Evolutionary Game Theory: network dynamics

Continuous assessment activity #5

Lorenzo Venieri - June 2022

Explanation:

Two-players games can be defined by their payoff matrix **W**:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 1 & S \\ T & 0 \end{pmatrix} \end{array}$$

A player can be either a collaborator (C) or a defector (D). The payoff matrix tells what is the payoff of a player playing a certain strategy (C or D) when he plays against another player with a given strategy. If a C plays another C, it's payoff will be 1, if a C plays a D, it's payoff will be S ("suckers payoff"). When a D plays with a C, his payoff is T ("temptation payoff"), but when he plays another D, his payoff is 0.

If we look at different combinations of T and S we can define different games that have different conceptual characteristics. From a game theoretic point of view, it is possible to classify all possible 2x2 symmetric games into 3 categories:

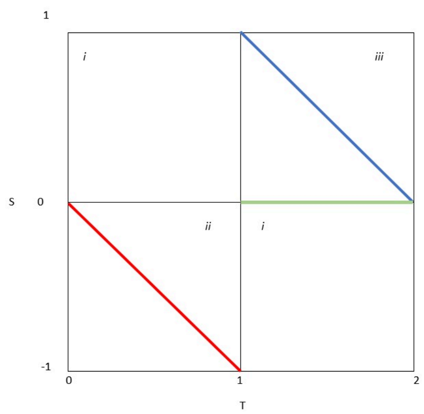
i) T < 1, S > 0 or T > 1, S < 0: their unique Nash equilibrium corresponds to the dominant strategy (C in the first case and D in the second case). Examples: Harmony game and Prisoner's Dilemma game.

ii) T < 0, S < 0: several Nash equilibria (one of them is a mixed strategy, unstable equilibrium of the replicator dynamics and therefore acts as a separator of the basins of attractions of two Nash equilibria in pure strategies, which are the attractors). Example: Stag Hunt

iii) T > 1, S > 0: several Nash equilibria (one of them is a mixed strategy, but in this case this is the global attractor of the replicator dynamics. Example: Snow Drift

This project focuses on three different evolutionary games:

- Weak prisoner's dilemma: focusing on the line 1 < T < 2, S = 0. (green)
- Stag hunt: focusing on the diagonal line between (T,S) = (0,0) and (T,S) = (1,-1). (red)
- Snow drift: focusing on the diagonal line between (T,S) = (1,1) and (T,S) = (2,0). (blue)

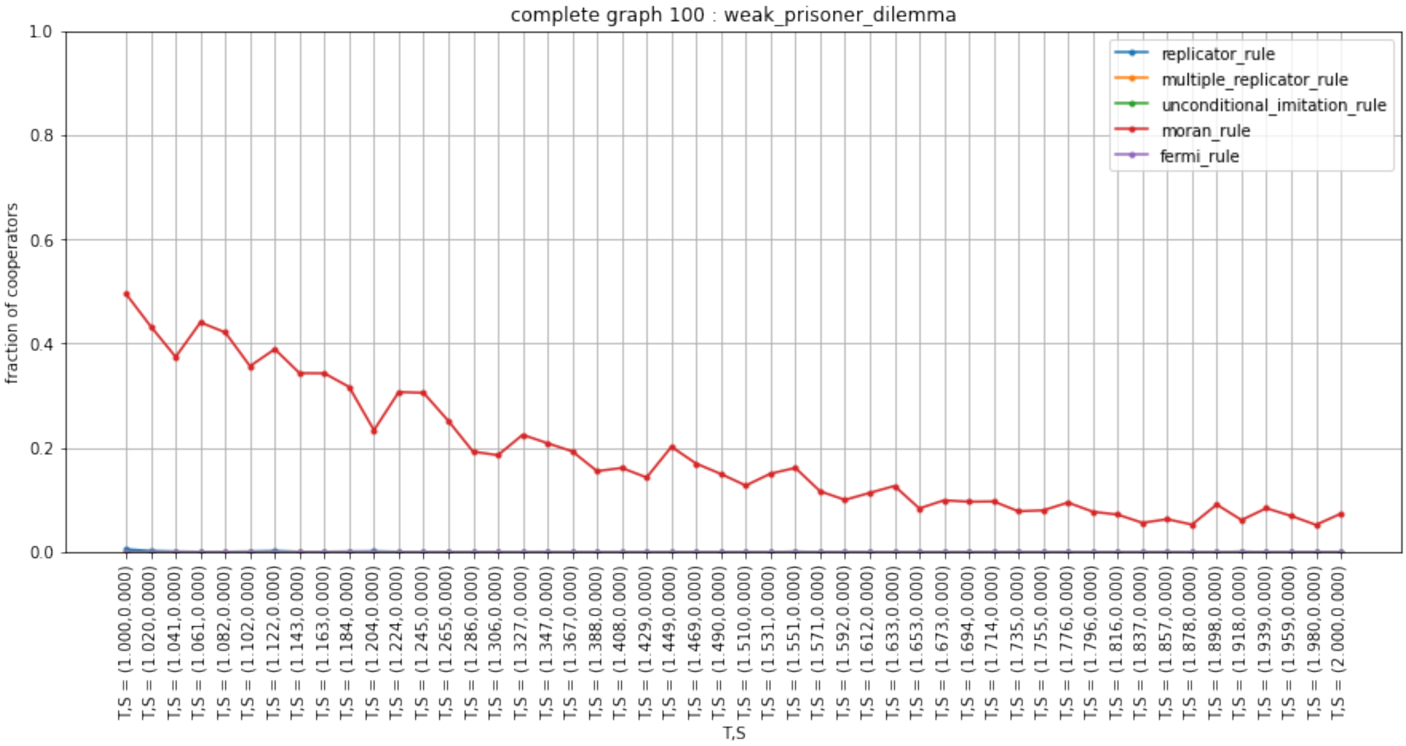


During the simulation were analyzed 50 points over these lines, each Monte Carlo simulation consists of 50 repetitions.
Before doing the full simulations, I checked how many steps were needed to reach a stationary state for the graphs analyzed, for different combinations of S and T. The stationary state was almost always reached before step 400. I decided to pick Ttrans = 400 and Tmax = 500 as, respectively, number of steps in the transitory state and number of steps of the simulation. Unfortunately, I'm not proficient enough in any compiled language and python is quite slow: given more time or more computational power, it would be better to increase Ttrans and Tmax to have a better evaluation of those cases when the stationary state is not reached before the 400th timestep.
All networks analyzed started from a configuration where each node is in C or D with equal probability (0.5 each).

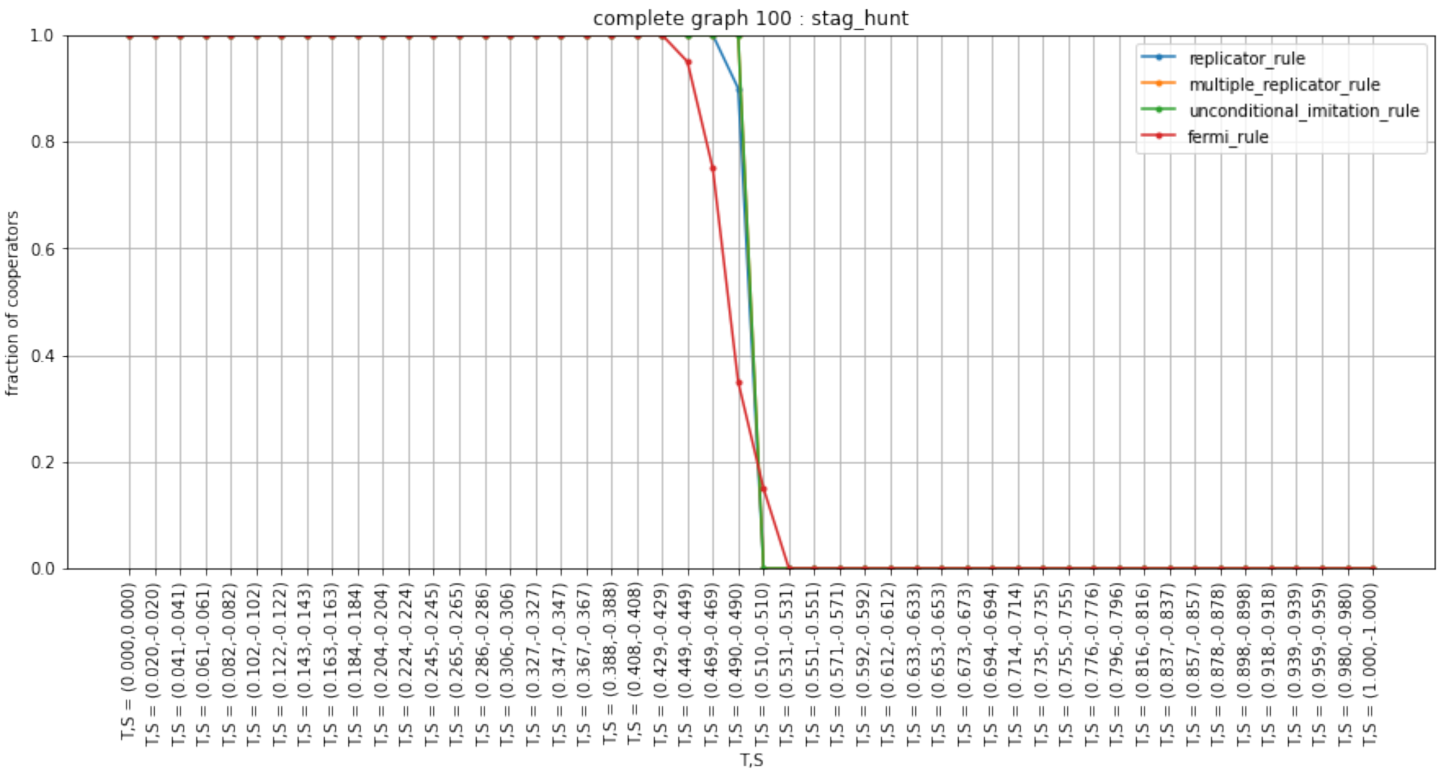
Starting from this configuration, each node plays the game defined by the T,S values analyzed with each one of its neighbors and calculates its total payoff.
After all payoffs are computed, each node may change its strategy comparing its payoffs with the payoffs of its neighbors, following a chosen update rule.
I implemented different update rules: replicator rule, multiple replicator rule, unconditional imitation rule, Moran rule and Fermi rule, their description can be found in [1].

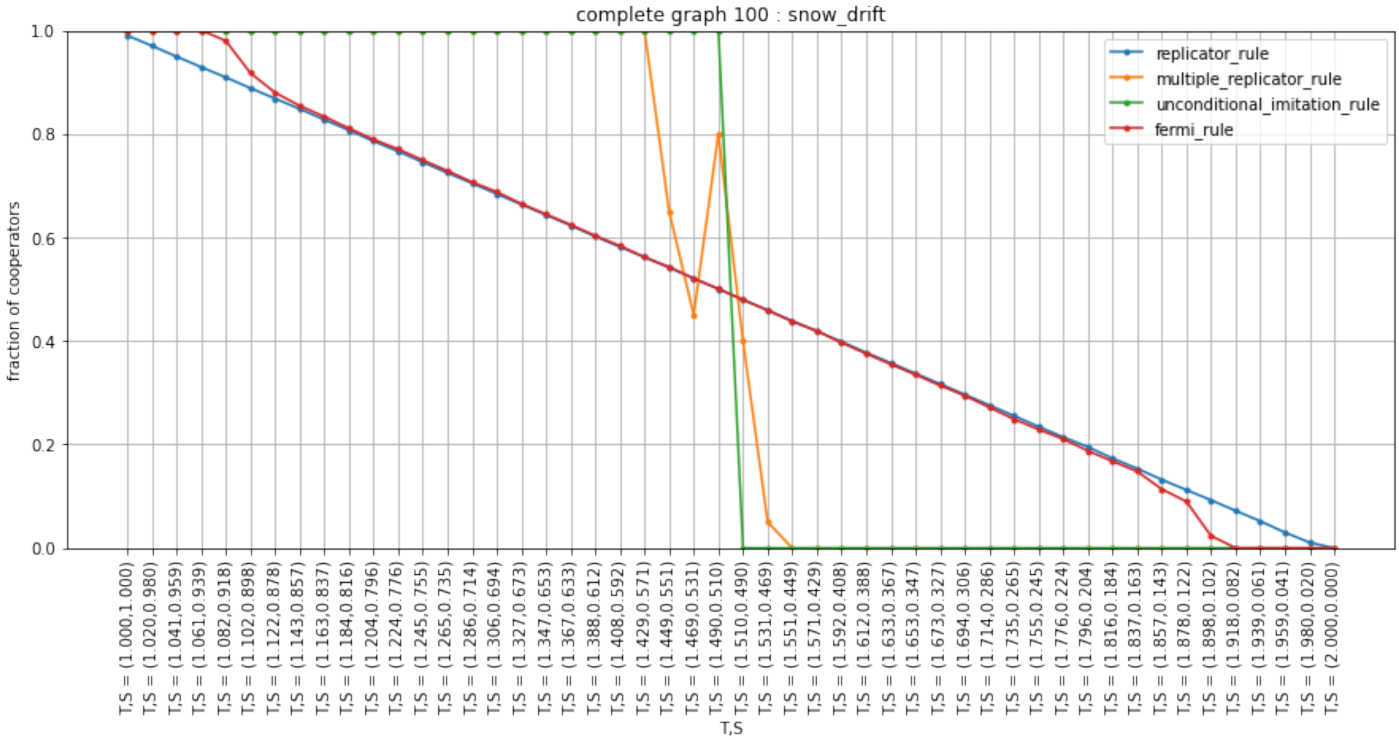
I studied different networks, with limited size (just 100 nodes) to make computing time manageable:
- Complete network
- Homogeneus random graph, with degrees = 4
- Erdos Renyi network, with average degree = 4
- Barabasi Albert network, with K = 4
- Lancichinetti–Fortunato–Radicchi (LFR) community graph

Complete network:



For all the rules except the Moran rule, the equilibrium, as expected, is with a fraction of cooperators close to zero. The different behavior with Moran rule may be explained by the fact that, with this rule, a player can adopt, with low probability, the strategy of a neighbor that has done worse than herself.
Moran rule is very computationally expensive, especially for a complete network, so I didn't simulate the behavior of this network with Moran rule for the others two games.





We can observe that the results for the replicator rule are in agreement with what can be seen in [1], fig.9:
For weak prisoner's dilemma, x* stays very close to 0. For stag hunt game, x* shifts rapidly from 1 to 0 when S ~ -0.5 and T ~ 0.5. For snow drift game x* slowly and steadily decreases from 1 to 0.

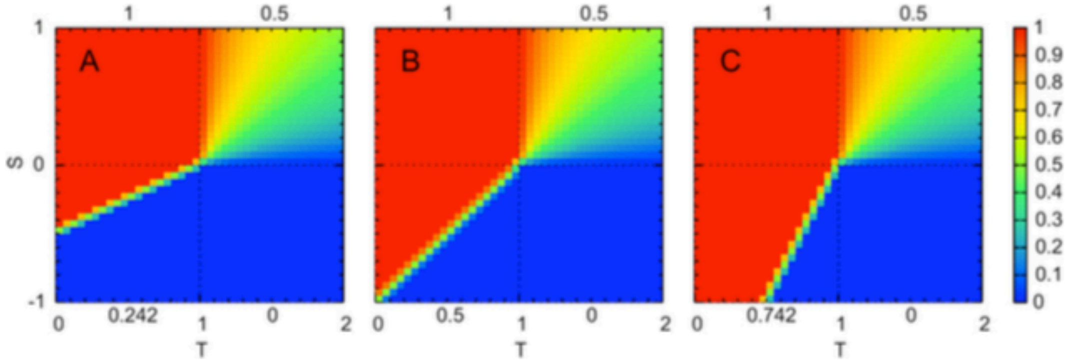


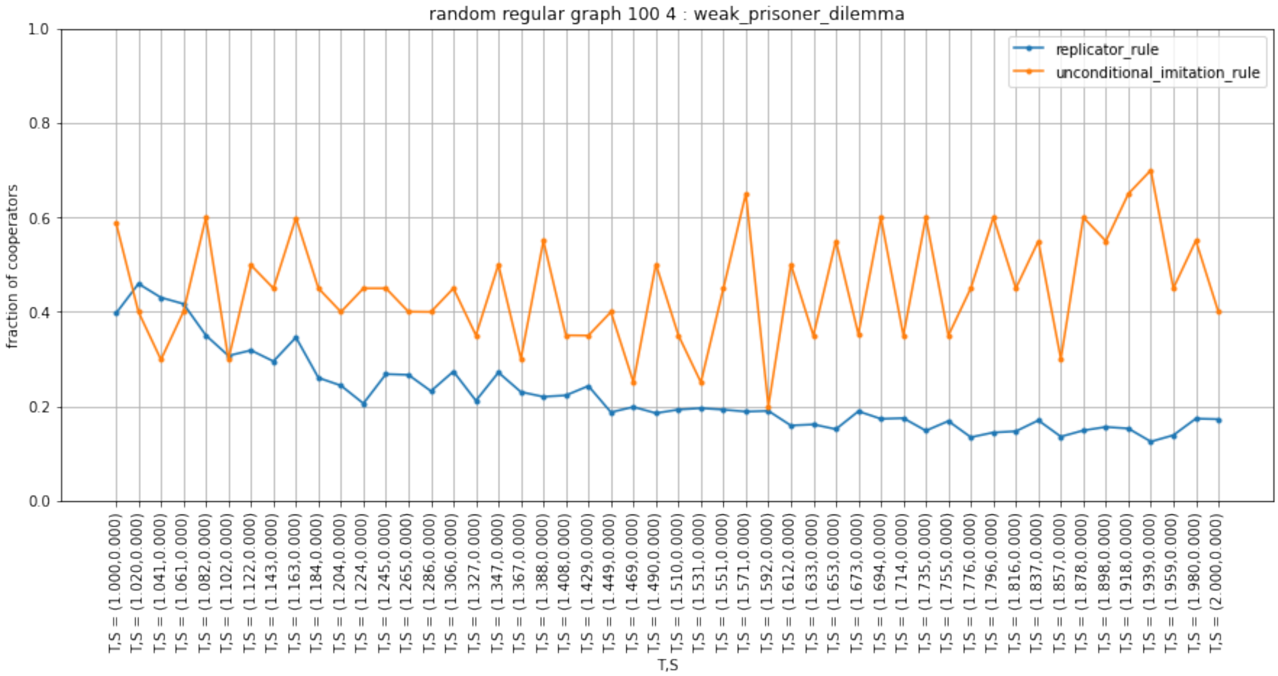
Fig. 9. Asymptotic density of cooperators x^* with the replicator rule on a complete network, when the initial density of cooperators is $x^0 = 1/3$ (left, A), $x^0 = 1/2$ (middle, B) and $x^0 = 2/3$ (right, C). This is the standard outcome for a well-mixed population with replicator dynamics, and thus constitutes the reference to assess the influence of a given population structure (see main text for further details).

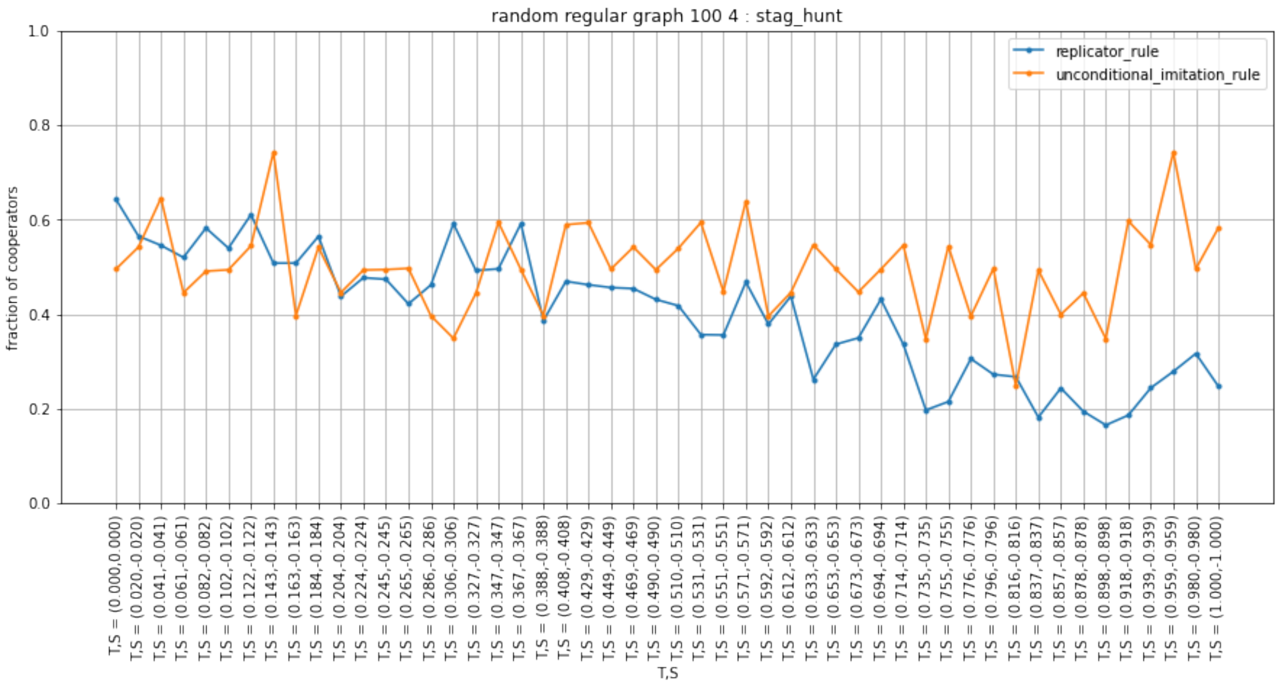
As stated in [1], "for a complete network, i.e. for a well-mixed or unstructured population, the differences between update rules may be not relevant, as far as they do not change in general the evolutionary outcome. These differences, however, become crucial when the population has some structure".
We will see that for other kinds of networks the choice of the update rule has greater consequences.

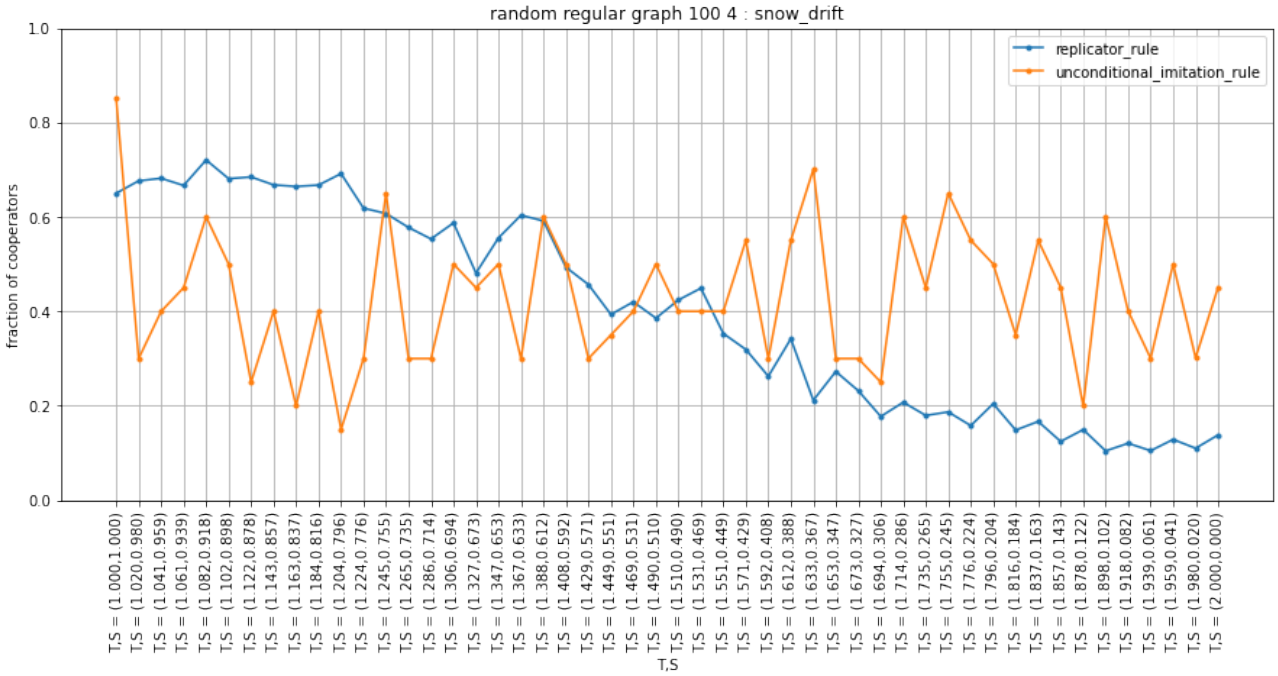
Homogeneous random network, k = 4

For homogeneous random networks the time needed to reach a stationary state is considerably higher than the one for the other models. So we will increase Tmax and Ttrans accordingly: Ttrans = 9000, Tmax = 10000.

The computational cost now is considerably higher so I'll show the plots only for replicator rule and unconditional imitation rule, to compare the results with the next networks with different degree heterogeneity. I also lowered the number of repetitions for the MC simulation to 20.

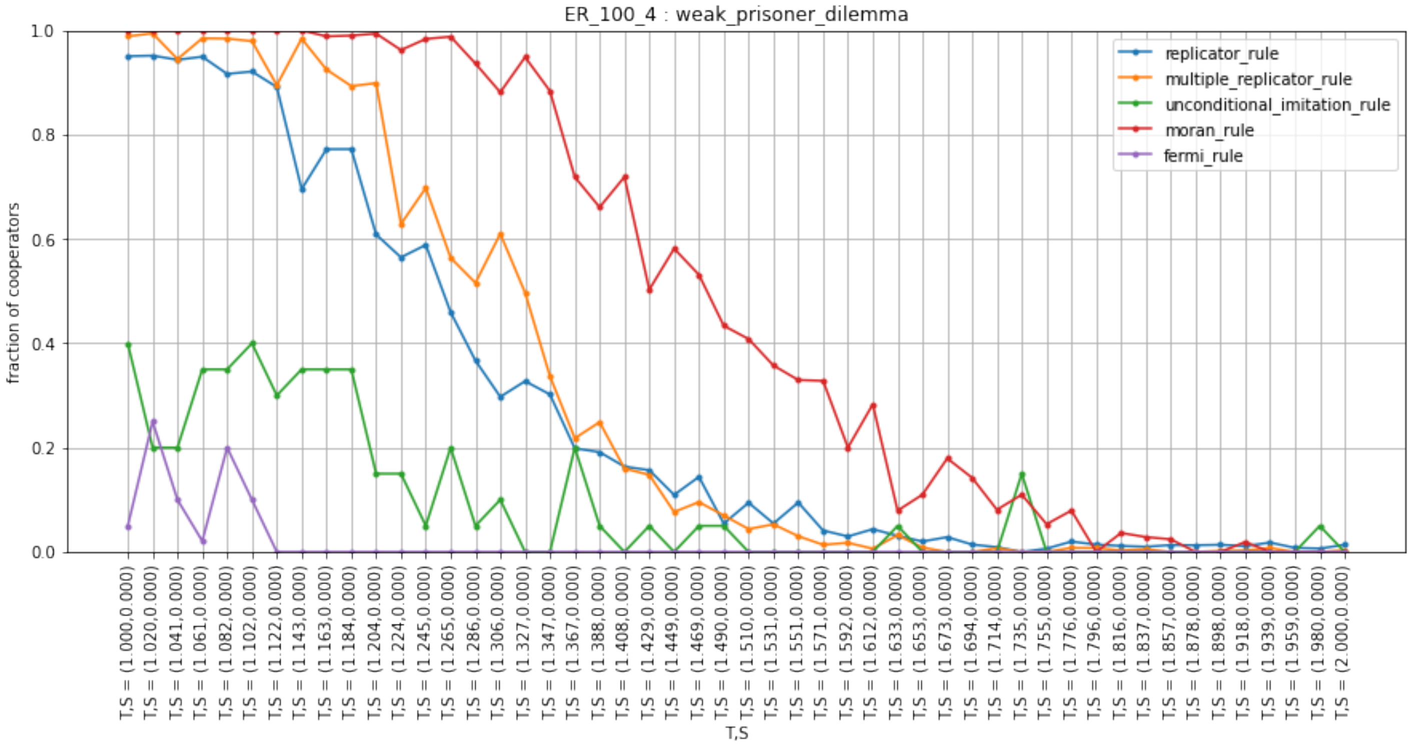


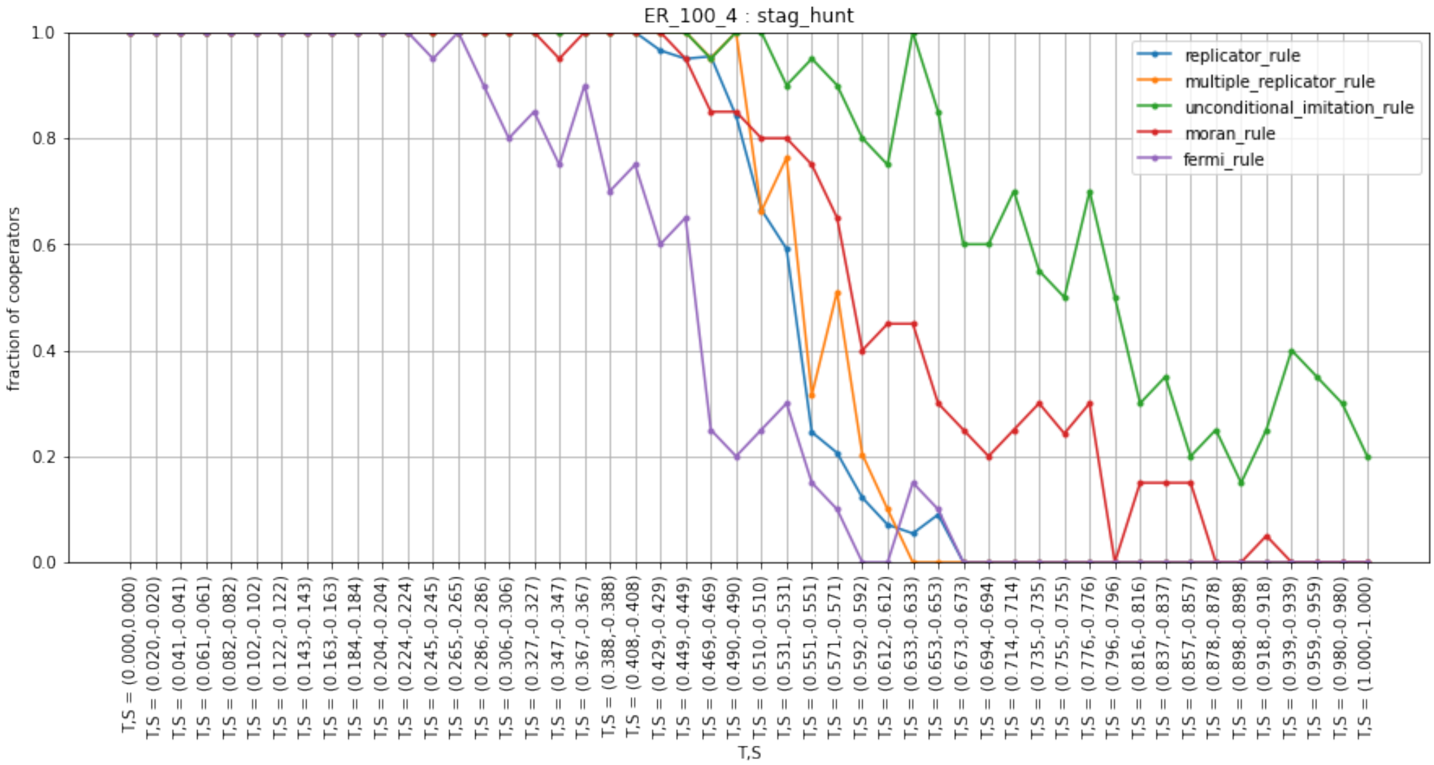


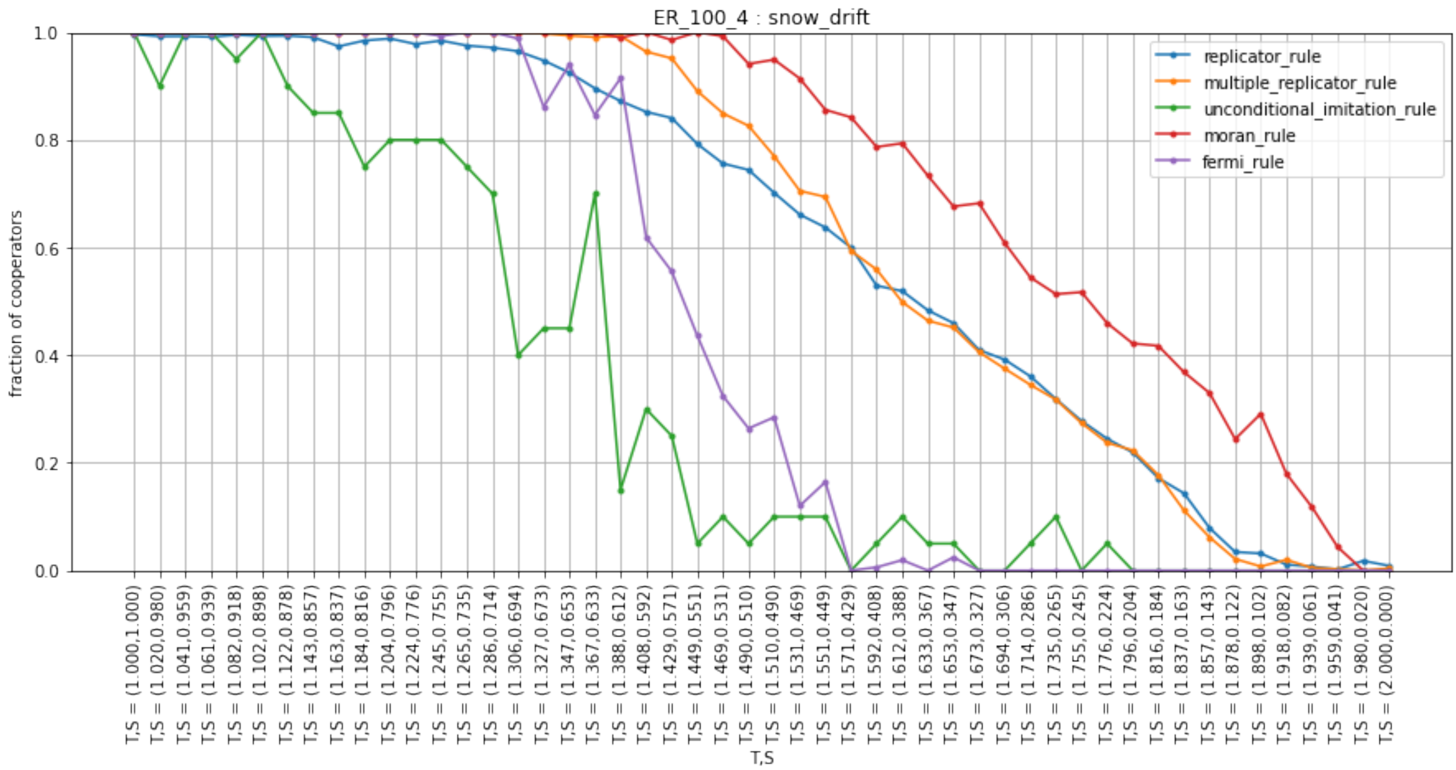


The unconditional imitation rule line oscillates heavily without a clear tendency to go up or down moving through the ST configurations analyzed. The unconditional imitation rule is completely deterministic and the fraction of cooperators reached at stationary state is completely determined by the initial configuration of cooperators and defector nodes. In fact, for the same S and T values and the same network, depending on the random initial configuration, the final fraction of cooperators may vary from 0 to 1.

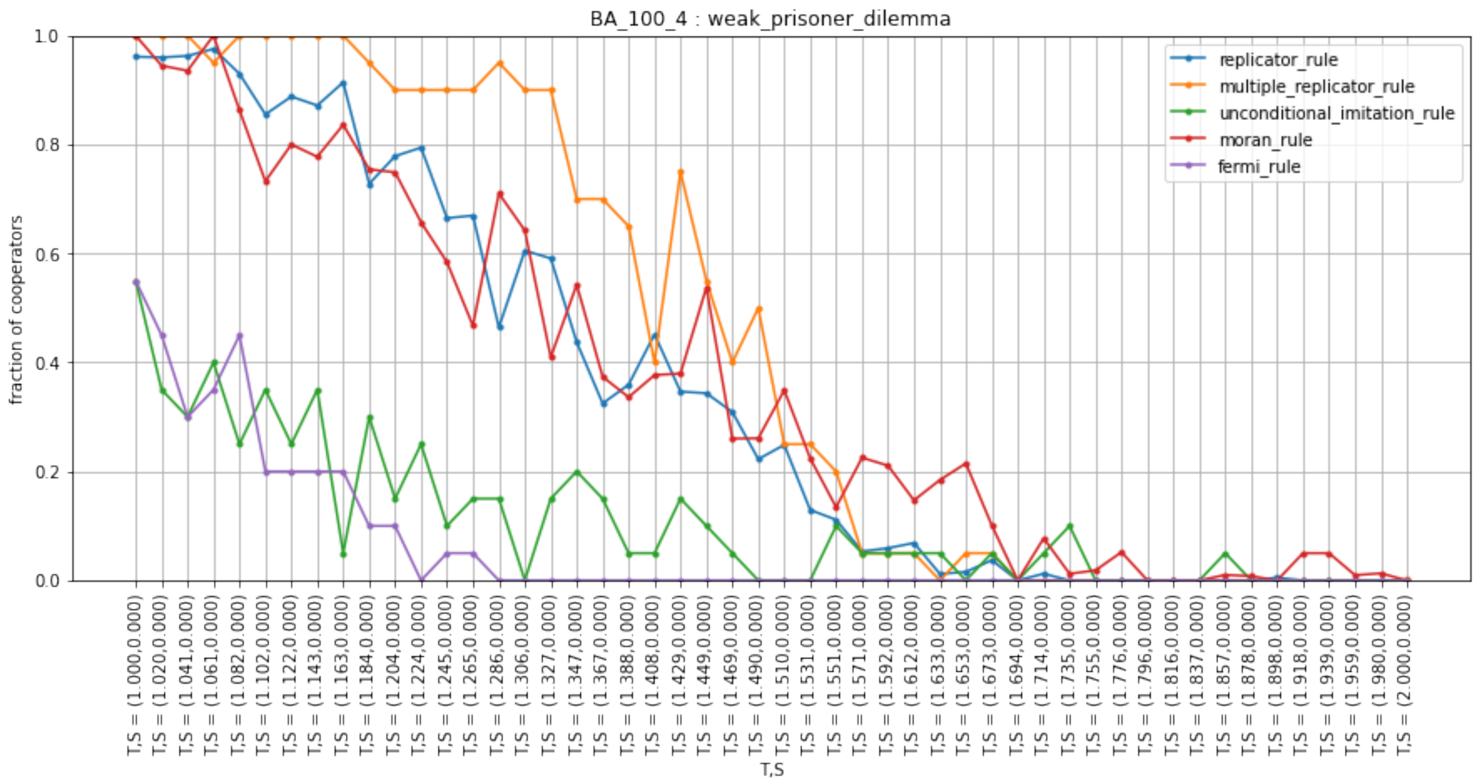
Erdos Renyi graph, k = 4:

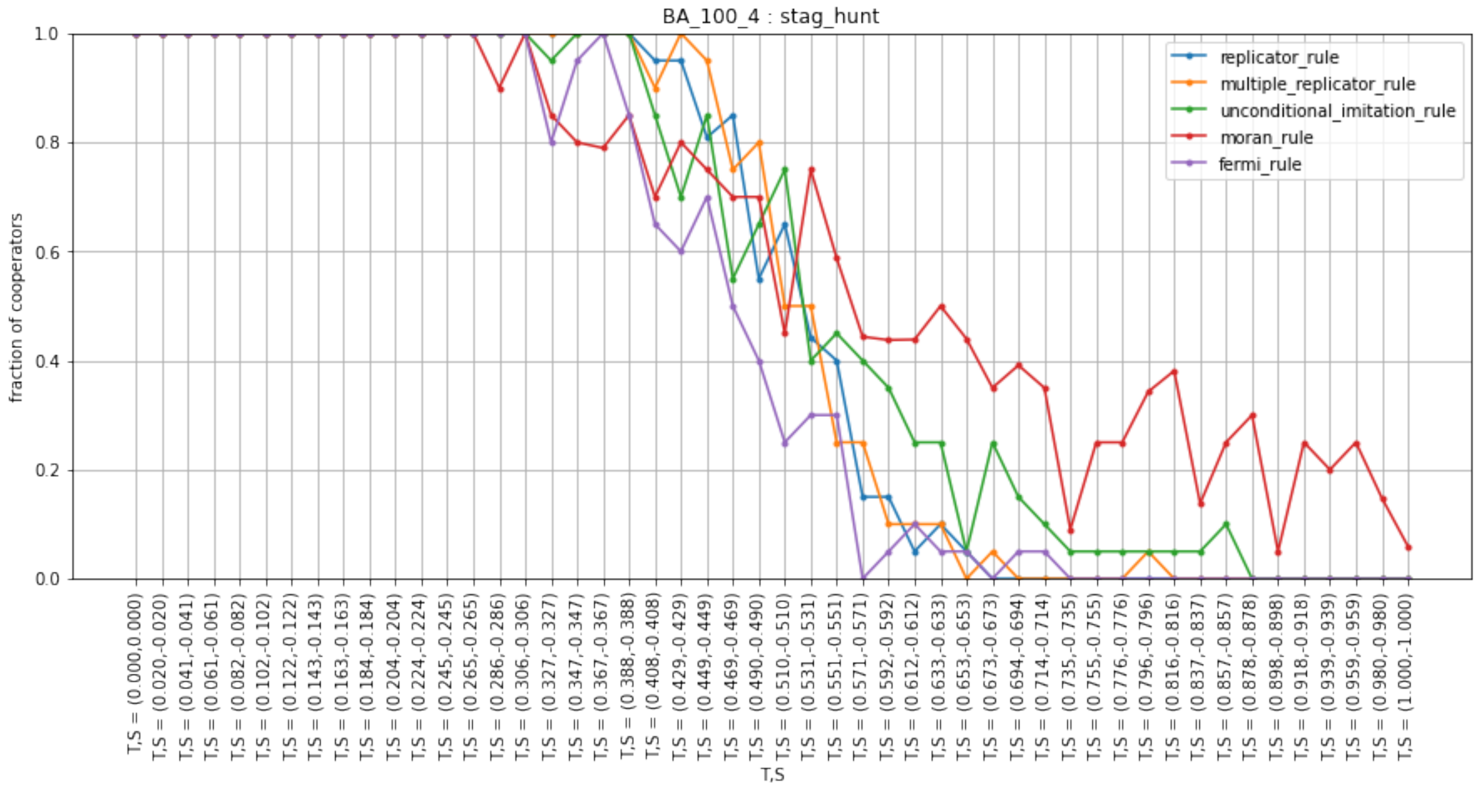


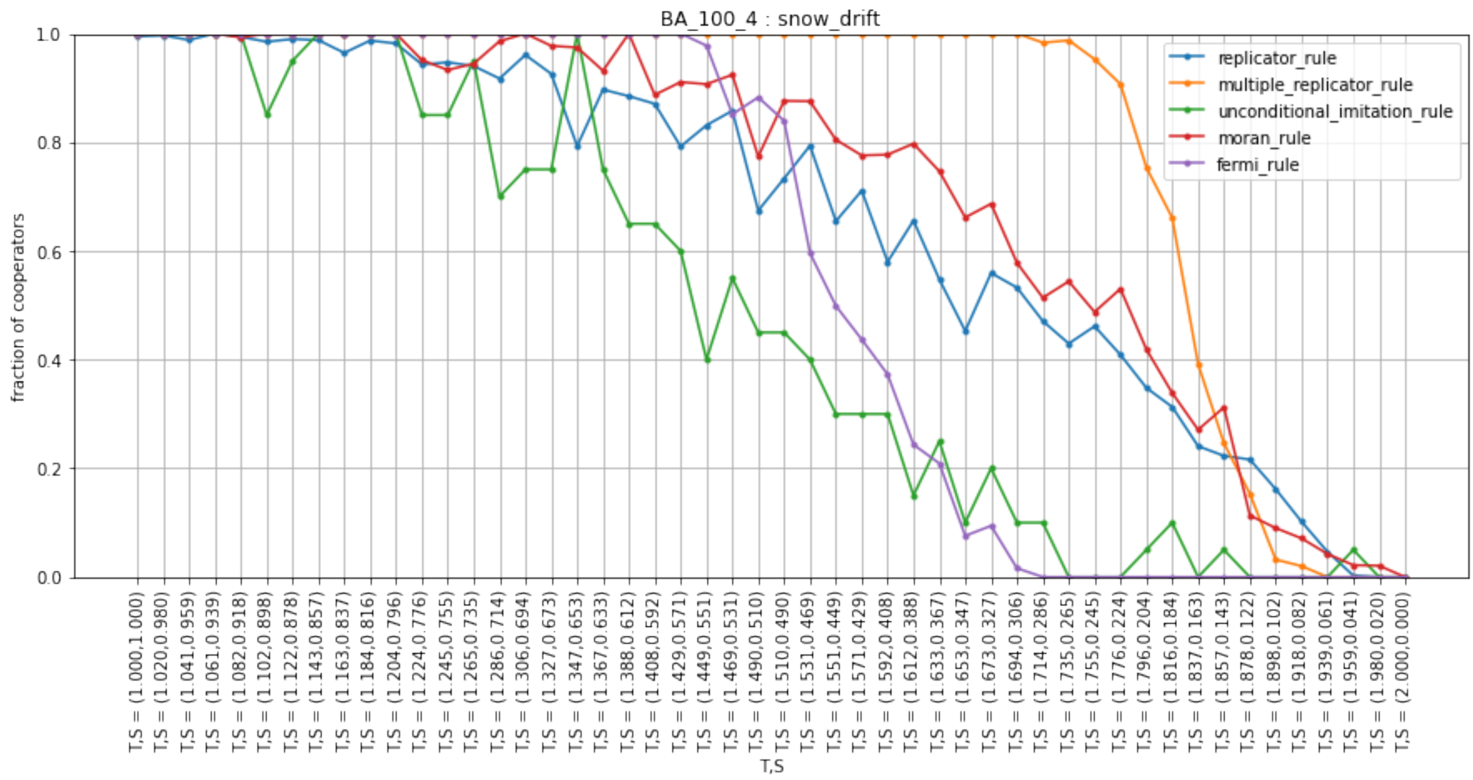




Barabasi Albert graph, k = 4:

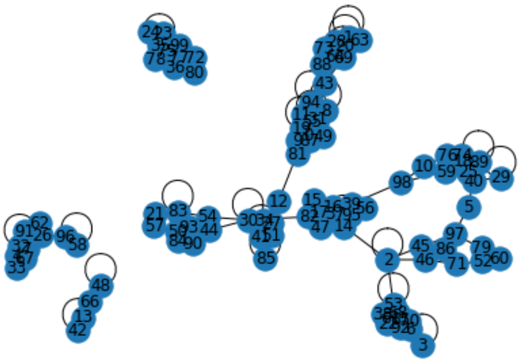


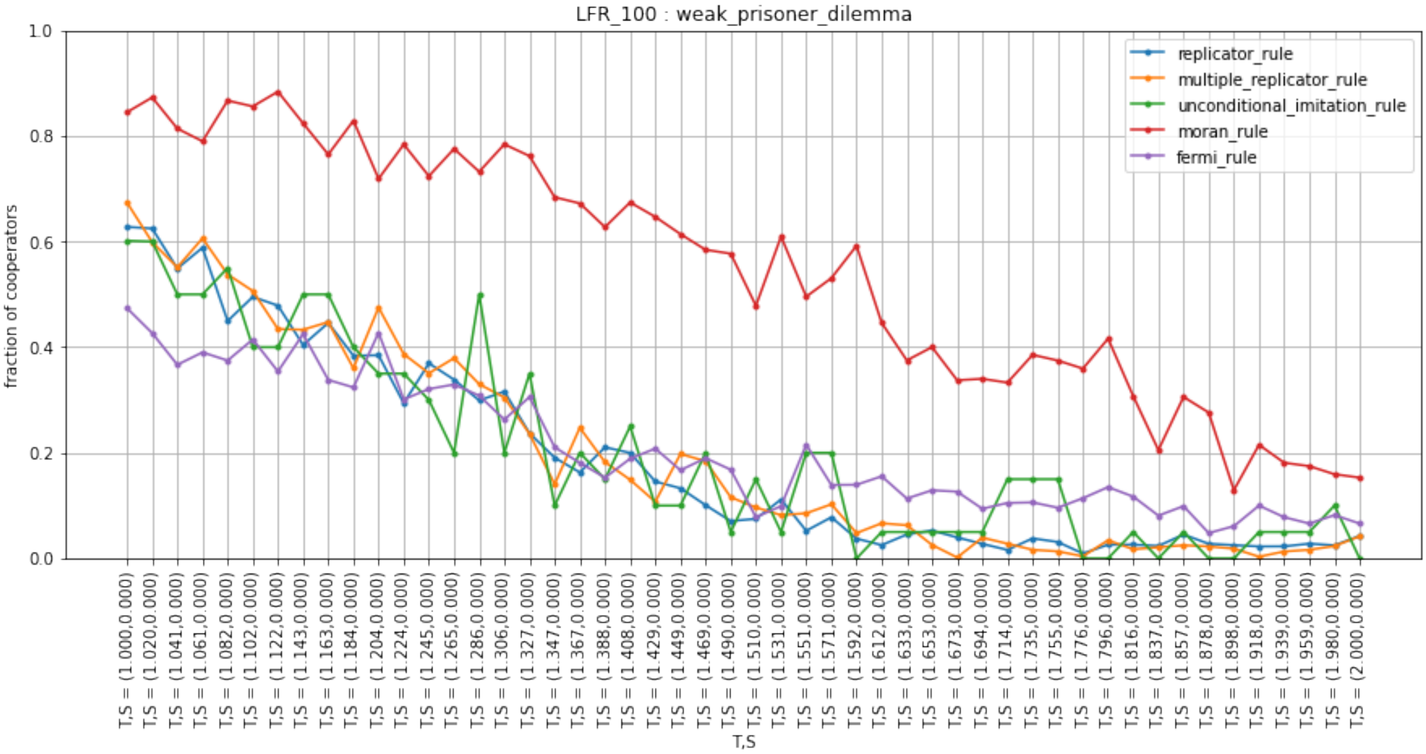

BA_100_4 : stag_hunt
fraction of cooperators
T,S
replicator_rule
multiple_replicator_rule
unconditional_imitation_rule
moran_rule
fermi_rule

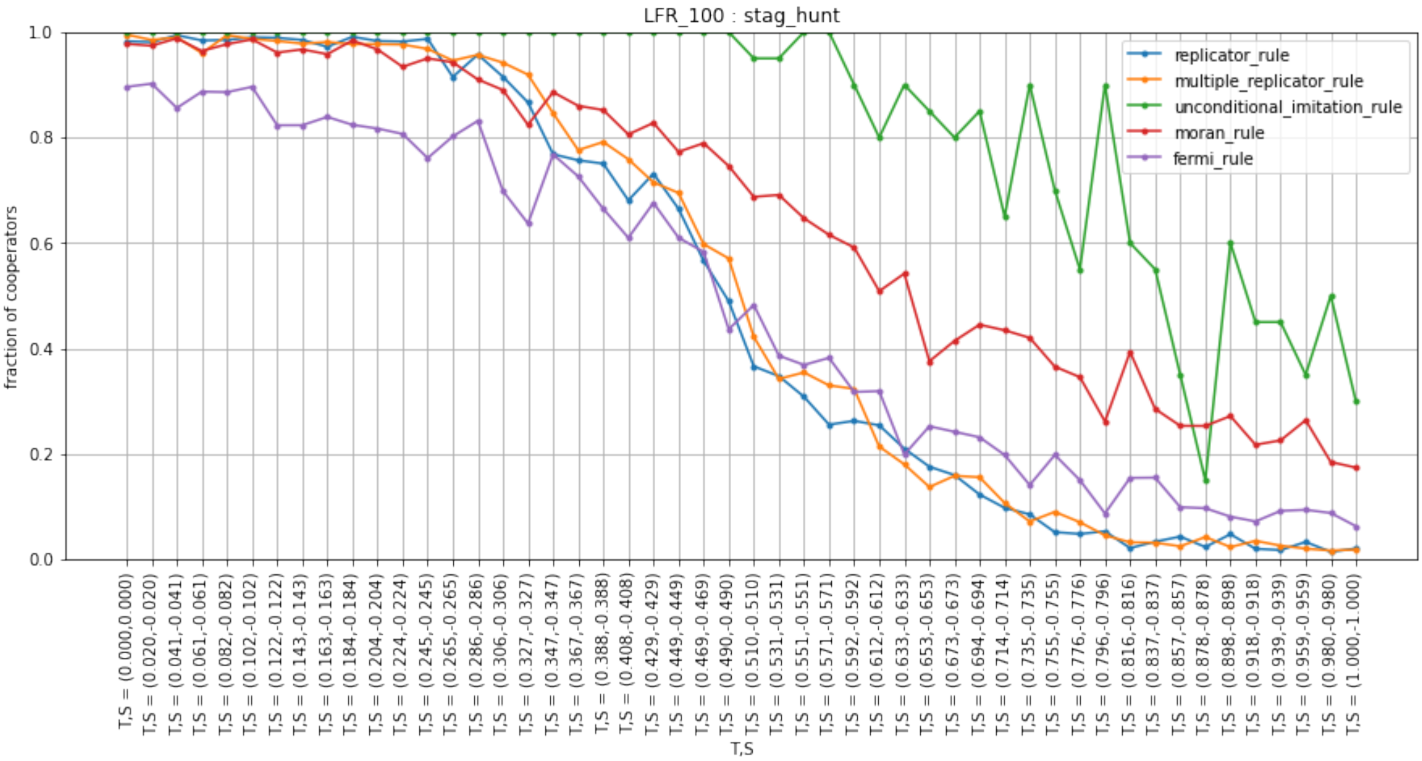

BA_100_4 : snow_drift
fraction of cooperators
T,S
replicator_rule
multiple_replicator_rule
unconditional_imitation_rule
moran_rule
fermi_rule

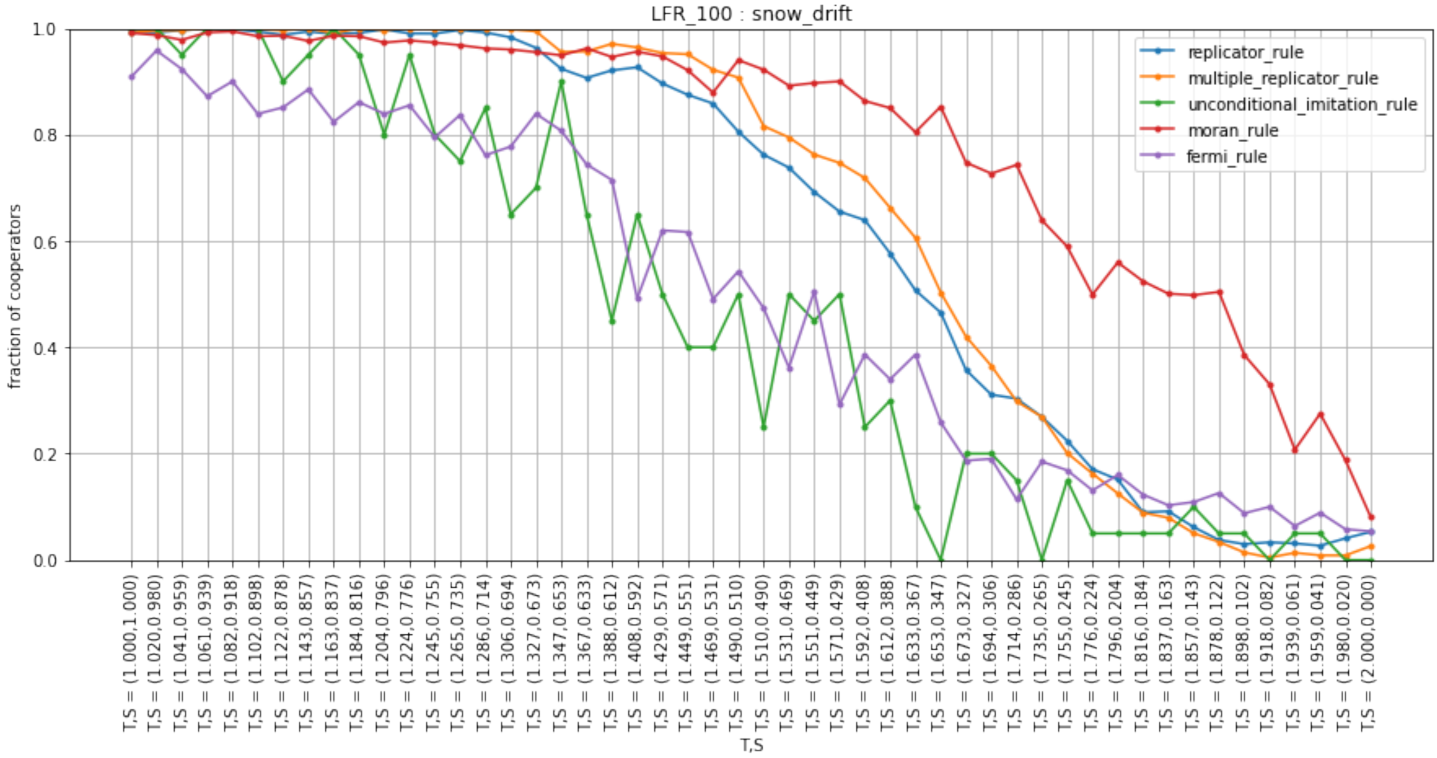
Lancichinetti–Fortunato–Radicchi (LFR) community graph

Parameters:
n = 100
tau1 = 3
tau2 = 1.5
mu = 0.1
average_degree=5, min_community=10









Conclusions:

I tried to see if my results with networks with different degree heterogeneity (homogenous random network, Erdos-Renyi and Barabasi-Albert) replicate some results obtained in [1]. Unlike what was done in [1], the networks studied here have average degree = 4.

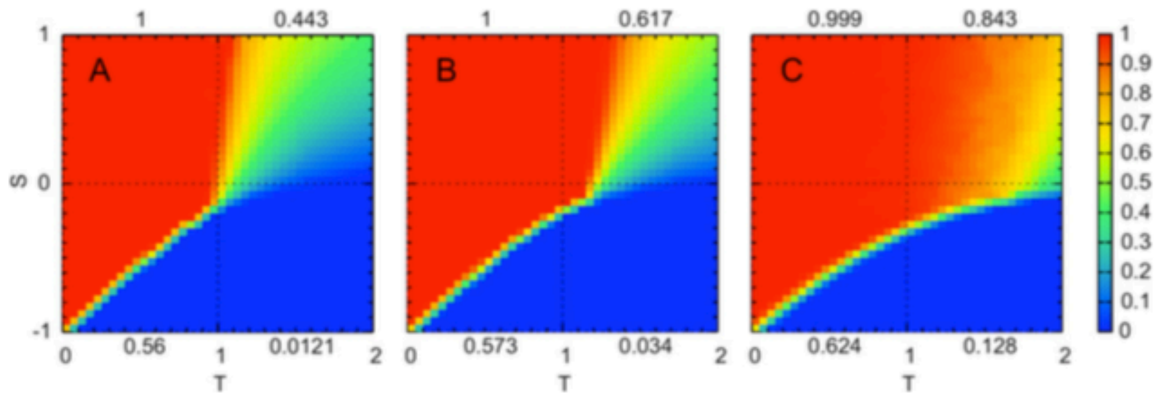


Fig. 20. Asymptotic density of cooperators x^* with the replicator update rule, for model networks with different degree heterogeneity: homogeneous random networks (left, A), Erdős–Rényi random networks (middle, B) and Barabási–Albert scale-free networks (right, C). In all cases the average degree is $\bar{k} = 8$ and the initial density of cooperators is $x^0 = 0.5$. As degree heterogeneity grows, from left to right, cooperation in Snowdrift games is clearly enhanced.

Also in our networks, as degree heterogeneity grows, cooperation in snowdrift games is promoted: the area under the replicator rule curve is smaller for the homogeneous random network, bigger for the ER network and grows just slightly for the BA network.
It can also be observed the same behavior seen in [1] for the weak prisoner's dilemma: for the random regular network the fraction of cooperators in small (~0.5) when (T,S) = (1,0) and slowly decreases as T approaches 2; for the ER network the C fraction starts close to 1

to start decreasing quite rapidly when T ~ 1.1; for the BA network the C fraction starts close to 1 and decreases more slowly with respect to what happens for the ER network (differently from what happens in [1] we reach the value 0 for the C fraction).

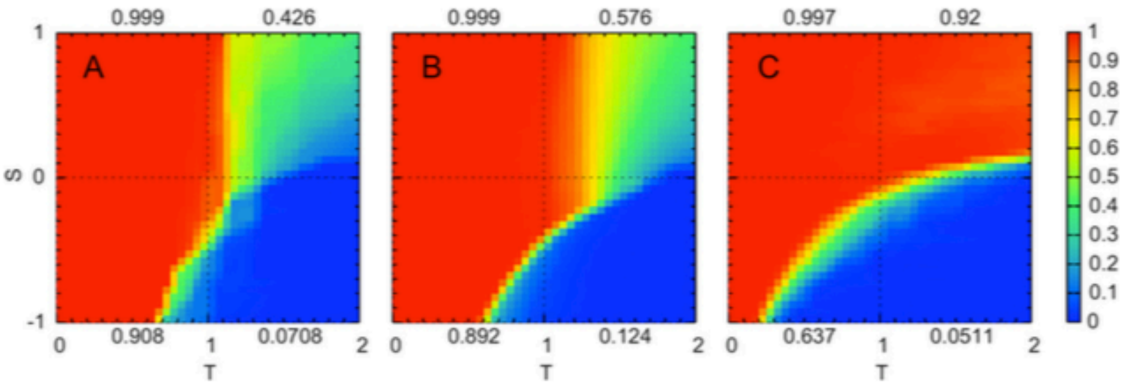


Fig. 21. Asymptotic density of cooperators x^* with unconditional imitation as update rule, for model networks with different degree heterogeneity: homogeneous random networks (left, A), Erdős–Rényi random networks (middle, B) and Barabási–Albert scale-free networks (right, C). In all cases the average degree is $\bar{k} = 8$ and the initial density of cooperators is $x^0 = 0.5$. As degree heterogeneity grows, from left to right, cooperation in Snowdrift games is enhanced again. In this case, however, cooperation is inhibited in Stag Hunt games and reaches a maximum in Prisoner's Dilemmas for Erdős–Rényi random networks.

We can see that also in our simulation, for unconditional imitation rule, the inhibition of cooperation for stag hunt games takes place when degree heterogeneity grows, at least for BA and ER networks: the area under the unconditional imitation rule line is greater in the ER network. The results for the homogeneous random network suffer too much from randomness of the initial configuration, maybe with higher number of repetitions for the MC simulation we would obtain more meaningful results.
As seen in [1], the opposite effect is visible for snow drift game: the area under the unconditional imitation rule curve is greater in the BA scale-free network.
Regarding weak prisoner's dilemma, there's not much difference between the BA and ER networks analyzed.

Bibliography:

[1] “Evolutionary game theory: Temporal and spatial effects beyond replicator dynamics” , Roca,Cuesta,Sánchez