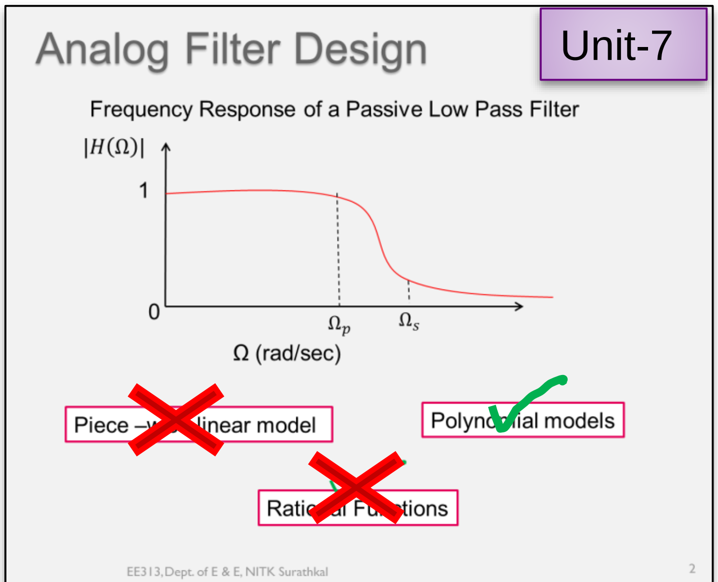
Digital Signal Processing (EE313): Introduction to Finite Impulse Response (FIR) Filters

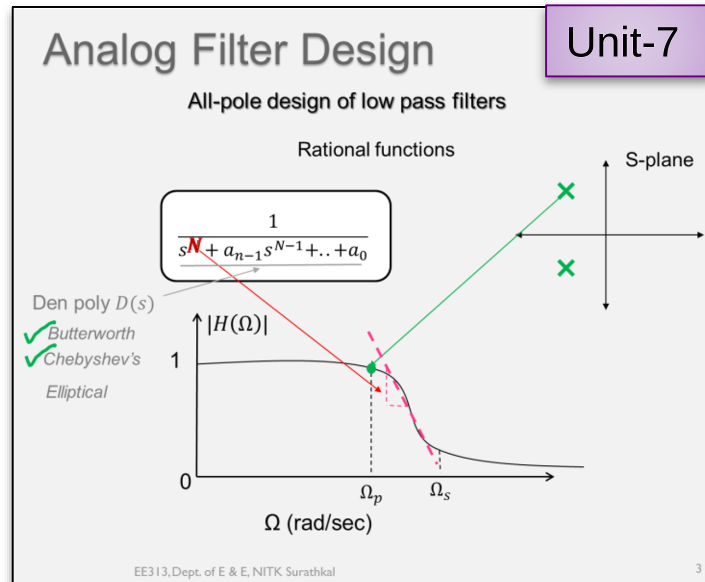
Krishnan C.M.C
Assistant Professor, E&E,
NITK Surathkal

FIR Filters: what are they?

In IIR Filters,

- Butterworth and Chebyshev polynomials
- All pole rational models.



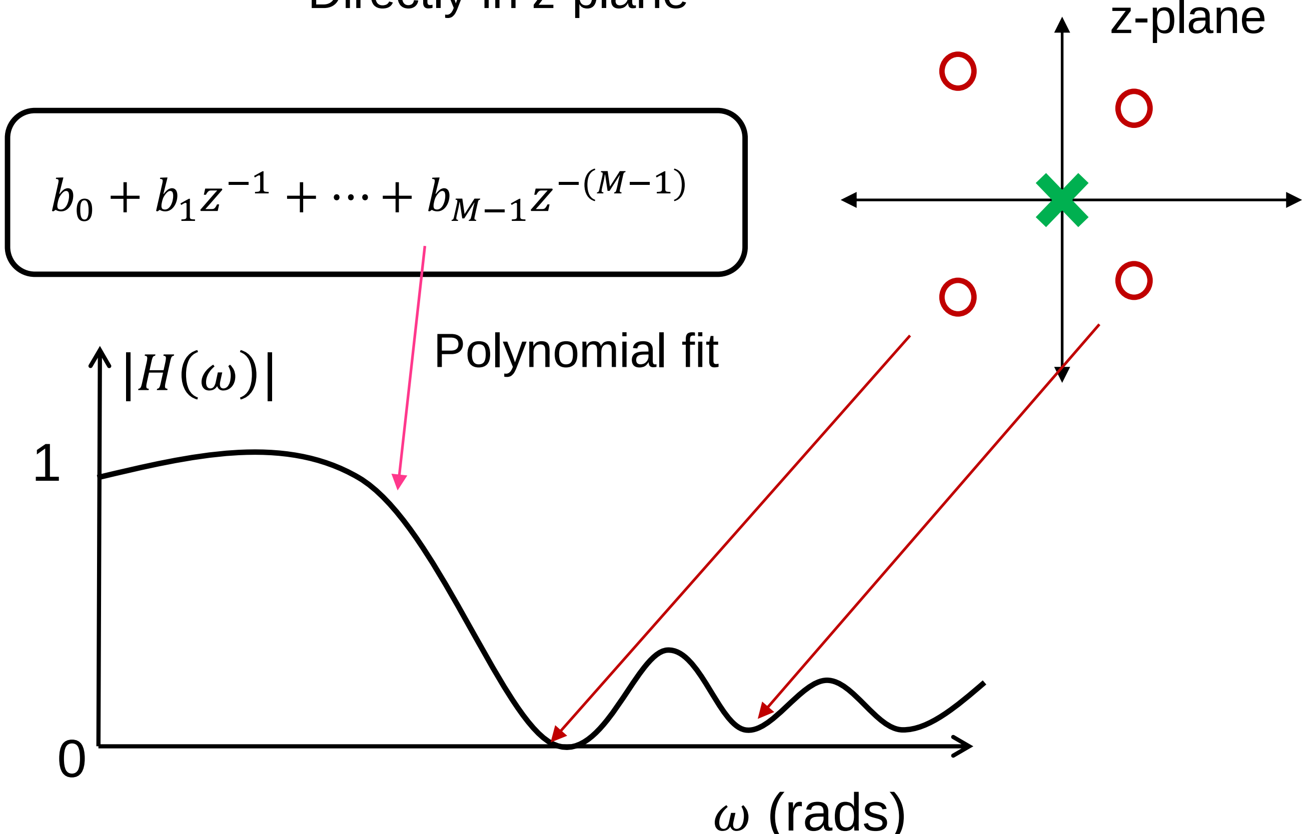


FIR filters: we use all-zero (polynomial) models.

FIR Filters: what are they?

FIR filters: we use all-zero (polynomial) models.

Directly in z-plane



Not strictly all-zero because

$$b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{-(M-1)} \Rightarrow \frac{b_0 z^{(M-1)} + b_1 z^{(M-2)} + \cdots + b_{M-1}}{z^{(M-1)}}$$

Simple FIR Filters

Simple FIR Low pass filter

$$H(z) = \frac{1}{2}(1 + z^{-1})$$

$$h(n) = \left\{\frac{1}{2}, \frac{1}{2}\right\}$$

Bode:

$$H(\omega) = \frac{1}{2}\left(1 + e^{-j\omega}\right) \Rightarrow \underbrace{e^{-j\frac{\omega}{2}} \cos\left(\frac{\omega}{2}\right)}$$

Pseudo magnitude

Mag

$$|H(\omega)| = \left|\cos\left(\frac{\omega}{2}\right)\right|$$

Phase

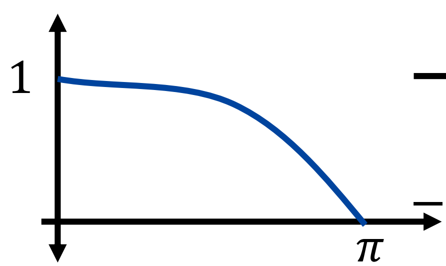
$$\angle H(\omega) = -\frac{\omega}{2} + \angle\left[\cos\left(\frac{\omega}{2}\right)\right]$$

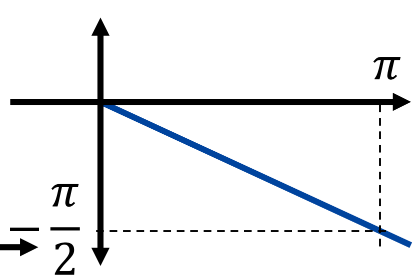
From 0 *to* π

$$0 \text{ if } \cos\left(\frac{\omega}{2}\right) \geq 0$$

$$\pi \text{ if } \cos\left(\frac{\omega}{2}\right) < 0$$

$$\omega_c = \frac{\pi}{2}$$





Difference eqn.:

$$y(n) = \frac{1}{2}\left(x(n) + x(n-1)\right)$$

Simple averaging filter

Pole-zero analysis

Zero at $z = -1$

$\omega = \pi$

Low pass

Simple FIR Filters

Simple FIR High pass filter

$$H(z) = \frac{1}{2}\left(1 - z^{-1}\right)$$

$$h(n) = \left\{\frac{1}{2}, -\frac{1}{2}\right\}$$

Bode:

$$H(\omega) = \frac{1}{2}\left(1 - e^{-j\omega}\right) \Rightarrow j e^{-j\frac{\omega}{2}} \underbrace{\sin\left(\frac{\omega}{2}\right)}_{\text{Pseudo magnitude}}$$

Mag

$$|H(\omega)| = \left|\sin\left(\frac{\omega}{2}\right)\right|$$

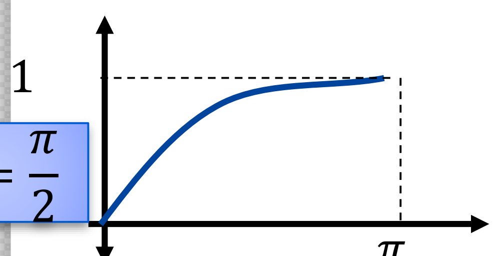
Phase

$$\angle H(\omega) = -\frac{\omega}{2} + \frac{\pi}{2} + \angle\left[\sin\left(\frac{\omega}{2}\right)\right]$$

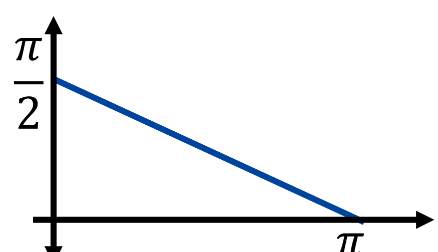
From 0 *to* π

$$0 \text{ if } \sin\left(\frac{\omega}{2}\right) \geq 0$$

$$\pi \text{ if } \sin\left(\frac{\omega}{2}\right) < 0$$



$$\omega_c = \frac{\pi}{2}$$



Difference eqn.:

$$y(n) = \frac{1}{2}\left(x(n) - x(n-1)\right)$$

Finds the change in the signal

Pole-zero analysis

Zero at $z = 1$

$\omega = 0$

High pass

Simple FIR Filters

Simple FIR Band pass filter

Cascade LP and HP

$$H(z) = \frac{1}{2}\left(1 - z^{-2}\right)$$

$$h(n) = \left\{\frac{1}{2}, 0, -\frac{1}{2}\right\}$$

Bode:

$$H(\omega) = \frac{1}{2}\left(1 - e^{-j2\omega}\right) \Rightarrow je^{-j\omega}\underbrace{\sin(\omega)}_{\text{Pseudo magnitude}}$$

Mag $|H(\omega)| = |\sin(\omega)|$

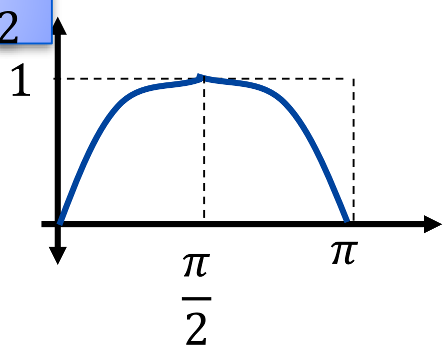
Phase $\angle H(\omega) = -\omega + \frac{\pi}{2} + \angle[\sin(\omega)]$

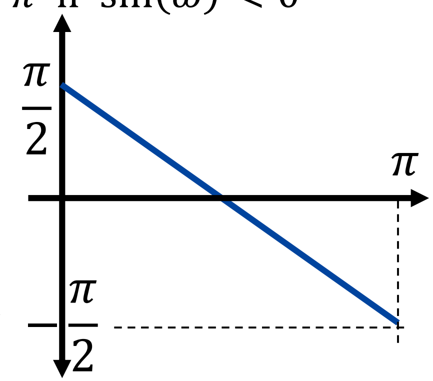
From 0 *to* π

$0 \;\text{ if }\; \sin(\omega) \geq 0$

$\pi \;\text{ if }\; \sin(\omega) < 0$

$\omega_{BW} = \frac{\pi}{2}$





Pole-zero analysis

Zero at $z = -1$ $\omega = \pi$

Zero at $z = 1$ $\omega = 0$

Band pass

Simple FIR Filters

Simple FIR Band reject filter

$$H(z) = \frac{1}{2}(1 + z^{-2})$$

$$h(n) = \left\{\frac{1}{2}, 0, \frac{1}{2}\right\}$$

Bode:

$$H(\omega) = \frac{1}{2}\left(1 + e^{-j2\omega}\right) \Rightarrow \underbrace{e^{-j\omega}\cos(\omega)}_{\text{Pseudo magnitude}}$$

Mag

$$|H(\omega)| = |\cos(\omega)|$$

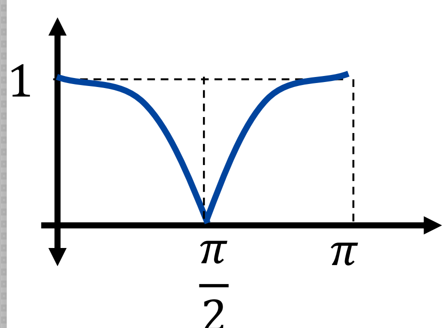
Phase

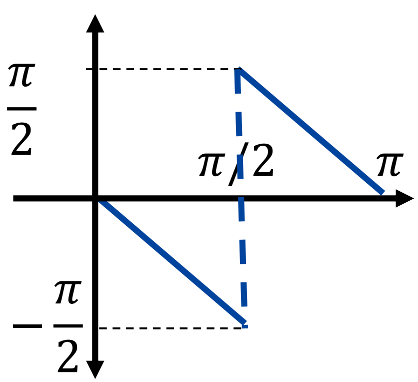
$$\angle H(\omega) = -\omega + \angle[\cos(\omega)]$$

$$\begin{cases} 0 & \text{if } \cos(\omega) \geq 0 \\ \pi & \text{if } \cos(\omega) < 0 \end{cases}$$

From 0 to $\frac{\pi}{2}$

From $\frac{\pi}{2}$ to π





Pole-zero analysis

Zero at $z = j$ — $\omega = \frac{\pi}{2}$

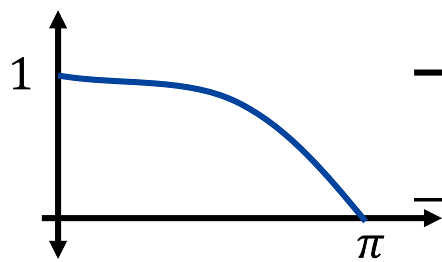
Zero at $z = -j$ — $\omega = -\frac{\pi}{2}$

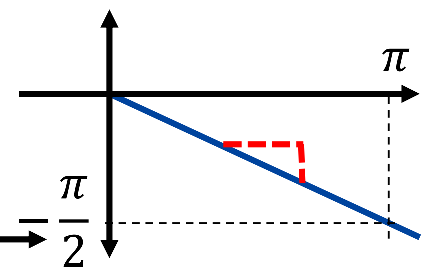
Band reject

Why polynomial approximation?

$$H(\omega) = \frac{1}{2}\left(1 + e^{-j\omega}\right) \Rightarrow e^{-j\frac{\omega}{2}} \cos\left(\frac{\omega}{2}\right)$$

$$t_p = t_g = \frac{1}{2} \text{ samples}$$



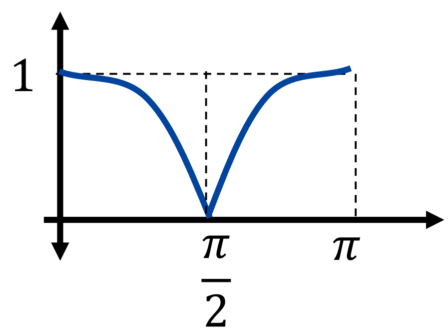


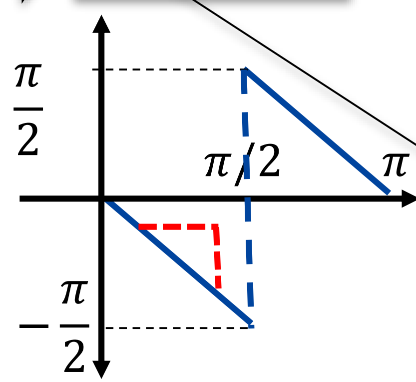
$$\frac{rad}{rad/sample}$$

Observations:

- Poor selectivity
- But linear phase
- Always stable.

$$H(\omega) = \frac{1}{2}\left(1 + e^{-j2\omega}\right) \Rightarrow e^{-j\omega} \cos(\omega)$$





$$t_p = t_g = 1 \text{ sample}$$

Digital Signal Processing (EE313): Introduction to Finite Impulse Response (FIR) Filters: Ensuring Linear phase

Krishnan C.M.C
Assistant Professor, E&E,
NITK Surathkal

Ensuring Linear Phase

- An FIR Filter is described by the difference-equation

$$y(n) = \underbrace{h(0)}_{b_0}x(n) + \underbrace{h(1)}_{b_1}x(n-1) + \dots + \underbrace{h(M-2)}_{b_{M-2}}x(n-M+2) + \underbrace{h(M-1)}_{b_{M-1}}x(n-M+1)$$

- It has a linear phase characteristic provided

Even

$$h(n) = \pm h(M-1-n), n = 0,1,\dots,M-1$$

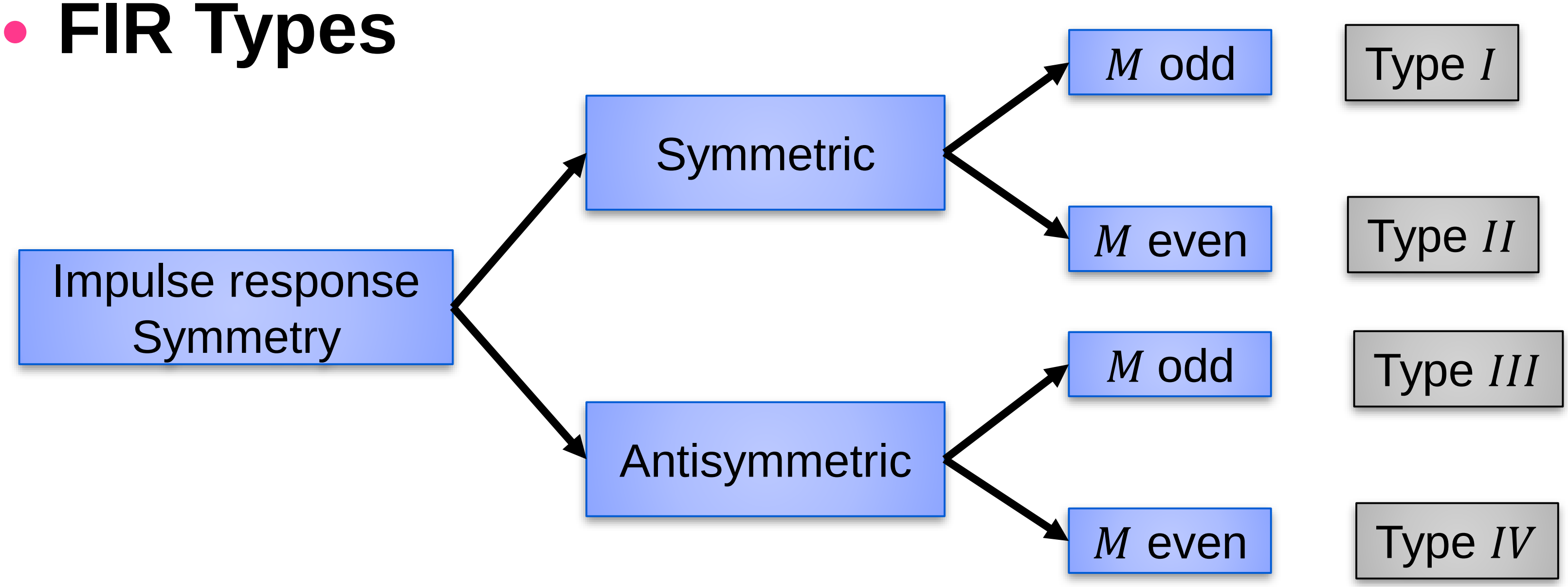
odd

- This means that the FIR filter is of linear phase when the symmetry conditions are enforced on its impulse response

Terminology

- M is the length of the filter
  - As in Proakis (different textbooks follow different conventions)

- **FIR Types**



Symmetry properties:

- The Z-transform of the impulse response is given by:

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \dots + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)}$$

$$h(n) = \pm h(M-1-n), n = 0,1,\dots,M-1$$

$$H(z) = h\left(\frac{M-1}{2}\right)z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[z^{-n} \pm z^{-(M-1-n)}\right]$$

Assume M is odd

Type I or III

Taking $z^{-\left(\frac{M-1}{2}\right)}$ outside

$$H(z) = z^{-\left(\frac{M-1}{2}\right)}\left\{h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[z^{\left(\frac{M-1-2n}{2}\right)} \pm z^{-\left(\frac{M-1-2n}{2}\right)}\right]\right\}$$

Symmetry properties:

If M is odd

Type I or III

$$H(z) = z^{-\left(\frac{M-1}{2}\right)} \left\{ h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} h(n) \left[z^{\left(\frac{M-1-2n}{2}\right)} \pm z^{-\left(\frac{M-1-2n}{2}\right)} \right] \right\}$$

If M is even

Type II or IV

$$H(z) = z^{-\left(\frac{M-1}{2}\right)} \left\{ \sum_{n=0}^{\frac{M}{2}-1} h(n) \left[z^{\left(\frac{M-1-2n}{2}\right)} \pm z^{-\left(\frac{M-1-2n}{2}\right)} \right] \right\}$$

Symmetric response (M odd), Type I

$$H(z) = z^{-\left(\frac{M-1}{2}\right)}\left\{h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[z^{\left(\frac{M-1-2n}{2}\right)} + z^{-\left(\frac{M-1-2n}{2}\right)}\right]\right\}$$

$z = e^{j\omega}$

$$H(e^{j\omega}) = e^{-j\alpha\omega}\left\{h(\alpha) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega(\alpha - n)\right\}$$

$$\alpha = \frac{M-1}{2}$$

Real part

phase part

$$H_r(\omega) = h(\alpha) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega(\alpha - n)$$

$$\theta(\omega) = \begin{cases} -\alpha\omega & if \quad H_r(\omega) \geq 0 \\ -\alpha\omega + \pi & if \quad H_r(\omega) < 0 \end{cases}$$

Not Magnitude

Pseudo-magnitude

Symmetric response (M even), Type II

$$H(z) = z^{-\left(\frac{M-1}{2}\right)}\left\{\sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{\left(\frac{M-1-2n}{2}\right)} + z^{-\left(\frac{M-1-2n}{2}\right)}\right]\right\}$$

$z = e^{j\omega}$

$$H\left(e^{j\omega}\right) = e^{-j\alpha\omega}\left\{2\sum_{n=0}^{\frac{M}{2}-1} h(n)\cos\omega(\alpha - n)\right\}$$

$$\alpha = \frac{M-1}{2}$$

Real part

$$H_r(\omega) = 2\sum_{n=0}^{\frac{M}{2}-1} h(n)\cos\omega(\alpha - n)$$

phase part

$$\theta(\omega) = \begin{cases} -\alpha\omega & if \quad H_r(\omega) \geq 0 \\ -\alpha\omega + \pi & if \quad H_r(\omega) < 0 \end{cases}$$

Antisymmetric response:

If M is odd   Type III

$$H_r(\omega) = 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\omega(\alpha - n)$$

$$\theta(\omega) = \begin{Bmatrix} -\alpha\omega + \frac{\pi}{2} & if \quad H_r(\omega) \geq 0 \\ -\alpha\omega + \frac{3\pi}{2} & if \quad H_r(\omega) < 0 \end{Bmatrix}$$

If M is even   Type IV

$$H_r(\omega) = 2\sum_{n=0}^{\frac{M}{2}-1} h(n)\sin\omega(\alpha - n)$$

$$\theta(\omega) = \begin{Bmatrix} -\alpha\omega + \frac{\pi}{2} & if \quad H_r(\omega) \geq 0 \\ -\alpha\omega + \frac{3\pi}{2} & if \quad H_r(\omega) < 0 \end{Bmatrix}$$

Frequency response $H(\omega)$ for different symmetry conditions on impulse response $h(n)$

$H(\omega)$ for Symmetric $h(n)$

Type I M odd

$$H_r(\omega) = h(\alpha) + 2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \cos \omega(\alpha - n)$$

Type II M even

$$H_r(\omega) = 2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \cos \omega(\alpha - n)$$

$$\theta(\omega) = \begin{cases} -\alpha\omega & if \quad H_r(\omega) \geq 0 \\ -\alpha\omega + \pi & if \quad H_r(\omega) < 0 \end{cases}$$

$H(\omega)$ for Antisymmetric $h(n)$

Type III M odd

$$H_r(\omega) = h(\alpha) + 2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \sin \omega(\alpha - n)$$

Type IV M even

$$H_r(\omega) = 2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \sin \omega(\alpha - n)$$

$$\theta(\omega) = \begin{cases} -\alpha\omega + \dfrac{\pi}{2} & if \quad H_r(\omega) \geq 0 \\ -\alpha\omega + \dfrac{3\pi}{2} & if \quad H_r(\omega) < 0 \end{cases}$$