Hands-on 3: Dynamic Programming

Eduardo Venturini

Problem 01: Holyday planning

The solution to this problem uses dynamic programming.

For $n' \leq n$ and $D' \leq D$, let $r_{n',D'}$ be the maximum number of attractions the tourist can visit traveling in the first n' cities for exactly D' days.

We notice that $r_{n',D'}$ satisfy the following recursive relation:

$$r_{n'+1,D'} = \max_{i \in \{0 \ldots D'\}} \left(r_{n',D'-i} + \sum_{j=1}^{i} a_{n',j} \right)$$

($a_{n',j}$ is the attractions available in the n' city at day j).

Using this formula, we can easily calculate $r_{n',D'}$ for every value of n' and D', and then return as solution $r_{n,D}$.

We need to calculate nD different values of $r_{n',D'}$, and for each value, we need to find the maximum of D' numbers. Since we can precalculate all the sums $\sum_{j=1}^{i} a_{n',j}$ with nD operations, the total time complexity is $\Theta(nD^2)$.

The space complexity is $\Theta(D)$: we just need to store $r_{n',D'}$ for $0 < D' \leq D$, starting from $n' = 0$ and updating the vector at each iteration of n' until $n' = n$.

Problem 02: Xmas Lights

The solution to this problem uses dynamic programming.

Let $s_{n'}$ be the color of the n'-th house.

For $n' \leq n$ and $0 \leq D' \leq 3$, let $r_{n',D'}$ be the number of D'-uples of houses among the first n' houses and among all colorings of such n' houses, such that the first house (if $D' \geq 1$) is red, the second one (if $D' \geq 2$) is white and the third one (if $D' \geq 3$) is green.

We notice that $r_{n',D'}$ satisfy the following recursive relations:

$$r_{0,0} = 1 \qquad r_{n'+1,0} = \begin{cases} 3r_{n',0} & \text{if } s(n'+1) = X \\ r_{n',0} & \text{if } s(n'+1) \neq X \end{cases}$$

$$r_{0,1} = 0 \qquad r_{n'+1,1} = \begin{cases} r_{n',1} + r_{n',0} & \text{if } s(n'+1) = R \\ 3r_{n',1} + r_{n',0} & \text{if } s(n'+1) = X \\ r_{n',1} & \text{if } s(n'+1) \neq \text{R, X} \end{cases}$$

$$r_{0,2} = 0 \qquad r_{n'+1,2} = \begin{cases} r_{n',2} + r_{n',1} & \text{if } s(n'+1) = W \\ 3r_{n',2} + r_{n',1} & \text{if } s(n'+1) = X \\ r_{n',2} & \text{if } s(n'+1) \neq \text{W, X} \end{cases}$$

$$r_{0,3} = 0 \qquad r_{n'+1,3} = \begin{cases} r_{n',3} + r_{n',2} & \text{if } s(n'+1) = G \\ 3r_{n',3} + r_{n',2} & \text{if } s(n'+1) = X \\ r_{n',3} & \text{if } s(n'+1) \neq \text{G, X} \end{cases}$$

Using this formula, we can easily calculate $r_{n',D'}$ for every value of n' and D', and then return as solution $r_{n,3}$.

Both time and space complexity are $\Theta(n)$, although space complexity can easily be reduced to $\Theta(1)$ by keeping the values of $r_{n',D'}$ for only the current n'.

We also notice that this approach is easily generalizable to arbitrary subsequences of houses and not only RWG.