4th Chapter:-

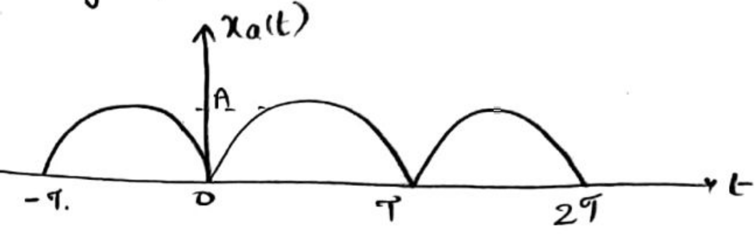
4.1 Consider the full-wave rectified sinusoid

a. Determine its spectrum $X_a(F)$

b. Compute the power of signal.

c. Plot the power spectral density.

d. Check the validity of Parseval's relation for this signal.



sol:- The frequency of sinusoid $= \frac{1}{2T.}$

Hence. exp $= \sin(2\pi * \frac{1}{2T} * t) = \sin(\pi t/T)$.

As it is a (continuous) periodic waveform. To find F.T let us write its fourier series expression

a] $x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T.}$ $\left[as\ F = \frac{1}{T}\right]$

$$C_k = \frac{1}{T.}\int_0^{T.} A.\sin(\pi t/T)e^{-j2\pi kt/T} dt$$

$$= \frac{A}{2jT}\int_0^T \left[e^{j\pi t/T} - e^{-j\pi t/T}\right] e^{-j2\pi kt/T} dt$$

$$= \frac{A}{2jT.}\int_0^T \left[e^{-j(2k-1)\pi t/T} - e^{-j(2k+1)\pi t/T}\right] dt$$

$$= \frac{A}{2jT}\left[\frac{e^{-j(2k-1)\pi t/T}}{-j(2k-1)\frac{\pi}{T}} - \frac{e^{-j(2k+1)\pi t/T}}{-j(2k+1)\pi/T.}\right]_0^T$$

$$= \frac{A}{2j*(-j)\pi}\left[\frac{e^{-j(2k-1)\pi t/T}}{(2k-1)} - \frac{e^{-j(2\pi+1)\pi t/T}}{(2k+1)}\right.$$

$$= \frac{A}{2\pi}\left[\frac{e^{-j(2k-1)\pi}}{2k-1} - \frac{e^{0}}{2k-1} - \left[\frac{e^{-j(2k+1)\pi}}{2k+1} - \frac{e^{0}}{2k+1}\right]\right]$$

$$= \frac{A}{2\pi}\left[\frac{(-1)^{2k-1}-1}{2k-1} - \frac{(-1)^{2k+1}-1}{2k+1}\right]$$

$$= \frac{A}{2\pi}\left[\frac{-2}{2k-1} - \frac{(-2)}{2k+1}\right] = \frac{A}{\cancel{2}\pi}(-\cancel{2})\left[\frac{1}{2k-1} - \frac{1}{2k+1}\right]$$

$$= \frac{-A}{\pi}\left[\frac{\cancel{2k}+1 - \cancel{2k}+1}{(2k-1)(2k+1)}\right]$$

$$= \frac{-2A}{\pi(4k^2-1)} = \frac{2A}{\pi(1-4k^2)}$$

Hence according the expression.

$$x_a(t) = \sum_{k=-\infty}^{\infty} C_k \cdot e^{j2\pi kt/T}$$

Hence $X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi Ft} dt$ [F → unknown]

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C_k \cdot e^{j2\pi kt/T} \cdot e^{-j2\pi Ft} dt.$$

$$= \sum_{k=-\infty}^{\infty} C_k \int_{-\infty}^{\infty} e^{-j2\pi(F-k/T)t} dt.$$

$$= \sum_{k=-\infty}^{\infty} C_k \cdot \delta(F-k/T)$$

$1 \rightarrow \delta(F)$

as F.T. $e^{j2\pi kt/T} \rightarrow \delta(F-k/T)$.

Hence the spectrum of $x_a(t)$ consists of spectral lines of frequencies k/T; $k = 0, \pm 1, \pm 2, \ldots$ with amplitude $|C_k|$ and phase $\angle C_k$.

$$X_a(F) = \sum^{\infty} \frac{2A}{\pi(1-4k^2)} \cdot \delta(F-k/T).$$

b]. As $x_a(t) = \sin(\pi t/T)$.

$$P_x = \frac{1}{T}\int_0^T A^2 \sin^2(\pi t/T). = \frac{A^2}{T}\int_0^T \left(\frac{1-\cos(2\pi t/T)}{2}\right)$$

$$= \frac{A^2}{2T}\left[t - \frac{\sin(2\pi t/T)}{2\pi/T}\right]_0^T$$

$$= \frac{A^2}{2T}\left[T - 0 - \left[\frac{\sin(2\pi)}{2\pi/T} - 0\right]\right]$$

$P_x = A^2/2 \; W$.

c] The power spectral density spectrum is $|C_k|^2$;

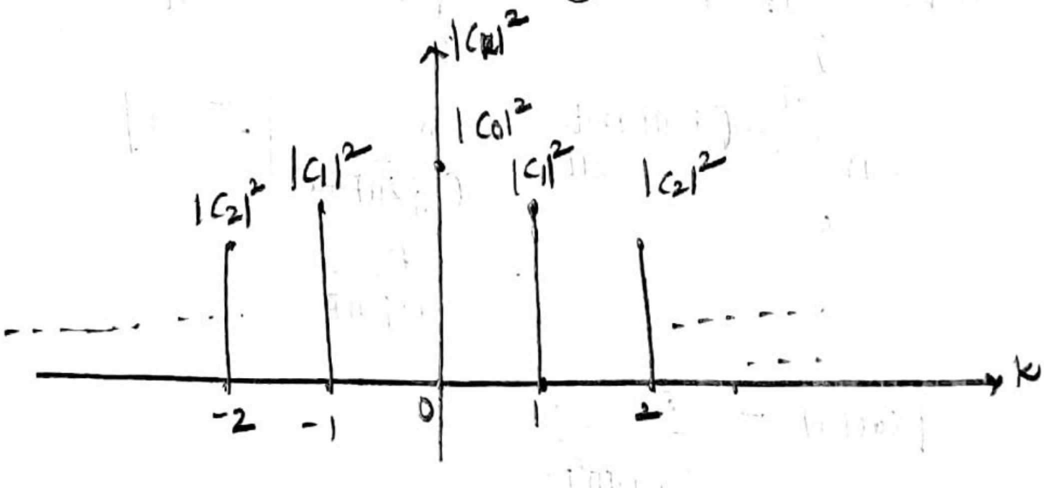
at $k = 0, \pm 1, \pm 2, \ldots$

$$C_k = \frac{2A}{\pi(1-4k^2)}$$

18-1

$|C_0|^2 = \left(\frac{2A}{\pi}\right)^2$; $|C_1|^2 = \left(\frac{2A}{3\pi}\right)^2$; $|C_2|^2 = \left(\frac{2A}{15\pi}\right)^2$ ...

Hence $C_0 = \frac{2A}{\pi}$; $C_1 = \frac{2A}{3\pi}$; $C_2 = \frac{2A}{15\pi}$ ...



d]. Proof of Parseval's theorem.

$$P_x = \frac{1}{T}\int_0^T x_a^2(t)\,dt = \sum_{k=-\infty}^{\infty} |C_k|^2$$

$$\sum_{k=-\infty}^{\infty} |C_k|^2 = \frac{4A^2}{\pi^2}\left[1 + \frac{1}{\;}\right. \quad C_0^2 + C_1^2 + C_{-1}^2 + C_2^2 + C_{-2}^2 \ldots$$

$$= C_0^2 + 2C_1^2 + 2C_2^2 + 2C_3^2 + \ldots$$

$$= \frac{4A^2}{\pi^2} \sum_{k=\alpha}^{\alpha} \frac{1}{(4k^2-1)^2}$$

$$= \frac{4A^2}{\pi^2}\left[1+\frac{2}{3^2}+\frac{2}{15^2}+\cdots\right]. \text{ as } \sum \; 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+$$

$$= \frac{4A^2}{\pi^2}[1.2337] \quad . \left[\text{as infinite series sum is } \pi^2/8\right.$$

$$= A^2/2 \qquad 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}$$

1.231

23

3×5, 1, 5×7

2. Sketch and compute the magnitude and phase spectra for the following signals ($a>0$).

a. $$x_a(t) = \begin{cases} Ae^{-at}; & t>0. \\ 0 & ; t<0. \end{cases}$$

sol:- $$X_a(F) = \int_{-\alpha}^{\alpha} x_a(t) e^{-j2\pi Ft}\,dt = \int_{0}^{\alpha} A\cdot e^{-at} e^{-j2\pi Ft}\,dt$$

$$= A\int_0^{\alpha} e^{-(j2\pi F+a)t}\,dt. = \frac{A}{(-j2\pi F+a)}\left[e^{-\alpha}-1\right]$$

$$= \frac{A}{a+j2\pi F}.$$

$$|X_a(F)| = \frac{A}{\sqrt{a^2+4\pi^2F^2}}$$

$$\angle X_a(F) = -\tan^{-1}\left(\frac{2\pi F}{a}\right).$$

(b) $x_a(t) = Ae^{-a|t|}$.

Sol:-

$$X_a(F) = \int_{-\alpha}^{\alpha} Ae^{-a|t|} e^{-j2\pi Ft} dt$$

$$= A\int_{0}^{\alpha} Ae^{-at} e^{-j2\pi Ft} + A\int_{-\alpha}^{0} e^{at} e^{-j2\pi Ft} dt$$

$$= A \cdot \left[\frac{e^{-(a+j2\pi Ft)t}}{-(+a+j2\pi F)} \right]_0^{\alpha} + A \cdot \left[\frac{e^{+(-j2F+a)t}}{(a-j2\pi F)} \right]_{-\alpha}^{\alpha}$$

$$= \frac{A}{a+j2\pi F} + \frac{A}{a-j2\pi F}$$

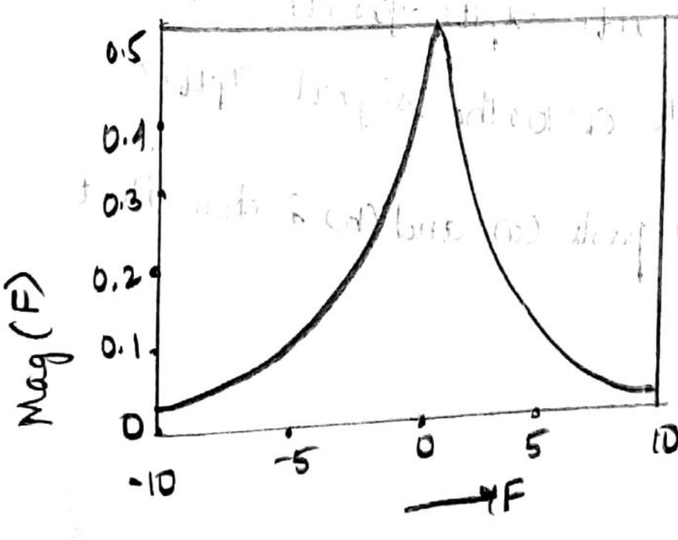
$$= \frac{A(a-j2\pi F) + A(a+j2\pi F)}{a^2+4\pi^2F^2}$$

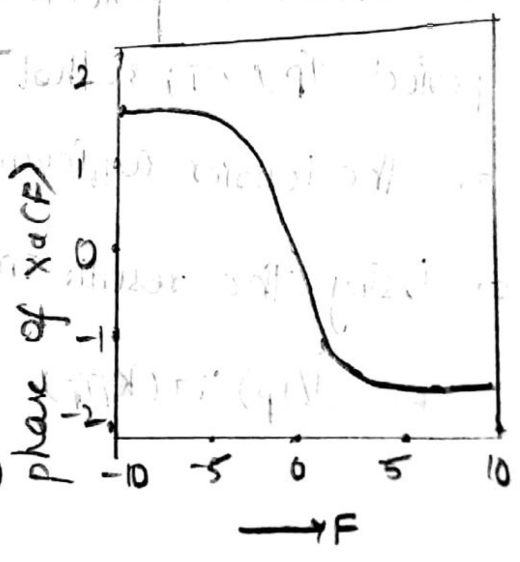
$$= \frac{A(2a)}{a^2+4\pi^2F^2}$$

$$|X_a(F)| = \frac{2aA}{a^2+4\pi^2F^2}$$

$$\angle X_a(F) = 0. \qquad \tan^{-1}\left(\frac{2\pi F}{4}\right)$$

(a) $A=2;\ a=4$ $\Rightarrow |X_a(F)| = \dfrac{2}{\sqrt{4^2+4\pi^2F^2}} = \dfrac{2}{\sqrt{1}}$ $\tan^{-1}\left(\frac{\pi F}{2}\right)$





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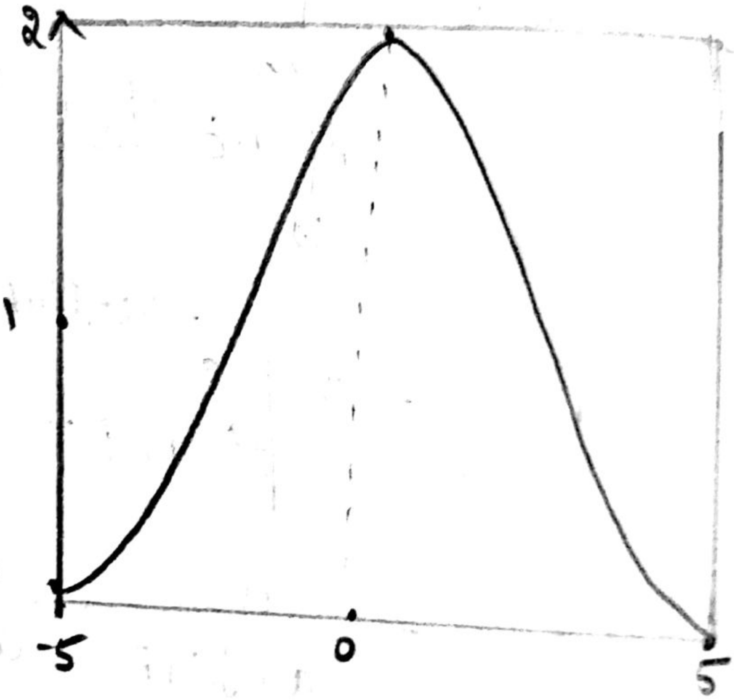
(b) $\angle X_a(F) = \dfrac{2aA}{a^2 + 4\pi^2 F^2} =$ $a = 1;\ A = 1$

$$= \frac{2}{1 + 4\pi^2 F^2}$$



$a = 6$

Magnitude spectrum

at $0 \Rightarrow 2$

$F = \pm 1 \Rightarrow \dfrac{2}{1 + 4\pi^2} = 0.049$

$F = \pm 2 \Rightarrow \dfrac{2}{1 + 16\pi^2} = 0.0125$

$F = \pm 3 = \dfrac{.2}{1 + 36\pi^2} = 0.00561$

4.3 Consider the signal.

$$x(t) = \begin{cases} 1-|t|/\tau ; & |t| \le \tau \\ 0 ; & \text{elsewhere.} \end{cases}$$

(a) Determine and sketch its magnitude and phase spectra ; $|X_a(F)|$ and $\measuredangle X_a(F)$ respectively.

(b) Create a periodic signal $x_p(t)$ with fundamental period $T_p \ge 2\tau$; so that $x(t) = x_p(t)$ for $|t| < T_p/2$. What are the fourier coeffecients c_k for the signal $x_p(t)$?

(c) Using the results in parts (a) and (b) show that $c_k = (1/T_p) X_a(k/T_p)$.

Sol:-

$$X_a(F) = \int_{-\tau}^{0} (1 + t/\tau) e^{-j2\pi F t} dt + \int_{0}^{\tau} (1 - t/\tau) e^{-j2\pi F t} dt$$

Alternatively we may find the fourier transform of

$$y(t) = x'(t) = \begin{cases} \frac{1}{\tau} & -\tau < t \le 0 \\ -1/\tau & 0 < t \le \tau . \end{cases}$$

$$Y(F) = \int_{-\tau}^{\tau} y(t) e^{-j2\pi F t} dt = \int_{-\tau}^{0} \frac{1}{\tau} e^{-j2\pi F t} dt + \int_{0}^{\tau} \left(\frac{-1}{\tau}\right) e^{-j2\pi F t} dt$$

$$= \frac{1}{\tau} \left[\frac{e^{-j2\pi F t}}{-j2\pi F} \right]_{-\tau}^{0} - \frac{1}{\tau} \left[\frac{e^{-j2\pi F t}}{-j2\pi F} \right]_{0}^{\tau}$$

$$= \frac{1}{\tau} \left[\frac{e^{+j2\pi F \tau} + e^{-j2\pi F \tau} - 2}{j2\pi F} \right]$$

$$= \frac{1}{j\pi \tau F} \left[\cos(2\pi F \tau) - 2 \right]$$

$$= \frac{2\sin^2 \pi F T}{j\pi F T}$$

and $X(F) = \frac{1}{j2\pi F} Y(F) = T\left[\frac{\sin \pi F \tau}{\pi F \tau}\right]^2$

$$|X(F)| = T\left(\frac{\sin \pi F \tau}{\pi F \tau}\right)^2$$

$$\angle X_a(F) = 0.$$

(b) $c_k = \frac{1}{T_P} \int_{-T_P/2}^{T_P/2} x(t) e^{-j2\pi k t/T_P} dt$

$$= \frac{1}{T_P}\left[\int_{-\tau}^{0} (1+t/\tau) e^{-j2\pi k t/T_P} dt + \int_{0}^{\tau} (1-t/\tau) e^{-j2\pi k t/T_P} dt\right].$$

$$= \frac{\tau}{T_P}\left[\frac{\sin(\pi k \tau/T_P)}{\pi k \tau/T_P}\right]^2$$

i.e. $\frac{1}{T_P}\left[\frac{1}{\tau}\left[\int_{-\tau}^{0} e^{-j2\pi k t/T_P} dt + \int_{0}^{\tau} e^{-j2\pi k t/T_P} dt\right]\right]$

$$= \frac{1}{T_P}\left[\frac{1}{\tau}\left[\frac{1 - e^{-j2\pi k \tau/T_P} + 1 - e^{-j2\pi k \tau/T_P}}{-j2\pi k/T_P}\right]\right]$$

$$\Rightarrow \frac{1}{T_P}\left[\frac{\sin^2(\pi k \tau/T_P)}{\pi k/T_P \cdot \tau}\right]$$

$$\Rightarrow \frac{\tau}{T_P}\left[\frac{\sin(\pi k \tau/T_P)}{\pi k \tau/T_P}\right]^2$$

(c) from (a) and (b); we have $c_k = \frac{1}{T_p} X_a(k/T_p)$

4.4. Consider the following periodic signal:

$$x(n) = \{\ldots, 1, 0, 1, 2, 3, 2, 1, 0, 1, \ldots\}$$

(a) Sketch the signal $x(n)$ and its magnitude and phase spectra.

(b) Using the results in part (a) verify Parseval's relation by computing the power in time and frequency domains.

Sol: $x(n) = \{\ldots, 1, 0, 1, 2, \underset{\uparrow}{3}, 2, 1, 0, 1, \ldots\}$ $N=6$;

$$c_k = \frac{1}{6} \sum_{n=0}^{5} x(n) e^{-j2\pi kn/6}$$

$$= \frac{1}{6}\left[3 + 2e^{-j2\pi\frac{k}{6}} + e^{-j2\pi\cdot\frac{2k}{6}} + e^{-j2\pi\frac{3k}{6}} + 2e^{-j2\pi\cdot\frac{4k}{6}}\right]$$

$$= \frac{1}{6}\left[3 + 4\cos\frac{\pi k}{3} + 2\cos\frac{2\pi k}{3}\right]$$

Hence $c_0 = 9/6$; $c_1 = \frac{4}{6}$; $c_2 = 0$; $c_3 = 1/6$; $c_4 = 0$; $c_5 = 4/6$.

(b) $$P_t = \frac{1}{6} \sum_{n=0}^{5} |x(n)|^2 = \frac{1}{6}(3^2 + 2^2 + 1^2 + 0 + 1^2 + 2^2) = 19/6.$$

$$P_f = \sum_{n=0}^{5} |c_n|^2 = (9/6)^2 + (4/6)^2 + (1/6)^2 + (4/6)^2 = 19/6.$$

4.5] Consider the signal.

$$x(n) = 2 + 2\cos \pi n/4 + \cos \pi n/2 + \tfrac{1}{2}\cos 3\pi n/4$$

(a) Determine and sketch its power density spectrum

(b) Evaluate the power of signal.

$N = 8$.

Sol:-

$$C_k = \frac{1}{8}\sum_{n=0}^{7} x(n) e^{-j2\pi kn/8}$$

2×2

$2 + 2 \; 4/_2$

$$= \frac{1}{8}\Big[$$

$$x(n) = \left\{ 11/2,\ 2+\tfrac{3}{4}\sqrt{2},\ 1,\ 2-\tfrac{3}{4}\sqrt{2},\ 1/2,\ 2-\tfrac{3}{4}\sqrt{2},\ 1,\ 2+\tfrac{3}{4}\sqrt{2} \right\}$$

$$C_k = \frac{1}{8}\sum_{n=0}^{7} x(n) e^{-j2\pi kn/8}$$

$e^{-j2\pi n/8}$

$$C_0 = 2; \qquad C_1 = \frac{1}{8}\sum_{n=0}^{7} x(n) e^{-j2\pi n/8}$$

$$C_1 = \frac{1}{8}x(0) + \frac{1}{8}x(1)e^{-j\pi/4} + \frac{1}{8}x(2)e^{-j\pi/2} + \frac{1}{8}x(3)e^{-j2\pi\times 3/8} + \frac{1}{8}x(4)e^{-j2\pi\times 4/8} + \frac{1}{8}x(5)e^{-j2\pi\times 5/8} + \frac{1}{8}x(6)e^{-j2\pi\times 6/8} + \frac{1}{8}x(7)e^{-j2\pi\times 7/8}$$

$$= \frac{1}{8}(11/2) + \frac{1}{8}\left(2+\tfrac{3}{4}\sqrt{2}\right)\left(\underbrace{\cos\pi/4}_{1/\sqrt{2}} - j\underbrace{\sin\pi/4}_{1/\sqrt{2}}\right) + \frac{1}{8}(1)(-j) + \frac{1}{8}\left(2-\tfrac{3}{4}\sqrt{2}\right)\left(\underbrace{\cos(2\pi\times 3/8)}_{-1/\sqrt{2}} - j\underbrace{\sin(2\pi\times 3/8)}_{1/\sqrt{2}}\right)$$

$+\frac{1}{8}\left(\left(\frac{1}{2}\right) \cdot e^{-j2\pi k}\left[\cos\left(2\pi * 4/8\right) - j\sin\left(2\pi * 4/8\right)\right]\right.$

$-1 \quad 0 \quad -1/\sqrt{2}$

$-1/\sqrt{2}$

$+\frac{1}{8}\left(2 - \frac{3}{4}\sqrt{2}\right)\left(\cos\left(2\pi * 5/8\right) - j\sin\left(2\pi * 5/8\right)\right]$

$+\frac{1}{8}\left(1\right)\left(\cos\left(2\pi * 6/8\right) - j\sin\left(2\pi * 6/8\right)\right] +$

$0 \quad -1$

$\frac{1}{8}\left(2 + 3/4\sqrt{2}\right)\left(\cos\left(2\pi * 7/8\right) - j\sin\left(2\pi * 7/8\right)\right)\right]$

$1/\sqrt{2} \quad -1/\sqrt{2}$

$\cos(\pi n/4) \Rightarrow \frac{1}{2} e^{j2\pi n/8} + e^{-j2\pi n/8}$

$-1+8$

$2 + e^{j6\pi n/8}$

$-6+8=2$

$k=1; \quad k=7$

$6;$

$8 \quad 1/4 \quad k_k = \quad 2 \quad k=0$

$1/2 \quad k=1$

$k=7$

$\cos\left(4\pi n/8\right) \Rightarrow \frac{1}{2}\left[e^{j4\pi n/8} + e^{-j4\pi n/8}\right] \quad 1$

$1 \quad 4$

$1/4\left[e^{j6\pi n/8} + e^{-j6\pi n/8}\right] \quad 1/4 \quad 6$

$1/4 \quad 2$

$-6+8$

$\{2, 1/2, 2, 0, 1, 0, 1/4, 1\}$

1

$2 + 2\cos \pi n/4 + \cos \pi n/2 \ldots \cos 3\pi n/4 \quad N=8$

$2\cos(2\pi r/8)$

$C_k = \frac{1}{8}\sum_{n=0}^{7} x(n) e^{-j\pi k n/4}$

$1/2 \quad \cos(\ldots)$

$C_k = \ldots$

$x(0), x(1), x(2), x(3)$

(b) $P = \sum_{k=0}^{7} |c(k)|^2 = 4+1+1+1/4+1/4+1/16+1/16 = 5\ldots$

4.6] Determine and sketch the magnitude and phase spectra of following periodic signals

(a) $x(n) = 4\sin\left(\frac{\pi(n-2)}{3}\right) = 4\sin\left(\frac{2\pi(n-2)}{6}\right)$

sol:- $C_k = \frac{1}{6} \sum_{n=0}^{5} x(n) e^{-j2\pi kn/6}$

$= \frac{4}{6} \sum_{n=0}^{5} \sin\left(\frac{\pi(n-2)}{3}\right) e^{-j2\pi kn/6}$

$= \frac{2}{3} \left[\sin\left(\frac{-2\pi}{3}\right)(1) + \sin\left(-\pi/3\right) e^{-j2\pi k/6} + 0 + \sin(\pi/3) e^{-j2\pi k \cdot 3/6} + \sin(2\pi/3) e^{-j2\pi k \cdot 4/6} + \cancel{\sin(3\pi/3)} \right]$

$C_k = \frac{2}{3}\left[-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} e^{-j\pi k/3} + \frac{\sqrt{3}}{2} e^{-j\pi k} + \frac{\sqrt{3}}{2} e^{-j4\pi k/3} \right]$

$C_k = \frac{1}{\sqrt{3}}\left[-1 - e^{-j\pi k/3} + e^{-j\pi k} + e^{-j4\pi k/3} \right]$

$C_0 = \frac{1}{\sqrt{3}}[0]$; $C_1 = \frac{1}{\sqrt{3}}\left[-1 - \left[1/2 - j\sqrt{3}/2\right] + 1 + \left(-1/2 + j\sqrt{3}/2\right) \right]$

$C_1 = \frac{1}{\sqrt{3}}\left[-1 - 1/2 + j\sqrt{3}/2 + 1 - 1/2 + j\sqrt{3}/2 \right]$

$\frac{1}{\sqrt{3}}\left[-1+j\sqrt{3}\right]$ $\quad = \frac{\pi}{2}$

$2\pi/3$

$$c_2 = \frac{1}{\sqrt{3}}\left[-1-e^{-j2\pi/3}+e^{-j2\pi}+e^{-j4\pi\times 2/3}\right]$$

$8\pi/3$

$2\pi + 2\pi/3$

$$= \frac{1}{\sqrt{3}}\left[-1+1-e^{-j2\pi/3}+e^{-j(2\pi+\pi/3)}\right] = 0$$

$$c_3 = \frac{1}{\sqrt{3}}\left[\cancel{-1}-e^{\cancel{-j\pi\times 3/3}}+e^{\cancel{-j3\pi}}+\cancel{e^{-j4\pi}}\right]$$

$$= \frac{1}{\sqrt{3}}\left[0\right]$$

$$c_4 = \frac{1}{\sqrt{3}}\left[\cancel{-1}-e^{\cancel{-j4\pi/3}}+e^{\cancel{-j4\pi}}+e^{\cancel{-j4\pi\times 4/3}}\right] = 0$$

$16\pi/3$

$$c_5 = \frac{1}{\sqrt{3}}\left[-1-e^{-j5\pi/3}+e^{\cancel{-j5\pi}}+e^{-j4\pi\times 5/3}\right]$$

$4\pi + 4\pi/3$

$$= \frac{1}{\sqrt{3}}\left[-2+\frac{1}{2}+j\frac{\sqrt{3}}{2}+\frac{1}{2}+\frac{j\sqrt{3}}{2}\right]$$

$-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$$= \frac{1}{\sqrt{3}}\left[-3+j\sqrt{3}\right] \Rightarrow$$

$\left(-\frac{\sqrt{3}}{2}+\frac{j}{2}\right)\cdot 2j \quad 2j\left(e^{j2\pi/3}\right)$

$\Rightarrow$

$\Rightarrow$

$\angle c_1 = \pi + \pi/2 = 2\pi/3 = 5\pi/6 \qquad \angle c_5 = -5\pi/6;$

$\angle c_0 = \angle c_2 = \angle c_3 = \angle c_4 = 0.$

(b) $x(n) = \cos(2\pi/3\, n) + \sin(2\pi/5\, n) \Rightarrow N = 15.$

$C_k = C_{1k} + C_{2k}.$

Sol:-

where C_{1k} is the DTFS coeffecients of $\cos(2\pi n/3)$ and C_{2k} is DTFS coeffecients $\sin(2\pi n/3)$. But.

$$\cos(2\pi n/3) = \frac{1}{2}\left(e^{j2\pi n/3} + e^{-j2\pi n/3}\right)$$

$$C_{1k} = \begin{cases} 1/2 ; & k = 5, 10 \\ 0 ; & \text{otherwise.} \end{cases}$$

Similarly,

$$\sin(2\pi n/5) = \frac{1}{2j}\left(e^{j2\pi n/5} - e^{-j2\pi n/5}\right)$$

Hence

$$C_{2k} = \begin{cases} 1/2j & k = 3 \\ -1/2j & k = 12 \\ 0 & \text{otherwise} \end{cases}$$

$$C_k = C_{1k} + C_{2k} = \begin{cases} 1/2j ; & k = 3 \\ 1/2 ; & k = 5 \\ 1/2 & k = 10 \\ -1/2j. & k = 12 \\ 0 & \text{otherwise.} \end{cases}$$

(c) $x(n) = \cos\left(\frac{2\pi n}{3}\right)\sin\left(\frac{2\pi n}{5}\right)$ $\quad -8+15$ $\quad ,2+$

$$= \frac{1}{2}\left[\sin\left(\frac{16\pi n}{15}\right) - \sin\left(\frac{4\pi n}{15}\right)\right]$$

Hence $N = 12$

$$= \frac{1}{2}\left[\frac{e^{j\frac{16\pi n}{15}} - e^{-j\frac{16\pi n}{15}} - e^{+j\frac{4\pi n}{15}} + e^{-j\frac{4\pi n}{15}}}{2j}\right.$$

$$C_k = \begin{cases} -1/4j & k=2,7 \\ 1/4j & k=8,13 \\ 0 & \text{otherwise.} \end{cases}$$

(d) $x(n) = \{ \ldots, -2, -1, \underset{\uparrow}{0}, 1, 2, -2, -1, 0, 1, 2, \ldots \}$

Sol: $N = 5;$ $C_k = \frac{1}{5} \sum_{k=0}^{5} x(n) e^{-j\frac{2\pi kn}{5}}$

2) $C_k = \frac{1}{5} \left[e^{-j\frac{2\pi k}{5}} + 2e^{-j\frac{2\pi k * 2}{5}} - 2e^{-j\frac{2\pi k * 3}{5}} - 1e^{-j 2\pi k * 4/5} \right]$

$$= \frac{1}{5} \left[e^{-j\frac{2\pi}{5}(1)} + 2e^{-j\frac{2\pi k}{5}(2)} - 2e^{-j\frac{2\pi k}{5}(-2)} - 1e^{-j\frac{2\pi k}{5}(-1)} \right]$$

$-\frac{4\pi}{5}; -\frac{6\pi}{5}$

$-\frac{2\pi}{5}\ -\frac{8\pi}{5}$

$$= \frac{2j}{5} \left[-\sin\left(\frac{2\pi k}{5}\right) - 2\sin\left(\frac{4\pi k}{5}\right) \right]$$

(e) $x(n) = \{ \ldots -1, -2, \underset{\uparrow}{1}, 2, -1, 0, -1, 2, 1, 2 \ldots \}$

$N = 6;$

$$C_k = \frac{1}{6} \sum_{n=0}^{5} x(n) e^{-j\frac{2\pi nk}{6}}$$

$$= \frac{1}{6} \left[1 + 2e^{-j\frac{2\pi k}{6}} - e^{-j\frac{4\pi k}{6}} - 1e^{j\frac{8\pi k}{6}} + 2e^{j 5\pi k/6} \right]$$

$$= \frac{1}{6} \left[1 + e^{j2\pi}\ 4\cos\left(\frac{\pi k}{3}\right) - 2\cos\left(\frac{2\pi k}{3}\right) \right]$$

Therefore $c_0 = 1/2$; $c_1 = 2/3$; $c_2 = 0$

$c_3 = -5/6$; $c_4 = 0$; $c_5 = 2/3$

(f) $x(n) = \{\ldots, 0, 0, \underset{\uparrow}{1}, 1, 0, 0, \overbrace{0, 1, 1, 0, 0}^{5} \ldots\}$

$N = 5$

$$C_k = \frac{1}{5} \sum_{n=0}^{4} x(n) e^{-j2\pi kn/5}$$

$$= \frac{1}{5}\left[1 + e^{-j2\pi k/5}\right]$$

$$= \frac{2}{5} e^{-j\pi k/5} \left[\cos(\pi k/5)\right]$$

$C_0 = 2/5$; $C_1 = \frac{2}{5}\cos(\pi/5) e^{-j\pi/5}$

$$C_2 = \frac{2}{5}\cos(2\pi/5) e^{-j2\pi/5}$$

$$C_3 = \frac{2}{5}\cos(3\pi/5) e^{-j3\pi/5}$$

$$C_4 = \frac{2}{5}\cos(4\pi/5) e^{-j4\pi/5}$$

(g) $x(n) = 1$; $-\alpha < n < \alpha$

$$C_k = \sum_{n=-\alpha}^{\alpha} x(n) e^{-j2\pi kn/N}$$

$$X(\omega) = \sum_{n=-\alpha}^{\alpha} x(n) e^{-j\omega n}$$

$$= \sum_{n=\alpha}$$

$C_0 = 1$

$\underline{N = 1}$

$$\sum_{N=}$$

$C_0 = 1$

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(b) $x(n) = (-1)^n; \quad -\alpha < n < \alpha.$

$N = 2$

$$\Rightarrow \quad c_k = \frac{1}{2} \sum_{n=0}^{1} x(n) e^{-j\frac{2\pi k n}{2}}$$

$$= \frac{1}{2}\left[1 - e^{-j\frac{2\pi k}{2}}\right]$$

$$\Rightarrow \frac{1}{2}\left(1 - e^{-j\pi k}\right).$$

$$\Rightarrow \quad c_0 = 0; \quad c_1 = 1 \,\mathcal{N}.$$

4.7] Determine the periodic signals $x(n)$; with fundamental period $N=8$; if their fourier coeffecients are given by:-

(a) $c_k = \cos k\pi/4 + \sin 3k\pi/4$

(b) $c_k = \begin{cases} \sin k\pi/3 \; ; & 0 \le k \le 6 \\ 0 & k = 7 \end{cases}$

(c) $\{c_k\} = \{\ldots 0, 1/4, \underbrace{1/2, 1, 2, 1, 1/2, 1/4, 0}_{7} \ldots\}$
$\qquad\qquad\qquad\qquad\;\; \uparrow$

Sol:- $$c_k = \frac{1}{N} \sum_{n=0}^{7} x(n) e^{-j\frac{2\pi k n}{N}}$$

$$= \frac{1}{8}\left[\frac{e^{j2\pi k/8} + e^{-j\pi k/8}}{2} + \frac{e^{j6\pi/8} - e^{-j6\pi/8}}{2j}\right]$$

$$= \frac{1}{8} \quad 4\delta(n+1) + 4\delta(n-1).$$
$$-4j\delta(n+3) + 4j\delta(n-3); \quad -3 \le n \le 5.$$

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(b) $c_k = \begin{cases} \sin k\pi/3 \, ; & 0 \le k \le 6 \\ 0 \, ; & k = 7 \end{cases}$ $\frac{1}{2j}$

Sol:- $c_0 = 0; \; c_1 = \sqrt{3}/2; \; c_2 = \sqrt{3}/2; \; c_3 = 0;$

$c_4 = -\sqrt{3}/2; \; c_5 = -\sqrt{3}/2; \; c_6 = c_7 = 0$

$$x(n) = \sum_{k=0}^{7} c_k e^{\frac{j2\pi kn}{8}}$$

$$= \frac{\sqrt{3}}{2}\left[e^{j\pi n/4} + e^{\frac{j2\pi n}{4}} - e^{\frac{j4\pi n}{4}} - e^{\frac{j5\pi n}{4}}\right]$$

(c) $\{c_k\} = \{ \cdots, 0, 1/4, 1/2, 1, \overbrace{2, 1, 1/2, 1/4, 0, \ldots}^{8}$
$\uparrow$

$2 + 2\cos \pi n/4 + \cos \pi n/4 + 1/2 \cos 3\pi n/4$

4.8] Two DT signals, $x_k(n)$ and $s_\ell(n)$; are said to be orthogonal over an integral interval $[N_1, N_2]$ if

$$\sum_{n=N_1}^{N_2} s_k(n) s_\ell^*(n) = \begin{cases} A_k \, ; & k = \ell \\ 0 \, ; & k \ne \ell . \end{cases}$$

If $A_k = 1$; the signals are called orthonormal

(a) Prove the relation.

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \begin{cases} N & k = 0, \pm N, \pm 2N, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

Sol:- If $k = 0, \pm N, \pm 2N, \ldots$

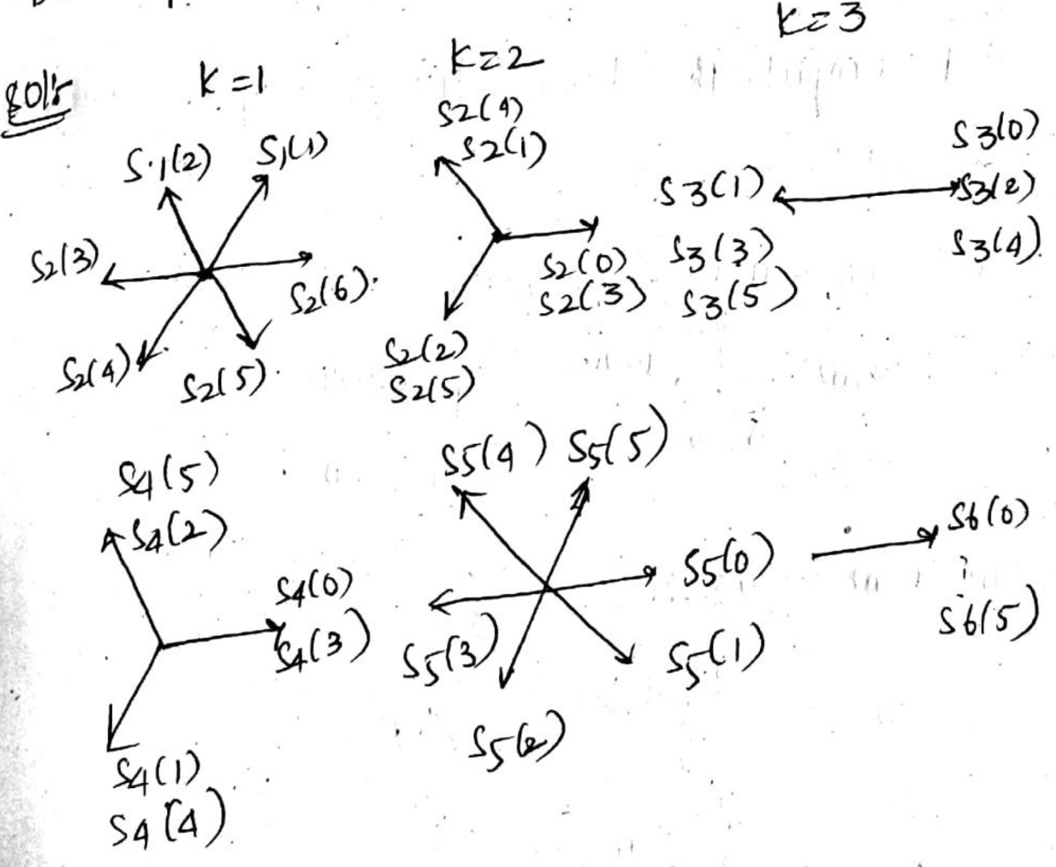
$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \text{sum}_{n=0}^{N-1} 1 = N.$$

If $k \neq 0, \pm N, \pm 2N, \ldots$

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \frac{1-e^{j2\pi k}}{1-e^{j2\pi k/N}} = 0.$$

(b) Illustrate the validity of the relation in part (a) by plotting for every value of $k = 1, 2, \ldots, 6$ the signals $s_k(n) = e^{j(2\pi/6)kn}$; $n = 0, 1, \ldots, 5$ [Note: for a given k, n the signal $s_k(n)$ can be represented as a vector in the complex plane].

Sol:-



(c) Show that the harmonically related signals

$s_k(n) = e^{j(2\pi/N)kn}$ are orthogonal

over any interval of length N.

$$\sum_{n=0}^{N-1} s_k(n) s_l^*(n) = \sum_{n=0}^{N-1} e^{j2\pi kn/N} e^{-j2\pi ln/N}$$

$$= \sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N}$$

$= N, k = l;$

$= 0; k \neq l.$

Therefore the $\{s_k(n)\}$ are orthogonal.

4.9 Compute the Fourier transform of the following signals.

(a) $x(n) = u(n) - u(n-6)$.

$$X(\omega) = \sum_{n=0}^{5} e^{-j\omega n} = \frac{1-e^{-j6\omega}}{1-e^{j\omega}}$$

(b) $x(n) = 2^n u(-n)$

$$= \sum_{-\alpha}^{0} 2^n e^{j\omega n} = \sum_{0}^{\alpha} (\ldots)^{-n} \frac{1}{1-\ldots}$$

$$= \sum_{0}^{\alpha} 2^{-n} e^{j\omega n} = \sum_{0}^{\alpha} \left(\frac{e^{j\omega}}{2}\right)^n$$

$$= \frac{1}{1-e^{j\omega}/2} = \frac{2}{2-e^{j\omega}}$$

(c) $x(n) = (1/4)^n u(n+4)$.

$$X(\omega) = (1/4)^{-4}\delta(n+4) + (1/4)^{-3}\delta(n+3) + (1/4)^{-2}\delta(n+2)$$

$$+ (1/4)^{-1}\delta(n+1) + \sum_{n=0}^{\alpha}(1/4)^n$$

$$\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \sum_{n=-4}^{\alpha}(1/4)^n e^{-j\omega n}$$

$$= \left(\sum_{m=0}^{\alpha}(1/4)^m e^{-j\omega n}\right) 4^4 e^{j4\omega}$$

$$= \frac{4^4 e^{j4\omega}}{1-\frac{1}{4}e^{-j\omega}}$$

$$\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^{m-4} e^{-j\omega(m-4)}$$

$n+4=0 = m=0$

$n+4=m$

$m=m$

(d) $x(n) = (\alpha^n \sin\omega_0 n)u(n)$; $|\alpha|<1$

$x(n) = \alpha^n \sin\omega_0 n\, u(n)$; $|\alpha|<1$.

Sol:- $x(n) = \alpha^n \sin\omega_0 n\, u(n)$;

$$X(\omega) = \sum_{n=0}^{\alpha}\alpha^n\left[\frac{e^{j\omega_0 n}-e^{-j\omega_0 n}}{2j}\right]e^{-j\omega n}$$

$$\sum_{m=-\infty}^{\infty}\alpha^{|m|}$$

$$= \frac{1}{2j}\sum_{n=0}^{\alpha}\left[\alpha e^{-j(\omega-\omega_0)}\right]^n$$

$$\sum_{m=0}^{\infty} \quad - \frac{1}{2j}\sum_{n=0}^{\alpha}\left[\alpha e^{-j(\omega+\omega_0)}\right]^n$$

$$= \frac{1}{2j}\left[\frac{1}{1-\alpha e^{-j(\omega-\omega_0)}} - \frac{1}{1-\alpha e^{-j(\omega+\omega_0)}}\right]$$

$$= \frac{1}{2j}\left[\frac{-\alpha e^{-j\omega}\cdot e^{-j\omega_0} + \alpha e^{-j\omega}\cdot e^{j\omega_0}}{1-2\alpha\cos\omega_0 e^{-j\omega}+\alpha^2 e^{-j2\omega}}\right]$$

$$= \frac{\alpha\sin\omega_0 e^{-j\omega}}{1-2\alpha\cos\omega_0 e^{-j\omega}+\alpha^2 e^{-j2\omega}}$$

(e) $x(n) = |\alpha|^n \sin w_0 n$; $|\alpha| < 1$.

Note that $\sum_{n=-\alpha}^{\alpha} |x(n)| = \sum_{n=-\alpha}^{\alpha} |\alpha|^n |\sin w_0 n|$.

Suppose that $w_0 = \pi/2$; so that $|\sin w_0 n| = 1$

$$\sum_{n=-\alpha}^{\alpha} (a)^n = \sum_{n=-\alpha}^{\alpha} |x(n)| \to \alpha.$$

Therefore ; the fourier transform doesnot exist.

(f) $x(n) = \begin{cases} 2-(1/2)^n & ; |n| \leq 4. \\ 0 & ; \text{elsewhere.} \end{cases}$

Sol:- $X(\omega) = \sum_{-4}^{4} (2-1/2)^n e^{-j\omega n}$

$$= \sum_{-4}^{4} 2e^{-j\omega n} - \sum_{n=-4}^{4} (1/2)^n e^{-j\omega n}$$

$$= 2\left[e^{-4j\omega} + e^{-3j\omega} + \dots + 1 + e^{j\omega} + e^{2j\omega} + \cancel{\dots} e^{4j\omega}\right]$$

$$- \left[(1/2)^{-4} e^{j\omega 4} + (1/2)^{-3} e^{3j\omega} + (1/2)^{-2} e^{2j\omega} + 2e^{j\omega} + 1 + (1/2)e^{-j\omega} + (1/2)^2 e^{-2j\omega} + (1/2)^3 e^{-3j\omega} + (\tfrac{1}{2})^4 e^{-4j\omega}\right]$$

$$= 2 \cdot \frac{e^{+4j\omega}}{1-e^{j\omega}}$$

(g) $x(n) = \{-2, -1, \underset{\uparrow}{0}, 1, 2\}$

$$X(\omega) = -2e^{j2\omega} - e^{j\omega} + e^{j\omega} + 2e^{-j2\omega}$$

$$= -2j\,[2\sin(2\omega) + \sin n\omega].$$

(h) $x(n) = \begin{cases} A(2M+1-|n|) ; & |n| \le M \\ 0 ; & |n| > M. \end{cases}$

sol:- $X(\omega) = \sum_{n=-M}^{M} A(2M+1-|n|)\, e^{-j\omega n}$

$$= \sum_{n=-M}^{-1} A(2M+1+n)\, e^{+j\omega n} + \sum_{n=0}^{M} A(2M+1-n)\, e^{-j\omega n}$$

$$= (2M+1)A + A\sum_{k=1}^{M}(2M+1-k) * (e^{-j\omega k} + e^{j\omega k})$$

$$= (2M+1)A + 2A\sum_{k=1}^{M}(2M+1-k)\cos(\omega k)$$

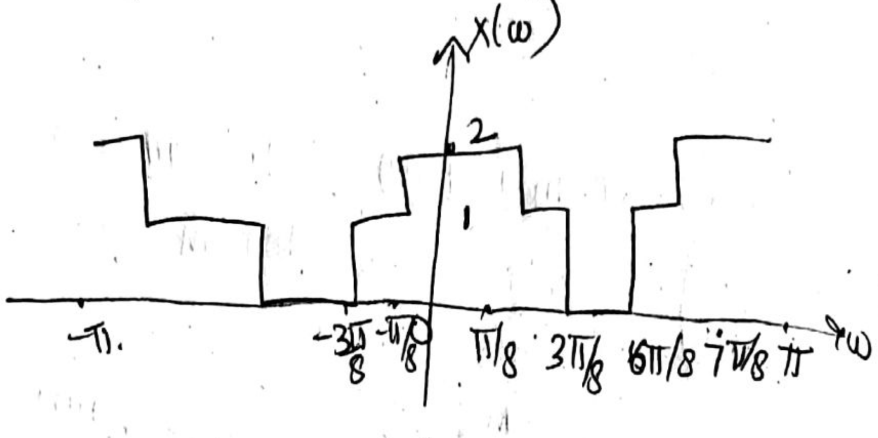
4.10] Determine the signals having the following Fourier transforms.

(a) $X(\omega) = \begin{cases} 0 ; & 0 \le |\omega| \le \omega_0 \\ 1 & \omega_0 < |\omega| \le \pi \end{cases}$

(b) $X(\omega) = \cos^2\omega$

(c) $X(\omega) = \begin{cases} 1, & \omega_0 - \delta\omega/2 \le |\omega| \le \omega_0 + \delta\omega/2 \\ 0; & \text{elsewhere.} \end{cases}$

(d) The signal shown in figure.



Sol:- (a) $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\,d\omega$

$$= \frac{1}{2\pi}\left[\int_{\omega_0}^{\pi} e^{j\omega n}\,d\omega + \int_{-\pi}^{-\omega_0} e^{+j\omega n}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\Bigg|_{\omega_0}^{\pi} + \frac{e^{j\omega n}}{jn}\Bigg|_{-\pi}^{-\omega_0}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\cancel{1} - e^{j\omega_0 n} + e^{-jn\omega_0} \cancel{-1}}{jn}\right]$$

$$= \frac{-1}{2\pi jn}\left[e^{j\omega_0 n} - e^{-jn\omega_0}\right] = -\frac{\sin(n\omega_0)}{n\pi};\quad n \neq 0$$

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(b) $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1+\cos 2\omega}{2}\right)e^{j\omega n}d\omega$

$$= \frac{1}{4\pi}\left[\int_{-\pi}^{\pi} e^{j\omega n} + \frac{1}{2}\int_{-\pi}^{\pi} e^{2j\omega}e^{j\omega n}d\omega + \frac{1}{2}\int_{-\pi}^{\pi} e^{-2j\omega}e^{j\omega n}d\omega\right]$$

$$= \frac{1}{4\pi}\left[\frac{e^{j\omega n}}{jn}\Big|_{-\pi}^{\pi} + \frac{1}{2}\,\frac{e^{j\omega(2+n)}}{j(2+n)}\Big|_{-\pi}^{\pi} + \frac{1}{2}\,\frac{e^{j\omega(-2+n)}}{j(n-2)}\Big|_{\pi}^{\pi}\right]$$

$$= \frac{1}{4\pi}\left[6n\right.$$

$$X(\omega) = 1/4\,(e^{j2\omega}+2+e^{-2j\omega})$$

$$x(n) = \frac{1}{4}\left[\delta(n+2)+2\delta(n)+\delta(n-2)\right].$$

(c) - $X(\omega) = \{$

$$x(n) = \frac{1}{2\pi}\left[\int_{\omega_0-\frac{\delta\omega}{2}}^{\omega_0+\delta\omega/2} 1.\,e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{-\omega_0+\delta\omega/2}^{-\omega_0-\frac{\delta\omega}{2}} e^{j\omega n}d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\Big|_{\omega_0-\frac{\delta\omega}{2}}^{\omega_0+\delta\omega/2} + \frac{e^{j\omega n}}{jn}\Big|_{-\omega_0+\delta\omega/2}^{-\omega_0-\delta\omega/2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_0 n}\sin(n\delta\omega/2)}{n\delta\omega/2)}\right]$$

(d) $x(n) = \frac{1}{2\pi} Re\left\{ \int_0^{\pi/8} 2 e^{j\omega n} d\omega + \int_{6\pi/8}^{7\pi/8} e^{j\omega n} d\omega + \int_{\pi/8}^{\pi} 2 e^{j\omega n} d\omega \right\}$

$$= \frac{1}{\pi}\left[\int_0^{\pi/8} 2\cos\omega n\, d\omega + \int_{\pi/8}^{3\pi/8} \cos\omega n\, d\omega + \int_{6\pi/8}^{7\pi/8} \cos\omega n\, d\omega + \int_{7\pi/8}^{\pi} 2\cos\omega n\, d\omega \right]$$

$$= \frac{1}{\pi n}\left[\sin\frac{\pi}{8}n + \sin\frac{6\pi n}{8} - \sin\frac{3\pi n}{8} - \sin\frac{7\pi n}{8} \right].$$

4.11. Consider the signal $x(n) = \{1, 0, -1, \underset{\uparrow}{2}, 3\}$

with fourier transform $X(\omega) = X_R(\omega) + jX_I(\omega)$. Determine and sketch the signal $y(n)$ with fourier transform

$$Y(\omega) = X_I(\omega) + X_R(\omega) e^{j2\omega}.$$

sol:-

$$x_e(n) = \frac{x(n) + x(-n)}{2}$$

$$= \{\tfrac{1}{2}, 0, 1, \underset{\uparrow}{2}, 1, 0, \tfrac{1}{2}\}$$

$$x_o(n) = \frac{x(n) - x(-n)}{2}$$

$$= \{\tfrac{1}{2}, 0, -2, \underset{\uparrow}{0}, 2, 0, \tfrac{1}{2}\}$$

Then $X_R(\omega) = \sum_{n=-3}^{3} x_e(n) e^{-j\omega n}$

$jX_I(\omega) = \sum_{n=-3}^{3} x_o(n) e^{-j\omega n}$

Now; $Y(\omega) = X_I(\omega) + X_R(\omega) e^{j2\omega}$; Therefore

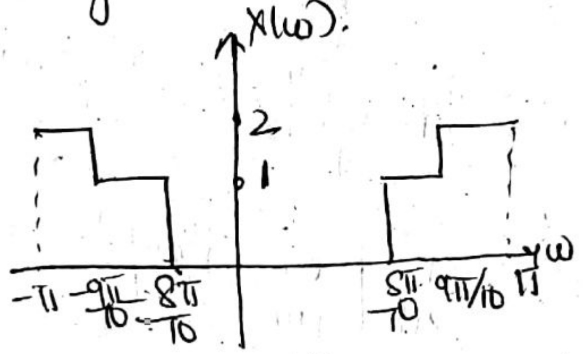
$y(n) = F^{-1}\{X_I(\omega)\} + F^{-1}\{X_R(\omega) e^{j2\omega}\}$.

$= -jx_o(n) + x_e(n+2)$

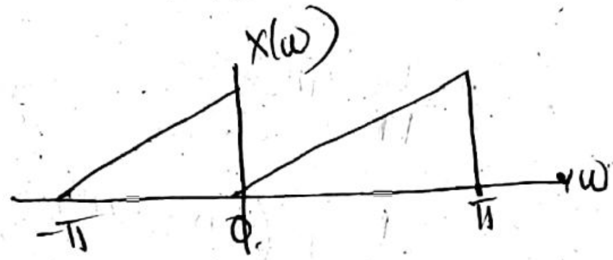
$= \{1/2, 0, 1-j/2, 2, 1+j/2, \underset{\uparrow}{0}, \frac{1}{2}-2j, 0, 1/2\}$

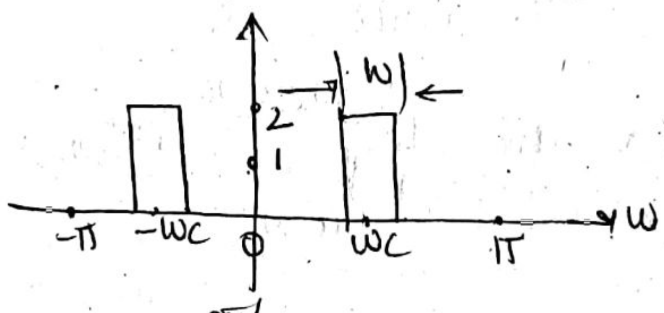
4.12] Determine the signal $x(n)$ if its Fourier transform is as given.

Sol:- (a)



$x(n) =$





$$x(n) = \frac{1}{2\pi}\left[\int_{8\pi/10}^{9\pi/10} e^{j\omega n} d\omega + \int_{-9\pi/10}^{-8\pi/10} e^{j\omega n} d\omega + 2\int_{9\pi/10}^{\pi} e^{j\omega n} d\omega + 2\int_{-\pi}^{-9\pi/10} e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{jn}\left(e^{j9\pi n/10} - e^{-j9\pi n/10} - e^{j8\pi n/10} + e^{-j8\pi n/10}\right)\right] +$$

$$\frac{2}{jn}\left(-e^{-j9n\pi/10} + e^{-j9\pi n/10} + e^{j\pi n} - e^{-j\pi n}\right)\Big]$$

$$= \frac{1}{n\pi}\left[\sin \pi n - \sin 8\pi n/10 - \sin 9\pi n/10\right]$$

$$= \frac{-1}{n\pi}\left[\sin \frac{4\pi n}{5} + \sin \frac{9\pi n}{10}\right]$$

(b) $$x(n) = \frac{1}{2\pi}\int_{-\pi}^{0} X(\omega)e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{0}^{\pi} X(\omega)e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0}\left(\frac{\omega}{\pi}+1\right)e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{0}^{\pi}\frac{\omega}{\pi}e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{\omega}{jn\pi}e^{j\omega n}\Big|_{-\pi}^{\pi} + \frac{e^{j\omega n}}{jn}\Big|_{-\pi}^{0}\right]$$

$-\pi$

$$= \frac{1}{\pi n}\sin\frac{\pi n}{2} \qquad \frac{\times 1}{2\pi}\,\frac{\pi e^{j\pi n} + \pi e^{-jn\pi}}{jn\pi} \qquad 1+e$$

$$\frac{1}{\pi n}\sin\left(\frac{\pi n}{2}\right)e^{-jn\pi/2}\ \checkmark. \qquad \underline{1-e^{-jn\pi}}$$

(c) $$x(n) = \frac{1}{2\pi}\int_{\omega_c - W/2}^{\omega_c + W/2} 2e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{-\omega_c - W/2}^{-\omega_c + W/2} 2e^{j\omega n}d\omega$$

$$= \frac{2}{2\pi}\left[\frac{e^{j\omega n}}{jn}\Big|_{\omega_c - W/2}^{\omega_c + W/2} + \frac{2}{2\pi}\,\frac{e^{j\omega n}}{jn}\Big|_{-\omega_c - W/2}^{-\omega_c + W/2}\right.$$

$$= \frac{2}{\pi n}\left[\sin(\omega_c + W/2)n - \sin(\omega_c - W/2)n\right].$$

13. The Fourier transform of the signal.

$$x(n) = \begin{cases} 1; & -M \le n \le M \\ 0; & \text{otherwise} \end{cases}$$

was shown to be.

$$X(\omega) = 1 + 2 \sum_{n=1}^{M} \cos(\omega n).$$

was shown that the Fourier transform of

$$x_1(n) = \begin{cases} 1; & 0 \le n \le M \\ 0; & \text{otherwise} \end{cases}$$ and

$$x_2(n) = \begin{cases} 1; & -M \le n \le -1 \\ 0 & \text{otherwise} \end{cases}$$

are respectively;

$$X_1(\omega) = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}}$$

$$X_2(\omega) = \frac{e^{j\omega} - e^{j\omega(M+1)}}{1 - e^{j\omega}}$$

Thus prove that $X(\omega) = X_1(\omega) + X_2(\omega)$

$$= \frac{\sin(M + 1/2)\omega}{\sin(\omega/2)}$$

and therefore

$$1 + 2 \sum_{n=1}^{M} \cos\omega n = \frac{\sin(M + 1/2)\omega}{\sin(\omega/2)}$$

and therefore

$$1 + 2 \sum_{n=1}^{M} \cos\omega n = \frac{\sin(M + 1/2)\omega}{\sin(\omega/2)}$$

Sol:- $X_1(\omega) = \int_0^M 1 . e^{-j\omega n} d\omega$

$$= \frac{e^{-j\omega n}}{-jn}\Big|_0^M = \frac{1}{-jn}\left[e^{-jnM} - 1\right]$$

$$\Rightarrow \frac{2}{n} e^{-jnM/2}\left[\frac{e^{jnM/2} - e^{-jnM/2}}{2j}\right]$$

$$\Rightarrow \frac{2}{n} e^{-jnM/2} \sin n$$

$$= \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}}$$

$$X_2(\omega) = \sum_{n=-M}^{-1} 1 . e^{-jn\omega}$$

$$= \sum_{n=1}^{M} e^{jn\omega} = \frac{e^{j\omega}(1 - e^{j\omega(M)})}{1 - e^{j\omega}}$$

$$X_1(\omega) + X_2(\omega) = \frac{(1 - e^{-j\omega M} e^{-j\omega})(1 - e^{j\omega}) + (1 - e^{-j\omega}) e^{j\omega}(1 - e^{j\omega M})}{(1 - e^{j\omega})(1 - e^{-j\omega})}$$

$$= \frac{\cancel{1} - e^{-j\omega M} e^{-j\omega} - \cancel{e^{j\omega}} + e^{-j\omega M} + \cancel{e^{j\omega}} - e^{j\omega(M+1)} - \cancel{1} + e^{j\omega M}}{1 - e^{-j\omega} - e^{j\omega} + 1}$$

$$\Rightarrow \frac{e^{j\omega M} + e^{-j\omega M} - e^{j\omega(M+1)} - e^{-j\omega(M+1)}}{2 - 2\cos\omega}$$

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$$\frac{2\cos(wm) - 2\cos(w(m+1))}{2(1-\cos w)}$$

$$\Rightarrow \frac{\cos wm - \cos(wm+w)}{1-\cos w} \qquad \cos A - \cos B \Rightarrow 2\sin\frac{(A+B)}{2}$$

$$\Rightarrow \frac{\cos w\cancel{m} - \cos wm \cos w - \sin wm \sin w}{2\sin^2 w/2}$$

$$\Rightarrow \frac{2\sin(wm + w/2)\cos w/2}{2\sin^2 w/2}$$

$$= \frac{\sin(m+1/2)w}{\sin w/2)}$$

4.14] Consider the signal

$$x(n) = \{-1, 2, \underset{\uparrow}{-3}, 2, -1\}$$

with Fourier transform $X(\omega)$. Compute the following quantities; without explicitly computing $X(\omega)$:-

(a) $X(0)$ (b) $\angle X(\omega)$ (c) $\int_{-\pi}^{\pi} X(\omega)\,d\omega$ (d) $X(\pi)$ (e) $\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$

Sol:- $X(\omega) = \sum_{n=-2}^{2} x(n) e^{-jn\omega}$ $\Rightarrow X(0) = \sum_{n=-2}^{2} x(n)$

$$= \{-1\}$$

(b) $\angle X(\omega) = \pi$ for all ω.

(c) $x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)d\omega$. Hence $\int_{-\pi}^{\pi} X(\omega)d\omega = 2\pi x(0)$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)d\omega = 2\pi(-3) = -6\pi$$

(d) $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$

$$X(\pi) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\pi n} = \sum_{n=-\infty}^{\infty} (-1)^n x(n)$$

$$= (-1)(+1) + (2)(-1) + -3 + (2)(-1) + (-1)$$

$$= -9$$

(e) $\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega = 2\pi \cdot \sum_n |x(n)|^2 = (2\pi)\left[1+4+9+4+1\right.$

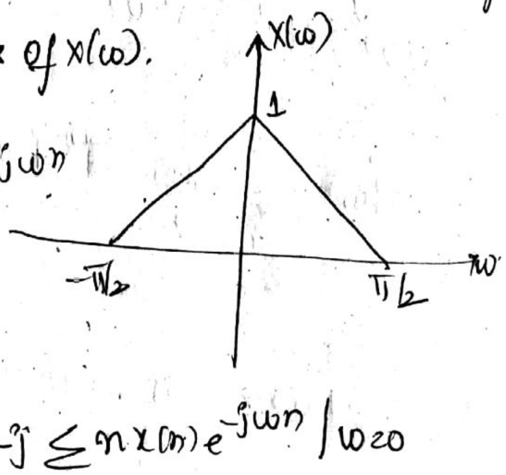
$$= 38\pi W.$$

15] The center of gravity of a signal is defined as

$$c = \sum_{n=-\infty}^{\infty} n x(n) \Big/ \sum_{n=-\infty}^{\infty} x(n).$$

and provides the measure of "time delay" of the signal.

(a) Express c in terms of $X(\omega)$.



Sol:- $X(\omega) = \sum_n x(n)e^{-j\omega n}$

$$X(0) = \sum_{n=-\infty}^{\infty} x(n)$$

$$\left.\frac{dX(\omega)}{d\omega}\right|_{\omega=0} = -j\sum n x(n)e^{-j\omega n}\Big|_{\omega=0}$$

$$= -j\sum_n n x(n).$$

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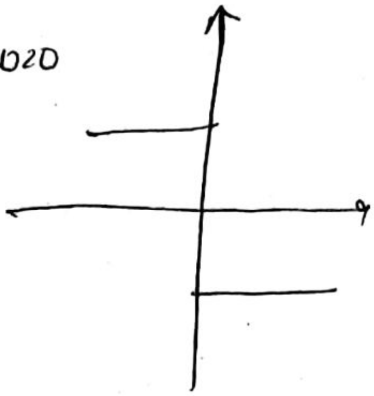
Therefore $c = \dfrac{j\frac{dX(\omega)}{d\omega}\Big|_{\omega=0}}{X(\omega)}$



(b) $X(0)=1$; $c = \frac{0}{1} = 0$

16] Consider the Fourier transform pair

$$a^n u(n) \xleftrightarrow{F} \frac{1}{1-ae^{-j\omega}} \;;\quad |a|<1.$$

Use the differentiation and in frequency domain and induction to show that

$$x_k(n) = \frac{(n+l-1)!}{n!\,(l-1)!} a^n u(n) \xleftrightarrow{F} X(\omega) = \frac{1}{(1-ae^{-j\omega})^l}.$$

Sol:- $x_1(n) \equiv a^n u(n)$; $\xleftrightarrow{F} \dfrac{1}{1-ae^{-j\omega}}$

Now suppose that.

$$x_k(n) = \frac{(n+k-1)!}{n!\,(k-1)!} a^n u(n)$$

proving this through $x_{k+1}(n)$.

Let $k = k+1$

$$\Rightarrow x_{k+1}(n) = \frac{(n+k+1-1)!}{n!\,(k+1-1)!} a^n u(n)$$

$$= \frac{(n+k)!}{n!\,k!} a^n u(n)$$

17 Let $x(n)$ be an arbitrary signal, not necessarily real valued, with Fourier transform $X(\omega)$. Express the Fourier transforms of the following signals in terms of $X(\omega)$.

(a) $x^*(n)$.

$$\sum_n x^*(n) e^{-j\omega n} = \left(\sum_n x(n) e^{-j(-\omega)n} \right)^*$$

$$= X^*(-\omega).$$

(b) $x^*(-n)$

$$\sum_n x^*(-n) e^{-j\omega n} = \sum_n x^*(n) e^{j\omega n}$$

$$= X^*(\omega).$$

(c) $y(n) = x(n) - x(n-1)$ $Y(\omega) = X(\omega)(1 - e^{-j\omega})$

(d) $y(n) = \sum_{k=-\infty}^{n} x(k)$

$x(n) = y(n) - y(n-1)$

$\Rightarrow \quad X(\omega) = (1-e^{-j\omega}) Y(\omega)$

$\Rightarrow \quad Y(\omega) = \dfrac{X(\omega)}{1-e^{-j\omega}}$

(e) $y(n) = x(2n)$; $Y(\omega) = \sum_{n} x(2n) e^{-j\omega n}$

$Y(\omega) = \sum_{n} x(n) e^{-j\frac{\omega}{2} n} = X(\omega/2)$

(f) $y(n) = \begin{cases} x(n/2) ; & n \text{ even} \\ 0 ; & n \text{ odd} \end{cases}$

$Y(\omega) = \sum_{n} x(n/2) e^{-j\omega n} = \sum_{n} x(n) e^{-j2\omega n}$

$= X(2\omega)$

4.18] Determine and sketch the Fourier transforms $X_1(\omega)$, $X_2(\omega)$ and $X_3(\omega)$? What is its physical of the following signals

(a) $x_1(n) = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$

(b) $x_2(n) = \{1, 0, 1, 0, \underset{\uparrow}{1}, 0, 1, 0, 1\}$

(c) $x_3(n) = \{1, 0, 0, 1, 0, 0, \underset{\uparrow}{1}, 0, 0, 1, 0, 0, 1\}$

(d) Is there any relation b/n $X_1(\omega)$, $X_2(\omega)$ & $X_3(\omega)$? What is its physical meaning?

(e) Show that if $x_k(n) = \begin{cases} x(n/k) & \text{if } n/k \text{ integer} \\ 0 & \text{otherwise} \end{cases}$

then $X_k(\omega) = X(k\omega)$.

Sol:- $X_1(\omega) = 1 + 2\cos(\omega) + 2\cos(2\omega)$

$X_2(\omega) = 1 + 2\cos(2\omega) + 2\cos(4\omega)$

$X_3(\omega) = 1 + 2\cos(3\omega) + 2\cos(6\omega)$.

(d) $X_2(\omega) = X_1(2\omega)$; $X_3(\omega) = X_1(3\omega)$

(e) If $x_k(n) = \begin{cases} x(n/k) ; & n/k \text{ an integer} \\ 0 ; & \text{otherwise.} \end{cases}$

then $X_k(\omega) = \sum_{n,\ n/k \text{ an integer}} x_k(n) e^{-j\omega n}$

$= \sum_{n} x(n) e^{-jk\omega n} = X(k\omega)$

16 sol: Let $x(n) = \alpha^n u(n)$. Then

$$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$. Using the differentiation in the frequency property;

$$n\alpha^n u(n) \longleftrightarrow j\cdot\frac{dX(e^{j\omega})}{d\omega} = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

$$(n+1)\alpha^n u(n) \xleftrightarrow{F.T.} j\frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega})$$

$$= \frac{1}{(1-\alpha e^{-j\omega})^2}$$

Let us assume that the given expression is true for $k = \cancel{r}-1$; [as it is true for $r=2$ and $r=1$].

$$x_{r-1}[n] = \frac{(n+r-2)!}{n!\,(r-2)!}\alpha^n u(n)$$

$$\xleftrightarrow{F.T.} X_{r-1}(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^{\cancel{r-1}}}$$

From the differentiation in frequency domain;

$$n x_{r-1}(n) \xleftrightarrow{F.T} \frac{\alpha(r-1)e^{-j\omega}}{(1-\alpha e^{-j\omega})^{r}}$$

and $$\frac{n x_{r-1}(n) + x_{r-1}(n)}{r-1} \xleftrightarrow{F.T} 1-\alpha e^{-j\omega} + \frac{\alpha(r-1)e^{-j\omega}}{r-1}$$

QS

Therefore ;

$$\frac{(n+1)x_{r-1}[n+1]}{\alpha(r-1)} \overset{F.T}{\longleftrightarrow} \frac{1}{(1-\alpha e^{-j\omega})^{r}}$$

The left hand side of above expression is

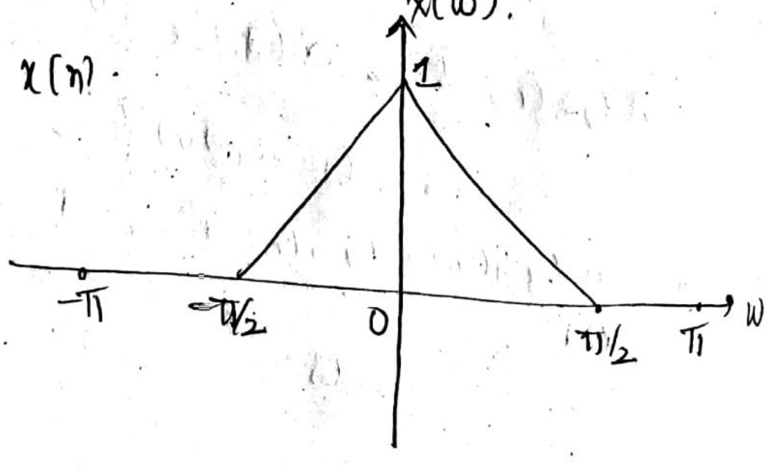
$$\frac{(n+1)\,x_{r-1}(n+1)}{\alpha(r-1)} = \frac{(n+r-1)!}{n!\,(r-1)!}\alpha^{n}u(n) = x_r(n)$$

Therefore ; we have shown that the result is valid for r if it is valid for r-1. Since we know that the result is valid for r=2; we may conclude that it is valid for r=3,4 and so....

4.19] Let x[n] be a signal with Fourier transform as shown. Determine and sketch the Fourier transforms of the following signals.

Sol: (a) $x_1(n) = x(n)\cos(\pi n/4)$

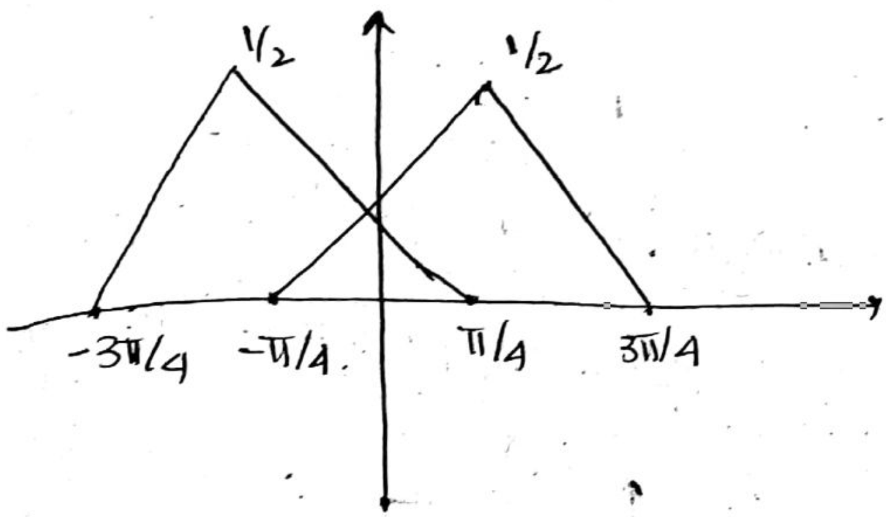
Sol:- x(n).



[Note that these signals are obtained by the amplitude modulation of a carrier $\cos\omega_c n$ or $\sin\omega_c n$ by the sequence $x(n)$].

Sol: $x(n)\cos(\pi n/4) \Rightarrow \frac{x(n)}{2}\left[e^{j\pi n/4} + e^{-j\pi n/4}\right]$

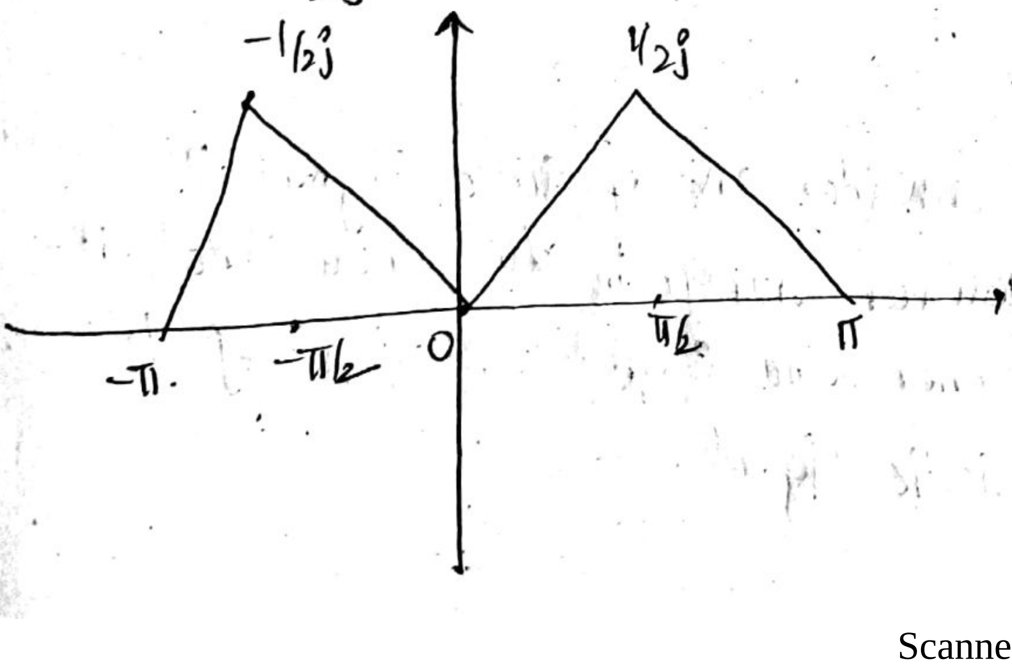
$\xleftrightarrow{F.T} \frac{1}{2}\left[X(\omega - \pi/4) + X(\omega + \pi/4)\right]$.



$x_2(n) = x(n)\sin(n\pi/2)$.

Sol: $x_2(n) = \frac{1}{2j}\left[e^{j\pi n/2} - e^{-j\pi n/2}\right] x(n)$.

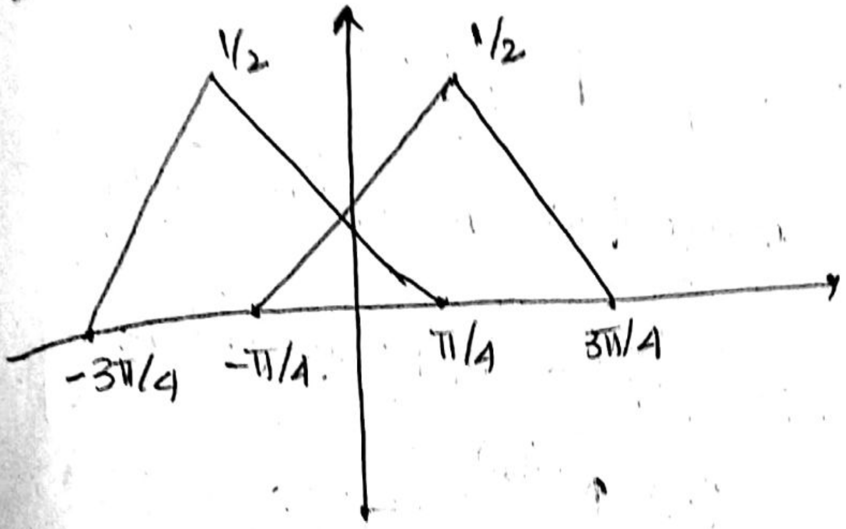
$= \frac{1}{2j}\left[X(\omega - \pi/2) - X(\omega + \pi/2)\right]$.



[Note that these signals are obtained by the amplitude modulation of a carrier cosine or sine n by the sequence $x(n)$].

sol: $x(n)\cos(\pi n/4) \rightleftharpoons \frac{x(n)}{2}\left[e^{j\pi n/4} + e^{-j\pi n/4}\right]$

$\xleftrightarrow{F.T} \frac{1}{2}\left[X(\omega-\pi/4) + X(\omega+\pi/4)\right]$.

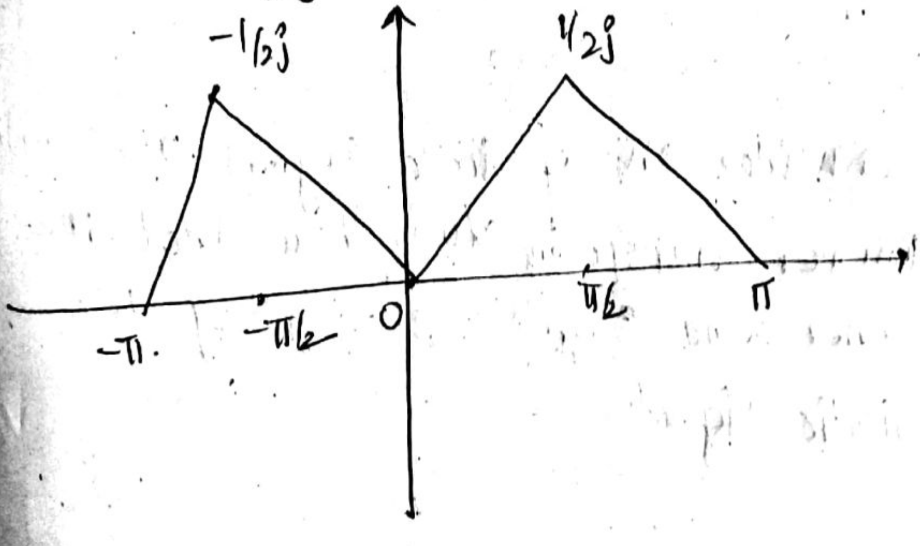


$x_2(n) = x(n)\sin(n\pi/2)$.

$= \left[e^{j\pi n/2} - e^{-j\pi n/2}\right] x(n)$.

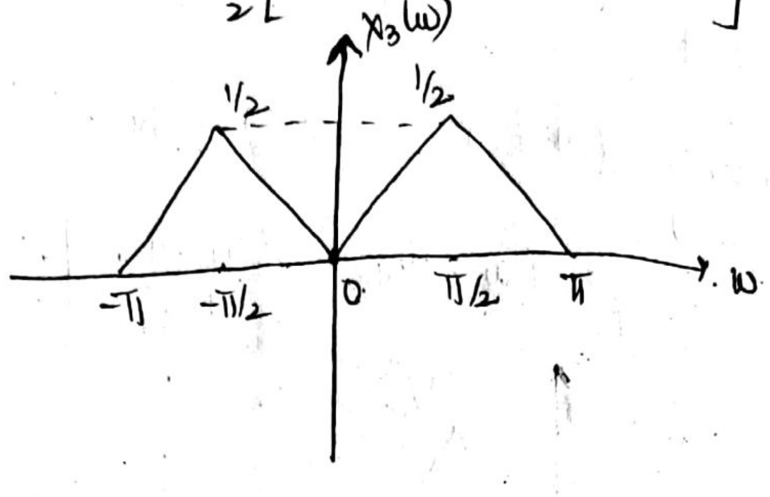
sol: $x_2(n) = \frac{1}{2j}\left[e^{j\pi n/2} - e^{-j\pi n/2}\right]$

$= \frac{1}{2j}\left[X(\omega-\pi/2) - X(\omega+\pi/2)\right]$.



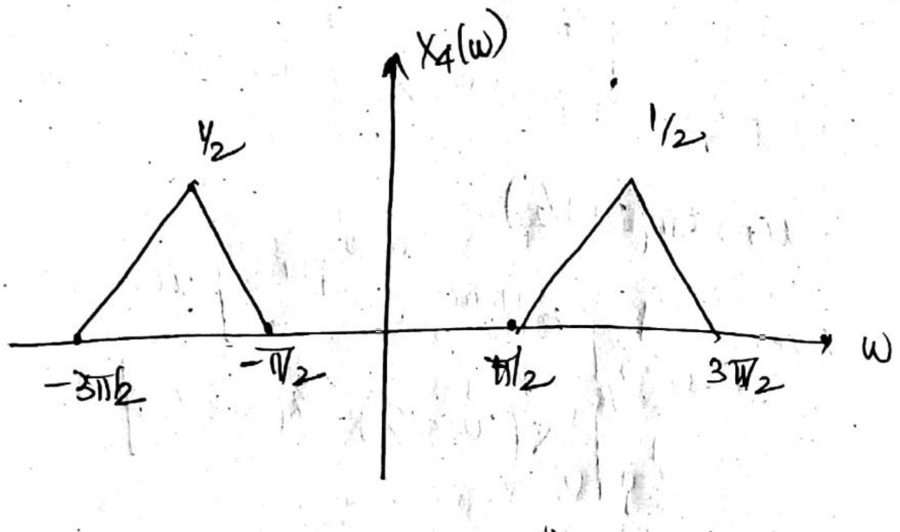
$$x_3(n) = x(n)\cos(\pi n/2) = \frac{x(n)}{2}\left[e^{j\pi n/2} + e^{-j\pi n/2}\right]$$

$$\xleftrightarrow{F.T} \frac{1}{2}\left[X(\omega - \pi/2) + X(\omega + \pi/2)\right]$$



$$x_4(n) = x(n).\cos(\pi n) = \frac{x(n)}{2}\left[e^{j\pi n} + e^{-j\pi n}\right]$$

$$\xleftrightarrow{F.T} \frac{1}{2}\left[X(\omega - \pi) + X(\omega + \pi)\right]$$



20]. Consider an aperiodic signal $x(n)$ with Fourier transform $X(\omega)$. Show that the Fourier series coeffecients of C_k of the periodic signal.

$$y(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

are given by

$$c_k^y = \frac{1}{N} X\left(\frac{2\pi}{N} \cdot k\right); \quad k=0,1,\dots,N-1.$$

Sol:-

$$c_k^y = \frac{1}{N} \sum_{n=0}^{N-1} y(n) e^{-j2\pi kn/N}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN) \right] e^{-j2\pi kn/N}$$

$$= \frac{1}{N} \sum_{l=-\infty}^{\infty} \sum_{m=-lN}^{N-1-lN} x(m) e^{-j2\pi k(m+lN)/N}.$$

But $\sum_{l=-\infty}^{\infty} \sum_{m=-lN}^{N-1-lN} x(m) e^{-j\omega(m+lN)} = X(\omega) \rightarrow (1)$

Therefore $c_k^y = \frac{1}{N} X(2\pi k/N)$.

Reason:- from (1);

let $l=0$; $\sum_{m=0}^{N-1} x(m) e^{-j\omega m} = X(\omega)$

$l=1$; $\sum_{m=-N}^{N-1-N} x(m) e^{-j\omega(m+N)} = X(\omega)$

as $X(\omega)$ is periodic and infinite duration; it always remains same.

21] Prove that $X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$.

may be expressed as

$$X_N(\omega) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{\sin[(2N+1)(\omega-\theta/2)]}{\sin(\omega-\theta)/2}$$

Sol:- Let $x_N(n) = \frac{\sin \omega_c n}{\pi n}$; $-N \le n \le N$.

$= x(n)\, w(n)$

where $x(n) = \frac{\sin \omega_c n}{\pi n}$ $\quad -\alpha \le n \le \alpha$

$w(n) = 1$; $\quad -N \le n \le N$.

$= 0$; $\quad$ otherwise.

Then $\frac{\sin \omega_c n}{\pi n} \xleftrightarrow{F} X(\omega) = 1$; $\quad |\omega| \le \omega_c$

$= 0$; otherwise.

as for $w(n)$ is rectangle

$$W(\omega) = \frac{\sin((2N+1)\omega)}{\sin(\omega/2)}$$

Now $X_N(\omega) = X(\omega) * W(\omega)$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta) \cdot W(\omega-\theta)\, d\theta$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} W(\omega-\theta)\, d\theta$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{\sin[(2N+1)(\omega-\theta)/2]}{\sin[(\omega-\theta)/2]}\, d\theta$$

22]. A signal $x(n)$ has following fourier transform

$$X(\omega) = \frac{1}{1-ae^{-j\omega}}.$$

Determine the fourier transforms of the following signals.

(a) $x(2n+1)$.

$$x(n+1) \longrightarrow X(\omega)e^{j\omega} \qquad \omega \Rightarrow \omega/2$$

$$x(2n+1) \longrightarrow X(\omega/2)e^{j\omega/2}$$

$$= \frac{e^{j\omega/2}}{1-ae^{-j\omega/2}}$$

doubt

(b) $e^{\pi n/2}x(n+2)$

$$x(n+2) \xrightarrow{DTFT} X(\omega)\, e^{j\omega 2}.$$

(or)

$$X_2(\omega) = \sum_n x(n+2) e^{\pi n/2} e^{-jn\omega}$$

Let $n+2 = k$ &

$$\sum_k x(k) e^{\pi\frac{(k-2)}{2}} e^{-j(k-2)\omega}$$

$$= \sum_k x(k) e^{-jk(\omega + j\pi/2)} \cdot e^{-\pi} e^{j2\omega}$$

$$= e^{-\pi} e^{j2\omega} X\left(\omega + j\pi/2\right)$$

(c) $x(-2n) \Rightarrow \sum_{n} x(-2n) e^{-j\omega n}$

Let $-2n = l$ $\quad \sum_{l} x(l) e^{-j\omega(-l/2)}$

$\Rightarrow \sum_{l} x(l) e^{-jl(-\omega/2)}$.

$\Rightarrow X(-\omega/2)$.

(d) $x(n)\cos(0.3\pi n) \Rightarrow \frac{x(n)}{2}\left[\cos(0.3\pi n)\right]$

$\Rightarrow \frac{x(n)}{2}\left[e^{j0.3\pi n} + e^{-j0.3\pi n}\right]$

$\Rightarrow \frac{1}{2}\left[X(\omega - 0.3\pi) + X(\omega + 0.3\pi)\right]$.

(e) $x(n) * x(n-1) \Rightarrow X_5(\omega) = X(\omega) \cdot \left[X(\omega) \cdot e^{-j\omega}\right]$

$= X^2(\omega) e^{-j\omega}$.

(f) $x(n) * x(-n) \Rightarrow X_6(\omega) = X(\omega) \cdot X(-\omega)$

$= \frac{1}{1 - ae^{-j\omega}} \cdot \frac{1}{1 - ae^{j\omega}}$

$\Rightarrow \frac{1}{1 + a^2 - 2a\cos\omega}$

23] From a discrete-time signal $x(n)$ with the Fourier transform $X(\omega)$; shown in figure: determine and sketch the Fourier transform

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Of the following signals:-

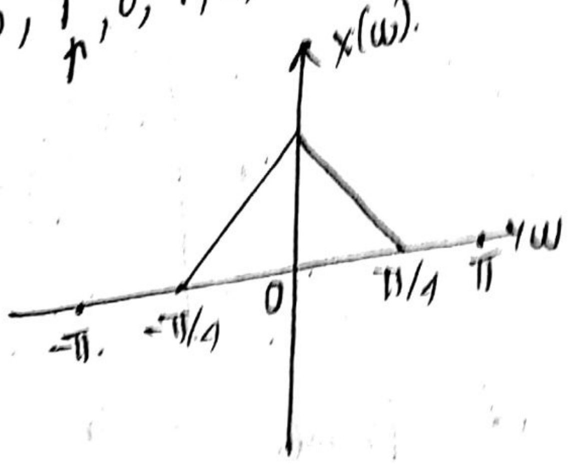
a]. $y_1(n) = \begin{cases} x(n); & n\text{ even} \\ 0; & n\text{ odd.} \end{cases}$

b] $y_2(n) = x(2n)$.

c] $y_3(n) = \begin{cases} x(n/2) & n\text{ even} \\ 0; & n\text{ odd.} \end{cases}$

Note that $y_1(n) = x(n) \cdot s(n)$ where

$s(n) = \{\ldots, 0, 1, 0, \underset{\uparrow}{1}, 0, 1, 0, 1 \ldots\}$



Sol:-

Let us first consider b.

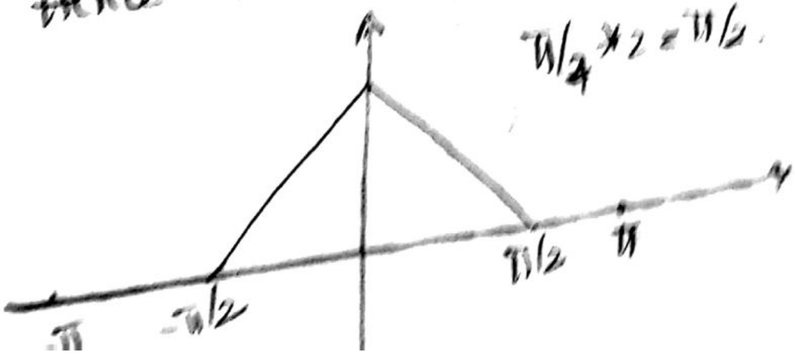
(b) $y_2(n) = x(2n)$

$$Y_2(\omega) = \sum_{n=-\infty}^{\infty} x(2n) e^{-j\omega n}$$

$$= \sum_{n} x(m) e^{-j\omega m/2}.$$

Hence $Y_2(\omega) = X(\omega/2)$. Hence expanding

$\pi/4 \times 2 = \pi/2$.

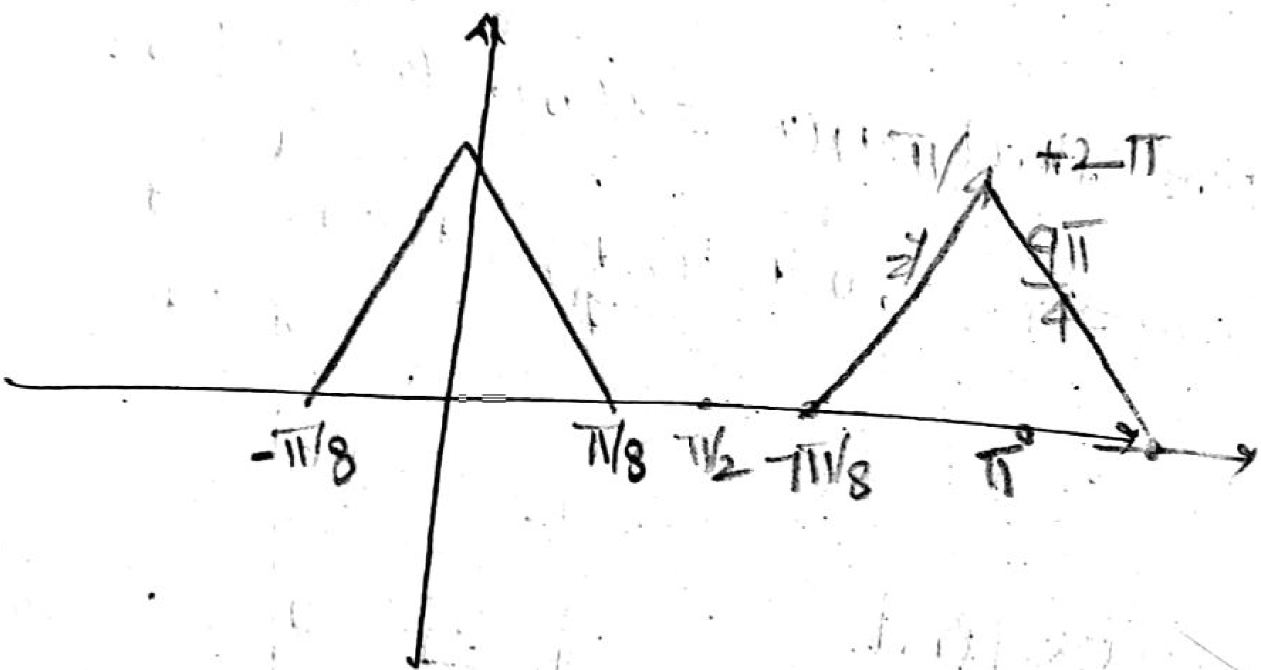


(c) $y_3(n) = x(n/2)$ even (n)

$$\Rightarrow \sum_{n=-\alpha \text{ even}}^{\alpha} x(n/2)\, e^{-jwn}.$$

Let $n/2 \Rightarrow l$ $$\Rightarrow \sum_{n=-\alpha \langle \text{all} \rangle}^{\alpha} x(l)\, e^{-jw2l}$$

$\Rightarrow X(2w) \rightarrow$ compression



doubt

(a) from (c) $\Rightarrow$ $y_B(n) = x(n)$ n even

$\Rightarrow x(n/2)$. ; n even

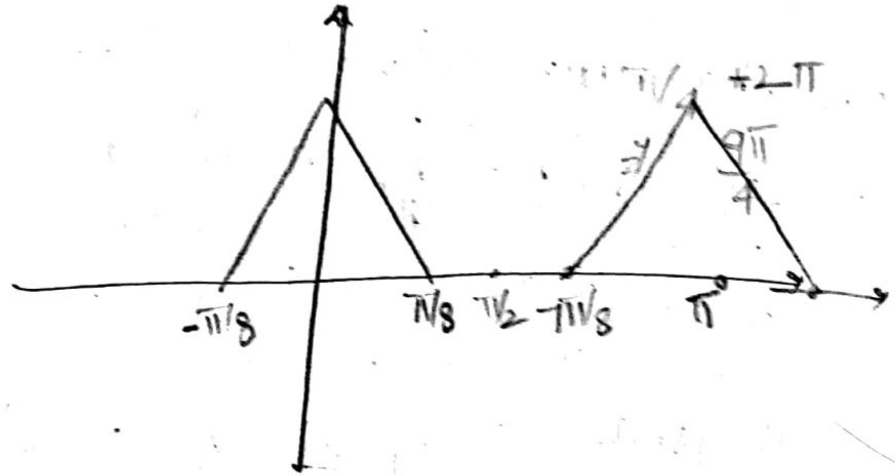
0 ; n odd

(c) $y_2(n) = x(n/2)$ even (n)

$\Rightarrow \sum_{n=-\infty \text{ even}}^{\infty} x(n/2)\, e^{-j\omega n}$

Let $n/2 \Rightarrow l$ $\quad \Rightarrow \sum_{n=-\infty}^{\infty} x(l)\, e^{-j\omega 2l}$

$\Rightarrow X(2\omega) \rightarrow$ compression



doubt

(a) from (c) $\Rightarrow y_B(n) = x(n)$ n even

$\Rightarrow x(n/2)$; n even

0 ; n odd