) = x\hat{i} + 3xy\,\hat{j} - (x+z)\hat{k}$ on a particle moving along the line segment from (1, 4, 2) to (0, 5, 1).

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(B) Applications of Surface Integrals

Let $\vec{F}$ be a continuous vector field on a finite surface and $\vec{n}$ is unit outward normal to S. The integral of $\vec{F}.\vec{n}$ on S is called flux of $\vec{F}$ on S is positive direction, i.e. flux is the scalar component on $\vec{n}$ on S.

Flux = $\iint_S \vec{F}.\vec{n}\,dS$

8.3.9 Example

Evaluate $\iint_S \vec{F}.\vec{n}\,dS$ if $\vec{F} = z\hat{i} + x\,\hat{j} - yz\,\hat{k}$, and S is a surface of cylinder $x^2 + y^2 = 9$ between $z = 0,\ z = 4$ in the first octant.

Solution: S is a surface of cylinder $x^2 + y^2 = 9$ between $z = 0, z = 4$ in first octant we have unit normal

$$\nabla\left(x^2+y^2\right)=2x\hat{i}+2y\,\hat{j}$$

$$\hat{n}=\frac{\nabla\left(x^2+y^2\right)}{\left|\nabla\left(x^2+y^2\right)\right|}=\frac{2x\hat{i}+2y\,\hat{j}}{\sqrt{4x^2+4y^2}}=\frac{x\hat{i}+y\,\hat{j}}{3}$$

$$\vec{F}.\vec{n}=\left(z\,\hat{i}+x\,\hat{j}-yz\,\hat{k}\right).\frac{x\hat{i}+y\hat{j}}{3}$$

$$=\left(xz+xy\right)\frac{1}{3}=\frac{1}{3}x\left(x+y\right).$$

dS = elementary area on the surface of the cylinder using cylindrical coordinates

$$dS=3\,d\theta\,dz$$

$$\iint_S \vec{F}.\vec{n}\,dS=\iint_S \frac{1}{3}x\left(z+y\right)3\,d\theta\,dz$$

Put $x=3\cos\theta,\ y=3\sin\theta$

$$=\int_{z=0}^{4}\int_{\theta=0}^{\pi/2}3\cos\theta\left(z+3\sin\theta\right)d\theta dz$$

$$=3\int_0^{\pi/2}\cos\theta\left[\frac{1}{2}z^2+3z\sin\theta\right]_0^4 d\theta$$

$$=3\int_0^{\pi/2}\cos\theta\left(8+12\sin\theta\right)d\theta$$

$$=12\left[2\int_0^{\pi/2}\cos\theta\,d\theta+3\int_0^{\pi}\sin\theta\cos\theta\,d\theta\right]$$

$$=12\left[2+\frac{3}{2}\right]=42\,.$$

Check Your Progress:

5. If $\vec{F} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and S is the surface of cylinder $x^2 + y^2 = 16$, $z = 0, z = 5$ in the first octant, then evaluate $\iint\limits_S \vec{F}.\vec{n}\, dS$.

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6. If $\vec{F} = yz\hat{j} + z^2\hat{k}$ and $x = 0, x = 1$ are planes and $y^2 + z^2 = 1$, $z \geq 0$ is the cylinder, then find flux of $\vec{F}$.

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8.4 SUMMARY

In this unit, we have evaluated vector triple integrals. We have explained how to find velocity and time when acceleration is given using the line integral. We also have explained how to find flux of a vector field using surface integrals.

8.5 CHECK YOUR PROGRESS - MODEL ANSWERS

1. The given cylinder $z = 4 - x^2$ meets the x - axis $[y = 0, z = 0]$ at $x^2 = 4$ or $x = 2$ i.e. at the point (2, 0, 0). It meets z - axis at z = 4 i.e. at (0, 0, 4).

   Therefore the limits of integration are as follows.

   $z = 0$ to $z = 4 - x^2$, $y = 0$ to $y = 2$, x = 0 to x = 2.

   Also $dV = dx\,dy\,dz$

$$\iiint\limits_V (2x + y)\, dV = \iiint\limits_V (2x + y)\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2} (2x + y)\, dx\, dy\, dz = \int_{x=0}^{2}\int_{y=0}^{2} (2x + y)[z]_0^{4-x^2}\, dx\, dy$$

$$= \int_0^2\int_0^2 (2x + y)(4 - x^2)\, dx\, dy = \int_0^2 \left[2x(4 - x^2)y + (4 - x^2)\frac{y^2}{2}\right]_0^2 dx$$

$$= \int_0^2 \left[4x\left(4-x^2\right)+2\left(4-x^2\right)\right]dx \quad = 2\int_0^2 \left(4+8x-x^2-2x^3\right)dx$$

$$= 2\left[4x+4x^2-\frac{x^3}{3}-\frac{x^4}{2}\right]_0^2 = \frac{80}{3}.$$

2. The first integration w.r.t. z is from 0 to $1-y^2$, thee next integration w.r.t. x from $\frac{y}{2}$ to 3 and the last integration w.r.t. y is from 0 to 1.

$$\therefore V = \iiint_V dV = \int_0^1 \int_{y/2}^3 \int_0^{1-y^2} dz\,dx\,dy$$

$$= \int_0^1 \int_{y/2}^3 \left(1-y^2\right)dx\,dy = \int_0^1 \left[x-xy^2\right]_{y/2}^3 dy$$

$$= \int_0^1 \left(3-3y^2-\frac{y}{2}+\frac{1}{2}y^3\right)dy = \left[3y-y^3-\frac{1}{4}y^2+\frac{1}{8}y^4\right]_0^1$$

$$= 3-1-\frac{1}{4}+\frac{1}{8} \quad = \frac{15}{8} \text{ cubic units.}$$

3. $6\hat{i}+1.348\hat{k}$

4. The vector of line through (1, 4, 2) and (0, 5, 1) is

$$\vec{r}(t) = (1,4,2)+\left[(0,5,1)-(1,4,2)\right]t \qquad \left[\because \vec{r} = \vec{a}+t\left(\vec{b}-\vec{a}\right)\right]$$

$$= (1-t, 4+t, 2-t)$$

$$\vec{r}(t) = (1-t)\hat{i}+(4+t)\hat{j}+(2-t)\hat{k}$$

$$r'(t) = -\hat{i}+\hat{j}-\hat{k}$$

$$\vec{F}.\frac{d\vec{r}}{dt} = -x+3xy+x+z \quad = 3xy+z$$

$$= 3(1-t)(4+t)+(2-t) \quad = -3t^2-10t+14$$

$$\int_0^1 \vec{F}.d\vec{r} = \int_0^1 \left(-3t^2-10t+14\right)dt = \left(-t^3-5t^2+14t\right)_0^1 = 8.$$

5. 90

6. 2

8.6 MODEL EXAMINATION QUESTIONS

1. If $\vec{F} = 2xz\,\hat{i} - x\,\hat{j} + y^2\hat{k}$, evaluate $\iiint_V \vec{F}\,dV$, where V is the region bounded by the surfaces $x=0, x=2,\ y=0,\ y=6,\ z=x^2,\ z=4$.

2. If $\vec{F} = \left(2x^2 - 3z\right)\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}$, evaluate $\iiint_V \left(\nabla \times \vec{F}\right)dV$, where V is the closed region bounded by $x=0, y=0, z=0,\ 2x+2y+z=4$.

3. Evaluate $\iiint_V div\,\vec{F}\,dV$ where $\vec{F} = 2xy\,\hat{i} + 3yz\,\hat{j} + 5x^2\hat{k}$ and V is the region bounded by the equations $x=0,\ y=6,\ z=4$ and the parabola $z=x^2$.

4. Use line integral to compute the work done by the force $\vec{F} = (2y+3)\hat{i} + xz\,\hat{j} + (yz-x)\hat{k}$ when it moves a particle from (0, 0, 0) to (2, 1, 1) along the curve $x=2t^2,\ y=t, z=t^3$.

5. Evaluate $\iint_S curl\,\vec{F}.\vec{n}\,dS$, where $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + 10x\,\hat{k}$, S is the surface of the plane $2x+y=6$ in the first octant cut off by the plane $z=4$.

6. If velocity vector is $\vec{F} = y\,\hat{i} + 2\,\hat{j} + xz\,\hat{k}$ m/sec. Show that the flux of water through parabolic cylinder $y=x^2,\ 0 \le x \le 3,\ 0 \le z \le 2$ is $69\,m^3/\sec$.

7. Find whether $\int_C 2xyz^2\,dx + \left(x^2z^2 + z\cos yz\right)dy + \left(2x^2yz + y\cos yz\right)dz$ is independent of path joining $\left(0, \dfrac{\pi}{2}, 1\right)$ and (1, 0, 1). If so evaluate line integral.

8. Show that the vector field defined by

   $\vec{F} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + \left(xy\cos z + y^2\right)\hat{k}$ is irrotational and find its velocity potential.

Answers:

1. $128\hat{i} - 24\hat{j} + 384\hat{k}$

2. $\frac{8}{3}\left(\hat{j} - \hat{k}\right)$

3. 24

4. $\frac{288}{35}$

5. -192

7. -1

8. $xy\sin z + \cos x + y^2 z + c$

UNIT - 9: CURVILINEAR COORDINATES

Contents

9.0 OBJECTIVES

After studying this unit, you will be able to:

- Define fundamental triad of mutually orthogonal unit vectors for curvilinear coordinate system.
- Derive gradient, divergence and curl interms of curvilinear coordinate system.

9.1 INTRODUCTION

A Cartesian coordinate system is a coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed distances to the point from two fixed perpendicular directed lines, measured in the same unit of length. Each reference link is called a coordinate axis or just axis of the system and the point where they meet is its origin, (0, 0). The coordinates can also be defined as the positions of the perpendicular projections of the point on to the two axes, expressed as signed distances from the origin.

Curvilinear coordiante systems are general ways of locating points in Euclidean space using coordinate functions that are invertible functions of the usual Cartesian coordinates. Their utility arises in problems with obvious geometric symmetries such as cylindrical or spherical symmetry.

Although Cartesian orthogonal coordinates are very intuitive and easy to use, it is more convenient to work with other coordinate systems. In this unit, a general method to express any variable and expression in an arbitrary curvilinear coordinate system is explained. This unit mainly focuses on finding the general expressions for the gradient, the divergence and the curl of scalar and vector fields. Specific applications to the cylindrical and spherical systems were discussed.

9.2 CURVILINEAR COORDINATE SYSTEM

9.2.1 Definition

A curvilinear coordinate system can be defined starting from the orthogonal cartesian coordinate system. Let the coordinates (x, y, z) of any point be expressed as functions of (u_1, u_2, u_3), so that $x = x(u_1, u_2, u_3)$, $y = y(u_1, u_2, u_3)$, $z = z(u_1, u_2, u_3)$ then u, v, w can be expressed in terms of x, y, z as $u_1 = u(x, y, z)$, $u_2 = v(x, y, z)$ and $u_2 = w(x, y, z)$.

And also if $\dfrac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \neq 0$ then the system of coordinates (u_1, u_2, u_3) are called the **curvilinear coordinates of the point**.

If one of the coordinates is kept constant say $u_1 = c$, then the locus of (x, y, z) is a surface which is called a **coordinate surface**. Thus we have three families of coordinate system corresponding to $u_1 = c$, $u_2 = c, u_3 = c$ and each pair of these surfaces intersect in curves called **coordinate curves or lines**.

If the coordinate surfaces intersect at right angles, the curvilinear coordinate system is called orthogonal and $(u, v, w) = (u_1, u_2, u_3)$ are called **orthogonal curvilinear coordinates**.

9.2.2 Unit Vectors in Curvilinear Systems

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point $P(x, y, z)$ where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the three coordinate curves.

[i.e. $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$; $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$]

Now $\vec{r}(u_1, u_2, u_3) = x(u_1, u_2, u_3)\hat{i} + y(u_1, u_2, u_3)\hat{j} + z(u_1, u_2, u_3)\hat{k}$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

Then $\vec{r}(u_1, u_2, u_3)$ is a vector point function of the variables u, v, w.

The unit tangent vector $\hat{i}$ along the tangent to u - curve at point P is

$$\hat{i} = \frac{\dfrac{\partial \vec{r}}{\partial u_1}}{\left|\dfrac{\partial \vec{r}}{\partial u_1}\right|}$$

If $\left|\dfrac{\partial \vec{r}}{\partial u_1}\right| = h_1$, then $\dfrac{\partial \vec{r}}{\partial u_1} = h_1 \hat{i}$

Similarly, unit tangent vectors along v - curve and w - curves are

$$\frac{\partial \vec{r}}{\partial u_2} = h_2 \hat{j}, \ \frac{\partial \vec{r}}{\partial u_3} = h_3 \hat{k}$$

$$\therefore \ d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$= h_1 \hat{i}\, du_1 + h_2 \hat{j}\, du_2 + h_3 \hat{k}\, du_3$$

The length of the arc ds is given by

$$dS^2 = d\vec{r}.d\vec{r} = h_1^{\,2} du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

9.2.3 Example

Find the square of the element of arc length in cylindrical coordinates.

Solution: The position vector $\vec{r}$ in cylindrical coordinates is

$$\vec{r} = r\cos\theta \hat{i} + r\sin\theta\, \hat{j} + z\,\hat{k}$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \theta} d\theta + \frac{\partial \vec{r}}{\partial z} dz$$

$$= \left(\cos\theta\, \hat{i} + \sin\theta\, \hat{j}\right) dr + \left(-r\sin\theta\, \hat{i} + r\cos\theta\, \hat{j}\right) d\theta + \hat{k}\, dz$$

$$= \left(\cos\theta dr - r\sin\theta\, d\theta\right)\hat{i} + \left(\sin\theta\, dr + r\cos\theta\, d\theta\right)\hat{j} + \hat{k}\, dz$$

$$dS^2 = d\vec{r}.d\vec{r} = h_1^{\,2} du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

$$dS^2 = \left(\cos\theta\, dr - r\sin\theta\, d\theta\right)^2 + \left(\sin\theta\, dr + r\cos\theta\, d\theta\right)^2 + dz^2 .$$

$$= (dr)^2 + r^2 (d\theta)^2 + dz^2$$

Here $h_1 = 1,\ h_2 = r,\ h_3 = 1$ which are also known as Lommel's constants.

9.2.4 Example

Find the volume element dV in cylindrical coordinates.

Solution: The volume element in orthogonal curvilinear coordinates u_1, u_2, u_3 is given by

$$u_1 = r,\ u_2 = \theta,\ u_3 = z$$

$$h_1 = 1,\ h_2 = r,\ h_3 = 1$$

Then $dv = h_1\, h_2\, h_3\, du_1\, du_2\, du_3$

$$u_1 = r \Rightarrow du_1 = dr$$

$$u_2 = \theta \Rightarrow du_2 = d\theta$$

$$u_3 = z \Rightarrow du_3 = dz$$

$$\therefore dV = r\, dr\, d\theta\, dz.$$

9.2.5 Example

Describe the coordinate surfaces and coordinate curves for cylindrical and spherical coordinates.

Solution:

Cylindrical coordinates

The coordinate surfaces are

$\rho = c_1$ Cylinders coaxial with the z - axis or z-axis if $c_1 = 0$.

$\phi = c_2$ Planes through z- axis.

$z = c_3$ Planes perpendicular to the z - axis.

The coordinate curves are

Intersection of $\rho = c_1$ and $\phi = c_2$ (z curve) is a straight line.

Intersection of $\rho = c_1$ and $z = c_3$ (ϕ curve) is a circle or (a point).

Intersection of $\phi = c_2$ and $z = c_3$ (ρ curve) is a straight line.

Spherical coordinates

The coordinate surfaces are

$r = c_1$ Spheres having centre at the origin.

$\theta = c_2$ Cones having vertex at the origin or lines if $c_2 = 0$ or π and the xy - plane if $c_2 = \frac{\pi}{2}$.

$\phi = c_3$ Planes through the z - axis.

The coordinate curves are

Intersection of $r = c_1$ and $\theta = c_2$ (ϕ curve) is a circle or a point.

Intersection of $r = c_1$ and $\phi = c_3$ (θ curve) is a semi circle $(c_1 \neq 0)$.

Intersection of $\theta = c_2$ and $\phi = c_3$ (r curve) is a line.

9.3 FUNDAMENTAL TRIAD OF MUTUALLY ORTHOGONAL UNIT VECTORS THROUGH ANY POINT

We know that the vectors $\nabla u, \nabla v, \nabla w$ lie along the normals to the coordinate surfaces which are the level surfaces of the functions u, v, w.

Because of the assumed orthogonality of the curvilinear coordinate system, we have $\nabla u . \nabla v = 0,\ \nabla v . \nabla w = 0,\ \nabla w . \nabla u = 0$.

We suppose that the mutually orthogonal unit vectors $\frac{\nabla u}{|\nabla u|}, \frac{\nabla v}{|\nabla v|}, \frac{\nabla w}{|\nabla w|}$ in this order form a right handed system. As $|\nabla u|$ is the directional derivative of u along the direction of the normal to the surface $u = u_0$ i.e. along the tangent to the curve $v = v_0,\ w = w_0$.

dS_1 denotes the differential to length along the curve, we see that

$$|\nabla u| = \frac{du}{dS_1} = \frac{1}{\sqrt{\left(\frac{\partial r}{\partial u}\right)^2}}$$

we write

$$h_1 = \frac{1}{|\nabla u|} = \sqrt{\left(\frac{\partial r}{\partial u}\right)^2}$$

$$h_2 = \frac{1}{|\nabla v|} = \sqrt{\left(\frac{\partial r}{\partial v}\right)^2}$$

$$h_3 = \frac{1}{|\nabla w|} = \sqrt{\left(\frac{\partial r}{\partial w}\right)^2}$$

Clearly $\vec{a} = h_1 \nabla u, \vec{b} = h_2 \nabla v, \vec{c} = h_3 \nabla w$ is a right handed system of mutually orthogonal unit vectors. These systems of vectors are different for different points.

9.4 DIFFERENTIAL OPERATORS INTERMS OF ORTHOGONAL CURVILINEAR COORDINATES

In this section, we express the gradient, divergence and curl interms of orthogonal curvilinear coordinates u, v, w. Then the Laplacian can be expressed as the divergence of a gradient.

9.4.1 Gradient

We have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}.\frac{\partial w}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}.\frac{\partial w}{\partial y}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}.\frac{\partial w}{\partial z}$$

In rectangular Cartesian coordinate system, we have

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial v}{\partial z}\hat{k}\right] + \frac{\partial f}{\partial w}\left[\frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k}\right]$$

$$= \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w$$

But $\nabla u = \frac{1}{h_1}\vec{a}, \ \nabla v = \frac{1}{h_2}\vec{b}, \ \nabla w = \frac{1}{h_3}\vec{c}$.

$\therefore$ The gradient of f in orthogonal curvilinear coordinates is

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u}\vec{a} + \frac{1}{h_2}\frac{\partial f}{\partial v}\vec{b} + \frac{1}{h_3}\frac{\partial f}{\partial w}\vec{c}\ .$$

The components of grad ϕ along the vectors $\vec{a}, \vec{b}, \vec{c}$ are $\frac{1}{h_1}\frac{\partial f}{\partial u}, \frac{1}{h_2}\frac{\partial f}{\partial v}, \frac{1}{h_3}\frac{\partial f}{\partial w}$ respectively.

9.4.2 Divergence

Consider any vector point function $\vec{f}(u,v,w)$.

Let $\vec{f} = f_1\vec{a} + f_2\vec{b} + f_3\vec{c}$ so that f_1, f_2, f_3 are the components of $\vec{f}$ along $\vec{a}, \vec{b}, \vec{c}$.

$$\vec{f} = f_1\left(\vec{b}\times\vec{c}\right) + f_2\left(\vec{c}\times\vec{a}\right) + f_3\left(\vec{a}\times\vec{b}\right)$$

$$= f_1 h_2 h_3 \nabla u \times \nabla w + f_2 h_3 h_1 \nabla w \times \nabla u + f_3 h_1 h_2 \nabla u \times \nabla v$$

$$\therefore \nabla.\vec{f} = \nabla.\left(f_1 h_2 h_3 \nabla v \times \nabla w\right) + \nabla.\left(f_2 h_3 h_1 \nabla w \times \nabla u\right) + \nabla.\left(f_3 h_1 h_2 \nabla u \times \nabla v\right) \quad \text{... (1)}$$

Consider $\nabla.\left(f_1 h_2 h_3 \nabla u \times \nabla w\right) = f_1 h_2 h_3 \nabla\left(\nabla v \times \nabla w\right) + \nabla v \times \nabla w.\nabla\left(f_1 h_2 h_3\right)$ ... (2)

$$\left(\because div\left(\phi\,\vec{f}\right) = \phi\, div\,\vec{f} + \vec{f}.grad\,\phi\right)$$

Now $\nabla.\left(\nabla v \times \nabla w\right) = \nabla w.\ curl\,\nabla v - \nabla v.curl\ \nabla w = 0$ ... (3)

($\because$ curl ∇v = curl grad $v = \vec{0}$ and curl $\nabla w = \vec{0}$)

$$\nabla\left(f_1 h_2 h_3\right) = \frac{\partial}{\partial u}\left(f_1 h_2 h_3\right)\nabla u + \frac{\partial}{\partial v}\left(f_1 h_2 h_3\right)\nabla v + \frac{\partial}{\partial w}\left(f_1 h_2 h_3\right)\nabla w \quad \text{... (4)}$$

From (2), (3) and (4), we get

$$\nabla.\left(f_1 h_2 h_3 \nabla u \times \nabla w\right) = \nabla v \times \nabla w.\nabla u.\frac{\partial}{\partial u}\left(f_1 h_2 h_3\right)$$

$$= \frac{\vec{b}\times\vec{c}.\vec{a}}{h_1 h_2 h_3}.\frac{\partial}{\partial u}\left(f_1 h_2 h_3\right)$$

$$= \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u}\left(f_1 h_2 h_3\right) \quad \text{... (5)}$$

Finally from (1) and (5), we get

$$\nabla.\vec{f} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(f_1 h_2 h_3\right) + \frac{\partial}{\partial v}\left(f_2 h_3 h_1\right) + \frac{\partial}{\partial w}\left(f_3 h_1 h_2\right)\right].$$

9.4.3 Curl

Consider a vector point function $\vec{f}\left(u,v,w\right)$.

Let $\vec{f} = f_1\vec{a} + f_2\vec{b} + f_3\vec{c} \ = h_1\, f_1 \nabla u + h_2\, f_2 \nabla v + h_3\, f_3 \nabla w$

curl $\vec{f} = \nabla \times \vec{f} = \nabla \times \left[h_1 f_1 \nabla u + h_2 f_2 \nabla v + h_3 f_3 \nabla w\right]$ ... (1)

Now $\quad \nabla \times \left(h_1 f_1 \nabla u\right) = \nabla\left(h_1 f_1\right) \times \nabla u + h_1 f_1 \nabla \times \nabla u$

$$= \nabla\left(h_1 f_1\right) \times \nabla u \qquad \left(\because \nabla \times \nabla u = \vec{0}\right)$$

$$= \left[\frac{\partial}{\partial u}\left(h_1 f_1\right)\nabla u + \frac{\partial}{\partial v}\left(h_1 f_1\right)\nabla v + \frac{\partial}{\partial w}\left(h_1 f_1\right)\nabla w\right] \times \nabla u$$

$$= \frac{\partial}{\partial v}\left(h_1 f_1\right)\nabla v \times \nabla u + \frac{\partial}{\partial w}\left(h_1 f_1\right)\nabla w \times \nabla u$$

$$= \frac{1}{h_1 h_2}\frac{\partial}{\partial v}\left(h_1 f_1\right)\vec{b} \times \vec{a} + \frac{1}{h_3 h_1}\frac{\partial}{\partial w}\left(h_1 f_1\right)\vec{c} \times \vec{a}$$

$$= -\frac{1}{h_1 h_2}\frac{\partial}{\partial v}\left(h_1 f_1\right)\vec{c} + \frac{1}{h_3 h_1}\frac{\partial}{\partial w}\left(h_1 f_1\right)\vec{b} \qquad \text{... (2)}$$

From (1) and (2), we get

$$\nabla \times \vec{f} = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial v}\left(h_3 f_3\right) - \frac{\partial\left(h_2 f_2\right)}{\partial w}\right]\vec{a} + \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial w}\left(h_1 f_1\right) - \frac{\partial}{\partial u}\left(h_3 f_3\right)\right]\vec{b}$$

$$+ \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u}\left(h_2 f_2\right) - \frac{\partial}{\partial v}\left(h_1 f_1\right)\right]\vec{c}$$

$$= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\vec{a} & h_2\vec{b} & h_3\vec{c} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}.$$

9.4.4 Laplacian Operator ∇^2

We have $\nabla^2\phi = \nabla.\nabla\phi$

$$= \nabla.\left[\frac{1}{h_1}\frac{\partial\phi}{\partial u}\vec{a} + \frac{1}{h_2}\frac{\partial\phi}{\partial v}\vec{b} + \frac{1}{h_3}\frac{\partial\phi}{\partial w}\vec{c}\right]$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial w}\right)\right].$$

9.4.5 Example

Suppose u_1, u_2, u_3 are orthogonal curvilinear coordinates. Find the Jacobian of x, y, z with respect to u_1, u_2, u_3.

Solution: $J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial z_3} \end{vmatrix}$$

$$= \left(\frac{\partial x}{\partial u_1}\hat{i} + \frac{\partial y}{\partial u_1}\hat{j} + \frac{\partial z}{\partial u_1}\hat{k}\right).\left(\left(\frac{\partial x}{\partial u_2}\hat{i} + \frac{\partial y}{\partial u_2}\hat{j} + \frac{\partial z}{\partial u_2}\hat{k}\right).\left(\frac{\partial x}{\partial u_3}\hat{i} + \frac{\partial y}{\partial u_3}\hat{j} + \frac{\partial z}{\partial u_3}\hat{k}\right)\right)$$

$$= \frac{\partial\vec{r}}{\partial u_1}.\left(\frac{\partial\vec{r}}{\partial u_2} \times \frac{\partial\vec{r}}{\partial u_3}\right)$$

$$= h_1\vec{a}.\left(h_2\vec{b} \times h_3\vec{c}\right) = h_1\,h_2\,h_3\,\vec{a}.\left(\vec{b} \times \vec{c}\right) = h_1\,h_2\,h_3\,.$$

If the Jacobian equals to zero identically then $\frac{\partial\vec{r}}{\partial u_1}, \frac{\partial\vec{r}}{\partial u_2}, \frac{\partial\vec{r}}{\partial u_3}$ are coplanar vectors and the curvilinear coordinate transformation breaks down and we have a relation of the form $F(x, y, z) = 0$.

9.4.6 Example

Let u_1, u_2, u_3 be general coordinates. Show that $\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ and $\nabla u_1, \nabla u_2, \nabla u_3$ are reciprocal system of vectors.

Solution: We have to show that $\frac{\partial \vec{r}}{\partial u_p}.\nabla u_q = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases}$

where p = 1, 2, 3 and q = 1, 2, 3.

We have $d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$

$$\Rightarrow \nabla u_1 . d\vec{r} = du_1$$

$$= \left(\nabla u_1 . \frac{\partial \vec{r}}{\partial u_1} \right) du_1 + \left(\nabla u_1 . \frac{\partial \vec{r}}{\partial u_2} \right) du_2 + \left(\nabla u_1 . \frac{\partial \vec{r}}{\partial u_3} \right) du_3$$

Now $\nabla u_1 . \frac{\partial \vec{r}}{\partial u_1} = 1, \nabla u_1 . \frac{\partial \vec{r}}{\partial u_2} = 0, \ \nabla u_1 . \frac{\partial \vec{r}}{\partial u_3} = 0 \cdot$

$$\therefore \ \frac{\partial \vec{r}}{\partial u_1} = \frac{1}{\nabla u_1}.$$

Similarly, upon multiplying by $\nabla u_2 . \frac{\partial \vec{r}}{\partial u_2}, \ \nabla u_3 . \frac{\partial \vec{r}}{\partial u_3}$ we can prove the result.

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. Prove that $\left\{ \frac{\partial \vec{r}}{\partial u_1} . \frac{\partial \vec{r}}{\partial u_2} \times \frac{\partial \vec{r}}{\partial u_3} \right\} . \left(\nabla u_1 . \nabla u_2 \times \nabla u_3 \right)$ = 1.

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9.5 WORKED OUT EXAMPLES

9.5.1 Exercise

Prove that a cylindrical coordinate system is orthogonal.

Solution: The position vector of any point in cylindrical coordinates is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$$

The tangent vectors to ρ, ϕ, z curves are given by $\frac{\partial\vec{r}}{\partial\rho}, \frac{\partial\vec{r}}{\partial\phi}, \frac{\partial\vec{r}}{\partial z}$.

Now $$\frac{\partial\vec{r}}{\partial\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\frac{\partial\vec{r}}{\partial\phi} = -\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}$$

$$\frac{\partial\vec{r}}{\partial z} = \hat{k}$$

The unit vectors in these directions are

$$\vec{a} = \vec{e}_\rho = \frac{\partial\vec{r}/\partial\rho}{|\partial\vec{r}/\partial\rho|} = \frac{\cos\phi\,\hat{i} + \sin\phi\,\hat{j}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\vec{b} = \vec{e}_\phi = \frac{-\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}}{\sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi}} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\vec{c} = \vec{e}_z = \frac{\partial\vec{r}/\partial z}{|\partial\vec{r}/\partial z|} = \hat{k}$$

$$\vec{a}.\vec{b} = \left(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}\right).\left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right) = 0$$

$$\vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$$

$\therefore$ $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular.

Hence cylindrical coordinate system is orthogonal.

9.5.2 Exercise

Express $\nabla\Phi, \nabla.\vec{A}, \nabla\times\vec{A}$ and $\nabla^2\Phi$ in cylindrical coordinates.

Solution: For cylindrical coordinates (ρ, ϕ, z),

we have $u_1=\rho,\ u_2=\phi,\ u_3=z$.

$h_1=h_\rho=1,\ h_2=h_\phi=\rho,\ h_3=h_z=1$.

$\vec{a}=e_\rho,\ \vec{b}=e_\phi,\ \vec{c}=e_z$

$$\nabla\Phi=\frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\vec{a}+\frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\vec{b}+\frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\vec{c}$$

$$=\frac{\partial\Phi}{\partial u_1}e_\rho+\frac{1}{\rho}\frac{\partial\Phi}{\partial u_2}e_\phi+\frac{\partial\Phi}{\partial u_3}e_z$$

$$=\frac{\partial\Phi}{\partial\rho}e_\rho+\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}e_\phi+\frac{\partial\Phi}{\partial z}e_z$$

$$\nabla.\vec{A}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1)+\frac{\partial}{\partial u_2}(h_3h_1A_2)+\frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

$$=\frac{1}{(1)(\rho)(1)}\left[\frac{\partial}{\partial\rho}((\rho)(1)A_\rho)+\frac{\partial}{\partial\phi}((1)(1)A_\phi)+\frac{\partial}{\partial z}((1)(\rho)A_z)\right]$$

$$=\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\rho)+\frac{\partial A_\phi}{\partial\phi}+\frac{\partial}{\partial z}(\rho A_z)\right]$$

where $\vec{A}=A_1\vec{a}+A_2\vec{b}+A_3\vec{c}$ i.e., $A_1=A_\rho,\ A_2=A_\phi,\ A_3=A_z$.

$$\nabla\times\vec{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{a} & h_2\vec{b} & h_3\vec{b} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$=\frac{1}{\rho}\begin{vmatrix} e_\rho & \rho e_\phi & e_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\left(\frac{\partial A_z}{\partial \phi} - \frac{\partial}{\partial z}(\rho A_\phi)\right)e_\rho + \left(\rho\frac{\partial A_\rho}{\partial z} - \rho\frac{\partial A_z}{\partial \rho}\right)e_\phi + \left(\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right)e_z\right]$$

$$\nabla^2\Phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \Phi}{\partial u_3}\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}.$$

9.5.3 Exercise

Represent the vector $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical coordinates.

Solution: The position vector of any point in cylindrical coordinates is

$$r = x\hat{i} + y\hat{j} + z\hat{k} \;\; = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$$

$$\frac{\partial \vec{r}}{\partial \rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\frac{\partial \vec{r}}{\partial \phi} = -\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}$$

$$\frac{\partial \vec{r}}{\partial z} = \hat{k}$$

$$\vec{e}_1 = \vec{e}_\rho = \frac{\partial r/\partial \rho}{|\partial r/\partial \rho|} \;\; = \frac{\cos\phi\,\hat{i} + \sin\phi\,\hat{j}}{\sqrt{\cos^2\phi + \sin^2\phi}} \;\; = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} \qquad \text{... (1)}$$

$$\vec{e}_2 = \vec{e}_\phi = \frac{d\vec{r}/d\phi}{|d\vec{r}/d\phi|} = \frac{-\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}}{\sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi}} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \qquad \text{... (2)}$$

$$\vec{e}_3 = \vec{e}_z = \frac{\partial \vec{r}/\partial z}{|\partial \vec{r}/\partial z|} = \hat{k} \qquad \text{... (3)}$$

The expression $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical coordinates be

$$\vec{F} = f_1\vec{e}_\rho + f_2\vec{e}_\phi + f_3\vec{e}_z$$

$$f_1 = \vec{F}.\vec{e}_\rho = \left(z\hat{i} - 2x\hat{j} + y\hat{k}\right).\left(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}\right) = z\cos\phi - 2x\sin\phi$$

$$f_2 = \vec{F}.\vec{e}_\phi = -z\sin\phi - 2x\cos\phi$$

$$f_3 = \vec{F}.\vec{e}_z = y$$

Now $\quad \vec{F} = (z\cos\phi - 2x\sin\phi)\vec{e}_\rho - (z\sin\phi + 2x\cos\phi)\vec{e}_\phi + y\vec{e}_z$

$$(z\cos\phi - \rho\sin 2\phi)\vec{e}_\rho - \left(2\sin\phi + 2\rho\cos^2\phi\right)\vec{e}_\phi + \rho\sin\phi\ \vec{e}_z$$

9.5.4 Exercise

Express the vector $\vec{F} = x\hat{i} + 2y\hat{j} + yz\hat{k}$ in spherical polar coordinates.

Solution: In spherical polar coordinates we have

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow \vec{R} = r\sin\theta\cos\phi\hat{i} + r\sin\theta\sin\phi\ \hat{j} + r\cos\theta\hat{k}$$

$$\frac{\partial\vec{R}}{\partial r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\ \hat{j} + \cos\theta\hat{k}$$

$$\left|\frac{\partial\vec{R}}{\partial r}\right| = \sqrt{(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

$$\vec{e}_r = \frac{\partial\vec{R}/\partial r}{\left|\partial\vec{R}/\partial r\right|} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\ \hat{j} + \cos\theta\hat{k} \qquad \text{... (1)}$$

$$\frac{\partial\vec{R}}{\partial\theta} = r\cos\theta\cos\phi\hat{i} + r\cos\theta\sin\phi\ \hat{j} - r\sin\theta\hat{k}$$

$$\left|\frac{\partial\vec{R}}{\partial\theta}\right| = \sqrt{(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + r^2\sin^2\theta}$$

$$= \sqrt{r^2 \cos^2\theta\left(\cos^2\phi + \sin^2\phi\right) + r^2 \sin^2\theta}$$

$$= \sqrt{r^2 \cos^2\theta + r^2 \sin^2\theta} \quad = \sqrt{r^2\left(\cos^2\theta + \sin^2\theta\right)} \; = r.$$

$$\vec{e}_\theta = \frac{\partial\vec{R}/\partial\theta}{\left|\partial\vec{R}/\partial\theta\right|} \; = \frac{r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}}{r}$$

$$= \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k} \qquad \ldots (2)$$

$$\vec{e}_\phi = \frac{\partial\vec{R}/\partial\phi}{\left|\partial\vec{R}/\partial\phi\right|} \; = \frac{-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}}{\sqrt{\left(r\sin\theta\sin\phi\right)^2 + \left(r\sin\theta\cos\phi\right)^2}}$$

$$= \frac{-r\sin\theta\left(\sin\phi\,\hat{i} - \cos\phi\,\hat{j}\right)}{r\sin\theta} \; = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\,.$$

Let the expression for $\vec{F} = x\,\hat{i} + 2y\,\hat{j} + yz\,\hat{k}$ in spherical coordinates be

$\vec{F} = f_1\vec{e}_r + f_2\vec{e}_\theta + f_3\vec{e}_\phi$, where

$$f_1 = \vec{F}.\vec{e}_r$$

$$= \left[r\sin\theta\cos\phi\,\hat{i} + 2r\sin\theta\sin\phi\,\hat{j} + r^2\sin\theta\cos\theta\sin\phi\,\hat{k}\right]$$

$$.\left[\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right]$$

$$= r\sin^2\theta\cos^2\phi + 2r\sin^2\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin\phi$$

$$f_2 = \vec{F}.\vec{e}_\theta$$

$$= \left[r\sin\theta\cos\phi\,\hat{i} + 2r\sin\theta\sin\phi\,\hat{j} + r^2\sin\theta\cos\theta\sin\phi\,\hat{k}\right].$$

$$\left[\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}\right]$$

$$= r\sin\theta\cos\theta\cos^2\phi + 2r\sin\theta\cos\theta\sin^2\phi - r^2\sin^2\theta\cos\theta\sin\phi\,.$$

$$f_3 = \vec{F}.\vec{e}_\phi$$

$$=\left[r\sin\theta\cos\phi\hat{i}+2r\sin\theta\sin\phi\hat{j}+r^2\sin\theta\cos\theta\sin\phi\hat{k}\right].\left[-\sin\phi\hat{i}+\cos\phi\hat{j}\right]$$

$$=-r\sin\theta\sin\phi\cos\phi+2r\sin\theta\sin\phi\cos\phi \quad =r\sin\theta\sin\phi\cos\phi.$$

Now $\vec{F} = f_1\vec{e}_r + f_2\vec{e}_\theta + f_3\vec{e}_\phi$

$$=\left(r\sin^2\theta\cos^2\phi+2r\sin^2\theta\sin^2\phi+r^2\sin\theta\cos^2\theta\sin\phi\right)\vec{e}_r$$

$$+\left(r\sin\theta\cos\theta\cos^2\phi+2r\sin\theta\cos\theta\sin^2\phi-r^2\sin^2\theta\cos\theta\sin\phi\right)\vec{e}_\theta$$

$$+\ \left(r\sin\theta\sin\phi\cos\phi\right)\vec{e}_\phi.$$

Check Your Progress:

2. Find the gradient of $f(x,y,z)=x^2y+y^2z+xz^2$ in curvilinear coordinates.

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9.5.5 Exercise

Prove that the spherical coordinate system is orthogonal.

Solution: If $\vec{R}$ is the position vector of a point $P(x,y,z)$, then $\vec{R}=x\hat{i}+y\hat{j}+z\hat{k}$.

In spherical coordinates, we know that

$$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta.$$

$$\vec{R}=r\sin\theta\cos\phi\hat{i}+r\sin\theta\sin\phi\hat{j}+r\cos\theta\hat{k}$$

$$\vec{a}=\vec{e}_r=\frac{\partial\vec{R}/\partial r}{\left|\partial\vec{R}/\partial r\right|}=\sin\theta\cos\phi\hat{i}+\cos\theta\sin\phi\hat{j}+\cos\theta\hat{k} \qquad \text{... (1)}$$

$$\vec{b}=\vec{e}_\theta=\frac{\partial\vec{R}/\partial\theta}{\left|\partial\vec{R}/\partial\theta\right|}=\cos\theta\cos\phi\hat{i}+\cos\theta\sin\phi\hat{j}-\sin\theta\hat{k} \qquad \text{... (2)}$$

$$\vec{c}=\vec{e}_\phi=\frac{\partial\vec{R}/\partial\phi}{\left|\partial\vec{R}/\partial\phi\right|}=-\sin\phi\hat{i}+\cos\phi\hat{j} \qquad \text{... (3)}$$

From (1), (2) and (3), we have $\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0$.

9.5.6 Exercise

Obtain expression for grand f, $\nabla.\vec{A}$, $\nabla\times\vec{A}$ in spherical coordinates.

Solution: For spherical coordinates

$u_1 = r, u_2 = \theta, u_3 = \phi$ and $h_1 = 1, h_2 = r, h_3 = r\sin\theta$.

Let $\vec{A} = A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3$

grad f:

$$\text{grad}\, f = \frac{\vec{e}_1}{h_1}\frac{\partial f}{\partial u_1} + \frac{\vec{e}_2}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3\frac{\partial f}{\partial u_3}$$

Substituting for h_1, h_2, h_3 and u_1, u_2, u_3, we have

$$\text{grad}\, f = \vec{e}_1\frac{\partial f}{\partial r} + \frac{\vec{e}_2}{r}\frac{\partial f}{\partial \theta} + \frac{\vec{e}_3}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\text{div}\ \vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(A_1h_2h_3\right) + \frac{\partial}{\partial u_2}\left(A_2h_3h_1\right) + \frac{\partial}{\partial u_3}\left(A_3h_1h_2\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(A_1r^2\sin\theta\right) + \frac{\partial}{\partial \theta}\left(A_2r\sin\theta\right) + \frac{\partial}{\partial \phi}\left(A_3r\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(A_1r^2\right) + r\frac{\partial}{\partial \theta}\left(A_2\sin\theta\right) + r\frac{\partial A_3}{\partial \phi}\right]$$

$$\text{curl}\ \vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_1 & A_2r & A_3r\sin\theta \end{vmatrix}.$$

9.6 SUMMARY

In this unit, we have discussed about orthogonal curvilinear coordinates and unit vectors in curvilinear systems. Also we have expressed gradient, divergence and curl interms of curvilinear coordinates and also we have expressed the same in cylindrical and spherical coordinates.

9.7 CHECK YOUR PROGRESS - MODEL ANSWERS

1. From the example (9.4.6), $\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ and $\nabla u_1, \nabla u_2, \nabla u_3$ are reciprocal system of vectors.

$$\nabla u_1.\left(\nabla u_2 \times \nabla u_3\right) = \begin{vmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{vmatrix} = J\left(\frac{u_1, u_2, u_3}{x, y, z}\right)$$

$$\therefore J\left(\frac{x, y, z}{u_1, u_2, u_3}\right).J\left(\frac{u_1, u_2, u_3}{x, y, z}\right) = 1.$$

2. In curvilinear coordinates $\left(u_1, u_2, u_3\right)$ is given function *f* takes the form

$$f = u_2 u_1^2 + u_3 u_2^2 + u_1 u_3^2$$

$$\frac{\partial f}{\partial u_1} = 2u_1 u_2 + u_3^2 \; ; \; \frac{\partial f}{\partial u_2} = 2u_3 u_2 + u_1^2 \; ; \; \frac{\partial f}{\partial u_3} = u_2^2 + 2u_1 u_3$$

Now $\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\vec{a} + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\vec{b} + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\vec{c}$

$$= \frac{1}{h_1}\left(2u_1 u_2 + u_3^2\right)\vec{a} + \frac{1}{h_2}\left(u_1^2 + 2u_3 u_2\right)\vec{b} + \frac{1}{h_3}\left(u_2^2 + 2u_1 u_3\right)\vec{c}$$

9.8 MODEL EXAMINATION QUESTIONS

1. Describe the coordinate surfaces and coordinate curves for cylindrical spherical coordinates.

2. Express gradient, curl and divergence in curvilinear coordinates.

3. Express gradient, curl and divergence in spherical coordinates.

4. Prove that a cylindrical coordinate system is orthogonal.

5. Prove that a spherical coordinate system is orthogonal.

6. Find the volume element dV in cylindrical coordinates.

Answers:

6. $dV = r\, dr\, d\theta\, dz$

BLOCK-IV: INTEGRAL THEOREMS

We have discussed line, surface and volume integrals in Block - III. In this block we will see how these integrals are related to one another. In practical applications, some times it is easier to calculate a surface integral rather than a volume integral or a line integral compared to a surface integral relative to a given problem. Integral theorems allow us to do the transformation from one form of the integral to another so that the computation becomes easy. In this block we study three such theorems, namely, the Green's theorem, the Stoke's theorem and the Gauss divergence theorem. Green's theorem in plane is established as a special case of Stoke's theorem. Verification and application of these theorems is illustrated with a number of examples.

This block includes the following units:

UNIT-10: GREEN'S THEOREM AND ITS APPLICATIONS

Contents

10.0 OBJECTIVES

After studying this unit, you will be able to:

- Understand the proof of Green's theorem.
- Apply Green's theorem to find area enclosed by simple closed curves in plane.
- Know how to transform a line integral to a double integral and vice-versa.

10.1 INTRODUCTION

In this unit we prove Green's theorem in plane. This theorem gives a relationship between the line integral of a two - dimensional vector field over a closed path in the plane and the double integral over the region enclosed by this path. Since this theorem establishes the equality between two types of integrals, it can be used to computer either line integral or double integral. Green's theorem is a special case of a more general theorem which will be proved in unit 12.

10.2 GREEN'S THEOREM

In the plane, a curve with no end points (or a curve whose initial point and end point coincide) and which completely encloses an area is called a closed curve. A closed curve which does not cross itself is called a simple closed curve. For example, circle, ellipse are simple closed curves.

10.2.1 Green's Theorem

Let A be a closed region in xy - plane bounded by a simple closed curve C. Let $M(x,y)$, $N(x,y)$ be continuous functions having continuous partial derivatives in R. Then

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in the positive (anti-clockwise) direction.

Proof:

Case (i): Suppose that the region R is such that any line parallel to the coordinate axes meets the boundary C in atmost two points. Suppose the region is included between the lines $x = a,\ x = b$ and $y = c,\ y = d$ (Fig. 10.1).

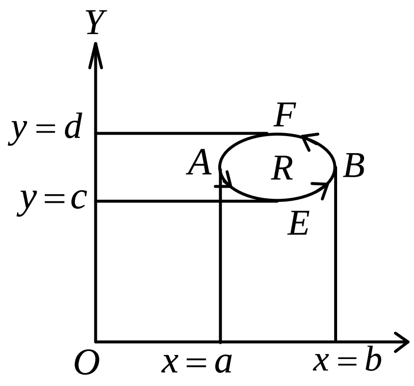

Fig: 10.1

Let C_1, C_2 denote the portions AEB and BFA respectively of the curve C. Let $y = f_1(x)$ on C_1 and $y = f_2(x)$ on C_2. Then

$$\iint_R \frac{\partial M}{\partial y} dx\,dy = \int_{x=a}^{x=b} \int_{y=f_1(x)}^{y=f_2(x)} \frac{\partial M}{\partial y} dx\,dy = \int_{x=a}^{x=b} \left[M(x,y)\right]_{y=f_1(x)}^{y=f_2(x)} dx$$

$$= \int_{x=a}^{x=b} \left[M\left(x, f_2(x)\right) - M\left(x, f_1(x)\right)\right] dx$$

$$= \int_a^b M\left(x, f_2(x)\right) dx - \int_a^b M\left(x, f_1(x)\right) dx$$

$$= -\int_b^a M\left(x, f_2(x)\right) dx - \int_a^b M\left(x, f_1(x)\right) dx$$

$$= -\left[\int_{C_2} M(x,y)\,dx + \int_{C_1} M(x,y)\,dx \right]$$

$$= -\int_C M\,dx \qquad \text{... (1)}$$

Now let C_1', C_2' denote the portions *FAE* and *EBF* of the curve *C*. Let $x = g_1(y)$ on C_1' and $x = g_2(y)$ on C_2'.

Then $\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=c}^{y=d} \int_{x=g_1(y)}^{x=g_2(y)} \frac{\partial N}{\partial x}\,dx\,dy$

$$= \int_{y=c}^{y=d} \left[N(x,y) \right]_{x=g_1(y)}^{x=g_2(y)} dy$$

$$= \int_{y=c}^{y=d} \left[N(g_2(y),\, y) - N(g_1(y), y) \right] dy$$

$$= \int_c^d N(g_2(y), y)\,dy - \int_d^c N(g_1(y), y)\,dy$$

$$= \int_c^d N(g_2(y), y)\,dy + \int_d^c N(g_1(y), y)\,dy$$

$$= \int_{C_2'} N(x,y)\,dy + \int_{C_1'} N(x,y)\,dy$$

$$= \int_C N\,dy \qquad \text{... (2)}$$

From (1) and (2), we have

$$\int_C M\,dx + \int_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy - \iint_R \frac{\partial M}{\partial y}\,dx\,dy$$

(or) $\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$.

Case (ii): Suppose the given plane region can be divided into a finite number of sub regions in such a way that any line parallel to the coordinate axes meets the boundary of each of these sub regions in atmost two points.

In this case, we apply case (i), to each of the sub regions and by adding the resulting integrals, Green's theorem can be proved.

For instance, consider the region shown in Fig. 10.2.

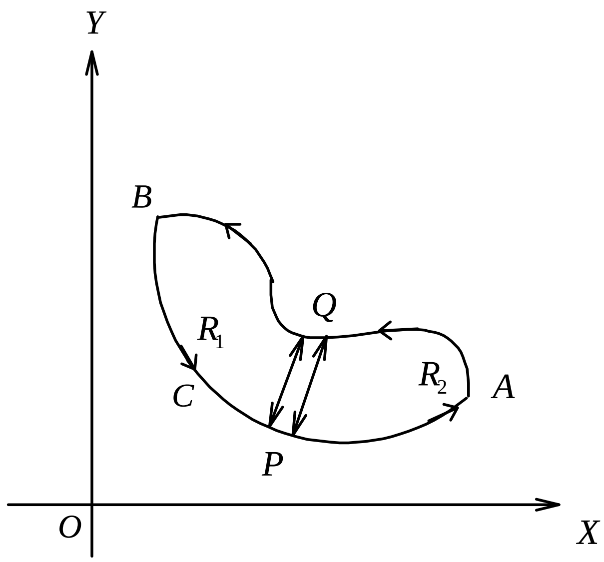

Fig: 10.2

We divide the region into two parts R_1 and R_2 with common boundary *PQ*. While travelling along *C*, the boundary of R_1, *QBPQ* and the boundary of R_2, *PAQP*, the common part of the boundary is covered twice in opposite directions. The line integrals along *PQ* and *QP* get cancelled and the required result is obtained.

In the same way the Green's theorem can be extended to multiply connected regions.

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. (a) Greens theorem in plane gives the relation between ______

(b) Green's theorem in plane is a special case of ______

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2. R is closed region in plane, shown by the shaded area in Fig 10.3. The boundary of the region consists of the circles C_1 and C_2 . *M, N* are functions defined on *R*, satisfying the conditions of Green's theorem. Describe the direction of integration along *C* in applying Green's theorem.

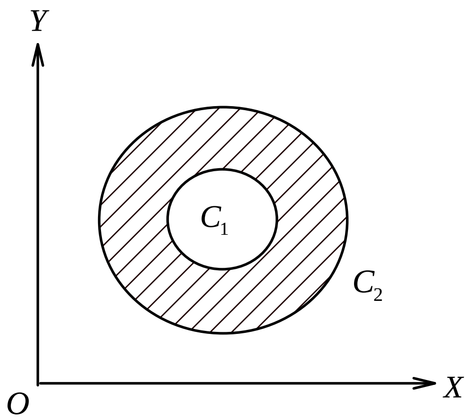

Fig: 10.3

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10.2.2 Vector Notation for Green's Theorem

We know that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$.

In *xy* - plane, since *z* = 0, $\vec{r} = x\hat{i} + y\hat{j}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$.

Let $M(x, y)$ and $N(x, y)$ be functions of *x* and *y* only.

If $\vec{F} = M\hat{i} + N\hat{j}$, then $\vec{F}.d\vec{r} = \left(M\hat{i} + N\hat{j}\right).\left(dx\,\hat{i} + dy\,\hat{j}\right) = M\,dx + N\,dy$.

Also, $curl\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \hat{i}\left(\dfrac{-\partial N}{\partial z}\right) - \hat{j}\left(-\dfrac{\partial M}{\partial z}\right) + \hat{k}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$.

Since *M* and *N* contain only *x* and *y* terms,

$\dfrac{\partial N}{\partial z} = 0, \dfrac{\partial M}{\partial z} = 0$.

$$\therefore \text{curl } \vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

$$\therefore \left(\nabla \times \vec{F}\right).\hat{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}.\hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

If $dR = dx\,dy$, then Green's theorem is given by

$$\int_C \vec{F}.d\vec{r} = \iint_R \left(\nabla \times \vec{F}\right).\hat{k}\,dR\,.$$

10.2.3 Remark

If s denotes the arc length of C and $\hat{n}$ is the unit tangent vector to the curve C, then we have,

$$d\vec{r} = \frac{d\vec{r}}{ds}ds = \hat{n}\,ds$$

$\therefore$ Green's theorem can also be written as

$$\int_C \vec{F}.\hat{n}\ ds = \iint_R \left(\nabla.\vec{F}\right)\hat{k}\,dR$$

10.2.4 Example

Verify Green's theorem in plane for the function $\vec{F}\left(x, y, z\right) = \left(x^2 - xy^3\right)\hat{i} + \left(y^2 - 2xy\right)\hat{j}$ when C is the boundary of the square with vertices (0, 0), (2, 0), (2, 2), (0, 2).

solution: Let O(0, 0), A(2, 0), B(2, 2) and C(0, 2) be the vertices of the square with boundary C and R be the plane region bounded by C.

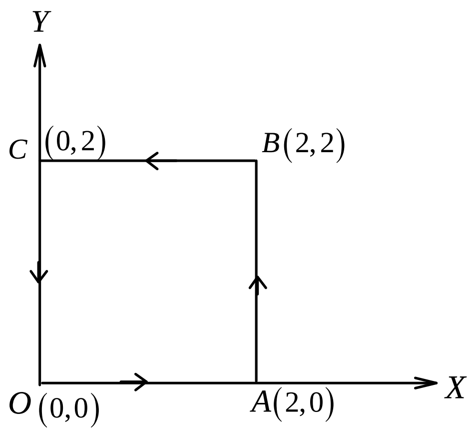

Fig: 10.4

By Green's theorem in plane, we have

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

From $\vec{F}$, $M = x^2 - xy^3$ and $N = y^2 - 2xy$ ... (1)

Now, RHS $= \int_{x=0}^{2} \int_{y=0}^{2} \left(-2y + 3xy^2\right) dx\,dy$

$$= \int_{x=0}^{2} \left[\frac{-2y^2}{2} + \frac{3xy^3}{3} \right]_0^2 dx = \int_{x=0}^{2} \left(-4 + 8x\right) dx$$

$$= \left[-4x + 8\frac{x^2}{2} \right]_0^2 = -8 + 16 = 8 \qquad \text{... (2)}$$

To evaluate LHS, we note that C is made of the 4 straight lines *OA, AB, BC, CO*. We evaluate the integral over each of these lines and add them up to find line integral over C.

$$\int_{OA} M\,dx + N\,dy = \int_{OA} \left(x^2 - xy^3\right) dx + \left(y^2 - 2xy\right) dy$$

$$= \int_{x=0}^{2} x^2\,dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3} \qquad \text{... (3)}$$

(since $y = 0$, $dy = 0$ along OA and x varies from 0 to 2)

$$\int_{AB} M\,dx + N\,dy = \int_{AB} \left(x^2 - xy^3\right) dx + \left(y^2 - 2xy\right) dy$$

$$= \int_{y=0}^{2} \left(x^2 - 2y^3\right).0 + \left(y^2 - 4y\right) dy$$

(since $x = 2$, $dx = 0$ along AB and y varies from 0 to 2)

$$= \left[\frac{y^3}{3} - \frac{4y^2}{2} \right]_0^2 = \frac{8}{3} - 8 = -\frac{16}{3} \qquad \text{... (4)}$$

$$\int_{BC} M\,dx + N\,dy = \int_{BC} \left(x^2 - xy^3\right)dx + \left(y^2 - 2xy\right)dy$$

$$= \int_{x=2}^{0} \left(x^2 - 8x\right)dx + \left(4 - 4x\right).0$$

(since $y = 2, dy = 0$ along BC and x varies from 0 to 2)

$$= \left[\frac{x^3}{3} - \frac{8x^2}{2}\right]_2^0 \quad = 0 - \left(\frac{8}{3} - 16\right) \quad = \frac{40}{3} \qquad \text{... (5)}$$

$$\int_{CO} M\,dx + N\,dy = \int_{CO} \left(x^2 - xy^3\right)dx + \left(y^2 - 2xy\right)dy$$

$$= \int_{y=2}^{0} \left(0 - 0\right)dx + y^2\,dy$$

(since $x = 0,\ dx = 0$ along CO and y varies from 0 to 2.)

$$= \left[\frac{y^3}{3}\right]_2^0 \quad = 0 - \frac{8}{3} = \frac{-8}{3} \qquad \text{... (6)}$$

Adding (3), (4), (5) and (6), we have

$$\int_C M\,dx + N\,dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8 \qquad \text{... (7)}$$

From (2) and (7), we have

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = 8.$$

Hence Green's theorem is verified.

10.2.5 Example

Verify Green's theorem in plane for $\int_C \left(xy + y^2\right)dx + x^2\,dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution: Solving $y = x$ and $y = x^2$, we get $x = x^2 \Rightarrow x = 0, x = 1$

$\therefore y = 0,\ y = 1$.

Thus, the points of intersection are (0, 0) and (1, 1).

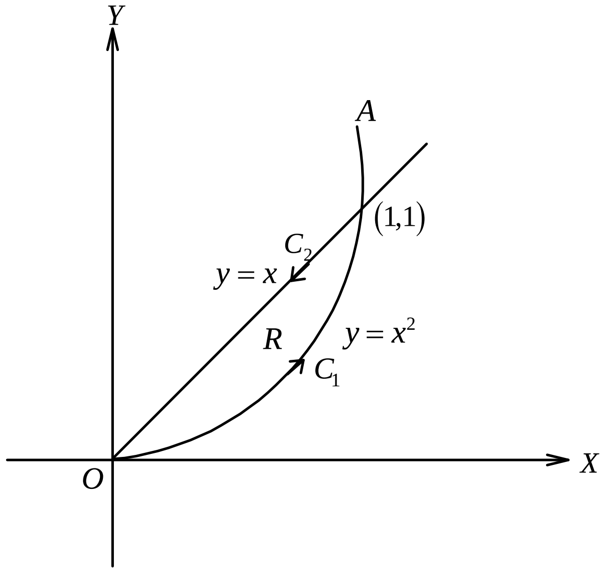

Fig: 10.5

On the curve $y = x^2$, $dy = 2x\,dx$ and x varies from 0 to 1.

$$\therefore \int_{C_1} M\,dx + N\,dy = \int_{x=0}^{1} \left(xy + y^2\right)dx + x^2\,dy$$

$$= \int_{x=0}^{1} \left(x^3 + x^4\right)dx + 2x^3 dx = \int_{0}^{1} \left(3x^3 + x^4\right)dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20} \qquad \text{... (1)}$$

On the straight line $y = x,\ dy = dx$, x varies from 1 to 0.

$$\therefore \int_{C_2} M\,dx + N\,dy = \int_{x=1}^{0} \left(xy + y^2\right)dx + x^2\,dy$$

$$= \int_{x=1}^{0} \left(x^2 + x^2\right) dx + x^2\, dx$$

$$= \int_{1}^{0} 3x^2\, dx = \left[\frac{3x^3}{3}\right]_1^0 = -1 \qquad \text{... (2)}$$

Adding (1) and (2), we have

$$\int_C M\, dx + N\, dy = \int_C \left(xy + y^2\right) dx + x^2\, dy$$

$$= \frac{19}{20} - 1 = \frac{-1}{20} \qquad \text{... (3)}$$

Again $\displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_{x=0}^{1} \int_{y=x^2}^{x} \left[\frac{\partial}{\partial x}\left(x^2\right) - \frac{\partial}{\partial y}\left(xy + y^2\right)\right] dx\, dy$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} \left[2x - \left(x + 2y\right)\right] dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} \left(x - 2y\right) dx\, dy = \int_{x=0}^{1} \left(xy - \frac{2y^2}{2}\right)_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[\left(x^2 - x^2\right) - \left(x^3 - x^4\right)\right] dx$$

$$= \int_{x=0}^{1} \left(x^4 - x^3\right) dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = \frac{-1}{20} \qquad \text{... (4)}$$

From (3) and (4), we have

$$\int_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Thus Green’s theorem is verified.

10.3 APPLICATIONS OF GREEN'S THEOREM

10.3.1 We know that the integral $\int_C \vec{F}.d\vec{r}$ will generally have different values along different paths joining two points A and B on the curve C. The line integral is said to be independent of path in a region R if for each pair of points A and B in R, the value of the integral is the same for all paths joining A and B in R.

A vector field $\vec{F}(x, y, z)$ defined and continuous in a region R is a conservative field if the line integral $\int_C \vec{F}.d\vec{r}$ is independent of path C in R.

The work done by a force $\vec{F}$ is moving a particle along a closed curve C is determined by $\int_C \vec{F}.d\vec{r}$. By Green's theorem this is equal to $\iint_R \nabla \times \vec{F}.d\vec{R}$, where R is the area (region) enclosed by the closed curve C.

If the force $\vec{F}$ is conservative, then there exists a potential function ϕ such that $\vec{F} = -\nabla\phi$ (negative sign by convention) and hence $\nabla \times \vec{F} = \vec{0}$.

$$\therefore \int_C \vec{F}.d\vec{r} = \iint_R \nabla \times \vec{F}.d\vec{R} = 0 .$$

If $\vec{F}$ represents the velocity of a fluid particle then $\int_C \vec{F}.d\vec{r}$ is called the circulation around the closed contour C. When the circulation of $\vec{F}$ around every closed curve, vanishes then that $\vec{F}$ is irrotational. In this case $\nabla \times \vec{F} = \vec{0}$.

10.3.2 Example

Let $\vec{F} = \dfrac{1}{x^2 + y^2}\left(-y\hat{i} + x\hat{j}\right)$. Calculate $\nabla \times \vec{F}$. Evalute $\int_C \vec{F}.d\vec{r}$ around any closed path not enclosing the origin.

Solution: Given that $\vec{F} = \dfrac{1}{x^2 + y^2}\left(-y\hat{i} + x\hat{j}\right)$

$$= -\frac{y}{x^2 + y^2}\hat{i} + \frac{x}{x^2 + y^2}\hat{j}$$

Now, in any region excluding (0, 0), we have

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix}$$

$$= \hat{i}\left(0 - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\right) - \hat{j}\left(0 + \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2}\right)\right)$$

$$+\hat{k}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]$$

$$= \hat{i}.0 + \hat{j}.0 + \hat{k}\left[\frac{\left(x^2+y^2\right).1 - x.2x}{\left(x^2+y^2\right)^2} + \frac{\left(x^2+y^2\right).1 - y.2y}{\left(x^2+y^2\right)^2}\right]$$

$$= \hat{k}\left[\frac{y^2 - x^2 + x^2 - y^2}{\left(x^2+y^2\right)^2}\right] = 0\hat{k} = \vec{0}.$$

Let $x = \rho\cos\theta$ and $y = \rho\sin\theta$, where $\left(\rho, \theta\right)$ are polar coordination.

Then $dx = -\rho\sin\theta\, d\theta + \cos\theta\, d\rho$ and $dy = \rho\cos\theta\, d\theta + \sin\theta\, d\rho$

$$\Rightarrow \vec{F}.d\vec{r} = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

$$= \frac{\left(-\rho\sin\theta\right)\left(-\rho\sin\theta\, d\theta + \cos\theta\, d\rho\right) + \left(\rho\cos\theta\right)\left(\rho\cos\theta\, d\theta + \sin\theta\, d\rho\right)}{\rho^2\cos^2\theta + \rho^2\sin^2\theta}$$

$$= \frac{\rho^2\sin^2\theta\, d\theta - \rho\sin\theta\cos\theta\, d\theta + \rho^2\cos^2\theta\, d\theta + \rho\sin\theta\cos\theta}{\rho^2\left(\cos^2\theta + \sin^2\theta\right)}$$

$$= \frac{\rho^2\left(\sin^2\theta + \cos^2\theta\right)d\theta}{\rho^2} = d\theta.$$

But $\frac{y}{x} = \frac{\rho\sin\theta}{\rho\cos\theta} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right).$

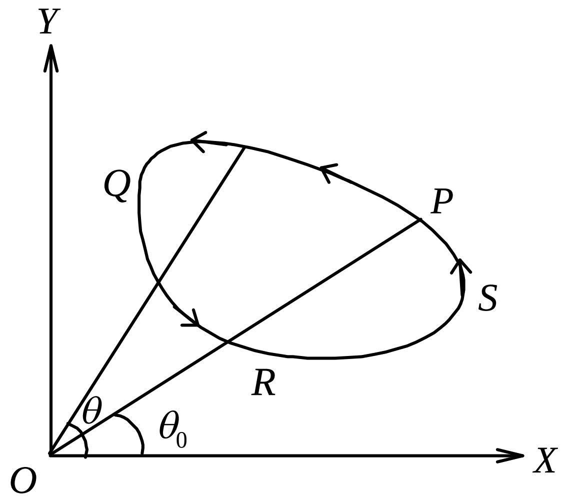

Fig: 10.6

Let $PQRSP$ be any closed curve in plane not enclosing the origin. At P, $\theta = \theta_0$ and after taking a full turn and coming back to P, $\theta = \theta_0$.

$$\therefore \int \vec{F}.d\vec{r} = \int_{\theta=\theta_0}^{\theta=\theta_0} d\theta = \left[\theta\right]_{\theta_0}^{\theta_0} = \theta_0 - \theta_0 = 0.$$

Thus $\vec{F}$ is conservative and the line integral is independent of path.

10.3.3 Calculating the Area Bounded by a Simple Closed Curve

By Green's theorem we have $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

Taking M = -y and $N = x$, we get $\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial y} = 1$.

$$\therefore \int_C x\,dy - y\,dx = \iint_R (1+1)\,dx\,dy = 2\iint_R dx\,dy = 2A$$

where A is area of the region *R*.

Thus area $A = \dfrac{1}{2}\int_C x\,dy - y\,dx$.

This result can be used to calculate the area bounded by the closed curve *C*.

10.3.4 Example

Find the area of the ellipse $x = a\cos\theta, y = b\sin\theta,\ 0 \le \theta \le 2\pi$.

Solution: $x = a\cos\theta, y = b\sin\theta$

$\Rightarrow dx = -a\sin\theta\,d\theta,\ dy = b\cos\theta\,d\theta$.

Area bounded by the ellipse

$$= A = \frac{1}{2}\int_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi} (a\cos\theta)(b\cos\theta)\,d\theta - (b\sin\theta)(-a\sin\theta)\,d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi} ab\left(\cos^2\theta + \sin^2\theta\right)d\theta$$

$$= \frac{1}{2}ab\int_{\theta=0}^{2\pi} d\theta = \frac{1}{2}ab[\theta]_0^{2\pi} = \frac{ab}{2}.2\pi = \pi\,ab\ .$$

10.3.5 Example

Use Green's theorem to find the area of the region bounded $y = x, y = x^{-1}, x = 4y$ in the first quadrant.

Solution: The area to be calculated is shown in Fig 10.7.

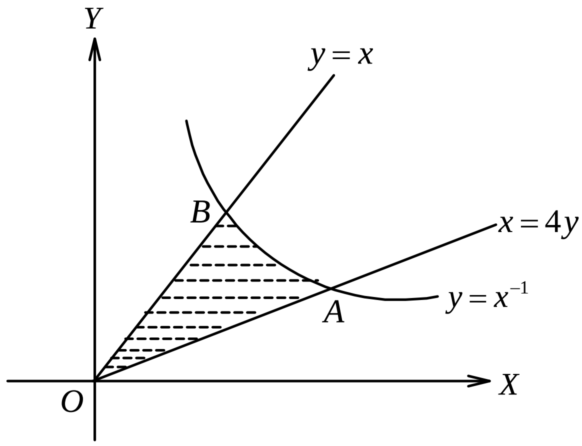

Fig: 10.7

The point of intersection of the curves $x = 4$ and $y = x^{-1}$ is A.

$x = 4y \Rightarrow x = 4x^{-1} \Rightarrow x^2 = 4 \Rightarrow x = 2$

$\therefore A = \left(2, \frac{1}{2}\right)$.

The point of intersection of $y = x^{-1}$ and $y = x$ is B.

$y = x, y = x^{-1} \Rightarrow x^{-1} = x \Rightarrow x^2 = 1 \Rightarrow x = 1$.

$\therefore B = (1,1)$.

Along OA, $x = 4y$, $dx = 4dy$ and x varies from 0 to 2.

$$\therefore \frac{1}{2}\int_{OA} x\,dy - y\,dx = \frac{1}{2}\int_{x=0}^{2} x.\frac{dx}{4} - \frac{x}{4}dx = 0 \qquad \text{... (1)}$$

Along AB, $y = x^{-1}$ or $xy = 1 \Rightarrow y\,dx + x\,dy = 0$

$\Rightarrow -y\,dx = x\,dy$ and x varies from 2 to 1.

$$\therefore \frac{1}{2}\int_{AB} x\,dy - y\,dx = \frac{1}{2}\int_{2}^{1} -y\,dx - y\,dx = -\int_{2}^{1} \frac{1}{x}dx$$

$$= -[\log x]_2^1 = -[\log 1 - \log 2] = \log 2 \qquad \text{... (2)}$$

Also BO, $y = x,\ dy = dx$ and x varies from 1 to 0.

$$\therefore \frac{1}{2}\int_{BO} x\,dy - y\,dx = \frac{1}{2}\int_{1}^{0} x\,dx - x\,dx = 0 \qquad \text{... (3)}$$

Adding (1), (2), (3), we get

area of the given region = $A = \frac{1}{2}\int x\,dy - y\,dx$ = 0 + log 2 + 0 = log 2.

Check Your Progress:

3. Using Green's theorem, find the area of the region enclosed by $y^2 = 4x, y = x$.

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4. Find the area bounded by one arc of the cycloid

$x = a(t - \sin t), y = a(1 - \cos t), a > 0$.

____

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10.3.6 Example

Evaluate $\int_{(0,0)}^{(2,1)} \left(10x^4 - 2xy^3\right)dx - 3x^2y^2dy$ along the path $x^4 - 6xy^3 = 4y^2$.

Solution: Comparing the given integral with $\int M\,dx + N\,dy$, we get

$M = 10x^4 - 2xy^3$ and $N = -3x^2y^2$.

So, $\frac{\partial M}{\partial y} = -6xy^2$ and $\frac{\partial N}{\partial x} = -6xy^2$.

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, by 10.3.1, the integral is independent of path.

Also, from theory of differential equations,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow M\,dx + N\,dy$$ is an exact differential.

We note that $\left(10x^4 - 2xy^3\right)dx - 3x^2y^2dy = d\left(2x^5 - x^2y^3\right)$

$$\therefore \int_{(0,0)}^{(2,1)} \left(10x^4 - 2xy^3\right)dx - 3x^2y^2\,dy = \int_{(0,0)}^{(2,1)} d\left(2x^5 - x^2y^3\right)$$

$$= \left[\left(2x^5 - x^2y^3\right)\right]_{(0,0)}^{(2,1)} = 2\left(2^5\right) - \left(2^2\right)\left(1^3\right) = 64 - 4 = 60.$$

10.3.7 Note

In this example, we observe that $\nabla \times \vec{F} = \vec{0}, \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence the integral is independent of the path. But the value of the integral is non-zero as the path is not a simple closed curve.

10.3.7 Example

Evaluate by Green's theorem $\int_C (\cos x \sin y - xy)\,dx + \sin x \cos y\,dy$, where C is the circle $x^2 + y^2 = 1$.

Solution:

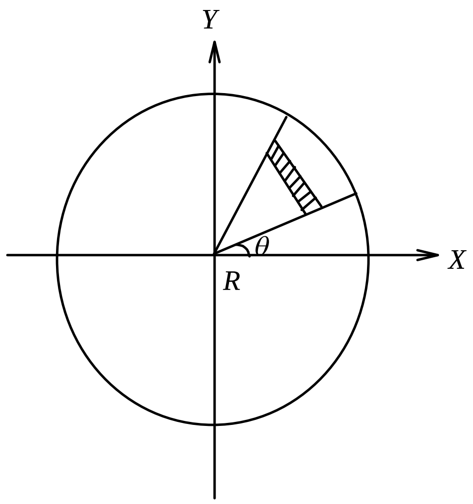


Fig: 10.8

Let R be the region bounded by the circle $C: x^2+y^2=1$ (Fig. 10.8)

By Greens theorem, $\int_C (\cos x \sin y - xy)dx + \sin x \cos y \, dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(\sin x \cos y) - \frac{\partial}{\partial y}(\cos x \sin y - xy) \right] dx\,dy$$

$$= \iint_R \cos x \cos y - (\cos x \cos y - x)\,dx\,dy$$

$$= \iint_R x\,dx\,dy \,.$$

Using polar coordinates $x = r\cos\theta, y = r\sin\theta$, we have $dx\,dy = r\,dr\,d\theta$.

$$\therefore \iint_R x\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cos\theta . r\,dr\,d\theta = \int_{r=0}^{1} r^2 .[\sin\theta]_{\theta=0}^{2\pi}\,dr$$

$$= (\sin 2\pi - \sin\theta)\left[\frac{r^3}{3}\right]_0^1$$

$$= 0\left(\frac{1}{3} - 0\right) = 0 \,.$$

10.3.8 Example

Apply Green's theorem in plane to evaluate, $\int_C (y - \sin x)dx + \cos x \, dy$, where C is the triangle enclosed by the lines $y = 0, x = \frac{\pi}{2}, \pi y = 2x$.

Solution: C is the triangle OAB

where O=(0, 0), $A = \left(\frac{\pi}{2}, 0\right)$ and $B = \left(\frac{\pi}{2}, 1\right)$. (Fig 10.9).

Let R be the region bounded by C.

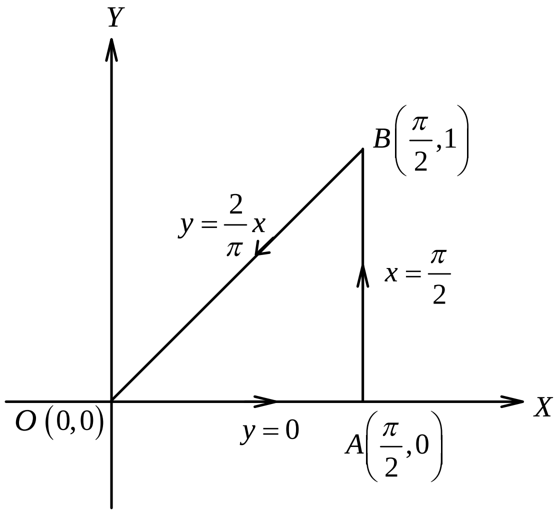


Fig: 10.9

Using Green's theorem,

$$\int_C \left[\left(y-\sin x\right)dx+\cos x\,dy\right]=\iint_R\left[\frac{\partial}{\partial x}\left(\cos x\right)-\frac{\partial}{\partial y}\left(y-\sin x\right)\right]dx\,dy$$

$$=\int_{x=0}^{\pi/2}\int_{y=0}^{2x/\pi}\left(-\sin x-1\right)dx\,dy$$

$$=\int_{x=0}^{\frac{\pi}{2}}\left(-\sin x-1\right)\left[y\right]_{y=0}^{\frac{2x}{\pi}}dx$$

$$=\int_{x=0}^{\frac{\pi}{2}}\frac{2x}{\pi}\left(-\sin x-1\right)dx$$

$$=-\frac{2}{\pi}\int_{x=0}^{\pi/2}x\sin x\,dx-\frac{2}{\pi}\int_{x=0}^{\pi/2}x\,dx$$

$$=-\frac{2}{\pi}\left[x\left(-\cos x\right)\right]_{x=0}^{\pi/2}-\int_{x=0}^{\pi/2}\left(-\cos x\right)dx-\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi/2}$$

$$=-\frac{2}{\pi}\left[\left(-\frac{\pi}{2}\cos\frac{\pi}{2}-0\right)+\left(\sin x\right)_0^{\pi/2}\right]-\frac{1}{\pi}\left(\frac{\pi^2}{4}\right)$$

$$= -\frac{2}{\pi}\left[\sin\frac{\pi}{2} - \sin 0\right] - \frac{\pi}{4} = \frac{-2}{\pi} - \frac{\pi}{4} = -\left(\frac{2}{\pi} + \frac{\pi}{4}\right).$$

10.3.9 Example

Verify Green's theorem in plane for $\int_C \left(x^2 - 2xy\right)dx + \left(x^2 y + 3\right)dy$ where C is the boundary of the region defined by $y = x$ and $y = x^2$.

Solution: Let R be the region bounded by the straight line $y = x$ and the parabola $y = x^2$ (Fig. 10.10)

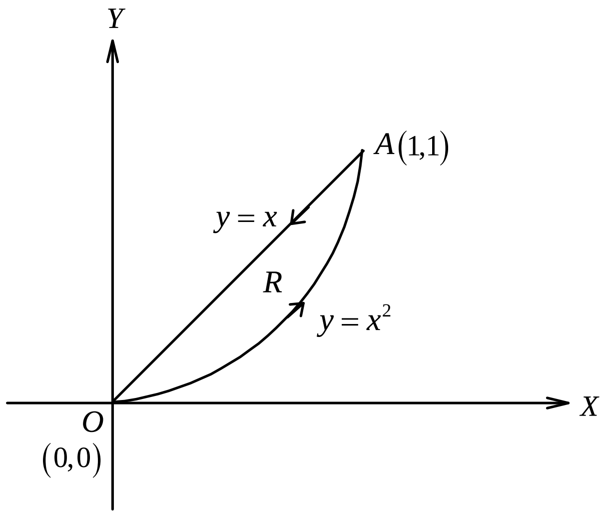


Fig: 10.10

Along OA, $y = x^2$, $dy = 2x\,dx$ and x varies from 0 to 1.

$$\therefore \int_{OA} \left(x^2 - 2xy\right)dx + \left(x^2 y + 3\right)dy = \int_{x=0}^{1} \left(x^2 - 2x.x^2\right)dx + \left(x^2.x^2 + 3\right)2x\,dx$$

$$= \int_{x=0}^{1} \left(x^2 - 2x^3 + 2x^5 + 6x\right)dx = \left[\frac{x^3}{3} - 2\frac{x^4}{4} + \frac{2x^6}{6} + \frac{6x^2}{2}\right]_{x=0}^{1}$$

$$= \left(\frac{1}{3} - \frac{2}{4} + \frac{2}{6} + \frac{6}{2}\right) - 0 = \frac{1}{3} - \frac{1}{2} + \frac{1}{3} + 3 = \frac{19}{6} \qquad \ldots (1)$$

Along AO, $y = x$, $dy = dx$, x varies from 1 to 0.

$$\therefore \int_{AO} \left(x^2-2xy\right)dx+\left(x^2y+3\right)dy = \int_{x=1}^{0} \left(x^2-2x^2\right)dx+\left(x^3+3\right)dx$$

$$= \int_{x=1}^{0} \left(-x^2+x^3+3\right)dx = \left[-\frac{x^3}{3}+\frac{x^4}{4}+3x\right]_1^0$$

$$= 0-\left(-\frac{1}{3}+\frac{1}{4}+3\right) = -\frac{35}{12} \qquad \text{... (2)}$$

Adding (1) and (2),

$$\int_C \left(x^2-2xy\right)dx+\left(x^2y+3\right)dy = \frac{19}{6}-\frac{35}{12}=\frac{1}{4} \qquad \text{... (3)}$$

Now, $\iint_R \left[\frac{\partial}{\partial x}\left(x^2y+3\right)-\frac{\partial}{\partial y}\left(x^2-2xy\right)\right]dx\,dy = \iint_R \left(2xy+2x\right)dx\,dy$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} \left(2xy+2x\right)dx\,dy = \int_{x=0}^{1}\left[2x\frac{y^2}{2}+2xy\right]_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[\left(x.x^2+2x.x\right)-\left(x.x^4+2x.x^2\right)\right]dx$$

$$= \int_{x=0}^{1} \left(2x^2-x^3-x^5\right)dx = \left[\frac{2x^3}{3}-\frac{x^6}{6}-\frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3}-\frac{1}{6}-\frac{1}{4} = \frac{8-2-3}{12} = \frac{1}{4} \qquad \text{... (4)}$$

From (3) and (4),

$$\int_C \left(x^2-2xy\right)dx+\left(x^2y+3\right)dy = \iint_R \left[\frac{\partial}{\partial x}\left(x^2y+3\right)-\frac{\partial}{\partial y}\left(x^2-2xy\right)\right]dx\,dy.$$

Hence Green’s theorem is verified.

10.3.10 Example

Verify Green’s theorem in plane for $\int_C \left(x^2+y^2\right)dx-2xy\,dy$, where C is the rectangle bounded by $x=0, x=a,\ y=0,\ y=b$.

Solution: Let R be the rectangular region bounded by C. (Fig. 10.11)

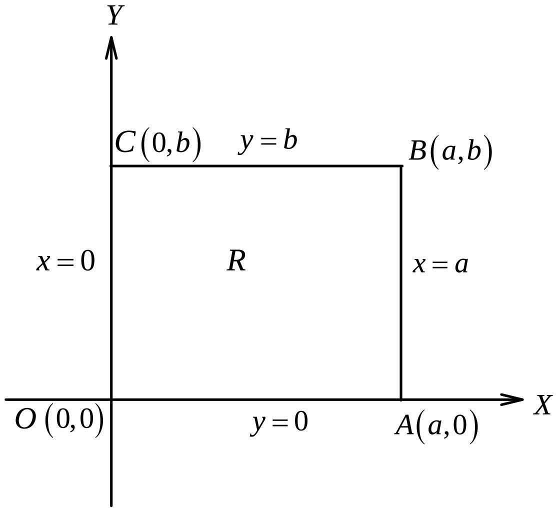


Fig: 10.11

Along OA, $y = 0,\ dy = 0$.

$$\int_{OA} \left(x^2 + y^2\right) dx - 2xy\, dy = \int_{x=0}^{a} x^2\, dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad \text{... (1)}$$

Along AB, $x = a,\ dx = 0,\ y$ changes from 0 to b.

$$\int_{AB} \left(x^2 + y^2\right) dx - 2xy\, dy = \int_{y=0}^{b} -2ay\, dy$$

$$= \left[-\frac{2ay^2}{2}\right]_0^b = -ab^2 \qquad \text{... (2)}$$

Along BC, $y = b,\ dy = 0$, x changes from a to 0.

$$\int_{BC} \left(x^2 + y^2\right) dx - 2xy\, dy = \int_{x=a}^{0} \left(x^2 + b^2\right) dx$$

$$= \left[\frac{x^3}{3} + b^2 x\right]_a^0 = 0 - \left(\frac{a^3}{3} + ab^2\right) = \frac{-a^3}{3} - ab^2 \qquad \text{... (3)}$$

Along CO, $x = 0,\ dx = 0$, y changes from b to 0.

$$\int_{CO} \left(x^2 + y^2\right) dx - 2xy\, dy = 0 \qquad \text{... (4)}$$

Adding (1), (2), (3) and (4), we get

$$\int_{C} \left(x^2 + y^2\right) dx - 2xy\, dy = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$= -ab^2 - ab^2 = -2ab^2 \qquad \text{... (5)}$$

$$\iint_R \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}\left(x^2 + y^2\right)\right] dx\, dy = \int_{x=0}^{a}\int_{y=0}^{b} (-2y - 2y)\, dx\, dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} -4y\, dx\, dy = -4[x]_0^a \left[\frac{y^2}{2}\right]_0^b$$

$$= -4a.\frac{b^2}{2} = -2ab^2 \qquad \text{... (6)}$$

From (5) and (6),

$$\int_{C} \left(x^2 + y^2\right) dx - 2xy\, dy = \iint_R \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}\left(x^2 + y^2\right)\right] dx\, dy$$

Hence Green's theorem is verified.

10.4 SUMMARY

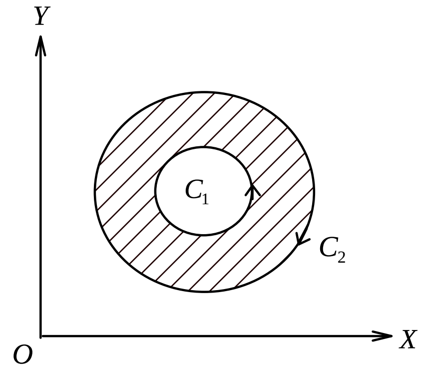
In this unit, you have learnt the Green's theorem in plane which helps in converting a line integral to a surface integral and vice-versa. The theorem is proved and examples are given to have a better understanding. Formula for finding the area of a simple closed curve in plane is derived from Green's theorem. Path independence line integrals is deduced as a consequence of Green's theorem. An important note that $\int_C M\, dx + N\, dy = 0$ around every simple closed curve C in a region R iff $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ everywhere in R is observed.

10.5 CHECK YOUR PROGRESS - MODEL ANSWERS

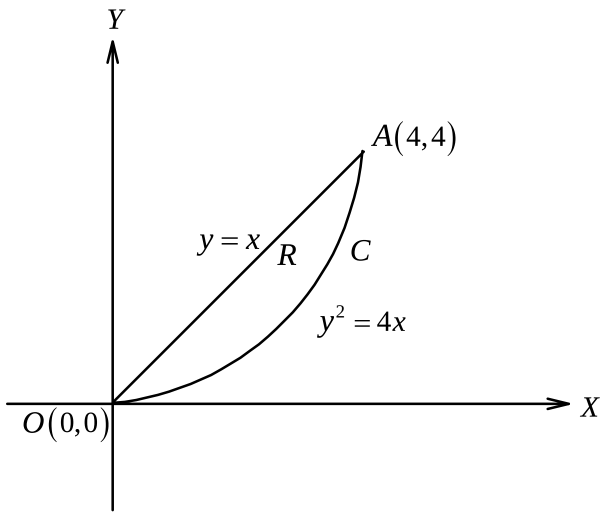
1. (a) A line integral around a simple closed curve C and a surface integral over the region R bounded by C.

   (b) Stoke's theorem.

2. The direction of integration is indicated by arrows in figure.



3.



By using Green's theorem, area enclosed by $C = \frac{1}{2}\int_C (x\,dy - y\,dx)$

Along OA, $y^2 = 4x \Rightarrow 2y\,dy = 4dx \Rightarrow dy = 2\frac{dx}{y} = \frac{2}{2\sqrt{x}}dx$

$\Rightarrow dy = \frac{1}{\sqrt{x}} dx$ and x varies from 0 to 4.

$$\therefore \frac{1}{2}\int_{OA} x\,dy - y\,dx = \frac{1}{2}\int_{x=0}^{4} x.\frac{1}{\sqrt{x}}\,dx - 2\sqrt{x}\,dx \;= \frac{1}{2}\int_{x=0}^{4}\left(\sqrt{x} - 2\sqrt{x}\right)dx$$

$$= \frac{1}{2}\int_{0}^{4} -\sqrt{x}\,dx \;= -\frac{1}{2}\left[\frac{x^{3/2}}{3/2}\right]_0^4 \;= -\frac{1}{3}\left(4^{3/2}\right) = -\frac{8}{3} \qquad \text{... (1)}$$

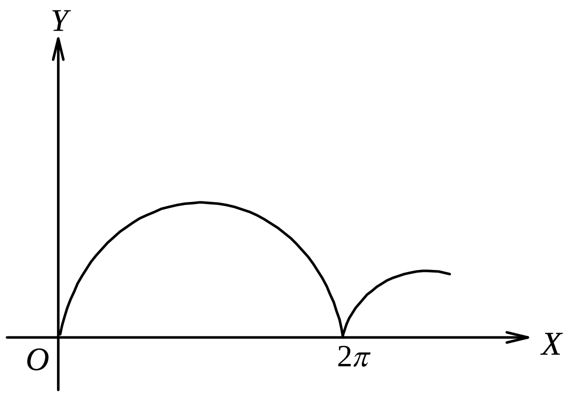
Along AO, $y = x,\ dy = dx$, x varies from 4 to 0.

$$\therefore \frac{1}{2}\int_{AO} x\,dy - y\,dx = \frac{1}{2}\int_{x=0}^{4} x\,dx - x\,dx = 0 \qquad \text{... (2)}$$

Adding (1) and (2)

Area bounded by C = $\frac{1}{2}\int_C x\,dy - y\,dx = \frac{-8}{3} + 0 \;= \frac{8}{3}$.

4. Cycloid is given by the parametric equations $x = a(t - \sin t), y = a(1 - \cos t), a > 0$ where t varies from 0 to 2π in figure.



By Green's theorem area under one arc of the cycloid = $\frac{1}{2}\int_C x\,dy - y\,dx$

$$= \frac{1}{2}\int_{t=0}^{2\pi} a(t - \sin t).a(0 + \sin t)\,dt - a(1 - \cos t)a(1 - \cos t)\,dt$$

$$= \frac{a^2}{2} \int_{t=0}^{2\pi} \left[t \sin t - \sin^2 t - (1 - \cos t)^2 \right] dt$$

$$= \frac{a^2}{2} \int_{t=0}^{2\pi} \left(t \sin t - \sin^2 t - 1 - \cos^2 t + 2 \cos t \right) dt$$

$$= \frac{a^2}{2} \int_{t=0}^{2\pi} \left(t \sin t + 2 \cos t - 2 \right) dt$$

$$= \frac{a^2}{2} \left[t(-\cos t) - (-\sin t) + 2 \sin t - 2t \right]_0^{2\pi}$$

$$= \frac{a^2}{2} \left[(-2\pi \cos 2\pi + \sin 2\pi + 2 \sin 2\pi - 4\pi) - 0 \right]$$

$$= \frac{a^2}{2} [-2\pi - 4\pi] \; = -3a^2 \pi \cdot$$

10.6 MODEL EXAMINATION QUESTIONS

1. State and prove Green's theorem in plane.

2. State Green's theorem using vector notation.

3. Interpret $\int x\,dy - y\,dx$ using Green's theorem.

4. Verify Green's theorem in plane for $\int_C \left(x^2 - 2xy \right) dx + \left(x^2 y + 3 \right) dy$ around the boundary of the region $y^2 = 8x$ and $x = 2$.

5. Verify Green's theorem in plane for $\int_C \left(xy + y^2 \right) dx + x^2 dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

6. Apply Green's theorem in plane to evaluate $\int_C \left(2x^2 - y^2 \right) dx + \left(x^2 + y^2 \right) dy$, where C is the boundary of the curve enclosed by the x-axis and the semi circle $y = \left(1 - x^2\right)^{\frac{1}{2}}$.

7. Evaluate $\int_C (y - \sin x)\,dx + \cos x\,dy$, where C is the triangle whose vertices are $(0,0), \left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 1\right)$.

8. Evaluate $\int_C \left(x^2 + y^2\right)dx + 3xy^2 dy$, where C is the circle $x^2 + y^2 = 4$ in xy - plane.

9. Evaluate $\int_C (3x + 4y)\,dx + (2x - 3y)\,dy$, where C is the circle of radius 2 with center at the origin in xy - plane and traversed in the positive direction.

10. Verify Green's theorem in plane for $\int_C \left(x^2 - xy^2\right)dx + \left(y^2 - 2xy\right)dy$, where C is the square with vertices (0, 0), (2, 0), (2, 2) and (0, 2).

11. Use Green's theorem to find the area of the region bounded by $y = x,\ y = -x,\ y = 4$.

12. Find the area of the region bounded by $x^2 + y^2 = 1$ using Green's theorem.

13. Find the area bounded by $x^{2/3} + y^{2/3} = a^{2/3},\ a > 0$.

(Hint: Use $x = a\cos^3\theta,\ y = a\sin^3\theta$).

14. Show that the integral $\int_{(1,2)}^{(3,4)} \left(xy^2 + y^2\right)dx + \left(x^2 y + 3xy^2\right)dy$ is independent of the path joining the points (1, 2) and (3, 4). Hence, evaluate the integral.

15. Evaluate $\int_{(1,0)}^{(-1,0)} \frac{-y\,dx + x\,dy}{x^2 + y^2}$ along the straight line segment from

(a) (1, 0) to (1, 1), then to (-1, 1) and then to (-1, 0).

(b) (1, 0) to (1, -1), then to (-1, -1) and then to (-1, 0).

Show that $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$. Explain why the integral is not path independent.

Answers

4. $\frac{128}{5}$ (common value)

5. $-\frac{1}{20}$ (common value)

6. $\frac{4}{3}$

7. $-\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$

8. 12π

9. -8π

10. Common value; 0

11. 16 sq. units

12. π sq. units

13. $\frac{3\pi a^2}{8}$ sq. units

14. 254

15. (a) π (b) $-\pi$

UNIT-11: STOKE'S THEOREM AND ITS APPLICATIONS

Contents

11.0 OBJECTIVES

After studying this unit, you will be able to:

- Convert a surface integral to a line integral and vice versa by using Stoke's theorem.
- Evaluate complex surface integrals easily.
- Understand that Green's theorem in plane in a particular case of Stoke's theorem.
- Derive formulae for various forms of surface integrals interms of line integrals.

11.1 INTRODUCTION

One of the very important theorems in vector calculus, Stoke's theorem states that the surface integral of the curl of a function are any surface bounded by a closed path is equal to the line integral of a particular vector function round that path. Stoke's theorem and its generalizations are often used in Physics, especially in electromagnetism. Many theorems in vector calculus can be simplified and generalized by using this theorem.

11.2 STOKE'S THEOREM

In this section we prove Stoke's theorem which gives a relation between surface and line integrals.

11.2.1 Stoke's Theorem

Let $\vec{F}$ be any continuously differentiable vector point function in a region containing a surface S bounded by a simple closed curve C. Then

$$\int_C \vec{F}.d\vec{r} = \iint_S Curl\ \vec{F}.\hat{n}\ dS,$$

where $\hat{n}$ is the unit normal vector at any point of S. The direction of $\hat{n}$ is taken as that in which a right handed screw rotated in the sense of description of C would move.

Proof: Let S be a surface such that its projection on xy - plane is a region bounded by a simple closed curve and the equation of the surface can be written as $z = f(x, y)$ where f is single valued, continuous and differentiable function.

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Then, Stoke's theorem can be written as

$$\int_C (F_1dx + F_2dy + F_3dz) = \iint_S \nabla \times \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right).\hat{n}\,ds \qquad \text{...... (1)}$$

Consider $\iint_S \left(\nabla \times F_1\hat{i}\right).\hat{n}\,dS$

$$\nabla \times F_1\hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix}$$

$$= 0.\hat{i} - \hat{j}\left(\frac{-\partial F_1}{\partial z}\right) + \hat{k}\left(-\frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial F_1}{\partial z}\hat{j} - \frac{\partial F_1}{\partial y}\hat{k}$$

$$\therefore \left(\Delta \times F_1\hat{i}\right).\hat{n}\,dS = \left(\frac{\partial F_1}{\partial z}\hat{j}.\hat{n} - \frac{\partial F_1}{\partial y}\hat{k}.\hat{n}\right)dS \qquad \text{...... (2)}$$

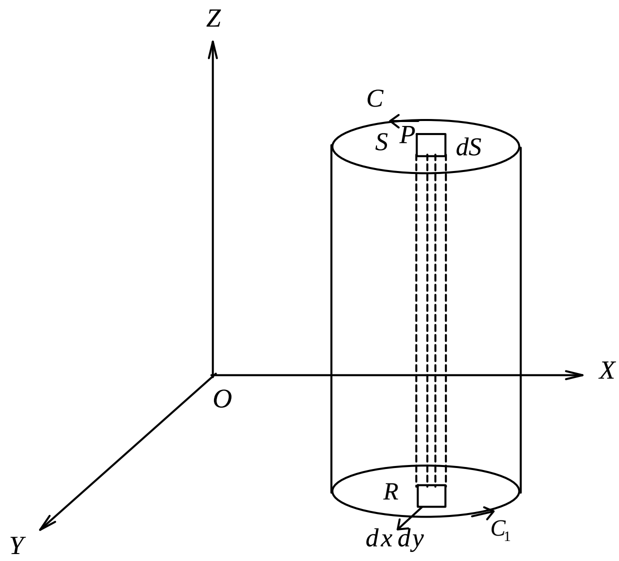


Fig: 11.1

Let *P* be a point on the surface *S* and *dS* be an element of area around *P*.

Let $\hat{n}$ be the unit normal to *dS* at *P*.

Let *R* be the orthogonal projection of *S* in the *xy* - plane.

Let C_1 be the boundary of *R*. Then C_1 becomes the orthogonal projection of *C*.

If $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of $\hat{n}$, we have

$$dx\,dy = \cos\gamma\, dS = \left(\hat{n}.\hat{k}\right) dS \qquad \text{... (3)}$$

Let the equation of *S* be $z = f(x, y)$.

If $\vec{r}$ denotes the position vector of any point on *S*, then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x\hat{i} + y\hat{j} + f(x, y)\hat{k}\,.$$

$$\therefore\ \frac{\partial \vec{r}}{\partial y} = \hat{j} + \frac{\partial z}{\partial y}\hat{k} = \hat{j} + \frac{\partial f}{\partial y}\hat{k}$$

Since $\dfrac{\partial \vec{r}}{\partial y}$ is a tangent vector to *S*, $\dfrac{\partial \vec{r}}{\partial y}$ is perpendicular to $\hat{n}$.

$$\therefore\ \frac{\partial \vec{r}}{\partial y}.\hat{n} = 0 \Rightarrow \hat{n}.\,\hat{j} + \frac{\partial z}{\partial y}\left(\hat{n}.\hat{k}\right) = 0 \Rightarrow \hat{n}.\,\hat{j} = -\frac{\partial z}{\partial y}\left(\hat{n}.\hat{k}\right) \qquad \text{... (4)}$$

From (2) and (4), we get

$$\left(\nabla \times F_1\hat{i}\right).\hat{n}\,dS = \left[\frac{\partial F_1}{\partial z}\left(-\frac{\partial z}{\partial y}\right)\left(\hat{n}.\hat{k}\right) - \frac{\partial F_1}{\partial y}\left(\hat{k}.\hat{n}\right)\right]dS$$

$$= -\left(\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F_1}{\partial y}\right)\left(\hat{k}.\hat{n}\right)dS \qquad \text{... (5)}$$

on *S*, since $z = f\left(x, y\right)$,

$$F_1\left(x, y, z\right) = F_1\left(x, y, f\left(x, y\right)\right) = F\left(x, y\right).$$

$$\therefore \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}.\frac{\partial z}{\partial y} = \frac{\partial F}{\partial y}.$$

$$\therefore \left(\nabla \times F_1\hat{i}\right).\hat{n}\,dS = \frac{\partial F}{\partial y}\left(\hat{k}.\hat{n}\right)dS$$

$$\Rightarrow \iint_S \left(\nabla \times F_1\hat{i}\right).\hat{n}\,dS = \iint_R \frac{\partial F}{\partial y}dx\,dy \qquad \text{...(6)}$$

By Green's theorem in plane,

$$\int_{C_1} F\,dx + 0\,dy = \iint_R \left(0 - \frac{\partial F}{\partial y}\right)dx\,dy \qquad \text{... (7)}$$

From (6) and (7), we get

$$\iint_S \left(\nabla \times F_1\hat{i}\right).\hat{n}\,dS = \int_{C_1} F\,dx$$

Since at each point $\left(x, y\right)$ of C_1, the value of *F* is the same as the value of F_1 at each point $\left(x, y, z\right)$ of *C*, we have

$$\int_{C_1} F\,dx = \int_C F_1 dx$$

$$\therefore \iint_S \left(\nabla \times F_1\hat{i}\right).\hat{n}\,dS = \int_C F_1\,dx \qquad \text{... (8)}$$

Similarly, by taking projection of *S* on *yz, zx* planes respectively, we get

$$\iint_S \left(\nabla \times F_2 \hat{j}\right).\hat{n}\, dS = \int_C F_2\, dy \qquad \text{... (9)}$$

and $$\iint_S \left(\nabla \times F_3 \hat{k}\right).\hat{n}\, dS = \int_C F_3\, dz \qquad \text{... (10)}$$

Adding (8), (9), (10), we get

$$\iint_S \left(\nabla \times F_1 \hat{i} + \nabla \times F_2 \hat{j} + \nabla \times F_3 \hat{k}\right).\hat{n}\, dS = \int_C \left(F_1 dx + F_2 dy + F_3\, dz\right)$$

$$\Rightarrow \iint_S \left(\nabla \times \vec{F}\right).\hat{n}\, dS = \int_C \vec{F}.d\vec{r}$$

$$\Rightarrow \int_C \vec{F}.d\vec{r} = \iint_S curl\, \vec{F}.\hat{n}\ dS$$ as required.

11.2.2 A Special case of Stoke's Theorem

Suppose the surface S in Stoke's theorem lies in the xy - plane so that the unit normal vector $\hat{n}$ lies along z - axis.

$$\therefore \int_C \vec{F}.d\vec{r} = \int_C \left(F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}\right).\left(\hat{i}\frac{dx}{dS} + \hat{j}\frac{dy}{dS} + \hat{k}\frac{dz}{dS}\right) dS$$

$$= \int_C \left(F_1 \frac{dx}{dx} + F_2 \frac{dy}{dS}\right) dS \qquad \left(\because \frac{dz}{dS} = 0\right)$$

$$= \int_C \left(F_1 dx + F_2 dy\right)$$

Now, curl $$\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$\therefore curl\ \vec{F}.\hat{n} = curl\ \vec{F}.\hat{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

$dS = dx\, dy$ is the element of area of S that lies in the xy-plane.

$$\therefore \iint_S curl\, \vec{F}.\hat{n}\, dS = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$

By Stoke's theorem,

$$\int_C \vec{F}.d\vec{r} = \iint_S curl\, \vec{F}.\hat{n}\, dS$$

$$\Rightarrow \int_C \left(F_1 dx + F_2 dy \right) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy \qquad \text{... (11)}$$

Observe that equation (11) is nothing but Green's theorem in plane.

Thus, for surfaces lying in *xy* - plane, Green's theorem and Stoke's theorem are just the same.

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. If C is a closed curve, then show that $\int_C \vec{r}.d\vec{r} = 0$.

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____

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2. If C is a closed curve, then show that $\int_C \phi \nabla \phi . d\vec{r} = 0$.

____

____

____

3. Prove that $\int_C \vec{r} \times d\vec{r} = 2\iint_S d\vec{S}$ by using Stoke's theorem.

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11.2.3 Example

Evaluate $\iint_S (\nabla \times \vec{F}).\hat{n}\, dS$, where $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$ and S is the sphere $x^2 + y^2 + z^2 = a^2$, above the xy - plane.

Solution:

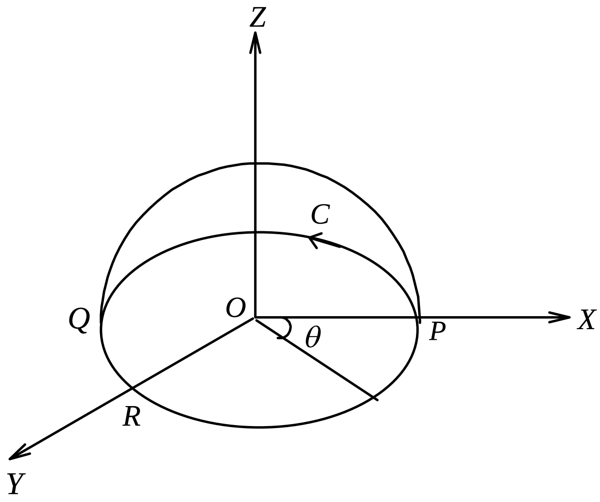

Fig: 11.2

Let C be the boundary of the surface S. Then C is given by $x^2 + y^2 = a^2$, $z = 0$.

The parametric equations of C are:

$x = a\cos\theta,\ y = a\sin\theta,\ z = 0,\ 0 \le \theta \le 2\pi$.

By Stoke's theorem, we have

$$\iint_S curl\, \vec{F}.\hat{n}\, dS = \int_C \vec{F}.d\vec{r}$$

$$= \int_C \left[y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k} \right].\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \right]$$

$$= \int_C y\,dx + (x - 2xz)\,dy - xy\,dz$$

$$= \int_C y\,dx + x\,dy \quad (\because z = 0,\ dz = 0 \ on\ C)$$

$$= \int_{\theta=0}^{2\pi} a\sin\theta(-a\sin\theta\,d\theta) + a\cos\theta(a\cos\theta)\,d\theta$$

$$= a^2 \int_0^{2\pi} \left(\cos^2\theta - \sin^2\theta\right) d\theta \ = a^2 \int_0^{2\pi} \cos 2\theta\, d\theta$$

$$= a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{2\pi} = a^2 (\sin 4\pi - \sin 0) = a^2.0 = 0 \ .$$

11.2.4 Example

If $\vec{F} = (x-z)\hat{i} + \left(x^3 + yz\right)\hat{j} - 3xy^2\,\hat{k}$ and S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the xy - plane, evaluate $\iint_S \left(\nabla \times \vec{F}\right).\hat{n}\,dS$ by converting into line integral.

Solution: Given surface S is : $z = 2 - \sqrt{x^2 + y^2}$.

This meets the xy - plane in a circle C.

The equation of this circle is: $x^2 + y^2 = 4,\ z = 0$.

$\therefore C$ is the boundary of S.

The parametric equation of C are:

$x = 2\cos\theta,\ y = 2\sin\theta, z = 0, 0 \le \theta \le 2\pi$ ... (1)

By Stoke's theorem, we have

$$\iint_S \left(\nabla \times \vec{F}\right).\hat{n}\,dS = \int_C \vec{F}.d\vec{r}$$

$$= \int_C \left[(x-2)\hat{i} + \left(x^3 + yz\right)\hat{j} - 3xy^2\,\hat{k}\right].\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

$$= \int_C (x-z)\,dx + \left(x^3 + yz\right)dy - 3xy^2 dz$$

$$= \int_C x\,dx + x^3 dy \quad (\because z = 0,\ dz = 0 \text{ on } C)$$

$$= \int_{\theta=0}^{2\pi} 2\cos\theta(-2\sin\theta)\,d\theta + (2\cos\theta)^3(2\cos\theta)\,d\theta$$

$$= -2\int_{\theta=0}^{2\pi} \sin 2\theta\,d\theta + 16\int_{\theta=0}^{2\pi} \cos^4\theta\,d\theta$$

$$= -2\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi} + 16.2\int_0^{\pi} \cos^4\theta\,d\theta$$

$$= 0 + 32\int_0^{\pi} \cos^4\theta\,d\theta = 0 + 32.2\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= 64.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = 12\pi\ .$$

11.2.5 Example

Evaluate $\iint_S (\nabla \times \vec{F}).\hat{n}\,dS$ where S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the *xy*- plane and $\vec{F} = (x^2 + y - 4)\hat{i} + 3xy\,\hat{j} + (2xy + z^2)\hat{k}$.

Solution: The given surface S and xy - plane intersect in the circle C with equation $x^2 + y^2 = 4$, $z = 0$. Hence, C is the boundary of S and its parametric equations are

$$x = 2\cos\theta,\ y = 2\sin\theta,\ z = 0, 0 \le \theta \le 2\pi \ \Rightarrow dx = -2\sin\theta\,d\theta,\ dy = 2\cos\theta\,d\theta\ .$$

By Stoke's theorem,

$$\iint_S (\nabla \times \vec{F}).\hat{n}\,dS = \int_C \vec{F}.d\vec{r}$$

$$= \int_C \left[(x^2 + y - 4)\hat{i} + 3xy\,\hat{j} + (2xy + z^2)\hat{k}\right].\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

$$= \int_C (x^2 + y - 4)dx + 3xy\,dy + (2xy + z^2)dz$$

$$= \int_C \left(x^2 + y - 4\right)dx + 3xy\,dy \quad (\because z = 0,\ dz = 0 \text{ on } C)$$

$$= \int_{\theta=0}^{2\pi} \left(4\cos^2\theta + 2\sin\theta - 4\right)\left(-2\sin\theta\,d\theta\right) + 3\left(2\cos\theta\right)\left(2\sin\theta\right)\left(2\cos\theta\,d\theta\right)$$

$$= \int_{\theta=0}^{2\pi} \left(-8\cos^2\theta\sin\theta - 4\sin^2\theta + 8\sin\theta + 24\cos^2\theta\sin\theta\right)d\theta$$

$$= 16\int_{\theta=0}^{2\pi} \cos^2\theta\sin\theta\,d\theta - 4\int_{\theta=0}^{2\pi} \sin^2\theta\,d\theta + 8\int_{\theta=0}^{2\pi} \sin\theta\,d\theta$$

$$= 16\left[\frac{-\cos^3\theta}{3}\right]_0^{2\pi} - 4\int_{\theta=0}^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta + 8\left[-\cos\theta\right]_0^{2\pi}$$

$$= -\frac{16}{3}\left[\cos^3 2\pi - \cos^3 0\right] - 2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} - 8\left[\cos 2\pi - \cos 0\right]$$

$$= -2\left[2\pi - 0\right] = -4\pi \qquad \left(\because \cos 2\pi = \cos 0 = 1,\ \sin 4\pi = 0\right).$$

11.2.6 Example

Evaluate $\iint_S \left(\nabla \times \vec{F}\right).\hat{n}\,dS$, where $\vec{F} = \left(y - z + 2\right)\hat{i} + \left(yz + 4\right)\hat{j} - xz\,\hat{k}$ over the surface of the cube $x = y = z = 0, x = y = z = 4$ above the *xy*- plane.

Solution: The given surface *S* is the cube formed by the planes $x = 0$, $x = 4$, $y = 0$, $y = 4$, $z = 0$, $z = 4$.

S meets *xy*- plane in the square *OABC* (Fig. 11.3)

$\therefore$ The boundary of the square *OABC* is the boudnary of *S*.

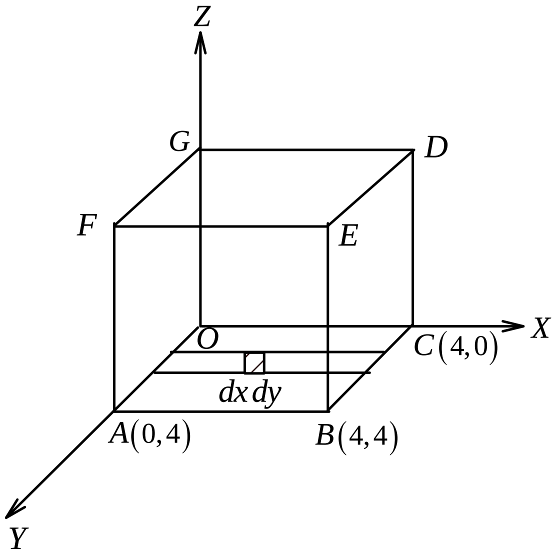

Fig: 11.3

By Stoke's theorem,

$$\iint_S \left(\nabla \times \vec{F}\right).\hat{n}\, dS = \int_C \vec{F}.d\vec{r}$$

$$= \int_C \left(\left(y - z + 2\right)\hat{i} + \left(yz + 4\right)\hat{j} - xz\,\hat{k}\right).\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= \int_C \left(y - z + 2\right)dx + \left(yz + 4\right)dy - xz\,dz$$

$$= \int_C \left(y + 2\right)dx + 4\,dy\ .$$

But C consists of the 4 straight lines OC, BC, BA, AO.
Along OC, $y = 0$, x varies from 0 to 4.

$$\therefore \int_{OC} \left(y + 2\right)dx + 4\,dy = \int_{x=0}^{4} 2\,dx = \left[2x\right]_0^4 = 8. \qquad \text{... (1)}$$

Along CB, $x = 4$, y varies from 0 to 4.

$$\int_{CB} \left(y + 2\right)dx + 4\,dy = \int_{y=0}^{4} 4\,dy = 4\left[y\right]_{y=0}^4 = 16 \qquad \text{... (2)}$$

Along BA, $y = 4$, x varies from 4 to 0.

$$\int_{BA} (y+2)\,dx + 4dy = \int_{x=4}^{0} (4+2)\,dx = 6[x]_4^0 = -24 \qquad \text{... (3)}$$

Along AO, $x = 0$, y varies from 4 to 0.

$$\therefore \int_{AO} (y+2)\,dx + 4\,dy = \int_{y=4}^{0} 4\,dy = 4[y]_4^0 = -16 \qquad \text{... (4)}$$

Adding (1), (2), (3) and (4), we get

$$\int_C (y+2)\,dx + 4\,dy = 8 + 16 - 24 - 16 = -16\,.$$

$$\therefore \iint_S \left(\nabla \times \vec{F}\right).\hat{n}\; dS = -16.$$

11.2.7 Example

Evaluate $\int_C y\,dx + z\,dy + x\,dz$ where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.

Solution: $y\,dx + z\,dy + x\,dz = \left(y\,\hat{i} + z\,\hat{j} + x\hat{k}\right).\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$

$$= \vec{F}.d\vec{r}\text{ , where } \vec{F} = y\,\hat{i} + z\,\hat{j} + x\hat{k}$$

By Stoke's theorem,

$$\int_C y\,dx + z\,dy + x\,dz = \int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\,dS$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \hat{i}\,(0-1) - \hat{j}\,(1-0) + \hat{k}\,(0-1) = -\left(i + \hat{j} + \hat{k}\right).$$

If the surface is projected on xy-plane, $\hat{n} = \hat{k}$.

$$\therefore \int_C y\,dx + z\,dy + x\,dz = \iint_S -\left(\hat{i} + \hat{j} + \hat{k}\right).\hat{k}\,dS$$

$$= -\iint_S dS = -S \text{ (where } S \text{ is the surface area of the sphere)}$$

$$= -4\pi a^2 \cdot$$

11.2.8 Example

Use Stoke's theorem to evaluate $\int_C \left(y\hat{i} + z\hat{j} + x\hat{k}\right).d\vec{r}$ where C is the intersection of $x^2 + y^2 + z^2 = 1$ and $x + y = 0$ traversed in clockwise direction when viewed from (1, 1, 0).

Solution: First note that the sphere has center (0, 0, 0) and radius 1. The plane passes through the centre (0, 0, 0) of the sphere. Hence C is a great circle of the sphere.

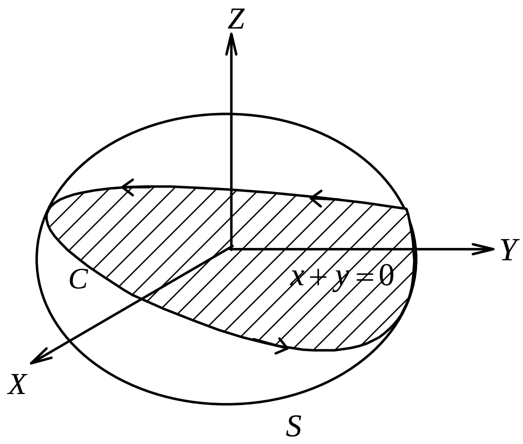

Fig: 11.4

Let $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \hat{i}(0-1) - \hat{j}(1-0) + \hat{k}(0-1) = -\left(i + \hat{j} + \hat{k}\right).$$

Let $\phi = (x + y)$. Then $\nabla\phi = \hat{i} + \hat{j}$ and $|\nabla\phi| = \sqrt{1+1} = \sqrt{2}$.

$\therefore$ A unit vector normal to the surface $x+y=0$ is $\hat{n}=\dfrac{\nabla\phi}{|\nabla\phi|}=\dfrac{\hat{i}+\hat{j}}{\sqrt{2}}$.

Hence, by Stoke's theorem,

$$\int_C \left(y\,\hat{i}+z\,\hat{j}+x\,\hat{k}\right).d\vec{r}=\int_C \vec{F}.d\vec{r} \;=\iint_S \left(\nabla\times\vec{F}\right).\hat{n}\,dS$$

$$=\iint_S -\left(\hat{i}+\hat{j}+\hat{k}\right).\frac{\hat{i}+\hat{j}}{\sqrt{2}}\,dS$$

$$=\iint_S \frac{1+1}{\sqrt{2}}\,dS \;=-\sqrt{2}\iint_S dS$$

$=-\sqrt{2}\left(S\right)$, where S is the area of the great circle with radius 1.

$$=-\sqrt{2}\left(\pi.1^2\right)=-\sqrt{2}\pi \;.$$

11.2.9 Example

Evaluate $\int_C \vec{F}.d\vec{r}$, where $\vec{F}=-y^2\hat{i}+x\,\hat{j}+z^2\,\hat{k}$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$.

Solution: Let $\phi=y+z-2$.

Then $\nabla\phi=\hat{j}+\hat{k}$ and $|\nabla\phi|=\sqrt{1+1}=\sqrt{2}$.

$\therefore$ A unit vector normal to the surface is

$$\hat{n}=\frac{\nabla\phi}{|\nabla\phi|}=\frac{\hat{k}+\hat{j}}{\sqrt{2}}.\;\; \vec{F}=-y^2\hat{i}+x\,\hat{j}+z^2\,\hat{k}\,.$$

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ -y^2 & x & z^2\end{vmatrix}$$

$$=\hat{i}\left(0-0\right)-\hat{j}\left(0-0\right)+\hat{k}\left(1+2y\right)=\left(1+2y\right)\hat{k}$$

$$\nabla\times\vec{F}.\hat{n}=(1+2y)\hat{k}.\frac{\hat{j}+\hat{k}}{\sqrt{2}}=\frac{1+2y}{\sqrt{2}}.$$

Let S be the surface having C as boundary.

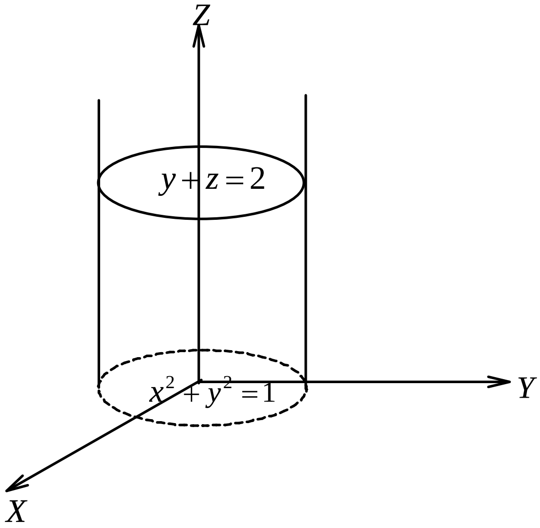

Fig: 11.5

By Stoke's theorem, $\int_C \vec{F}.d\vec{r}=\iint_S\left(\nabla\times\vec{F}\right).\hat{n}\,dS$

$$=\iint_S\frac{1+2y}{\sqrt{2}}dS=\iint_S\frac{1+2y}{\sqrt{2}}\frac{dx\,dy}{1/\sqrt{2}}$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left(1+2r\sin\theta\right)r\,dr\,d\theta$$

(Using $x=r\cos\theta$ and $y=r\sin\theta$)

$$=\int_{r=0}^{1}\left[\theta-2r\cos\theta\right]_0^{2\pi}r\,dr=\int_0^1\left(2\pi-2r\cos2\pi+2r\cos0\right)r\,dr$$

$$=\int_0^1 2\pi\,r\,dr=2\pi\left[\frac{r^2}{2}\right]_0^1=2\pi.\frac{1}{2}=\pi.$$

11.2.10 Example

Verify Stoke's theorem for the function $\vec{F} = (2y+z)\hat{i} + (x-z)\hat{j} + (y-x)\hat{k}$ over the triangle *ABC* cut from the plane $x+y+z=1$ by co-ordinate planes.

Solution: The plane $x+y+z=1$ intersects the co-ordinate axes at *A* (1, 0, 0), *B* (0, 1, 0) and *C*(0, 0, 1) respectively.

S is the triangle *ABC* and *C* is the boundary of *S* traversed in anticlockwise direction. (Fig 11.6).

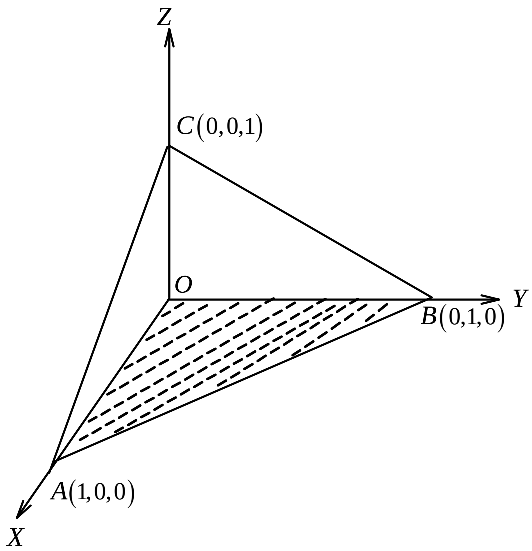

Fig: 11.6

(i) Equation of line *AB* is $y = 1 - x$,

$\therefore dy = -dx,\ z = 0$, *x* varies from 1 to 0.

$$\therefore \int_{AB} \vec{F}.d\vec{r} = \int_{AB} (2y+z)dx + (x-z)dy + (y-x)dz$$

$$= \int_1^0 2(1-x)dx + (x)(-dx) = \int_1^0 (2-3x)dx$$

$$= \left[2x - 3\frac{x^2}{2}\right]_1^0 = 0 - \left(2 - \frac{3}{2}\right) = -\frac{1}{2} \qquad \text{... (1)}$$

Along BC, $y+z=1,\ x=0, dy=-dz$, y varies from 1 to 0.

$$\int_{BC} \vec{F}.d\vec{r} = \int_1^0 -(1-y)dy + y(-dy)$$

$$= \int_1^0 -dy = [-y]_1^0 = 1. \qquad \text{... (2)}$$

Along CA, $x+z=1,\ y=0,\ dx=-dz$ and x varies from 0 to 1.

$$\therefore \int_{CA} \vec{F}.d\vec{r} = \int_0^1 (1-x)dx - x(-dx)$$

$$= \int_0^1 dx = [x]_0^1 = 1 \qquad \text{... (3)}$$

Adding (1), (2) and (3), we get

$$\int_C \vec{F}.d\vec{r} = -\frac{1}{2}+1+1=\frac{3}{2} \qquad \text{... (4)}$$

(ii) Let $\phi = x+y+z-1$.

Then $\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x) + \hat{j}\frac{\partial}{\partial y}(y) + \hat{k}\frac{\partial}{\partial z}(z) \ = \hat{i}+\hat{j}+\hat{k}$.

$|\nabla\phi| = \sqrt{1+1+1} = \sqrt{3}$.

A unit normal to the surface is $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$.

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y+z & x-z & y-x \end{vmatrix}$$

$$= \hat{i}(1+1) - \hat{j}(-1-1) + \hat{k}(1-2) = 2\hat{i}+2\hat{j}-\hat{k}$$

$$\nabla\times\vec{F}.\hat{n} = \left(2\hat{i}+2\hat{j}-\hat{k}\right).\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}} = \frac{2+2-1}{\sqrt{3}} \ = \sqrt{3}.$$

$$\therefore \iint_S \nabla\times\vec{F}.\hat{n}\,dS = \iint_S \sqrt{3}\,dS = \sqrt{3}\int_{x=0}^{1}\int_{y=0}^{1-x}\sqrt{3}\,dx\,dy$$

$$= \sqrt{3}\int_{x=0}^{1}\sqrt{3}\left[y\right]_{y=0}^{1-x}dx = 3\int_{x=0}^{1}(1-x)\,dx$$

$$= 3\left[x-\frac{x^2}{2}\right]_0^1 \quad = 3\left(1-\frac{1}{2}\right) = \frac{3}{2} \qquad \text{... (5)}$$

Since (4) and (5) yield the same reult,

$$\iint_S \nabla\times\vec{F}.\hat{n}\,dS = \int_C \vec{F}.d\vec{r}$$

Hence Stoke's theorem is verified.

11.2.11 Example

Verify Stoke's theorem for the hemisphere $x^2+y^2+z^2=9, z\geq 0$ and $\vec{F}=\left(z^2y-y\right)\hat{i}+\left(x-2yz\right)\hat{j}+\left(2xz-y^2\right)\hat{k}$.

Solution: *S* is the upper half of the sphere $x^2+y^2+z^2=9$, *C* is its boundary, $x^2+y^2+z^2=9$, $z = 0$, which is a circle with centre (0, 0, 0) and radius 3.

Let $x=3\cos\theta,\ y=3\sin\theta$ be the parametric equations of *C* where $0\leq\theta\leq 2\pi$.

$$\int_C \vec{F}.d\vec{r} = \int\left(z^2-y\right)dx+\left(x-2yz\right)dy+\left(2xz-y^2\right)dz$$

$$= \int -y\,dx+x\,dy \qquad (\because z=0,\ dz=0 \text{ on } C)$$

$$= \int_{\theta=0}^{2\pi}\left(-3\sin\theta\left(-3\sin\theta\right)+3\cos\theta\left(3\cos\theta\right)\right)d\theta$$

$$= 9\int_0^{2\pi}\left(\sin^2\theta+\cos^2\theta\right)d\theta = 9\int_0^{2\pi}d\theta$$

$$= 9\left[\theta\right]_0^{2\pi} = 9\left(2\pi-0\right) = 18\pi. \qquad \text{... (1)}$$

The unit vector normal to the surface is $\hat{n} = \hat{k}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 - y & x - 2yz & 2xz - y^2 \end{vmatrix}$$

$$= \hat{i}\left(-2y + 2y\right) - \hat{j}\left(2z - 2z\right) + \hat{k}\left(1+1\right) \; = 2\hat{k}$$

$\nabla \times \vec{F}.\hat{n} = 2\hat{k}.\hat{k} = 2 \cdot$

Let R be the projection of S on xy - plane.

$\therefore dS = dx\,dy$ and x varies from -3 to 3.

$$\therefore \iint_S \nabla \times \vec{F}.\hat{n}\, dS = \iint_S 2\, dS = 2\int_{x=-3}^{3} \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx\, dy$$

$$= 2\int_{-3}^{3} \left[y\right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx = 2\int_{-3}^{3} \left(\sqrt{9-x^2} + \sqrt{9-x^2}\right) dx$$

$$= 4\int_{-3}^{3} \sqrt{9-x^2}\, dx \; = 8\int_{0}^{3} \sqrt{9-x^2}\, dx$$

$$= 8\left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3 = 8 \times \frac{9}{2}\sin^{-1}\left(1\right)$$

$$= 36\frac{\pi}{2} = 18\pi \qquad \text{... (2)}$$

From (1) and (2), we get

$$\iint_S \nabla \times \vec{F}.\hat{n}\, dS = \int_C \vec{F}.d\vec{r}\,.$$

Check Your Progress:

4. Evaluate by Stoke's theorem $\int_C \left(e^x dx + 2y\,dy - dz\right)$ where C is the curve $x^2 + y^2 = 9,\ z = 2$.

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5. Evaluate $\iint_S (\nabla \times \vec{v}).\hat{n}\,dS$ over the surface of the paraboloid $z + x^2 + y^2 = 1,\ z \geq 0$ where $\vec{v} = y\hat{i} + z\,\hat{j} + x\hat{k}$

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11.3 SUMMARY

In this unit, we have proved Stoke's theorem which gives a relation between line and surface integrals. Stoke's theorem when applied to surfaces in plane yields the same results as Green's theorem. This result is established and examples are given to illustrate this. A number of example are given, where in surface integrals are evaluated by using Stoke's theorem and then makes an attempt to reduce the complexity of the problems. Few important results are derived as consequences of Stoke's theorem.

11.4 CHECK YOUR PROGRESS - MODEL ANSWERS

1. $\int_C \vec{r}.d\vec{r} = \iint_S (\nabla \times \vec{r}).\hat{n}\,dS$

$$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = \vec{0}$$

$$\therefore \iint_S (\nabla \times \vec{r}).\hat{n}\,dS = 0\,.$$

Hence $\int_C \vec{r}.d\vec{r} = 0$.

2. Using Stoke's theorem, we have

$$\int_C \phi\nabla\phi.d\vec{r} = \iint_S \nabla\times(\phi\nabla\phi).\hat{n}\,dS$$

$$= \iint_S \left[\nabla\phi\times\nabla\phi + \phi\nabla\times(\nabla\phi)\right].\hat{n}\,dS$$

$$= \iint_S \left[\nabla\phi\times\nabla\phi + \phi(curl\,grad\,\phi)\right].\hat{n}\,dS$$

But $\nabla\phi\times\nabla\phi = \vec{0}$ and curl grade $\phi = \vec{0}$ by vector identities.

$$\therefore \int_C \phi\nabla\phi.d\vec{r} = \iint_S \left[\vec{0} + \phi\vec{0}\right].\hat{n}\,dS = 0.$$

3. Let $\vec{a}$ be an arbitrary constant vector.

Applying Stoke's theorem for $\vec{a}\times\vec{r}$, we have

$$\int_C (\vec{a}\times\vec{r}).d\vec{r} = \iint_S curl(\vec{a}\times\vec{r}).\hat{n}\,dS \qquad \text{... (1)}$$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\therefore \vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \hat{i}(a_2z - a_3y) - \hat{j}(a_1z - a_3x) + \hat{k}(a_1y - a_2x)$$

$$Curl(\vec{a}\times\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2z - a_3y & a_3x - a_1z & a_1y - a_2x \end{vmatrix}$$

$$= \hat{i}[a_1 + a_1] - \hat{j}[-a_2 - a_2] + \hat{k}[a_3 + a_3] = 2(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = 2\vec{a}$$

$$\therefore \int_C \vec{a}\times\vec{r}.d\vec{r} = \iint_S 2\vec{a}.\hat{n}\,dS$$

$$\Rightarrow \int_C \vec{a}.\vec{r}\times d\vec{r} = 2\iint_S \vec{a}.\hat{n}\,dS = 2\vec{a}.\iint_S \hat{n}\,dS$$

$$\Rightarrow \int_C \vec{r}\times d\vec{r} = 2\iint_S d\vec{S} \qquad (\because \vec{a} \text{ is arbitrary})$$

4. By Stoke's theorem

$$\int_C \left(e^x dx + 2y\,dy - dz\right) = \int_C \left(e^x\hat{i} + 2y\,\hat{j} - \hat{k}\right)d\vec{r} = \iint_S \nabla\times\vec{F}.\hat{n}\,dS \qquad .... (1)$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = \hat{i}\left(0-0\right) - \hat{j}\left(0-0\right) + \hat{k}\left(0-0\right) = \vec{0}\,.$$

$$\therefore \iint_S \left(\nabla\times\vec{F}\right).\hat{n}\,dS = 0\,.$$

Hence by equation (1), $\int_C \left(e^x dx + 2y\,dy - dz\right) = 0$.

5. The boundary of the surface S is the circle $x^2 + y^2 = 1,\ z = 0$.

Its pararmetric equations are $x = \cos\theta,\ y = \sin\theta,\ z = 0,\ 0 \le \theta \le 2\pi$.

By Stoke's theorem, $\iint_S \left(\nabla\times\vec{v}\right).\hat{n}\,dS = \int_C \vec{v}.d\vec{r}$

$$= \int_C \left(y\hat{i} + z\,\hat{j} + x\,\hat{k}\right).\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= \int_C \left(y\,dx + z\,dy + x\,dz\right)$$

$$= \int_C y\ dx \qquad (\because z = 0, dz = 0 \text{ on } C)\,.$$

$$= \int_{\theta=0}^{2\pi} \sin\theta(-\sin\theta)\,d\theta = -\int_{\theta=0}^{2\pi} \sin^2\theta\,d\theta = -\int_{\theta=0}^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= -\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta=0}^{2\pi} = -\frac{1}{2}[2\pi - 0] = -\pi\ .$$

11.5 MODEL EXAMINATION QUESTIONS

1. Use Stoke's theorem to evaluate $\iint_S curl\,\vec{F}.\hat{n}\,dS$ over the open hemi-spherical surface $x^2 + y^2 + z^2 = a^2$, $z > 0$ where $\vec{F} = y\,\hat{i} + zx\,\hat{j} + y\,\hat{k}$.

2. Let C be the curve in R^3 given by $x^2 + y^2 = a^2$, $z = 0$ traced in anticlockwise direction and $\vec{F} = x^2y^3\,\hat{i} + \hat{j} + z\,\hat{k}$. Using Stoke's theorem, evaluate $\int_C \vec{F}.d\vec{r}$.

3. Evaluate $\int_C (2x - y)\,dx - yz^2\,dy - y^2z\,dz$, where C is the circle $x^2 + y^2 = 1$, corresponding to the surface of the sphere with unit radius.

4. Let C be the boundary of the triangle with vertices (0, 1, 0), (1, 0, 0) and (2, 1, 0). If $\vec{F} = -y\,\hat{i} + y^2z\,\hat{j} + zx\,\hat{k}$, then use Stoke's theorem to evaluate $\int_C \vec{F}.d\vec{r}$, where C is traversed in the counter clockwise direction.

5. Evaluate the surface integral $\iint_S \nabla \times \vec{F}.\hat{n}\,dS$,

   where $\vec{F} = \left(x^2 + y - 4\right)\hat{i} + 3xy\,\hat{j} + \left(2xz + z^2\right)\hat{k}$, where S is the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy - plane.

6. Compute the surface integral $\iint_S \left(ax^2 + by^2 + cz^2\right)dS$ over the sphere $x^2 + y^2 + z^2 = 1$.

7. Evaluate $\int_C \sin z\,dx - \cos x\,dy + \sin y\,dz$, where C is the boundary of the rectangle $0 \le x \le \pi, 0 \le y \le 1,\ z = 3$.

8. Evaluate $\int_C y\,dx + (1-2z)\,x\,dy - xy\,dz$, where C is the circle $x^2 + y^2 = a^2, z = 0$.

9. Evaluate the line integral $\int_C \vec{F}.d\vec{r}$ by Stoke's theorem, where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices (0, 0, 0), (1, 0, 0) and (0, 1, 0).

10. If $\vec{F} = (2x^2 + yz)\hat{i} + (y^2 + zx)\hat{j} + (3z^2 + xy)\hat{k}$, where S is the surface of the region bounded by $x = 0, y = 0, z = 0$ and $x + y + z = 1$, evaluate $\iint_S \nabla \times \vec{F}.\hat{n}\,dS$.

11. Verify Stoke's theorem for the function $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$, where C is the unit circle in xy - plane bounding the hemisphere $z = \sqrt{1 - x^2 - y^2}$.

12. Verify Stoke's theorem for $\vec{F} = -y^3\hat{i} + x^3\hat{j}$, where S is the circular disk $x^2 + y^2 \le 1,\ z = 0$.

Answers

1. $-\pi a^2$
2.
3. π
4.
5. -16π
6. $\frac{4}{3}(a+b+c)$
7. 2
8. 0
9. 0
10. 0
11. Common value π
12. Common value $\frac{3\pi}{2}$

UNIT-12: GAUSS'S DIVERGENCE THEOREM AND ITS APPLICATIONS

Contents

12.0 OBJECTIVES

After studying this unit, you will be able to:

- Know Gauss divergence theorem and understand its proof.
- Express divergence theorem and understand its proof.
- Apply divergence theorem to transform surface integrals to volume integrals and vice-versa.
- Verify divergence theorem for surfaces enclosing a volume and appreciate the use of divergence theorem in reducing the complexity of the problem.

12.1 INTRODUCTION

In this unit, we prove the Gauss's divergence theorem which states that the normal surface integral of a vector function $\vec{F}$ over the boundary of a closed region is equal to the volume integral of $\text{div}\,\vec{F}$ taken over the closed region. By proving this we relate a surface integral to a volume integral and while solving problems we can switch over from one to another. There are many important deductions from Gauss's theorem. It is useful in deriving the condition for conservation of mass or the continuity equation in Physics.

12.2 GAUSS'S DIVERGENCE THEOREM

Gauss's divergence theorem is one of the most important theorems of vector analysis. It relates a surface integral to a volume integral.

12.2.1 Theorem

If V is the volume bounded by a closed surface S and $\vec{F}$ is a vector point function with continuous derivatives in V, then $\iint\limits_S \vec{F}.\hat{n}\,dS = \iiint\limits_V div\,\vec{F}\,dV$, where $\hat{n}$ is the outward drawn unit normal to S.

Proof: Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$,

where F_1, F_2, F_3 and their derivatives in any direction are finite, uniform and continuous.

By definition of divergence;

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

To prove divergence theorem, we have to prove that

$$\iint\limits_S \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right).\hat{n}\,dS = \iiint\limits_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz \qquad \text{... (1)}$$

We assume that the closed surface is such that any line parallel to the coordinate axes cuts S in atmost 2 points.

For the surface S, let the line parallel to the z - axis cut it at two distinct points (x, y, z_1) and (x, y, z_2) where z_1 and z_2 are both functions of x and y and $z_1 < z_2$. Denote the lower portion of S containing the point (x, y, z_1) as S_1 and that containing the point (x, y, z_2) as S_2, then the closed surface for which $z_1 = z_2$ separates S_1 from S_2 (from fig. 12.1).

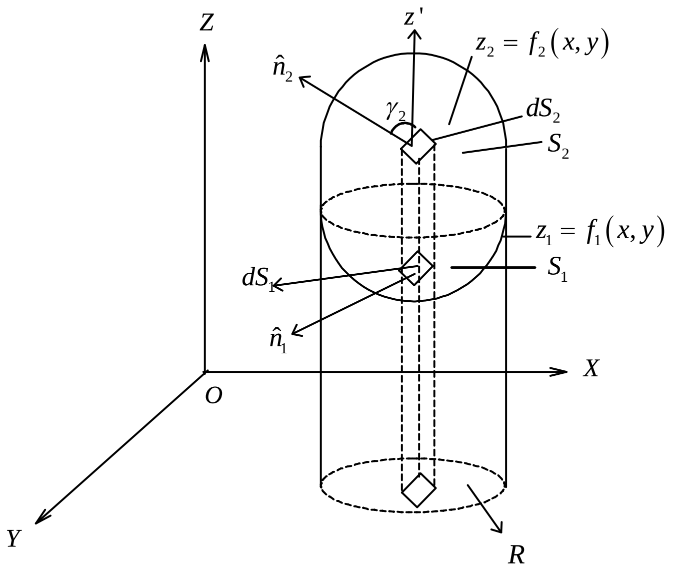

Fig: 12.1

Let the equation of S, S_1 and S_2 be $z = f(x,y)$, $z = f_1(x,y)$, $z = f_2(x,y)$ respectively. Hence we have $z_1 = f_1(x,y)$ and $z_2 = f_2(x,y)$.

Suppose R is the projection of S on the xy - plane. Then

$$\iiint_V \frac{\partial F_3}{\partial z} dx\, dy\, dz = \iint_R \left(\int_{z_1}^{z_2} \frac{\partial F_3}{\partial z} dz \right) dx\, dy$$

$$= \iint_R F_3(x,y,z)\Big|_{z_1 = f_1(x,y)}^{f_2(x,y)} dx\, dy$$

$$= \iint_R F_3(x,y,f_2)\, dx\, dy - F_3(x,y,f_1)\, dx\, dy \qquad \text{... (2)}$$

The outward drawn normal $\hat{n}_2$ to the surface S_2 makes an acute angle γ_2 with $\hat{k}$.

$$\therefore dx\, dy = \cos\gamma_2\, dS_2 = \hat{k}.\hat{n}_2\, dS_2 \qquad \text{... (3)}$$

Again the unit outward drawn normal $\hat{n}_1$ to the surface S_1 makes an acute angle γ_1 with $\hat{k}$.

$$\therefore dx\, dy = -\cos\gamma_1\, dS_1 = \hat{k}.\hat{n}_1 dS_1 \qquad \text{... (4)}$$

Using (3) and (4) in (2), we have

$$\iiint_V \frac{\partial F_3}{\partial z} dx\,dy\,dz = \iint_{S_2} F_3\,\hat{k}.\hat{n}_2\,dS_2 + \iint_{S_1} F_3\,\hat{k}.\hat{n}_1\,dS_1 = \iint_S F_3\,\hat{k}.\hat{n}\,dS \qquad \text{... (5)}$$

Similarly, by projecting S onto yz - plane and zx - plane we obtain

$$\iiint_V \frac{\partial F_1}{\partial x} dx\,dy\,dz = \iint_S F_1\,\hat{i}.\hat{n}\,dS \qquad \text{... (6)}$$

and $$\iiint_V \frac{\partial F_2}{\partial y} dx\,dy\,dz = \iint_S F_2\,\hat{j}.\hat{n}\,dS \qquad \text{... (7)}$$

Adding (5), (6), (7), we get

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz = \iint_S \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k} \right).\hat{n}\,dS$$

which is same as equation (1).

We can remove the constraint and extend the theorem to surfaces which are such that the lines parallel to the coordinate axes meet them in more than two points. For this, we divide the region into subregions whose surfaces satisfy the condition on each of these regions we apply the theorem and add the results. We observe that volume integrals over separate subregions add upto give volume integral over the entire volume v. The surface integral over the common boundaries of two subregions cancel each other and the remaining surface integrals combine to give a surface integral over the entire surface s. Thus the theorem holds in the general case also.

12.2.2 Cartesian form of Gauss's Divergence Theorem

Let $\cos\alpha, \cos\beta, \cos\gamma$ be the direction cosines of the unit normal $\hat{n}$, so that $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$.

Then $\cos\alpha\,dS = dy\,dz,\ \cos\beta\,dS = dz\,dx,\ \cos\gamma\,dS = dx\,dy$.

$$\therefore \iint_S \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k} \right).\hat{n}\,dS = \iint_S \left(F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma \right) dS$$

$$= \iint_S \left(F_1 dy\,dz + F_2 dz\,dx + F_3 dx\,dy \right)$$

Hence, the divergence theorem in Cartesian form can be written as

$$\iint_S F_1\,dy\,dz + F_2 dz\,dx + F_3 dx\,dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz\ .$$

12.2.3 Note

If $d\vec{S}$ denotes a vector whose magnitude is dS and direction is same as that of $\hat{n}$, then $d\vec{S} = \hat{n}\,dS$. Hence the Gauss's theorem may be written as

$$\iint_S \vec{F}.d\vec{S} = \iiint_V div\,\vec{F}\,dV = \iiint_V \nabla.\vec{F}\,dV \text{ as div } \vec{F} = \nabla.\vec{F} \text{ by notation.}$$

12.2.4 Deductions from Gauss's Divergence Theorem

(i) $$\iint_S \hat{n}\times\vec{F}\ dS = \iiint_V \nabla\times\vec{F}\,dV$$

(ii) $$\iint_S \hat{n}\phi\,dS = \iiint_V \nabla\phi\,dV$$

Proof: By divergence theorem we have, $\iint_S \vec{F}.\hat{n}\,dS = \iiint_V div\,\vec{F}\,dV$ ... (1)

(i) Replacing $\vec{F}$ by $\vec{a}\times\vec{F}$ in equation (1), where $\vec{a}$ is an arbitrary constant vector.

$$\iint_S \left(\vec{a}\times\vec{F}\right).\hat{n}\,dS = \iiint_V div\left(\vec{a}\times\vec{F}\right)dV$$

$$\Rightarrow \iint_S \vec{a}.\left(\vec{F}\times\hat{n}\right)dS = \iiint_V \nabla.\left(\vec{a}\times\vec{F}\right)dV = \iiint_V \vec{a}.\left(\vec{F}\times\nabla\right)dV$$

$$\Rightarrow -\vec{a}.\iint_S \hat{n}\times\vec{F}\,dS = -\vec{a}.\iiint_V \left(\nabla\times\vec{F}\right)dV$$

$$\Rightarrow \vec{a}.\left[\iint_S \hat{n}\times\vec{F}\,dS - \iiint_V \left(\nabla\times\vec{F}\right)dV\right] = 0$$

$$\Rightarrow \iint_S \hat{n}\times\vec{F}\,dS - \iiint_V \left(\nabla\times\vec{F}\right)dV = 0 \quad \text{as } \vec{a} \text{ is an arbitrary vector.}$$

Hence $\iint_S \hat{n}\times\vec{F}\,dS = \iiint_V \nabla\times\vec{F}\,dV$.

(ii) Replacing $\vec{F}$ with $\vec{a}\phi$ in (1) for an arbitrary constant vector $\vec{a}$

$$\iint_S \vec{a}\phi.\hat{n}dS = \iiint_V div\left(\vec{a}\,\phi\right)dV \qquad ... (2)$$

div $\left(\vec{a}\phi\right) = \phi \, div\,\vec{a} + \vec{a}.grad\,\phi = \vec{a}.\nabla\phi$ $\left(\because div\,\vec{a} = 0\right)$

$\therefore$ From (2),

$$\iint_S \vec{a}.\hat{n}\phi\,dS = \iiint_V \vec{a}.\nabla\phi\,dV$$

$$\Rightarrow \vec{a}.\iint_S \hat{n}\phi\,dS = \vec{a}.\iiint_V \nabla\phi\,dV$$

$$\Rightarrow \vec{a}.\left(\iint_S \hat{n}\phi\,dS - \iiint_V \nabla\phi\,dV\right) = 0$$

$$\Rightarrow \iint_S \hat{n}\phi\,dS - \iiint_V \nabla\phi\,dV = 0$$ ($\because$ $\vec{a}$ is arbitrary)

$$\Rightarrow \iint_S \hat{n}\phi\,dS - \iiint_V \nabla\phi\,dV.$$

Check Your Progress:

Note: (a) Space is given below for writing your answer.

(b) Compare your answer with the one given at the end of this unit.

1. If $\hat{n}$ is the unit outward drawn normal to any closed surface *S*, then show that $\iiint_V div\,\hat{n}\,dV = S$.

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____

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2. For any closed surface *s*, prove that $\iint_S curl\,\vec{F}.\hat{n}\,dS = 0$.

____

____

____

3. If V is the volume enclosed by a closed surface S and $\vec{F} = 2x\hat{i} + 3y\hat{j} + 4z\hat{k}$, evaluate $\iint_S \vec{F}.\hat{n}\,dS$.

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12.2.5 Example

Evaluate $\iint_S \vec{r}.\hat{n}\,dS$ where S is a closed surface.

Solution: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\text{div } \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

By divergence theorem,

$$\iint_S \vec{r}.\hat{n}\,dS = \iiint_V div\,\vec{r}\,dV = \iiint_V 3\,dV = 3\iiint_V dV = 3V .$$

12.2.6 Example

Evaluate $\iint_S \vec{F}.\hat{n}\,dS$, where $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$ and S is the surface of sphere $x^2 + y^2 + z^2 = 1$.

Solution: div $\vec{F} = \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) = a + b + c$.

By divergence theorem,

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \nabla.\vec{F}\,dV = \iiint_V (a+b+c)\,dV$$

$$= (a+b+c)V = (a+b+c)\frac{4\pi}{3}.$$

[$\because$ V = volume of the given sphere $= \frac{4}{3}\pi r^2 = \frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$.]

12.2.7 Example

Apply divergence theorem to evaluate $\iint_S (x+z)\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 4$.

Solution: By using Cartesian form of divergence theorem, we have

$$\iint_S (x+z)\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+y) \right] dx\,dy\,dz$$

$$= \iiint_V (1+1+0)\,dV = 2V$$

$$= 2 \times \frac{4}{3}\pi r^2 = 2 \times \frac{4}{3}\pi (2)^3 = \frac{64\pi}{3}.$$

12.2.8 Example

By using Gauss's divergence theorem, evaluate $I = \iint_S xz^2\,dy\,dz + \left(x^2 y - z^3\right) dz\,dx + \left(2xy + y^2 z\right) dx\,dy$, where S is the surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$.

Solution:

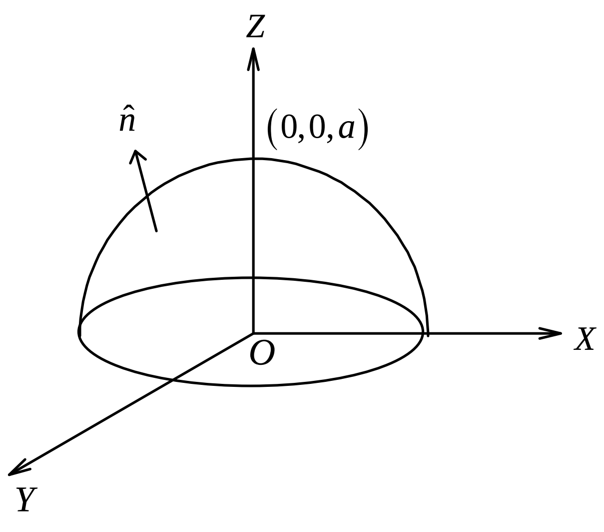

Fig: 12.2

Using the Cartesian form of divergence theorem,

$$I = \iiint_V \left[\frac{\partial}{\partial x}\left(xz^2\right) + \frac{\partial}{\partial y}\left(x^2 y - z^3\right) + \frac{\partial}{\partial z}\left(2xy + y^2 z\right) \right] dx\,dy\,dz$$

$$= \iiint_V \left(z^2 + x^2 + y^2\right) dx\,dy\,dz \qquad \text{... (1)}$$

Taking $x = r\sin\theta\sin\phi,\ y = r\sin\theta\cos\phi, z = r\cos\theta$, we have

$dx\,dy\,dz = r^2 \sin\theta\,dr\,d\theta\,d\phi$.

Also $z^2 + x^2 + y^2 = r^2\cos^2\theta + \left(r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi\right)$

$$= r^2\cos^2\theta + r^2\sin^2\theta\left(\sin^2\phi + \cos^2\phi\right)$$

$$= r^2\cos^2\theta + r^2\sin^2\theta = r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2 .$$

$$\therefore\ I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^4 \sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left(\frac{r^5}{5}\right)_{r=0}^{a} \sin\theta\,d\theta\,d\phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\frac{a^5}{5}\sin\theta\,d\theta\,d\phi$$

$$= \frac{a^5}{5}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\sin\theta\,d\theta\,d\phi = \frac{a^5}{5}\left[-\cos\theta\right]_0^{\pi/2}\left[\phi\right]_0^{2\pi}$$

$$= \frac{a^5}{5}\left[-\cos\frac{\pi}{2} + \cos 0\right]\left[2\pi - 0\right] = \frac{a^5}{5}.2\pi = \frac{2\pi a^5}{5} .$$

12.2.9 Example

Evaluate $\iint_S x^2\left(x\,dy\,dz + y\,dx\,dz + z\,dx\,dy\right)$, where S is the surface of the cylinder $x^2 + y^2 = a^2$ bounded by the planes $z = 0$ and $z = b$.

Solution: By Cartesian form of divergence theorem,

$$\iint_V \left(x^3 dy\,dz + x^2 y dx\,dz + x^2 z\,dx\,dy\right)$$

$$= \iiint_V \left[\frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(x^2 y\right) + \frac{\partial}{\partial z}\left(x^2 z\right) \right] dx\,dy\,dz$$

$$= \iiint_V \left(3x^2 + x^2 + x^2\right) dV = 5 \int_{z=0}^{b} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{x=-a}^{a} x^2\, dx\,dy\,dz$$

$$= 5 \int_{z=0}^{b} \int_{x=-a}^{a} x^2 \left[y\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx\,dz = 5 \int_{z=0}^{b} \int_{x=-a}^{a} 2x^2 \sqrt{a^2-x^2}\, dx\,dz$$

$$= 10 \int_{x=-a}^{a} x^2 \sqrt{a^2-x^2}\, dx \left[z\right]_0^b = 10b \int_{x=-a}^{a} x^2 \sqrt{a^2-x^2}\, dx \qquad \text{... (1)}$$

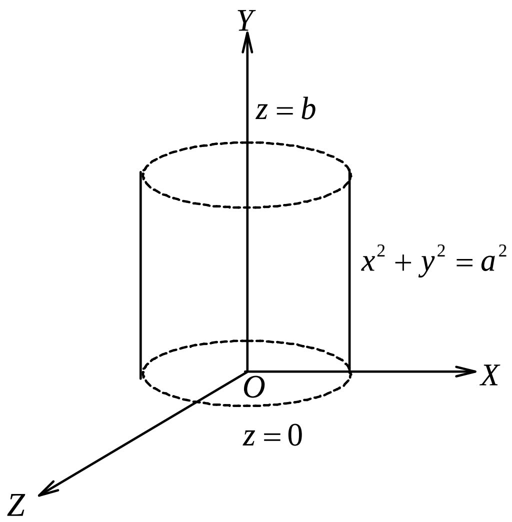

Fig: 12.4

Taking $x = a\sin\theta$,

$$\int_{x=-a}^{a} x^2 \sqrt{a^2-x^2}\, dx = 2\int_0^a x^2 \sqrt{a^2-x^2}\, dx \,.$$

$$= 2 \int_0^{\pi/2} a^2 \sin^2\theta . \sqrt{a^2 - a^2 \sin^2\theta} . a\cos\theta\, d\theta$$

$$= 2a^4 \int_0^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta = 2a^4 \int_0^{\pi/2} \frac{\sin^2 2\theta}{4} d\theta$$

$$=\frac{a^4}{2}\int_0^{\pi/2}\frac{1-\cos 4\theta}{2}d\theta=\frac{a^4}{4}\left[\theta-\frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$=\frac{a^4}{4}\left[\frac{\pi}{2}-0\right]=\frac{a^4\pi}{8}.$$

Substituting (2) in (1), the required surface integral $=10b.\frac{a^4\pi}{8}=\frac{5a^4b\pi}{4}$.

12.2.10 Example

Evaluate $\iint_S \vec{F}.\hat{n}\,dS$ over the surface of the region above the xy - plane bounded by the cone $z^2=x^2+y^2$ and the plane $z=4$ where $\vec{F}=4xz\,\hat{i}+2xyz^2\,\hat{j}+3z\,\hat{k}$.

Solution:

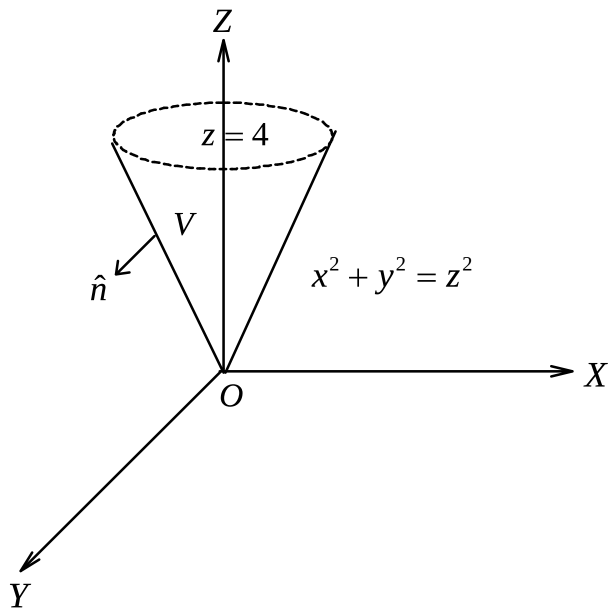

Fig: 12.4

Given $\vec{F}=4xz\,\hat{i}+2xyz^2\,\hat{j}+3z\,\hat{k}$

$$\nabla.\vec{F}=\frac{\partial}{\partial x}(4xz)+\frac{\partial}{\partial y}\left(2xyz^2\right)+\frac{\partial}{\partial z}(3z)=4z+2xz^2+3$$

By using divergence theorem,

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \nabla.\vec{F}\,dV = \iiint_V \left(4z + 2xz^3 + 3\right) dx\,dy\,dz$$

$$= \iint_{x^2+y^2=16} \int_{z=\sqrt{x^2+y^2}}^{4} \left[4z + 2xz^2 + 3\right] dx\,dy\,dz$$

$$= \iint_{x^2+y^2=16} \left[2z^2 + \frac{2xz^3}{3} + 3z\right]_{\sqrt{x^2+y^2}}^{4} dx\,dy$$

$$= \iint_{x^2+y^2=16} \left\{\left(44 + \frac{128}{3}x\right) - \left[2\left(x^2+y^2\right) + \frac{2}{3}x\left(x^2+y^2\right)^{3/2} + 3\sqrt{x^2+y^2}\right]\right\} dx\,dy$$

Let $x = r\cos\theta, y = r\sin\theta$. Then $dx\,dy = r\,dr\,d\theta$.

$$\therefore \int_{r=0}^{4} \int_{\theta=0}^{2\pi} \left[44 + \frac{128}{3} r\cos\theta - 2r^2 - \frac{2}{3} r\cos\theta . r^3 - 3r\right] r\,dr\,d\theta$$

$$= \int_{r=0}^{4} \left[44\theta + \frac{128}{3} r\sin\theta - 2r^2\theta - \frac{2}{3} r\sin\theta . r^3 - 3r\,\theta\right]_{\theta=0}^{2\pi} r\,dr$$

$$= \int_{r=0}^{4} \left[88\pi - \left(2r^2 + 3r\right)2\pi\right] r\,dr \quad = \left[88\pi . \frac{r^2}{2} - 2\pi\left(\frac{2r^4}{4} + \frac{3r^3}{3}\right)\right]_{r=0}^{4}$$

$$= 704\pi - 2\pi\left(128 + 64\right) = 704\pi - 384\pi = 320\pi .$$

12.2.11 Example

If V is the volume of a region bounded by a surface S, then prove that

$$V = \iint_S x\;dy\,dz = \iint_S y\,dz\,dx = \iint_S z\,dx\,dy = \frac{1}{3}\iint_S \left(x\,dy\,dz + y\,dz\,dx + z\,dx\,dy\right).$$

Solution: Using Gauss's divergence theorem,

$$\iint_S x\,dy\,dz = \iiint_V \frac{\partial}{\partial x}\left(x\right) dx\,dy\,dz = \iiint_V dV = V$$

Similarly, $\iint_S y\,dz\,dx = V$ and $\iint_S z\,dx\,dy = V$.

Adding all three, we get

$$\iint_S \left(x\,dy\,dz + y\,dz\,dx + z\,dx\,dy \right) = 3V \,.$$

$$\therefore\ \frac{1}{3}\iint_S \left(x\,dy\,dz + y\,dz\,dx + z\,dx\,dy \right) = V \,.$$

12.2.12 Example

Compute $\iint_S \left(2x^2 + 3y^2 + 5z^2 \right) dS$ over the sphere $x^2 + y^2 + z^2 = 1$.

Solution: By divergence theorem, $\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \nabla.\vec{F}\,dV$

we have to calculate $\iint_S \left(2x^2 + 3y^2 + 5z^2 \right) dS$.

By taking $\vec{F}.\hat{n} = 2x^2 + 3y^2 + 5z^2$, we find $\vec{F}$ and calculate $\nabla.\vec{F}$.

Then divergence theorem helps in finding the surface integral.

Let $\phi = x^2 + y^2 + z^2 - 1$. Then $\nabla\phi = 2x\hat{i} + 2y\hat{k} + 2z\hat{k}$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{4\left(x^2 + y^2 + z^2 \right)} = \sqrt{4} = 2 \,.$$

A unit normal to the surface is : $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{2x\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}}{2} = x\hat{i} + y\,\hat{j} + z\,\hat{k} \,.$$

$$\vec{F}.\hat{n} = 2x^2 + 3y^2 + 5z^2$$

$$\Rightarrow \vec{F}.\left(x\hat{i} + y\,\hat{j} + z\,\hat{k} \right) = 2x^2 + 3y^2 + 5z^2$$

$$\Rightarrow \vec{F} = 2x\hat{i} + 3y\,\hat{j} + 5z\,\hat{k}$$

$$\therefore \nabla.\vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(5z) = 2 + 3 + 5 = 10.$$

$$\iint_S \left(2x^2+3y^2+5z^2\right)dS=\iint_S \vec{F}.\hat{n}\,dS=\iiint_V \nabla.\vec{F}\,dV=\iiint_V 10\,dV$$

$$=10V=10.\frac{4}{3}\pi=\frac{40\pi}{3}\quad\left(\because V=\frac{4}{3}\pi(1)^3\right).$$

12.2.13 Example

Verify divergence theorem for $\vec{F}=\left(x^2-yz\right)\hat{i}+\left(y^2-zx\right)\hat{j}+\left(z^2-xy\right)\hat{k}$ taken over the rectangular parallelopiped $0\le x\le a, 0\le y\le b, 0\le z\le c$.

Solution:

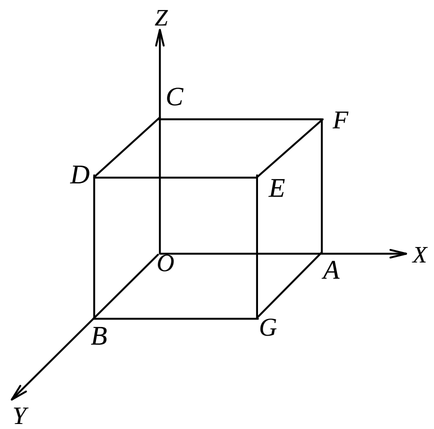

Fig. 12.5

Given $\vec{F}=\left(x^2-yz\right)\hat{i}+\left(y^2-zx\right)\hat{j}+\left(z^2-xy\right)\hat{k}$

$$\nabla.\vec{F}=\frac{\partial}{\partial x}\left(x^2-yz\right)+\frac{\partial}{\partial y}\left(y^2-2x\right)+\frac{\partial}{\partial z}\left(z^2-xy\right)$$

$$=2x+2y+2z=2\left(x+y+z\right).$$

$$\iiint_V \nabla.\vec{F}\,dV=2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}\left(x+y+z\right)dx\,dy\,dz$$

$$=2\int_{y=0}^{b}\int_{z=0}^{c}\left[\frac{x^2}{2}+xy+xz\right]_{x=0}^{a}dy\,dz=2\int_{y=0}^{b}\int_{z=0}^{c}\left(\frac{a^2}{2}+ay+az\right)dy\,dz$$

$$= 2\int_{z=0}^{c}\left[\frac{a^2}{2}y+\frac{ay^2}{2}+az\,y\right]_{y=0}^{b}dz \;= 2\int_{z=0}^{c}\left(\frac{a^2}{2}b+\frac{ab^2}{2}+ab\,z\right)dz$$

$$= 2\left[\frac{a^2b}{2}z+\frac{ab^2}{2}z+ab\frac{z^2}{2}\right]_{z=0}^{c} \;= a^2bc+ab^2c+abc^2$$

$$= abc\left(a+b+c\right) \qquad \text{... (1)}$$

To evaluate the surface integral, we evaluate the integral over the six faces of the parallelopiped and add them up.

For the face $OAFC,\;\; y=0,\, \hat{n}=-\hat{j}$.

$$\therefore \iint_{OAFC}\vec{F}.\hat{n}\,dS = \iint_{OAFC}\vec{F}.-\hat{j}\,dS$$

$$= \int_{x=0}^{a}\int_{z=0}^{c}-\left(y^2-zx\right)dz\,dx \;= \int_{x=0}^{a}\int_{z=0}^{c}zx\,dz\,dx \;\left(\because y=0 \text{ on } OAFC\right)$$

$$= \left[\frac{z^2}{2}\right]_0^c\left[\frac{x^2}{2}\right]_0^a = \frac{c^2a^2}{4} \qquad \text{... (2)}$$

For the face $DEGB,\;\; y=b,\;\; \hat{n}=\hat{j}$.

$$\vec{F}.\hat{n} = y^2 - zx$$

$$\therefore \iint_{DEGB}\vec{F}.\hat{n}\,dS = \int_{x=0}^{a}\int_{z=0}^{c}\left(y^2-zx\right)dz\,dx$$

$$= \int_{x=0}^{a}\int_{z=0}^{c}\left(b^2-zx\right)dz\,dx \quad \left(\because y=b \text{ on } DEGB\right)$$

$$= \int_{x=0}^{a}\left[b^2z-\frac{z^2x}{2}\right]_{z=0}^{c}dx \;= \int_{x=0}^{a}\left[b^2c-\frac{xc^2}{2}\right]dx$$

$$= \left[b^2cx-\frac{x^2c^2}{4}\right]_{x=0}^{a} = ab^2c-\frac{a^2c^2}{4} \qquad \text{... (3)}$$

For the face $OCDB,\;\; x=0, \hat{n}=-\hat{i}$.

$$\vec{F}.\hat{n} = \left(x^2 - yz\right)(-1) = yz - x^2$$

$$\therefore \iint_{OCDB} \vec{F}.\hat{n}\, dS = \int_{y=0}^{b} \int_{z=0}^{c} \left(yz - x^2\right) dy\, dz$$

$$= \int_{y=0}^{b} \int_{z=0}^{c} yz\ dy\, dz \qquad [\because\ x = 0 \text{ on } OCDB]$$

$$= \int_{y=0}^{b} \left[\frac{yz^2}{2}\right]_{z=0}^{c} dy = \int_{y=0}^{b} \frac{yc^2}{2} dy$$

$$= \left[\frac{y^2}{4}c^2\right]_0^b = \frac{b^2c^2}{4} \qquad \ldots (4)$$

For the face *AFEG*, $x = a,\ \hat{n} = \hat{i}$.

$$\vec{F}.\hat{n} = \left(x^2 - yz\right) = a^2 - yz \qquad \left(\because x = a \text{ on } AFEG\right)$$

$$\therefore \iint_{AFEG} \vec{F}.\hat{n}\, dS = \int_{y=0}^{b} \int_{z=0}^{c} \left(a^2 - yz\right) dy\, dz$$

$$= \int_{y=0}^{b} \left[a^2 z - \frac{yz^2}{2}\right]_{z=0}^{c} dy = \int_{y=0}^{b} \left(a^2 c - \frac{c^2 y}{2}\right) dy$$

$$= \left[a^2 cy - \frac{c^2 y^2}{4}\right]_{y=0}^{b} = a^2 bc - \frac{b^2 c^2}{4} \qquad \ldots (5)$$

For the face *OAGB*, $z = 0,\ \hat{n} = -\hat{k}$.

$$\vec{F}.\hat{n} = -\left(z^2 - xy\right) = xy - z^2 = xy \quad \left(\because z = 0 \text{ on } OAGB\right)$$

$$\iint_{OAGB} \vec{F}.\hat{n}\, dS = \int_{x=0}^{a} \int_{y=0}^{b} xy\, dx\, dy$$

$$= \left[\frac{x^2}{2} \right]_{x=0}^{a} \left[\frac{y^2}{2} \right]_{y=0}^{b} = \frac{a^2 b^2}{4} \qquad \text{... (6)}$$

For the face $CFED$, $z = c,\ \hat{n} = \hat{k}$.

$$\vec{F}.\hat{n} = \left(z^2 - xy\right) = c^2 - xy \qquad [\because z = c \text{ on } CFED]$$

$$\iint_{CFED} \vec{F}.\hat{n}\, dS = \int_{x=0}^{a} \int_{y=0}^{b} \left(c^2 - xy\right) dx\, dy$$

$$= \int_{x=0}^{a} \left[c^2 y - \frac{xy^2}{2} \right]_{y=0}^{b} dx = \int_{x=0}^{a} \left[c^2 b - \frac{xb^2}{2} \right] dx$$

$$= \left[c^2 bx - \frac{x^2 b^2}{4} \right]_{x=0}^{a} = abc^2 - \frac{a^2 b^2}{4} \qquad \text{... (7)}$$

Adding equations (2) to (7), we get

$$\iint_S \vec{F}.\hat{n}\, dS = \frac{c^2 a^2}{4} + ab^2 c - \frac{a^2 c^2}{4} + \frac{b^2 c^2}{4} + a^2 bc - \frac{b^2 c^2}{4} + \frac{a^2 b^2}{4} + abc^2 - \frac{a^2 b^2}{4}$$

$$= ab^2 c + a^2 bc + abc^2 = abc\left(b + a + c\right) \qquad \text{... (8)}$$

Comparing equations (1) and (8), we see that

$$\iint_S \vec{F}.\hat{n}\, dS = \iiint_V \nabla.\vec{F}\, dV = abc\left(a + b + c\right).$$

12.2.14 Example

Verify the Gauss divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

Solution: Let S denote the entire surface of the given cylinder and V be the volume enclosed by it.

Let S_1, S_2, S_3 denote the circular disc ($z = 3$), curved surface, the circular disc ($z = 0$) respectively.

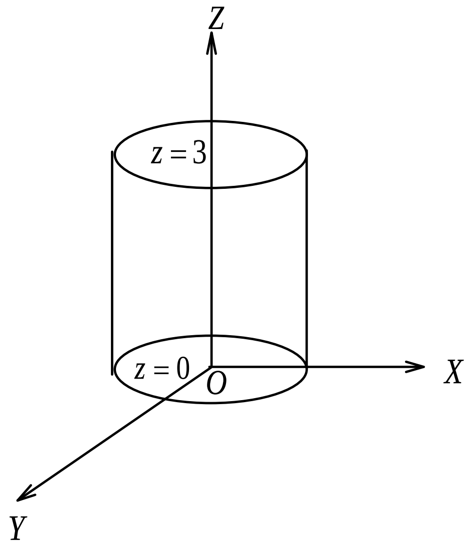

Fig: 12.6

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(ux) + \frac{\partial}{\partial y}\left(-2y^2\right) + \frac{\partial}{\partial z}\left(z^2\right) = 4 - 4y + 2z \,.$$

$$\iiint\limits_V \nabla.\vec{F}\, dV = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3} (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + z^2\right]_{z=0}^{3} dx\, dy$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\, dx\, dy$$

$$= \int_{x=-2}^{2} \left[21y - 12\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{x=-2}^{2} \left\{\left[21\sqrt{4-x^2} - 6\left(4-x^2\right)\right] - \left[-21\sqrt{4-x^2} - 6\left(4-x^2\right)\right]\right\} dx$$

$$= \int_{x=-2}^{2} 42\sqrt{4-x^2}\, dx \quad = 42\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$= 42\left[2\sin^{-1}(1) - 2\sin^{-1}(-1)\right]$$

$$= 84\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 84\pi \qquad \text{... (1)}$$

For the surface S_1, $z = 3$ and unit normal vector $\hat{n} = \hat{k}$.

$$\therefore \iint_{S_1} \vec{F}.\hat{n}\,dS = \iint_{S_1}\left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right).\hat{k}\,dS = \iint_{S_1} z^2 dS = \iint_{s_1} 9\,dS \qquad (\because z = 3)$$

$$= 9S_1 = 9 \times \pi(2)^2 = 36\pi$$

($\because$ S_1 = are of the circular disc with radius 2). ... (2)

For the curved surface S_2, let $\phi = x^2 + y^2 - 4$.

Then $\nabla\phi = 2x\hat{i} + 2y\,\hat{j}$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2} = \sqrt{4.4} = 4.$$

$\therefore$ Unit vector normal to S_2 is: $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{x\hat{i} + y\,\hat{j}}{2}$.

$$\iint_{S_2} \vec{F}.\hat{n}\,dS = \iint_{S_2}\left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right).\left(\frac{x\hat{i} + y\,\hat{j}}{2}\right)dS = \iint_{S_2}\left(2x^2 - y^3\right)dS.$$

For the surfce S_2, the parametric equations are $x = 2\cos\theta,\ y = 2\sin\theta$.

Let $dS = r\,d\theta\,dz = 2d\theta\,dz$ be an element of the curved surface. Then for the entire surface S_2, z varies from 0 to 3 and θ varies from 0 to 2π.

$$\therefore \iint_{S_2} \vec{F}.\hat{n}\,dS = \int_{\theta=0}^{2\pi}\int_{z=0}^{3}\left(8\cos^2\theta - 8\sin^3\theta\right)2\,d\theta\,dz$$

$$= 16\int_{\theta=0}^{2\pi}\left(\cos^2\theta - \sin^3\theta\right)d\theta \int_{z=0}^{3} dz$$

$$= 16\int_{\theta=0}^{2\pi}\left(\cos^2\theta - \sin^3\theta\right)d\theta\,[z]_0^3$$

$$= 48\int_{\theta=0}^{2\pi}\left[\frac{1+\cos 2\theta}{2}-\frac{3\sin\theta-\sin 3\theta}{4}\right]d\theta$$

$$= 48\left[\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)-\frac{1}{4}\left(-3\cos\theta+\frac{\cos 3\theta}{3}\right)\right]_0^{2\pi}$$

$$= 48\left[\left(\pi+\frac{3}{4}-\frac{1}{12}\right)-\left(\frac{3}{4}-\frac{1}{12}\right)\right]= 48\pi \qquad \text{... (3)}$$

For the surface S_3, $z = 0, \hat{n} = -\hat{k}$.

$$\therefore \iint_{S_3}\vec{F}.\hat{n}\,dS = \iint_{S_3}\left(4x\hat{i}-2y^2\hat{j}+z^2\hat{k}\right).\left(-\hat{k}\right)dS$$

$$= \iint_{S_3}-z^2 dS = \iint_{S_3}0\,dS = 0 \ \left(\because z=0\right) \qquad \text{... (4)}$$

Adding (2), (3) and (4), we get

$$\iint_{S}\vec{F}.\hat{n}\,dS = \iint_{S_1}\vec{F}.\hat{n}\,dS+\iint_{S_2}\vec{F}.\hat{n}\,dS+\iint_{S_3}\vec{F}.\hat{n}\,dS$$

$$= 36\pi + 48\pi + 0 = 84\pi \qquad \text{... (5)}$$

From (1) and (5),

$$\iint_{S}\vec{F}.\hat{n}\,dS = \iiint_{V}\nabla.\vec{F}\,dV$$

Hence divergence theorem is verified.

Check Your Progress:

4. By using Gauss divergence theorem, evaluate $\iint_S \left(x\,dy\,dz + y\,dz\,dx + z\,dx\,dy\right)$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 4$.

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5. Find $\iint_S \vec{F}.\hat{n}\,dS$, where $\vec{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + xz\,\hat{k}$ and S is the surface of the parallelopiped formed by $x=0,\ y=0,\ z=0, x=2,\ y=1,\ z=3$ by using divergence theorem.

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12.3 PHYSICAL INTERPRETATION OF DIVERGENCE THEOREM

Suppose a fluid moves so that its velocity at any point (x,y,z) is $\vec{v}(x,y,z)$ (Fig. 12.7).

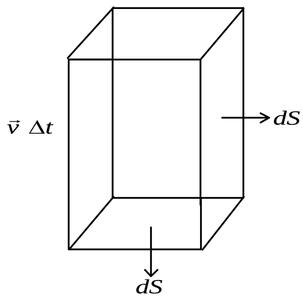

Fig: 12.7

Then volume of fluid crossing dS in Δt seconds = volume contained in the cylinder of base dS and slant height $\vec{v}\,\Delta t = \vec{v}\Delta t.\hat{n}\,dS = \vec{v}.\hat{n}\,dS\,\Delta t$

$\therefore$ Volume of fluid crossing dS in one second = $\vec{v}.\hat{n}\,dS$.

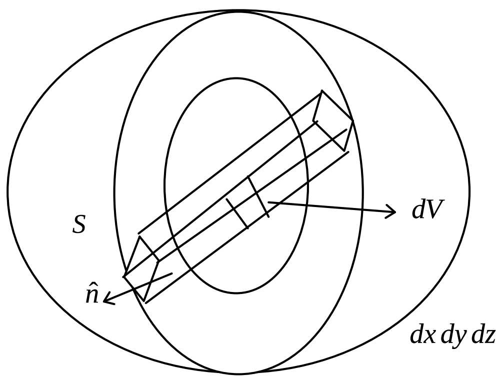

Fig: 12.8

From Fig. 12.8, total volume per second of the fluid emerging from closed surface $S = \iint_S \vec{v}.\hat{n}\,dS$

Now, $\nabla.\hat{n}\,dV$ is the volume per second of the fluid emerging from all volume elements in $S = \iiint_V \nabla.\vec{v}\,dV$

$$\therefore \iint_S \vec{v}.\hat{n}\,dS = \iiint_V \nabla.\vec{v}\,dV.$$

12.4 SUMMARY

In this unit we proved the Gauss's divergence theorem. The Cartesian form of the theorem is discussed and a number of important relations between surface and volume integrals are deduced as consequences of this theorem. Physical interpretation of the theorem is outlined so that interested readers may refer to applications in hydrodynamics and electromagnetic theory. A number of examples are discussed at the end of the unit to demonstrate the ease with which surface integrals can be converted to volume integrals. A few examples are worked out to verify the theorem.

12.5 CHECK YOUR PROGRESS - MODEL ANSWERS

1. By using Gauss's divergence theorem, we have

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \nabla.\vec{F}dV$$

Taking $\vec{F} = \hat{n}$, we get

$$\iint_S \hat{n}.\hat{n}\,dS = \iiint_V \nabla.\vec{f}\,dV$$

$$\Rightarrow \iint_S dS = \iint_V \nabla.\hat{n}\,dV \qquad (\because \hat{n} \text{ is a unit vector } \hat{n}.\hat{n} = 1)$$

$$\Rightarrow S = \iiint_V div\,\hat{n}\;\; dV$$

2. By divergence theorem, we have

$$\iint_S \vec{f}.\hat{n}\,dS = \iiint_V \nabla.\vec{f}\,dV$$

Taking $\vec{f} = curl\,\vec{F}$, we get

$$\iint_S curl\,\vec{F}.\hat{n}\,dS = \iiint_V \nabla.\left(curl\,\vec{F}\right)dV = \iiint_V div\left(curl\,\vec{F}\right)dV.$$

But from vector identities we know that div(curl $\vec{F}$) = 0.

$\therefore$ The integral on the right side is zero.

Hence, $\iint_S curl\ \vec{F}.\hat{n}\ dS = 0$.

3. Given that $\vec{F} = 2x\hat{i} + 3y\,\hat{j} + 4z\,\hat{k}$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{kdz}(4z) = 2 + 3 + 4 = 9.$$

By divergence theorem,

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \nabla.\vec{F}dV = \iiint_V 9\,dV = 9V.$$

Note that this result is applicable to every surface enclosing a volume *V*, on which $\vec{F}$ is defined.

4. By Cartesian form of divergence theorem, we have

$$\iint_S F_1\,dy\,dz + F_2 dz\,dx + F_3 dx\,dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz \quad \text{... (1)}$$

Comparing $\iint x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ with LHS of (1), we get

$F_1 = x,\ F_2 = y, F_3 = z$.

$$\therefore \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

From (1), $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \iiint_V 3\,dV = 3V = 3\frac{4}{3}\pi(2)^3 = 32\pi$.

($\because$ volume of the sphere $x^2 + y^2 + z^2 = 4$ is $\frac{4}{3}\pi(2)^3$)

5. Given $\vec{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + xz\,\hat{k}$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz) = 2y + z^2 + x.$$

By divergence theorem, $\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \nabla.\vec{F}dV$

$$= \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3}\left(2y+z^2+x\right)dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{1}\left[2yz+\frac{z^3}{3}+xz\right]_{z=0}^{3} dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{1}\left(6y+9+3x\right)dx\,dy = \int_{x=0}^{2}\left[\frac{6y^2}{2}+9y+3xy\right]_{y=0}^{1} dx$$

$$= \int_{x=0}^{2}\left(3+9+3x\right)dx = \left[12x+\frac{3x^2}{2}\right]_{x=0}^{2} = 24+6=30.$$

12.6 MODEL EXAMINATION QUESTIONS

1. Use divergence theorem to find $\iint_S \vec{F}.\hat{n}\,dS$ for $\vec{F} = x\hat{i} - y\hat{j} + 2z\hat{k}$ over the sphere $(x-2)^2+(y-3)^2+(z-1)^2=1$.

2. If $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, then evaluate $\iint_S \vec{F}.\hat{n}\,dS$, where S is the surface of the cube enclosed by $x=0, x=1, y=0, y=1, z=0$ and $z = 1$.

3. Use Gauss's theorem and evaluate $\iint_S \vec{F}.\hat{n}\,dS$ when $\vec{F} = xy^2\hat{i} + y^3\hat{j} + y^2z\hat{k}$ and S is the closed surface formed by the cylinder $x^2+y^2=9$ and the plane $z=0$ and $z=2$.

4. Find the value of $\iint_S x^2\,dy\,dz + y^2dx\,dz + 2z\left(xy-x-y\right)dx\,dy$ where S is the surface of the cube $0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1$.

5. Let V be the region enclosed by a surface S and $\hat{n}$ be the unit outward drawn normal to it. Prove that $\iint_S \left(\Delta\times\vec{A}\right).\hat{n}\,dS = 0$. Hence $\vec{A}$ is the function defined on S.

6. Find the value of $\iint_S \left[\left(x^2-yz\right)\hat{i} - 2x^2y\,\hat{j} + 2\hat{k}\right].\hat{n}\,dS$, where S is the surface of the cube formed by $0\le x\le a,\ 0\le y\le a,\ 0\le z\le a$.

7. Prove by using theorem that $\iint_S \left[y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right].\hat{n}\,dS = \frac{\pi}{12}$, where S is the part of the sphere $x^2+y^2+z^2=1$ above the xy - plane and bounded by the plane.

8. Evaluate $\iint_S \left(ax^2+by^2+cz^2\right)dS$ over the unit sphere.

9. Transform $\iint_S \left(x^3 dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy\right)$ into a volume integral and evaluate its value where S is the closed surface consisting of the cylinder $x^2+y^2=a^2$ and the circular discs $z=0$ and $z=b$.

10. Verify divergence theorem for $\vec{F}=x^2\hat{i}+z\,\hat{j}+yz\,\hat{k}$ taken over the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

11. Verify Gauss's divergence theorem for $\vec{F}=x\,\hat{i}+y\,\hat{j}+\left(z^2-1\right)\hat{k}$ over the cylinder $x^2+y^2=4$ bounded by the planes $z=0$ and $z=1$.

Answers:

1. $\dfrac{8\pi}{3}$

2. 3/2

3. $\dfrac{405\pi}{2}$

4. 1/2

5. $\dfrac{1}{3}a^5$

6. $\dfrac{4\pi}{3}(a+b+c)$

9. $\dfrac{5}{4}\pi a^4 b$

10. Common value 3/2

11. Common value 4π

Dr. B. R. Ambedkar Open University

Faculty of Science

III Year (3 Year Degree Programme) Semester - VI

MODEL QUESTION PAPER

MATHEMATICS

DISCIPLINE SPECIFIC ELECTIVE COURSE-D

VECTOR CALCULUS (BS 617 MAT DSE(D)-E)

[Time: 3 hours] **[Max. Marks: 80]**

Section - A

(Short Answer Questions)

[Marks: 4 x 5 = 20]

Note: (a) Answer any **FOUR** of the following questions

(b) Each question carries **5** marks.

1. [Block - I]: Show that the vectors $9\hat{i} + \hat{j} - 6\hat{k}$ and $4\hat{i} - 6\hat{j} + 5\hat{k}$ are orthogonal.

2. [Block - I]: If $\vec{a}, \vec{b}$ are constant vectors and $\vec{r} = e^{nt}\vec{a} + e^{-nt}\vec{b}$, show that $\frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$.

3. [Block - II]: Find the spherical equation for the hyperboloid of two sheets $x^2 - y^2 - z^2 = 1$.

4. [Block - II]: Evaluate $\int_{y=0}^{2}\int_{x=0}^{3} xy\,dx\,dy$.

5. [Block - III]: Evaluate the line integral $\int_C x^2 y\,dx + (x - z)\,dy + xyz\,dz$, where C is the arc of the parabola $y = x^2$ in the plane z = 2 from A (0, 0, 2) to B (1, 1, 2).

6. [Block - III]: Express gradient, curl and divergence in curvilinear coordinates.

7. [Block - IV]: Evaluate $\int_C (x^2 + y^2)\,dx + 3xy^2\,dy$, where C is the circle $x^2 + y^2 = 4$ in xy - plane.

8. [Block - IV]: Compute the surface integral $\iint_S \left(ax^2 + by^2 + cz^2\right) dS$ over the sphere $x^2 + y^2 + z^2 = 1$.

Section - B

[Long Answer Questions]

[Marks: 4 x 10 = 40]

Note: (a) Answer **FOUR** of the following four questions.

(b) Each question carries **10** marks.

9. [Block - I]: If $\vec{F} = t^3\hat{i} - 2t\,\hat{j} + (2t+1)\hat{k}$ and $\vec{G} = (3-2t)\hat{i} + t\,\hat{j} - \hat{k}$, then find $\frac{d}{dt}\left(\vec{F}.\vec{G}\right)$ and $\frac{d}{dt}\left(\vec{F} \times \vec{G}\right)$ when t = -1.

(OR)

10. [Block - I]: If $\vec{r} = x\hat{i} + y\,\hat{j} + z\hat{k}$, and $\vec{a}, \vec{b}$ are constant vectors, then show that

(i) $\nabla\phi(r) = \frac{\phi'(r)}{r}\vec{r}$ (ii) $\nabla\phi(r) \times \vec{r} = \vec{0}$.

11. [Block - II]: Evaluate $\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dx\,dy$ by changing into polar coordinates.

(OR)

12. [Block - II]: Using the double integration determine the area of the region bounded by the curves $y^2 = 4ax,\ x + y = 3a$ and y = 0.

13. [Block - III]: If $\vec{F} = \left(2x^2 - 3z\right)\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}$, evaluate $\iiint_V \left(\nabla \times \vec{F}\right) dV$, where V is the closed region bounded by $x = 0, y = 0, z = 0,\ 2x + 2y + z = 4$.

(OR)

14. [Block - III]: Prove that a spherical coordinate system is orthogonal.

15. [Block - IV]: State Green's theorem using vector notation.

(OR)

16. [Block - IV]: Find the value of $\iint_S x^2\,dy\,dz + y^2 dx\,dz + 2z(xy - x - y)\,dx\,dy$ where S is the surface of the cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$.

Section - C

(Objective Type Questions)

(Marks : 20 x 1 = 20)

I. Multiple Choice Questions. **(10 Marks)**

17. The projection of $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ on $\vec{b} = -6\hat{i} + 2\hat{j} - 3\hat{k}$ is

(a) 12 (b) -1 (c) 4 (d) 13

18. If $\vec{r} = e^{nt}\vec{a} + e^{-nt}\vec{b}$, where $\vec{a}, \vec{b}$ are constant vectors, then $\frac{d^2\vec{r}}{dt^2} =$

(a) $n\vec{r}$ (b) $-n^2\vec{r}$ (c) $n^2\vec{r}$ (d) $\vec{r}$

19. If $f(x, y, z) = x^3 - y^3 + xz^2$, then grad f at the point (1, -1, 2) is

(a) $7\hat{i} - 3\hat{j} + 4\hat{k}$ (b) $\hat{i} + 3\hat{j} + 2\hat{k}$ (c) $7\hat{i} + 3\hat{k} - 4\hat{k}$ (d) $\hat{i} - 3\hat{j} - 2\hat{k}$

20. div (curl $\vec{F}$) =

(a) 1 (b) 0 (c) $\vec{0}$ (d) -1

21. $\int_C \vec{r}.d\vec{r} =$

(a) 3 (b) 2 (c) 1 (d) 0

22. The value of the line integral $\int_C y^2 dx + x^2 dy$, where C is the boundary of the square $-1 \le x \le 1$, $-1 \le y \le 1$ is

(a) 0 (b) 2 (x + y) (c) 4 (d) 4/3

23. The spherical coordinate system is

(a) Orthogonal (b) coplanar (c) non-coplanar (d) not orthogonal

24. The work done by the force $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ in moving a particle from (1, 1, 1) to the point (3, 3, 2) along the path C is

(a) 17 (b) 10 (c) 0 (d) cannot be found

25. A necessary and sufficient condition that the line integral $\int_C \vec{F}.d\vec{r}$ for every closed C vanishes is

(a) curl $\vec{F} = \vec{0}$ (b) div $\vec{F} = 0$ (c) curl $\vec{F} \neq \vec{0}$ (d) div $\vec{F} \neq 0$

26. In the orthogonal curvilinear coordinates, the value of $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ is

(a) $h_1h_2h_3$ (b) $\dfrac{1}{h_1h_2h_3}$ (c) $\dfrac{h_1}{h_2h_3}$ (d) $\dfrac{h_1h_2}{h_3}$

II. Match the following. (5 Marks)

27. $\nabla.\vec{r}$ = [ ] (a) $\left(-1,\sqrt{3},1\right)$

28. $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ [ ] (b) $\left(2\sqrt{2},\frac{\pi}{4},\frac{\pi}{3}\right)$

29. Cartesian equation of $r\sin\theta = r\cos\theta + 4$ is [ ] (c) $y = x + 4$

30. Rectangular coordinates of $\left(2,\frac{2\pi}{3},1\right)$ are [ ] (d) 3

31. Spherical coordinates of $\left(\sqrt{6},\frac{\pi}{4},\sqrt{2}\right)$ are [ ] (e) Orthogonal

(f) Solenoidal

III. Fill in the blanks. (5 Marks)

32. $\dfrac{d}{dt}\left(\vec{F}\times\vec{G}\right) =$ ___________

33. curl (curl $\vec{F}$) = ___________

34. The polar equation of $x^2 - y^2 = 4$ is ___________

35. The cylindrical coordinates of (1, -3, 5) are ___________

36. The polar coordinates of the point (-1, -1) are ___________

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