1 Funzione di Green

Generica:

$$\begin{aligned}V(\mathbf{x}) = &\int \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d\nu(\mathbf{x}')\\&+ \frac{1}{4\pi}\oint (G(\mathbf{x},\mathbf{x}')\frac{\partial V(\mathbf{x}')}{\partial n'}\\&- V(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'})dS(\mathbf{x}')\end{aligned}$$

Condizioni Dirichlet:

$$\begin{aligned}&G_D(\mathbf{x},\mathbf{x}') = 0 \qquad \text{per } \mathbf{x}' \in S\\&V(\mathbf{x}) = \int \rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')d\nu(\mathbf{x}')\\&- \frac{1}{4\pi}\oint V(\mathbf{x}')\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}dS(\mathbf{x}')\end{aligned}$$

Condizioni Neumann:

$$\begin{aligned}&\frac{\partial G_N(\mathbf{x},\mathbf{x}')}{\partial n'} = -\frac{4\pi}{S} \qquad \text{per } \mathbf{x}' \in S\\&V(\mathbf{x}) = \langle V\rangle + \int \rho(\mathbf{x}')G_N(\mathbf{x},\mathbf{x}')d\nu(\mathbf{x}')\\&+ \frac{1}{4\pi}\oint G_N(\mathbf{x},\mathbf{x}')\frac{\partial V(\mathbf{x}')}{\partial n'}dS(\mathbf{x}')\end{aligned}$$

Sfera:

$$\begin{aligned}G(\mathbf{x},\mathbf{x}') &= \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x} - \frac{a^2}{x'^2}\mathbf{x}'\right|}\\\left.\frac{\partial G}{\partial n'}\right|_{x'=a} &= -\frac{x^2-a^2}{a(x^2+a^2-2ax\cos(\gamma))^{3/2}}\end{aligned}$$

2 Operatori differenziali

$$\begin{aligned}\nabla V(\rho,\phi,z) &= \frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}\\\nabla V(r,\theta,\phi) &= \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin(\theta)}\frac{\partial V}{\partial \phi}\hat{\phi}\\\nabla\cdot V(\rho,\phi,z) &= \frac{1}{\rho}\frac{\partial(\rho V_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}\\\nabla\cdot V(r,\theta,\phi) &= \frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(V_\theta\sin(\theta))}{\partial\theta}\\&+ \frac{1}{r\sin(\theta)}\frac{\partial V_\phi}{\partial\phi}\\\nabla\times V(\rho,\phi,z) &= \left(\frac{1}{\rho}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{\rho}\\&+ \left(\frac{\partial V_p}{\partial z} - \frac{\partial V_z}{\partial \rho}\right)\hat{\phi}\\&+ \frac{1}{\rho}\left(\frac{\partial(\rho V_\phi)}{\partial\rho} - \frac{\partial V_\rho}{\partial\phi}\right)\hat{z}\\\nabla^2 V(\rho,\phi,z) &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}\\\nabla^2 V(r,\theta,\phi) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)\\&+ \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial V}{\partial\theta}\right)\\&+ \frac{1}{r^2\sin(\theta)^2}\frac{\partial^2 V}{\partial\phi^2}\end{aligned}$$

3 Polinomi di Legendre

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

$$\int_{-1}^{1} P_{l'}(x)P_l(x)dx = \frac{2}{2l+1}\delta_{l'l}$$

$$\begin{aligned}P_0(x) &= 1\\P_1(x) &= x\\P_2(x) &= \tfrac{1}{2}(3x^2-1)\\P_3(x) &= \tfrac{1}{2}(5x^3-3x)\\P_4(x) &= \tfrac{1}{8}(35x^4-30x^2+3)\end{aligned}$$

4 Armoniche sferiche

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos(\theta))e^{im\phi}$$

$$Y_{l,-m}(\theta,\phi) = (-1)^m\overline{Y_{lm}(\theta,\phi)}$$

$$\begin{aligned}P_l^m(x) &= (-1)^m(1-x^2)^{m/2}\frac{\partial^m}{\partial x^m}P_l(x)\\&= (-1)^m(1-x^2)^{m/2}\frac{\partial^{l+m}}{\partial x^{l+m}}(x^2-1)^l\end{aligned}$$

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta)$$

$$Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{i\phi}$$

$$Y_{20}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}(\tfrac{3}{2}\cos(\theta)^2 - \tfrac{1}{2})$$

$$Y_{21}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin(\theta)\cos(\theta)e^{i\phi}$$

$$Y_{22}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin(\theta)^2 e^{2i\phi}$$

$$Y_{30}(\theta,\phi) = \sqrt{\frac{7}{4\pi}}(\tfrac{5}{2}\cos(\theta)^3 - \tfrac{3}{2}\cos(\theta))$$

$$Y_{31}(\theta,\phi) = -\sqrt{\frac{21}{64\pi}}\sin(\theta)(5\cos(\theta)^2 - 1)e^{i\phi}$$

$$Y_{32}(\theta,\phi) = \sqrt{\frac{105}{32\pi}}\sin(\theta)^2\cos(\theta)e^{2i\phi}$$

$$Y_{33}(\theta,\phi) = -\sqrt{\frac{35}{64\pi}}\sin(\theta)^3 e^{3i\phi}$$