Hints and Answers : Tutorial Sheet-1

Autumn 2024

1. Apply first the Intermediate Value Theorem, and then the Rolle's Theorem.

2. (a) The conditions of LMVT are not satisfied.
   (b) The conditions of LMVT are not satisfied.
   (c) The conditions of LMVT are not satisfied.

3. (a) $\xi = \frac{\pi}{2}$.
   (b) $\xi = \frac{\sqrt{6}-1}{6}$.

4. Limit $\lim_{h\to 0} \theta$ in each case is $1/2$.

5. (a) Apply LMVT to f on $[1, 2]$. The largest possible value of $f(1)$ is -3.
   (b) Consider $f(x) = x^{\frac{1}{3}}$ on $[27, 28]$. By LMVT, deduce that $3 < \sqrt[3]{28} < 3.037037037...$
   (c) Apply LMVT to f on $\left[x_1, \frac{x_1+x_2}{2}\right]$ and $\left[\frac{x_1+x_2}{2}, x_2\right]$ resectively.

6. (a) Consider $f(x) = \frac{\sin x}{x}$ on $0 < x < \pi/2$. By considering $f'(x)$ on $[\delta, \pi/2]$ for any $0 < \delta < \pi/2$, show that $f(x)$ is a strictly decreasing function on $[\delta, \pi/2]$.
   (b) Apply LMVT to $f(x) = x^n - a^n$ on $[a, b]$.
   (c) Consider $f(x) = \ln(1 + x) - \frac{x}{1+x}$ and $g(x) = x - \ln(1 + x)$ on $[0, \infty)$. Considering their derivatives, test the monotonicity.

7. (a) Let $P = (c, f(c))$. Apply LMVT to f on $[a, c]$ and $[c, b]$ respectively. Then further apply LMVT to $f'(x)$.
   (b) Consider the 2nd function $g(x) = x^2$ to apply CMVT.
   (c) Apply LMVT to f on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ respectively.

8. (a) Consider $g(x) = \begin{vmatrix} f(x) & f(b) \\ \phi(x) & \phi(b) \end{vmatrix}$ and apply LMVT.
   (b) Along with f, consider the 2nd function $g(x)$ as x, x^2 and x^3 separately, and apply CMVT in each case.

9. (a) Consider $f(x) = 1 - \cos x$ and $g(x) = \frac{x^2}{2}$ for all $x \in \mathbb{R}$.

   (b) Consider $g(x) = \frac{f(x)}{x^2}$ and $h(x) = \frac{1}{x^2}$ for all $x \in [a, b]$.

   (c) Consider $f(x) = \ln x$ for all $x > 0$ and $g(x) = \sin^{-1}(x)$ for all $x \in \mathbb{R}$. Apply CMVT on the interval $[x, 1]$ for some $0 < x < 1$.

10. Apply LMVT to $f(x)$ on two intervals $[a, x_0]$ and $[x_0, b]$ respectively, to get two points $c_1 \in (a, x_0)$ and $c_2 \in (x_0, b)$. Next apply LMVT to $f'(x)$ on $[c_1, c_2]$.